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Quasi-Uniformity on BL-algebras

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Abstract

In this paper, by using the notation of filter in a BL-algebra A, we introduce the quasi-uniformity Q and uniformity Q^* on A. Then we make the topologies $T(Q)$ and $T(Q^*)$ on A and show that $(A, \wedge, \vee, \odot, T(Q))$ is a compact connected topological BL-algebra and $(A, T(Q^*))$ is a topological BL-algebra. Also we study Q^*-cauchy filters and minimal Q^*-filters on BL-algebra A and prove that the bi-completion $(\widetilde{A}, \widetilde{Q})$ of quasi-uniform BL-algebra (A, Q) is a topological BL-algebra.

2010 MSC: 06B10, 03G10.
Keywords : BL-algebra, (semi)topological BL-algebra, filter, Quasi-uniforme space, Bicompletion

1 Introduction

BL-algebras have been introduced by Hájek [11] in order to investigate many-valued logic by algebraic means. His motivations for introducing BL-algebras

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were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general" many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0,1]. In 1973, André Weil [24] introduced the concept of a uniform space as a generalization of the concept of a metric space in which many non-topological invariant can be defined. This concept of uniformity fits naturally in the study of topological groups. The study of quasi-uniformities started in 1948 with Nachbin's investigations on uniform preordered spaces. In 1960, Á. Csaszar introduced quasi-uniform spaces and showed that every topological space is quasi-uniformizable. This result established an interesting analogy between metrizable spaces and general topological spaces. Just as a metrizable space can be studied with reference to particular compatible metric(s), a topological space can be studied with reference to particular compatible quasi-uniformity(ies). In this and some other respects, a quasi-uniformity is a more natural generalization of a metric than is a uniformity. Quasi-uniform structures were also studied in algebraic structures. In particular the study of paratopological groups and asymmetrically normed linear spaces with the help of quasi-uniformities is well known. See for example, [17], [18], [19], [20]. In the last ten years many mathematicians have studied properties of BL-algebras endowed with a topology. For example A. Di Nola and L. Leustean [9] studied compact representations of BL-algebras, L. C. Ciungu [7] investigated some concepts of convergence in the class of perfect BL-algebras, J. Mi Ko and Y. C. Kim [21] studied relationships between closure operators and BL-algebras.

In [2] and [4] we study (semi)topological BL-algebras and metrizability on BL-algebras. We showed that continuity the operations $\odot$ and $\rightarrow$ imply continuity $\wedge$ and $\vee$. Also, we found some conditions under which a locally compact topological BL-algebra become metrizable. But in there we can not answer some questions, for example:

(*i*) Is there a topology $\mathcal{U}$ on BL-algebra A such that $(A,\mathcal{U})$ be a (semi)topological BL-algebra?

(*ii*) Is there a topology $\mathcal{U}$ on a BL-algebra A such that $(A,\mathcal{U})$ be a compact connected topological BL-algebra?

(*iii*) Is there a topological BL-algebra $(A,\mathcal{U})$ such that T_0, T_1 and T_2 spaces be equivalent?

(*iv*) If $(A,\rightarrow,\mathcal{U})$ is a semitopological BL-algebra, is there a topology $\mathcal{V}$ coarsere than $\mathcal{U}$ or finer than $\mathcal{U}$ such that $(A,\mathcal{V})$ be a (semi)topological

BL-algebra?

Now in this paper, we answer to some above questions and get some interesting results as mentioned in abstract.

2 Preliminary

Recall that a set X with a family $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of its subsets is called a *topological space*, denoted by $(X,\mathcal{U})$, if $X, \emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of X and the complement of $X \in \mathcal{U}$, that is $X \setminus U$, is said to be a *closed set*. If B is a subset of X, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subset P of X is said to be a *neighborhood* of $x \in X$, if there exists an open set U such that $x \in U \subseteq P$. A subfamily $\{U_\alpha : \alpha \in J\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$ there exists an $\alpha \in J$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is a union of members of $\{U_\alpha\}$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in X. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a $fundamental\ system$ of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists a V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$. $(X,\mathcal{U})$ is said to be *compact*, if each open covering of X is reducible to a finite open covering. Also $(X,\mathcal{U})$ is said to be *disconnected* if there are two nonempty, disjoint, open subsets $U, V \subseteq X$ such that $X = U \cup V$, and connected otherwise. The maximal connected subset containing a point of X is called the *component* of that point. Topological space $(X,\mathcal{U})$ is said to be:
(i) T_0 if for each $x \neq y \in X$, there is one in an open set excluding the other,
(ii) T_1 if for each $x \neq y \in X$, each are in an open set not containing the other,
(iii) T_2 if for each $x \neq y \in X$, both are in two disjoint open set.(See [1])

