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Indeterminate Problems in Greek Primary Education

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Abstract

Indeterminate problems are problems that can be written with κ equations with more than κ unknowns and have been used since ancient times from many civilizations.
Problem solving constitutes a critical part of Mathematics Educations, in which emphasis is given on the Curricula of Mathematics. Open-ended problems may have several correct answers or differed ways of finding the correct answer.
In the present study, the way students of the 5^{th} grade manage an open-ended problem is examined and also elements of the way they solve it are presented.
Keywords: Indeterminate; Open-Ended Problems; Primary Education

1. Introduction

Indeterminate problems are a type of problems, whose theory is an important part of Number theory [1]. When the conditions of an indeterminate problem are such that its solutions may only be integers numbers, then the problem is called Diophantine [2].

In his *Arithmetic* Diophantus of Alexandria (3rd century A.D.) gave special emphasis on indeterminate problem solving, that is to say on problems which can be written with κ equations with more than κ unknowns [3]. For this reason, the branch of Mathematics which sometimes is referred in literature as indeterminate analysis, is also called Diophantine Analysis [4].

The solution of diophantine equations of first and of second degree is found in several Indian works, such as *Aryabhatiya* (5th century A.D.) In their works, Brahmagupta (7th century) and Bhaskara II (12th century) gave integer solutions of Diophantine equations of the first and of the second degree. Problems of linear equations with one or two unknowns are also contained in the Bakhshali manuscript, which was found in 1881 [5].

Problems of indeterminate analysis are also found in Chinese works, for example in *Sun-Tsu Suan-ching* (65 A.D) [6] while in the *Classic Arithmetic of Chang-chiu-chien* (468 A.D.) is emerged the well-known 'Hundred Fowls Problem' [7]. In Middle Ages we find many similar problems in Indian, Arabic and European works.

Specifically, Alcuin (703-854 A.D.) in *Propositiones ad Acuendos Juvenes* (775 A.D.) mentions the following indeterminate problem:

"A certain master of a household has 100 people in his service to whom he proposes to distribute 100 bushels of corn: 3 bushels per man, 2 bushels per woman and $\frac{1}{2}$ bushel per child. Can anyone say how many men, women and children there were?" [8].

One of the solutions that can be given in Alcuin;s problem is the following:
Let the number of men be *x*, the number of women *y* and that of children *z*. Then *x,y,z* $\in Z^{+}$, therefore we have:

$$\begin{cases} x+y+z=100 \\ 3x+2y+\frac{1}{2}z=100 \end{cases} \leftrightarrow \quad \begin{cases} z=100-x-y \\ 6x+4y+100-x-y=200 \end{cases} \leftrightarrow$$

$\begin{cases} z=100-x-y \\ 5x+3y=100 \end{cases}$ (1). We have $5x+3y=100 \leftrightarrow x=20-\frac{3y}{5}$. Since $x \in Z^{+}$, we conclude that $\frac{y}{5}=t \leftrightarrow y=5t$ with $t \in Z^{+}$ (1) because y>0.

If $y=5t,$ then $x=20-3t$ and $z=100-20+3t-5t \leftrightarrow z=80-2t.$ Since *x*>0, then $20-3t>0 \leftrightarrow t<\frac{20}{3}$ (2). From (1) and (2) we obtain t=1, 2, 3, 4, 5, 6.

Therefore, for t=1,2,3,4,5,6, from the relations $x = 20 - 3t$, $y = 5t$ and $z = 80 - 2t$, the corresponding solutions of the following table (Table 1) are given:

t	1	2	3	4	5	6
x	17	14	11	8	5	2
y	5	10	15	20	25	30
z	78	76	74	72	70	68

Table 1. The six solutions of Alcuin's indeterminate problem

As it follows from the above, there are six different solutions for the previous problem: (17, 5, 78), (14, 10, 76), (11, 15, 74), (8, 20, 72), (5, 25, 70), (2, 30, 68). Alcuin gave only one of them, the (11, 15, 74) [8].

Leonardo of Pisa (approximately 1180-1250), one of the most important mathematicians of the Middle Ages [4], in the work *Liber Abaci* mentions the next problem:

"... a partridge is worth 3 denari, a pigeon 2, a turtledove $\frac{1}{2}$ denaro, a sparrow $\frac{1}{4}$ denaro, and I wish from them 30 birds for 30 denari" [9], which can be solved as follows:

Let *x* be the partridges, *y* the pigeons, *z* the turtledoves and *w* the sparrows. Then *x,y,z,w* ∈ Z^+ , therefore we have:

$$\begin{cases} x+y+z+w=30 \\ 3x+2y+\frac{1}{2}z+\frac{1}{4}w=30 \end{cases} \leftrightarrow \begin{cases} w=30-x-y-z \\ 12x+8y+2z+30-x-y-z=120 \end{cases} \leftrightarrow$$

$$\begin{cases} w=30-x-y-z \\ 11x+7y+z=90 \end{cases} \leftrightarrow \begin{cases} w=30-x-y-z \\ z=90-11x-7y \end{cases}$$

Because z>0, it will be 90-11x-7y>0 ↔ 11x+7y<90 (1). Since *x,y,z,w* ∈ Z^+, from (1) we have $z = 90 - 11x - 7y$ and την w=30-x-y-z. The following table (Table 2) gives the corresponding values of x,y,z,w:

x	1	2	3	4	6	7
y	11	9	8	6	3	1
z	2	5	1	4	3	6
w	16	14	18	16	18	16

Table 2.The six solutions of Leonardo's indeterminate problem

In conclusion, we may have six different solutions: (1, 11, 2, 16), (2, 9, 5, 14), (3, 8, 1, 18), (4, 6, 4, 16), (6, 3, 3, 18), (7, 1, 6, 16) Leonardo mentions in Liber abaci mentions only the following two: (7, 1, 6, 16), (4, 6, 4, 16) [9].

In his *Elements of Algebra* (1770), L.Euler on 'Part II. Containing the Analysis of Indeterminate Quantities, Chapter I.Of the Resolutions of Equations of the First Degree, which contain more than one Unknown Quantity' defines Indeterminate Analysis as follows:

'When a question [...] does not furnish as many equations as there are unknown quantities to be determined, some of these must remain undetermined and depend on our will; for which reason, some questions are said to be indeterminate; forming the subject of a particular branch of algebra, which is called Indeterminate Analysis' [10].

Subsequently, he solves that type of indeterminate problems following a method called *Regula Coeci* ('Blind Man's Rule') in Chapter II which is entitled 'Of the Rule which is called Regula Coeci, for determining by means of Two Equations, Three or more Unknown Quantities' [10], [2].

One of the indeterminate problems Euler mentions is the following:

"*Thirty persons, men, women, and children, spend 50 crowns in a tavern; the share of a man is 3 crowns, that of a woman 2 crowns, and that of a child is 1 crown; how many persons were there of each class?"* [10].

One of its solution is the following:

Let *x* be the men, *y* the women, and *z* the children. Then *x,y,z* $\in Z^{+}$, therefore we have:

$$\begin{cases} x+y+z=30 \\ 3x+2y+Z=50 \end{cases} \leftrightarrow \begin{cases} z=30-x-y \\ 3x+2y+30-x-y=50 \end{cases} \leftrightarrow \begin{cases} z=30-x-y \\ 2x+y=20 \end{cases}$$

(1).

$\leftrightarrow \begin{cases} z=30-x-y \\ x=10-\frac{y}{2} \end{cases}$. Therefore, if we make $\frac{y}{2}=t \leftrightarrow y=2t$, then t>0, since y>0. For $y=2t$ from (1) we have: $x=10-t$ and *z=20 –t.* Because t>0 and *x>0* ↔ *10-t>0* ↔ *t<10*, it will be t=1, 2, 3, 4, 5, 6 , 7, 8, 9.

For t= 1, 2, 3, 4, 5, 6 , 7, 8, 9, x=10 –t, y=2t and z=20-t, therefore the problem has nine different solutions: (9, 2, 19), (8, 4, 18), (7, 6, 17), (6, 8, 16), (5, 10, 15), (4, 12, 14), (3, 14, 13), (2, 16, 12), (1, 18, 11).