Definition 2.1. [1] Let $(A, *)$ be an algebra of type 2 and $\mathcal{U}$ be a topology on A. Then $\mathcal{A} = (A, *, \mathcal{U})$ is called a
(i) *left (right) topological algebra* if for all $a \in A$, the map $*_a : A \to A$ is defined by $x \to a * x$ ($x \to x * a$) is continuous, or equivalently, for any x in A and any open set U of $a * x$ $(x * a)$, there exists an open set V of x such that $a * V \subseteq U$ $(V * a \subseteq U)$.
(ii) *semitopological algebra* if $\mathcal{A}$ is a right and left topological algebra.
(iii) *topological algebra* if the operation $*$ is continuous, or equivalently, if for any x, y in A and any open set (neighborhood) W of $x * y$, there exist two open sets (neighborhoods) U and V of x and y, respectively, such that $U * V \subseteq W$.

Proposition 2.2. [1] Let $(A, *)$ be a commutative algebra of type 2 and $\mathcal{U}$ be a topology on A. Then right and left topological algebras are equivalent. Moreover, $(A, *, \mathcal{U})$ is a semitopological algebra if and only if it is right or left topological algebra.

Definition 2.3. [1] Let A be a nonempty set and $\{*_i\}_{i\in I}$ be a family of operations of type 2 on A and $\mathcal{U}$ be a topology on A. Then
(i) $(A, \{*_i\}_{i\in I}, \mathcal{U})$ is a right(left) topological algebra if for any $i \in I$, $(A, *_i, \mathcal{U})$ is a right (left) topological algebra.
(ii) $(A, \{*_i\}_{i\in I}, \mathcal{U})$ is a semitopological (topological) algebra if for all $i \in I$, $(A, *_i, \mathcal{U})$ is a semitopological (topological) algebra.

Definition 2.4. [11] A *BL-algebra* is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid and for any $a, b, c \in A$,

$$c \leq a \to b \Leftrightarrow a \odot c \leq b, \quad a \wedge b = a \odot (a \to b), \quad (a \to b) \vee (b \to a) = 1.$$

Let A be a BL-algebra. We define $a' = a \to 0$ and denote $(a')'$ by a''. The map $c : A \to A$ by $c(a) = a'$, for any $a \in A$, is called the *negation map*. Also, we define $a^0 = 1$ and $a^n = a^{n-1} \odot a$, for all natural numbers n.

Example 2.5. *[11] (i) Let "$\odot$" and "$\to$" on the real unit interval $I = [0, 1]$ be defined as follows:*

$$x \odot y = \min\{x, y\} \qquad x \to y = \begin{cases} 1 & , x \leq y, \\ y & , otherwise. \end{cases}$$

Then $\mathcal{I} = (I, \min, \max, \odot, \to, 0, 1)$ is a BL-algebra.

(ii) Let $\odot$ be the usual multiplication of real numbers on the unit interval $I = [0, 1]$ and $x \to y = 1$ iff, $x \leq y$ and y/x otherwise. Then $\mathcal{I} = (I, \min, \max, \odot, \to, 0, 1)$ is a BL-algebra.

Proposition 2.6. [11] Let A be a BL-algebra. The following properties hold.
(B_1) $x \odot y \leq x, y$ and $x \odot 0 = 0$,
(B_2) $x \leq y$ implies $x \odot z \leq y \odot z$,
(B_3) $x \leq y$ iff $x \to y = 1$,
(B_4) $1 \to x = x$, $1 \odot x = x$,
(B_5) $y \leq x \to y$,
(B_6) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,
(B_7) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(B_8) $x \leq y \Rightarrow x \to z \geq y \to z,\ z \to x \leq z \to y$,

(B_9) $x \to y \leq (z \to x) \to (z \to y)$,
(B_{10}) $x \to y \leq (y \to z) \to (x \to z)$,
(B_{11}) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,
(B_{12}) $(y \wedge z) \to x = (y \to x) \vee (z \to x)$,
(B_{13}) $(y \vee z) \to x = (y \to x) \wedge (z \to x)$,
(B_{14}) $x \to y \leq x \odot z \to y \odot z$,
(B_{15}) $(x \to y) \odot (y \to z) \leq x \to z$,
(B_{16}) $(x \to y) \odot (a \to z) \leq (x \vee a) \to (y \vee z)$,
(B_{17}) $(x \to y) \odot (a \to z) \leq (x \wedge a) \to (y \wedge z)$,
(B_{18}) $(x \to y) \odot (a \to z) \leq (x \odot a) \to (y \odot z)$.

Definition 2.7. [11] A *filter* of a BL-algebra A is a nonempty set $F \subseteq A$ such that $x, y \in F$ implies $x \odot y \in F$ and if $x \in F$ and $x \leq y$ imply $y \in F$, for any $x, y \in A$.

It is easy to prove that if F is a filter of a BL-algebra A, then for each $x, y \in F$, $x \wedge y$, $x \vee y$ and $x \to y$ are in F

Proposition 2.8. [11] Let F be a subset of BL-algebra A such that $1 \in F$. Then the following conditions are equivalent.
(i) F is a filter.
(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$.
(iii) $x \to y \in F$ and $y \to z \in F$ imply $x \to z \in F$.

Proposition 2.9. [11] Let F be a filter of a BL-algebra A. Define $x \equiv^F y \Leftrightarrow x \to y, y \to x \in F$. Then $\equiv^F$ is a congruence relation on A. Moreover, if $x/F = \{y \in A : y \equiv^F x\}$, then
(i) $x/F = y/F \Leftrightarrow y \equiv^F x$,
(ii) $x/F = 1/F \Leftrightarrow x \in F$.

Definition 2.10. [2] (i) Let A be a BL-algebra and $(A, \{*_i\}, \mathcal{U})$ be a semitopological (topological) algebra, where $\{*_i\} \subseteq \{\wedge, \vee, \odot, \to\}$, then $(A, \{*_i\}, \mathcal{U})$ is called a semitopological (topological) BL-algebra.

Remark 2.11. If $\{*_i\} = \{\wedge, \vee, \odot, \to\}$, we consider $\mathcal{A} = (A, \mathcal{U})$ instead of $(A, \{\wedge, \vee, \odot, \to\}, \mathcal{U})$, for simplicity.

Proposition 2.12. [2] Let $(A, \{\odot, \to\}, \mathcal{U})$ be a topological BL-algebra. Then $(A, \mathcal{U})$ is a topological BL-algebra.

Notation. From now on, in this paper, we use of *BL-filter* instead of *filter* in BL-algebras.

Definition 2.13. [10] Let X be a non-empty set. A family $\mathcal{F}$ of nonempty subsets of X is called a $filter$ on X if (i) $X \in \mathcal{F}$, (ii) for each F_1, F_2 of elements of $\mathcal{F}$, $F_1 \cap F_2 \in \mathcal{F}$ and, (iii) if $F \in \mathcal{F}$ and $F \subseteq G$, then $G \in \mathcal{F}$.

A subset $\mathcal{B}$ of a filter $\mathcal{F}$ on X is said to be a *base* of $\mathcal{F}$ if every set of $\mathcal{F}$ contains a set of $\mathcal{B}$.
If $\mathcal{F}$ is a family of nonempty subsets of X, then there exists the smallest filter on X containing $\mathcal{F}$, denoted with $fil(\mathcal{F})$ and called generated filter by $\mathcal{F}$.

Definition 2.14. [10] A $quasi\text{-}uniformity$ on a set X is a filter Q on X such that
(i) $\triangle = \{(x, x) \in X \times X : x \in A\} \subseteq q$, for each $q \in Q$,
(ii) for each $q \in Q$, there is a $p \in Q$ such that $p \circ p \subseteq q$, where

$$p \circ p = \{(x, y) \in X \times X : \exists z \in A \ s.t \ (x, z), (z, y) \in p\}.$$

The pair (X, Q) is called a $quasi\text{-}uniform\ space$.