Other important mathematicians involved with indeterminate problem solving was J.H.Poincaré (1854-1912), who published a paper on number theory which was referred to the study of Diophantine equations as well as D.Hilbert (1862-1943) who dealt with the solvability's problem of Diophantine equations [4].

2. Problem Solving in Education

In his effort to define 'problem', Schoenfeld (1985) mentions the relativeness of the term, since it does not consist an inherent property in a mathematical task. "The same tasks that call for significant efforts for some students may well be routine exercises for others and answering them may just be a matter of recall for a given mathematician... being a 'problem' is particular relationship between the individual and the task that makes the task a problem for that person... if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem' [11]. For the above reason he used the following definition of the *Oxford English Dictionary*: Problem. A doubtful or difficult question; a matter of inquiry, discussion, or thought; a question that exercises the mind' [11].

According to G.Polya, mathematical problem solving is a process which can be divided in four phases: understanding the problem, devising a plan, carrying out the plan and looking back [12]. Solving a mathematical problem involves four serially initiative cognitive processes: translating, integrating, planning and execution [13]. Mathematical problem solving includes several activities, for example doing word problems, creating patterns, interpreting figures, developing geometric constructions, proving theorems, etc. Every mathematical activity can also be considered as a problem solving [14].

Students, using one or more different strategies, such as applying a solution rubric that has been presented in class, using logical-mathematical reasoning or a trial-and-error approach or a 'guess and check' approach [15], are led to a correct or an incorrect answer, which is a part of a larger set of cognitive processes [16].

According to NCSM (1989) *problem solving* can be understood as "*a process where previously acquired data are used in a new and unknown situation*" [17]. The National Council of Teachers of Mathematics includes problem solving in the process standards referring that: Solving problems is not only a goal of learning mathematics but also a major means of doing so. It is an integral part of mathematics, not an isolated piece of the mathematics program. Students require frequent opportunities to formulate, grapple with, and solve complex problems that involve a significant amount of effort. They are to be encouraged to reflect on their thinking during the problem-solving process so they can apply and adapt the strategies they develop to other problems and in other contexts. By solving mathematical problems, students acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that serve them well outside the mathematics classroom'[18].

In the Interdisciplinary Unified Context of Mathematics Curriculum (2003) of Greece, problem solving is one of the cognitive content's axes for all six classes of Primary Education, which aims: 'Students to inquire a situation, to construct, questions and problems based on specific data, to reformulate the same problem in a different way, to identify and describe similar situations, to investigate open problematic situations, to use mathematics in daily life and to become familiar with new technologies'[19]. "

Regarding the New Curriculum in Compulsory Education (2011) which is applied along with the previous one, problem solving is a main goal in Mathematics and is included in specific mathematical operations' development.

Particularly is mentioned that: ‘Problem solving constitutes the core of the development process of mathematical knowledge and mathematical thinking. Students learn better when they themselves have the opportunity to investigate mathematical ideas through problems solving, since their involvement in this procedure helps them to gradually ‘construct’ mathematical knowledge deepening conceptually in it and realizing not only its functional aspect but also its cultural and historical dimension’ [20].

3. Open-Ended Problems in Primary Education

As Euler characteristically mentions ‘this part of analysis (he means the indeterminate analysis) frequently requires artifices entirely appropriate to it, which are of great service in exercising the judgment of beginners, and giving them dexterity in calculation’ [10]. Although the majority of problems that are traditionally used in teaching Mathematics require students to give a predefined, specific and unique answer, the last years the approach of open-ended problems was developed [21]. 'Tasks are said to be open, if their starting or goal situation is not exactly given’[22]. The type of open-ended problems may have many solutions or different ways of finding the correct answer [23].