If Q is a quasi-uniformity on a set X, $q \in Q$ and $q^{-1} = \{(x, y) : (y, x) \in q\}$, then $Q^{-1} = \{q^{-1} : q \in Q\}$ is also a quasi-uniformity on X called the $conjugate$ of Q. It is well-known that if Q satisfies condition: $q \in Q$ implies $q^{-1} \in Q$, then Q is a $uniformity$. Furthermore, $Q^* = Q \vee Q^{-1}$ is a uniformity on X. If Q and R are quasi-uniformities on X and $Q \subseteq R$, then Q is called $coarser$ than R. A subfamily $\mathcal{B}$ of quasi-uniformity Q is said to be a base for Q if each $q \in Q$ contains some member of $\mathcal{B}$.(See [10])

Proposition 2.15. [22] Let $\mathcal{B}$ be a family of subsetes of $X \times X$ such that
(i) $\triangle \subseteq q$, for each $q \in \mathcal{B}$,
(ii) for $q_1, q_2 \in \mathcal{B}$, there exists a $q_3 \in \mathcal{B}$ such that $q_3 \subseteq q_1 \cap q_2$,
(iii) for each $q \in \mathcal{B}$, there is a $p \in \mathcal{B}$ such that $p \circ p \subseteq q$.
Then, there is the unique quasiuniformity $Q = \{q \subseteq X \times X : for\ some\ p \in \mathcal{B}, p \subseteq q\}$ on X for which $\mathcal{B}$ is a base.

The topology $T(Q) = \{G \subseteq X : \forall x \in G\ \exists q \in Q\ s.t\ q(x) \subseteq G\}$ is called the topology induced by the quasi-uniformity Q.

Definition 2.16. [10] (i) A filter $\mathcal{G}$ on quasi-uniform space (X, Q) is called Q^*-*cauchy* filter if for each $U \in Q$, there is a $G \in \mathcal{G}$ such that $G \times G \subseteq U$.
(ii) A quasi-uniform space (X, Q) is called *bicomplete* if each Q^*-*cauchy* filter converges with respect to the topology $T(Q^*)$.
(iii) A *bicompletion* of a quasi-uniform space (X, Q) is a bicomplete quasi-uniform space $(Y, \mathcal{V})$ that has a $T(\mathcal{V}^*)$-dense subspace quasi-unimorphic to

(X, Q).
(iv) A Q^*-cauchy filter on a quasi-uniform space (X, Q) is *minimal* provided that it contains no Q^*-cauchy filter other than itself.

Lemma 2.17. *[10] Let $\mathcal{G}$ be a Q^*-cauchy filter on a quasi-uniform space (X, Q). Then, there is exactly one minimal Q^*-cauchy filter coarser than $\mathcal{G}$. Furthermore, if $\mathcal{B}$ is a base for $\mathcal{G}$, then $\{q(B) : B \in \mathcal{B}$ and q is a symetric member of $Q^*\}$ is a base for the minimal Q^*-cauchy filter coarser than $\mathcal{G}$.*

Lemma 2.18. *[10] Let (X, Q) be a T_0 quasi-uniform space and $\widetilde{X}$ be the family of all minimal Q^*-cauchy filters on (A, Q). For each $q \in Q$, let*

$$\widetilde{q} = \{(\mathcal{G}, \mathcal{H}) \in \widetilde{X} \times \widetilde{X} : \exists G \in \mathcal{G} \text{ and } H \in \mathcal{H} \text{ s.t } G \times H \subseteq q\},$$

and $\widetilde{Q} = fil\{\widetilde{q} : q \in Q\}$. Then the following statements hold:
(i) $(\widetilde{X}, \widetilde{Q})$ is a T_0 bicomplete quasi-uniform space and (X, Q) is a quasi-uniformly embedded as a $T(\widetilde{(Q^*)})$-dense subspace of $(\widetilde{X}, \widetilde{Q})$ by the map $i : X \to \widetilde{X}$ such that, for each $x \in X$, $i(x)$ is the $T(Q^*)$-neighborhood filter at x. Furthermore, the uniformities $\widetilde{Q}^*$ and $\widetilde{(Q^*)}$ coincide.

Notation. From now on, in this paper we let A be a BL-algebra and $\mathcal{F}$ be a family of BL-filters in A which is closed under intersection , unless otherwise state.

3 Quasi-uniformity on BL-algebras

In this section, by using of BL-filters we introduce a quasi-uniformity Q on BL-algebra A and stay some properties it. We show that (A, Q) is not a T_1 and T_2 quasi-uniform space but it is a T_0 quasi-uniform space. Also we study Q^*-cauchy filters, minimal Q^*-cauchy filters and we make a quasi-uniform space $(\widetilde{A}, \widetilde{Q})$ of minimal Q^*-cauchy filters of (A, Q) which admits the structure of a BL-algebra.

Lemma 3.1. *Let F be a BL-filter of BL-algebra A and $F_\star(x) = \{y : y \to x \in F\}$, for each $x \in A$. Then for each $x, y \in A$, the following properties hold.*
(i) $x \leq y$ implies $F_\star(x) \subseteq F_\star(y)$,
(ii) $F_\star(x) \wedge F_\star(y) = F_\star(x \wedge y) = F_\star(x) \cap F_\star(y)$,
(iii) $F_\star(x) \vee F_\star(y) \subseteq F_\star(x \vee y)$,
(iv) $F_\star(x) \odot F_\star(y) \subseteq F_\star(x \odot y)$,
(v) If for each $a \in A$, $a \odot a = a$, then $F_\star(x) \odot F_\star(y) = F_\star(x \odot y)$,

(vi) $x \in F \Leftrightarrow 1 \in F_\star(x) \Leftrightarrow F_\star(x) = A$,
(vii) For $a, b \in A$, *if* $a \vee b \in F_\star(x)$, *then* $a, b \in F_\star(x)$,
(viii) If $y \in F_\star(x)$, *then* $F_\star(y) \subseteq F_\star(x)$.

Proof. (i) Let $x, y \in A$, such that $x \leq y$ and $z \in F_\star(x)$. Then by (B_8), $z \to x \leq z \to y$. Since F is a BL-filter and $z \to x \in F$, $z \to y$ is in F and so $z \in F_\star(y)$.
(ii) Let $x, y \in A$, such that $a \in F_\star(x)$ and $b \in F_\star(y)$. Then $a \to x \in F$ and $b \to y \in F$ and so $(a \to x) \odot (b \to y) \in F$. Since by (B_{17}), $(a \to x) \odot (b \to y) \leq (a \wedge b) \to (x \wedge y)$, we get $(a \wedge b) \to (x \wedge y) \in F$. Thus, $a \wedge b \in F_\star(x \wedge y)$. Now, if $a \in F_\star(x \wedge y)$, since $a \to (x \wedge y) \in F$ and by (B_{11}), $a \to (x \wedge y) = (a \to x) \wedge (a \to y)$, we conclude that $a \to x \in F$ and $a \to y \in F$. Hence $a \in F_\star(x) \cap F_\star(y)$. Finally, let $a \in F_\star(x) \cap F_\star(y)$. Since $a = a \wedge a$, then $a \in F_\star(x) \wedge F_\star(y)$.
$(iii), (iv)$ The proof is similar to the proof of (ii), by some modification.
(v) Let $x, y \in A$ such that $z \in F_\star(x \odot y)$. Then $z \to (x \odot y) \in F$. By (B_8), $z \to (x \odot y) \leq z \to x$ and $z \to (x \odot y) \leq z \to y$ which imply that $z \to x, z \to y \in F$. Hence z is in both $F_\star(x)$ and $F_\star(y)$ and so $z = z \odot z \in F_\star(x) \odot F_\star(y)$.
(vi) The proof is clear.
$(vii), (viii)$ The proof come from by (B_{13}) and (B_{15}). $\square$

Lemma 3.2. *Let F be a BL-filter of BL-algebra A. Define* $F_\star = \{(x, y) \in A \times A : y \in F_\star(x)\}$ *and* $F^*_\star = F_\star \cap F_\star^{-1}$. *Then*
(i) $F_\star^{-1} = \{(x, y) \in A \times A : x \to y \in F\}$,
(ii) $F^*_\star = \{(x, y) \in A \times A : x \equiv^F y\} = {F^*_\star}^{-1}$,
(iii) $F^*_\star(x) = \{y : x \equiv^F y\}$,
(iv) $F_\star^{-1}(x) \to y \subseteq F_\star(x \to y)$,
(v) If $\bullet \in \{\wedge, \vee, \odot, \to\}$, *then* $F^*_\star(x) \bullet F^*_\star(y) \subseteq F^*_\star(x \bullet y)$.