Some types of open-ended problems are: investigations (a starting point is given), problem posing, real-life situations (with roots in the everyday life), projects (larger study entities, which require independent working), problem fields (a collection of contextually connected problems), problems without a question, and problem variations (”what-if”-method) [24]. The last years, several studies have shown the effective use of these problems in students’ creativity and mathematical thought [25], [26]. Nowadays, open-ended problems are considered a useful tool in the development of Mathematics teaching at schools, since, apart from their creativity and mathematical thinking cultivation, their flexibility and freedom allow all students to participate in solving them. For that reason they are offered for differentiated teaching [27]. In addition, the introduction of these types of problems in classroom brings mathematical education closer to real life Mathematics [28].

Open-ended problems, however, seem to be difficulty manageable from teachers in classroom. ‘Teachers do not usually possess either the tools to evaluate the work of the different students or the tools for promoting higher levels of problem solving. There are also some other disadvantages such as difficulty of successful problem posing, difficulty of developing meaningful problem situations, and difficulty of summarizing the lesson’ [29].

Open-ended problems are found in Greek school textbooks of Primary Education from the first grade [30], however not frequently.

4. Research

Research took place in a set of 115 students of the fifth grade in five different areas of Attica. Mathematics' teachers of those five classrooms, when they read the worksheet, they mentioned that their students: a. were already familiar from the first grade to solve that type of problems, b. were usually give one correct answer, since similar problems are included in school textbooks. They also mentioned that they were rewarding their students who were giving one correct answer, as well as those who were giving more than one answer, in the same way as when they were solving problems in more than one way. However, all five teachers of the research, when asked if and whether they have managed with their students the whole set of the solutions of such type of problem, answered that they had never done something like that. The four of them who had 5,7,10 and 25 years of teaching experience, said that they considered that something like that was not important since their goal was many students to be rewarded for solving the problem. Only one of them, with 20 years of teaching experience, said that, although he intended to do so, he did not have the time. However, every time he was trying to record on the board and to discuss in his classroom the different answers of his students.

A two-page worksheet was given to 115 students, which included the following problem along with the picture of the five notes:

"I find different ways with which I can exchange a 100€ note with notes of 50€, 20 €, 10€ and 5 €"

Figure 1. The problem and the pictures of the worksheet.

The worksheet was divided in four parts, in each of which respectively were requested the answers for the exchange of notes of: A. one value, B. two different values, C. three different values and D four different values.

A discussion was preceded about whether students have met the specific problem in their daily life. Out of the 115 students, 93 replied affirmatively. The 68 mentioned that they had got as a gift a note of 100€ and they had to exchange it with notes of lower value. Other 25 students answered that they had also got as a gift a note of 100€ but they gave it to their parents because it was of great value and it was not possible for them to use it. Only 22 students said that they had never seen a note of 100€ before. However, all of them showed great interest in solving the problem, something that shows that, when dealing with open ended problems associated with real life, students are encouraged to be independent thinkers, to share, reflect on and value alternative responses, to be excited about learning, to be responsible for their learning and to complete tasks reflective of their true abilities [31].

5. Research's Results

Subsequently, the four tables (Table A-D) corresponding to the four parts in which the worksheet was divided, which they include the solutions of the problem, as well as the answers of the students, are presented.

Table A. Exchange with notes of one value					
Students' answers					
Solution's number	Value of note	Algebraic solution	Correct	Incorrect	No answer
1st	50 €	2x =50=100	115	0	0
2nd	20€	5x20=100	111	3	0
3rd	10 €	10x10=100	114	1	0
4th	5 €	20x5=100	93	17	5

In the case of the exchange of one value's notes, of the 115 students 108 gave answers of the type: '...notes of ...€', as well as in the cases of the exchange of notes of two, three or four different values '...notes of ...€ +...notes of ...€'. The other 7 students gave answers of the type: '50€+50€=100€'.

The above show that students' majority was able to calculate mentally. Mental calculation is the process in which the person accurately estimates the arithmetic result without the use of external means, for example specific objects, pencil and paper, etc. [32]. In the last decades special emphasis has been put, not only in the in the curriculum but also in the school textbooks, on mental calculations and improvised strategies. In parallel, on international level a reluctance to teach specific traditional algorithms in instructive textbooks is observed [33], [34].