Proof. The proof of $(i), (ii)$ and (iii) are clear.
(iv) Let $a \in F_\star^{-1}(x) \to y$. Then there exists a $z \in F_\star^{-1}(x)$ such that $a = z \to y$ and $x \to z \in F$. By (B_{10}), $(z \to y) \to (x \to y) \geq x \to z$. Since F is a filter, $(z \to y) \to (x \to y) \in F$. Hence $a = z \to y \in F_\star(x \to y)$.
(v) Let $a \in F^*_\star(x)$ and $b \in F^*_\star(y)$. Then by (iii), $a \equiv^F x$ and $b \equiv^F y$. By Proposition 2.9, $a \bullet b \equiv^F x \bullet y$. Therefore, $a \bullet b \in F^*_\star(x \bullet y)$. $\square$

Theorem 3.3. *Let $\mathcal{F}$ be a family of BL-filters of BL-algebra A which is closed under finite intersection. Then the set $\mathcal{B} = \{F_\star : F \in \mathcal{F}\}$ is a base for the unique quasi-uniformity* $Q = \{q \subseteq A \times A : \exists F \in \mathcal{F} \ s.t \ F_\star \subseteq q\}$. *Moreover,* $Q^* = \{q \subseteq A \times A : \exists F \in \mathcal{F} \ s.t \ F^*_\star \subseteq q\}$.

Proof. We prove that $\mathcal{B}$ satisfies in conditions $(i), (ii)$ and (iii) of Proposition 2.15. For (i), it is easy to see that for each $F \in \mathcal{F}$, $\triangle \subseteq F_\star$. Let $F_1, F_2 \in \mathcal{F}$

and $F = F_1 \cap F_2$. If $(x, y) \in F_\star$, then $y \to x \in F = F_1 \cap F_2$. Hence $(x, y) \in F_{1\star} \cap F_{2\star}$. This concludes that $F_\star \subseteq F_{1\star} \cap F_{2\star}$ and so (ii) is true. Finally for (iii), let $F \in \mathcal{F}$ and $(x, y) \in F_\star \circ F_\star$. Then there is a $z \in A$ such that (x, z) and (z, y) are both in $F_\star$. Hence $z \to x$ and $y \to z$ are in F. Since F is a filter and by (B_{15}), $(y \to z) \odot (z \to x) \leq y \to x$, we conclude that $y \to x \in F$. Hence $F_\star \circ F_\star \subseteq F_\star$ and so (iii) is true. Therefore, by Proposition 2.15, Q is a unique quasi-uniformity on A for which $\mathcal{B}$ is a base.

Now, we prove that

$$Q^* = \{q \subseteq A \times A : \exists F \in \mathcal{F} \ s.t \ F_\star^* \subseteq q\}.$$

First we prove that $\mathcal{P} = \{q \subseteq A \times A : \exists F \in \mathcal{F} \ s.t \ F_\star^* \subseteq q\}$ is a uniformity on A. With a similar argument as above, we get $\{F_\star^* : F \in \mathcal{F}\}$ is a base for the quasi-uniformity $\mathcal{P} = \{q \subseteq A \times A : \exists F \in \mathcal{F} \ s.t \ F_\star^* \subseteq q\}$. To prove that $\mathcal{P}$ is a uniformity we have to show that for each $q \in \mathcal{P}$, q^{-1} is in $\mathcal{P}$. Suppose $q \in \mathcal{P}$. Then there exists a $F \in \mathcal{F}$, such that $F_\star^* \subseteq q$. By Lemma 3.2(ii), $F_\star^* = F_\star^{*^{-1}}$. Hence $F_\star^* \subseteq q^{-1}$ and so $q^{-1} \in \mathcal{P}$. Thus $\mathcal{P}$ is a uniformity on A which contains Q. Since $Q^* = Q \vee Q^{-1}$, then $Q^* \subseteq \mathcal{P}$. On the other hand, if $q \in \mathcal{P}$, then there is a $F \in \mathcal{F}$ such that $F_\star^* \subseteq q$. Since $F_\star^* = F_\star \cap F_\star^{-1} \in Q^*$, we get that $q \in Q^*$. Therefore, $Q^* = \mathcal{P}$. □

In Theorem 3.3, we call Q is quasi-uniformity induced by $\mathcal{F}$, the pair (A, Q) is quasi-uniform BL-algebra and the pair (A, Q^*) is uniform BL-algebra.

Notation. From now on, $\mathcal{F}$, Q and Q^* are as in Theorem 3.3.

Example 3.4. *Let $\mathcal{I}$ be the BL-algebra in Example 2.5 (i), and for each $a \in [0, 1)$, $F_a = (a, 1]$. Then F_a is a BL-filter in $\mathcal{I}$ and easily proved that for each $a, b \in [0, 1)$, $F_a \cap F_b = F_{a \wedge b}$. Hence $\mathcal{F} = \{F_a\}_{a \in [0,1)}$ is a family of BL-filters which is closed under intersection. For each $a \in [0, 1)$,*

$$F_{a\star} = (a, 1] \times [0, 1], \ F_{a\star}^{-1} = [0, 1] \times (a, 1] \ \textit{and} \ F_{a\star}^* = (a, 1] \times (a, 1].$$

By Theorem 3.3, $Q = \{q : \exists a \in [0, 1) \ s.t \ (a, 1] \times [0, 1] \subseteq q\}$ and $Q^* = \{q : \exists a \in [0, 1) \ s.t \ (a, 1] \times (a, 1] \subseteq q\}$.

Recall that a map f from a (quasi)uniform space (X, Q) into a (quasi)uniform space (Y, R) is *(quasi) uniformly continuous*, if for each $V \in R$, there exists a $U \in Q$ such that $(x, y) \in U$ implies $(f(x), f(y)) \in V$. If $f : (X, Q) \hookrightarrow (Y, R)$ is a quasi-uniform continuous map between quasi-uniform spaces, then $f : (X, Q^*) \hookrightarrow (Y, R^*)$ is a uniform continuous map. (See [10])

Proposition 3.5. In BL-algebra A, for each $a \in A$, the mappings $t_a(x) = a \wedge x$, $r_a(x) = a \vee x$, $l_a(x) = a \odot x$ and $L_a(x) = a \to x$ of quasi-uniform BL-algebra (A, Q) into quasi-uniform BL-algebra (A, Q) are quasi-uniformly continuous. Moreover, they are uniformly continuous mappings of uniform BL-algebra (A, Q^*) into uniform BL-algebra (A, Q^*).

Proof. Let $q \in Q$. Then, there is a $F \in \mathcal{F}$ such that $F_\star \subseteq q$. If $(x, y) \in F_\star$, then $y \to x \in F$. By (B_{10}) $(a \wedge y) \to (a \wedge x) \geq y \to x$ which implies that $(a \wedge y) \to (a \wedge x) \in F \subseteq q$. Hence t_a is quasi-uniform continuous. Moreover, $t_a : (A, Q^*) \hookrightarrow (A, Q^*)$ is uniform continuous. In a similar fashion and by use of (B_{16}), (B_{14}) and (B_9), we can prove that, respectively, r_a, l_a and L_a are quasi-uniform continuous of $(A, Q) \hookrightarrow (A, Q)$ and are uniform continuous of $(A, Q^*) \hookrightarrow (A, Q^*)$. □

Let (X, Q) be a (quasi)uniform space and $\mathcal{B}$ be a base for it. Recall (X, Q) is
(i) T_0 quasi-uniform if (x, y) and (y, x) are in $\bigcap_{U \in \mathcal{B}} U$, then $x = y$, for each $x, y \in X$,
(ii) T_1 quasi-uniform if $\triangle = \bigcap_{U \in \mathcal{B}} U$,
(iii) T_2 quasi-uniform if $\triangle = \bigcap_{U \in \mathcal{B}} U^{-1} \circ U$. (See [10])

Theorem 3.6. *Quasi-uniform BL-algebra (A, Q) is not T_1 and T_2 quasi-uniform. If $\{1\} \in \mathcal{F}$, then (A, Q) is a T_0 quasi-uniform space and uniform BL-algebra (A, Q^*) is T_0, T_1 and T_2 quasi-uniform space.*