All the students of the sample wrote correctly the first answer of Table A. Regarding the second answer, 111 wrote it correctly and the other three gave an incorrect answer. Specifically, two of them wrote" '3 notes of 20€' and the one: '4 notes of 20€'. This shows that these students were not able to calculate that the exchanges they suggest were giving 60€ and 80€ respectively. 114 students wrote correctly the third answer while one of them, using notes of two different values, answered: '1 note of 50€ and 5 notes of 10€' (that is the 5th answer of Table B). The fourth answer wrote 93 students while 5 students did not answer at all. Of the 17 students who answered incorrectly, 12 answered '3 notes of

20€' and 5: '20 notes of 20€'. Answers like the last one, which is inappropriate in the context of solving the problem were included in incorrect ones [29].

In all four answers related to the exchange with notes of one value, 42 of the 93 students who wrote correctly all these four answers, followed descending order regarding to notes; value, as given in the worksheet. Concerning the others, 5 began with the notes of 5€ and followed the descending order of the notes' value. The rest 46 wrote the answers: 1st, 3rd, 2nd and 4th, which shows that the calculations with notes of value: 50€, 10€, 20 and finally of 5 €, were easier to them. It is also interesting that 17 students wrote a 5th solution: '1 note of 100€', which shows that they probably did not read carefully the problem.

Table B. Exchange with notes of two different values					
Students' answers					
Solution's number	Value of note	Algebraic solution	Correct	Incorrect	No answer
5th	50 € and 10 €	1x50+5x10=100	95	16	4
6th	50 € and 5 €	1x50+10x5=100	75	25	15
7th	20 € and 10 €	1x20+8x10=100	93	0	42
8th		2x20+6x10=100	86	5	24
9th		3x20+4x10=100	54	0	61
10th		4x20+2x10=100	63	0	52
11th	20 € and 5 €	1x20+16x5=100	65	0	50
12th		2x20+12x5=100	39	0	76
13th		3x20+8x5=100	48	0	67
14th		4x20+4x5=100	47	0	68
15th	10 € and 5 €	1x10+18x5=100	63	3	49
16th		2x10+16x5=100	52	0	63
17th		3x10+14x5=100	33	0	82
18th		4x10+12x5=100	45	0	70
19th		5x10+10x5=100	53	0	62
20th		6x10+8x5=100	51	0	64
21th		7x10+6x5=100	57	0	58
22th		8x10+4x5=100	50	0	65
23th		9x10+2x5=100	46	0	69

Regarding the exchange with notes with two different values, five combinations are distinguished: i.50 € and 10 €, ii. 50 € and 5 €, iii. 20 € and 10 €, iv. 20 € and 5 € and v. 10 € and 5 €. The combination of the notes of 50€ and 20€ does not constitute solution of the problem. It is interesting that there was not any of the students who wrote it. As far as the students answers are concerned, it seems that the majority of them start every combination taking the highest value note once and calculate accordingly that of the lower's. As it can be seen from Table B., students found more correct answers for the combinations involving notes of high value (i.50 € and 10 €, ii. 50 € and 5 €, iii. 20 € and 10 €). Almost half of the students did not write any answer for the combination of 10 € and 5€. Many students, having found one correct answer for each combination, which was usually the one with one note of the higher value, afterwards they did not try to find more answers.

Table C. Exchange with notes of three different values					
Students' answers					
Solution's number	Value of note	Algebraic solution	Correct	Incorrect	No answer
24th	50 € and 20 € and 10 €	1x50+1x20+3x10=100	68	12	35
25th		1x50+2x20+1x10=100	75	9	31
26th	50 € and 20 € and 5 €	1x50+1x20+6x5=100	49	0	66
27th		1x50+2x20+2x5=100	53	0	62
28th	50 € and 10 € and 5 €	1x50+1x10+8x5=100	36	0	79
29th		1x50+2x10+6x5=100	36	0	79
30th		1x50+3x10+4x5=100	29	0	86
31th		1x50+4x10+2x5=100	19	0	96
32th	20 € and 10€ and 5€	1x20+7x10+2x5=100	24	0	91
33th		1x20+6x10+4x5=100	15	0	100
34th		1x20+5x10+6x5=100	15	0	100
35th		1x20+4x10+8x5=100	24	0	91
36th		1x20+3x10+10x5=100	34	0	81
37th		1x20+2x10+12x5=100	15	0	100
38th		1x20+1x10+14x5=100	15	0	100
39th		2x20+5x10+2x5=100	45	0	70
40th		2x20+4x10+4x5=100	21	0	94
41th		2x20+3x10+6x5=100	9	0	106
42th		2x20+2x10+8x5=100	21	0	94
43th		2x20+1x10+10x5=100	21	0	94
44th		3x20+3x10+2x5=100	21	0	94
45th		3x20+2x10+4x5=100	34	0	84
46th		3x20+1x10+6x5=100	9	0	106
47th		4x20+1x10+2x5=100	39	0	76

Regarding the wrong answers, they were combinations of notes not of two but of one value or there were inappropriate. Furthermore, there were several students who repeated the same answer twice, either as it was or vice versa, as for example: '4 n.of 20€+2n.of 10€ along with 2n.of 10€+4n.of 20€.

Concerning the exchange with notes of three different values, four combinations are distinguished: i.50 € and 20 € and 10 €, ii. 50 € and 20 € and 5 €, 50 € and 10 € and 5 € and iv. 20 € and 10 and 5€. Students' answers show that the majority of them start each combination by taking the note of the highest value once, the largest number of the note of the immediate lower value and calculating that of the lowest accordingly. As it can be seen from Table C., students found more correct answers for the combinations involving high-value notes (i.50 €, 20€ and 10 €, ii. 50 €, 20€ and 5 € and iii. 50 €, 10 € and 5 €). Only 45 out of the 115 students gave at least one answer to the combination of the three smaller values (20€ and 10€ and 5€). The smaller was the value of notes they had to combine the fewer students was able to answer correctly. Many students, after having found one correct answer for the exchange with notes of three different values, did not go further. From their written results one can understand that they did not work systematically but they wrote correct answers randomly. Nevertheless, there were 9 students who worked systematically and wrote all the correct answers. Regarding the mistakes, these were relative only to the first combination since they combined notes not of three different values but of two. There were also several students who repeated the same answer.

Table D. Exchange with notes of four different values

Students' answers					
Solution's number	Value of note	Algebraic solution	Correct	Incorrect	No answer
48th	50€, 20€, 10€ and 5€	1x50+1x20+1x10+4x5=100	81	31	3
49th		1x50+1x20+2x10+2x5=100	70	29	16

In the case of the exchange with notes of four different values, 81 students wrote the 48th answer and 70 the 49th. It seems that these students were able to discern that this exchange was needed only one note of 50€ and one of 20€ was necessary.

Furthermore, it was probably more difficult to discern that they could then use either one or two notes of 10€ and two or four notes of 5€ respectively. In addition, students who did not write the 49th answer were more than those who did not write the 49th. From the wrong answers, 29 of students used notes of the same value twice, (as for example, '2n.of 20€+1n.of 10€+1n.of 20€+2n.of 5€'), that is they used notes with three different values. 2 of those students answered: ‘1n.of 50€+10n.of 5€’.

6. Conclusions

Indeterminate analysis’ problems are found in Primary Education and are included in open-ended problems, that is to say in problems amenable to many correct answers.

In the present research an open-ended problem of exchanging notes was given to students of the fifth grade of Elementary School. It was found that students showed a great interest in solving the problem, as it was related to real life situations and experiences of most students. All students participated in its solution and gave more than one correct answers in each of the four main different combinations of notes according to their abilities.

Most of student answers were the result of mental calculations. In a few cases they used formal algorithms especially addition. The different answers and the different way of approaching the problem showed that problems of that type give students the opportunity to promote their mathematical thinking and creativity. It is also of great interest the development of strategies students of this age used so as to find all of the correct answers and even more without such a problem’s management previously preceded in the past by teachers. However, there is no research data on whether Elementary School teachers in Greece identify that many of these open-ended problems in school textbooks are indeterminate problems.

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