Proof. Let $x, y \in A$ and $F \in \mathcal{F}$. Since $y \to 1 = 1 \in F$, we get that $(1, y) \in \bigcap_{F \in \mathcal{F}} F_\star$. Hence (A, Q) is not T_0 quasi-uniform. Also since $x \to 1 = y \to 1 \in F$, we conclude that $(1, x), (1, y) \in F_\star$. Hence $(x, y) \in F_\star^{-1} \circ F_\star$ which implies that $\triangle \neq \bigcap_{F \in \mathcal{F}} F_\star^{-1} \circ F_\star$. So (A, Q) is not T_2 quasi-unifom
Let $\{1\} \in \mathcal{F}$ and (x, y) and (y, x) be in $\bigcap_{F \in \mathcal{F}} F_\star$. Then for each $F \in \mathcal{F}$, $x \to y$ and $y \to x$ are in F. Hence $x \equiv^{\{1\}} y$, which implies that $x = y$. Therefore, (A, Q) is T_0 quasi-uniform. With a similar argument as above, we can prove that (A, Q^*) is a T_0 and T_1 quasi-uniform space. To verify T_2 quasi-uniformity, let $(x, y) \in \bigcap_{F \in \mathcal{F}} F_\star^{*^{-1}} \circ F_\star^*$. Then for each $F \in \mathcal{F}$ there is a $z \in A$ such that $(x, z) \in F_\star^{*^{-1}}$ and $(z, y) \in F_\star^*$. By Lemma 3.2(ii), $x \equiv^F y$. Since $\{1\} \in \mathcal{F}$, we get that $x = y$. Therefore, (A, Q^*) is a T_2 quasi-uniform space. □

Proposition 3.7. Let $\mathcal{B}$ be a base for a Q^*-cauchy filter $\mathcal{G}$ on quasi-uniform BL-algebra (A, Q). Then the set $\{F_\star^*(B) : F \in \mathcal{F},\ B \in \mathcal{B}\}$ is a base for the uniqe minimal Q^*-cauchy filter coarser than $\mathcal{G}$.

Proof. By Lemma 2.17, the set $\{q(B) : B \in \mathcal{B},\ q^{-1} = q \in Q^*\}$ is a base for the unique minimal Q^*-cauchy filter $\mathcal{G}_0$ coarser than $\mathcal{G}$. Let $q^{-1} = q \in Q^*$ and $B \in \mathcal{B}$. Then for some $F \in \mathcal{F}$, $F^*_\star \subseteq q$. So, $F^*_\star(B) \subseteq q(B)$. Now, it is easy to prove that the set $\{F^*_\star(B) : F \in \mathcal{F},\ B \in \mathcal{B}\}$ is a base for $\mathcal{G}_0$. □

Proposition 3.8. $\mathcal{F}$ is a base for a minimal Q^*-cauchy filter on quasi-uniform BL-algebra (A, Q).

Proof. Let $\mathcal{C} = \{S \subseteq A : \exists F \in \mathcal{F}\ s.t\ F \subseteq S\}$. It is easy to prove that $\mathcal{C}$ is a filter and $\mathcal{F}$ is a base for it. We prove that $\mathcal{C}$ is a Q^*-cauchy filter. For this, let $q \in Q$. There is a $F \in \mathcal{F}$ such that $F_\star \subseteq q$. Since F is a filter, clearly $F \times F \subseteq F_\star \subseteq q$. Hence $\mathcal{C}$ is a Q^*-cauchy filter. Now, by Proposition 3.7, the set $\{F^*_\star(F_1) : F, F_1 \in \mathcal{F}\}$ is a base for the unique minimal Q^*-cauchy filter $\mathcal{F}_0$ coarser than $\mathcal{C}$. To complete proof we show that for each $F, F_1 \in \mathcal{F}$, $F^*_\star(F_1) = F_1$. Let $F, F_1 \in \mathcal{F}$. If $y \in F^*_\star(F_1)$, then for some $x \in F_1$, $x \equiv^F y$. By Proposition 2.9, $y \in F_1$. Hence $F^*_\star(F_1) \subseteq F_1$. Clearly, $F_1 \subseteq F^*_\star(F_1)$. Therefore, $F_1 = F^*_\star(F_1)$. Thus proved that $\mathcal{F}$ is a base for $\mathcal{F}_0$. □

Proposition 3.9. The set $\mathcal{B} = \{F^*_\star(0) : F \in \mathcal{F}\}$ is a base for a minimal Q^*-cauchy filter on quasi-uniform BL-algebra (A, Q).

Proof. Let $\mathcal{C} = \{S \subseteq A : \exists F \in \mathcal{F}\ s.t\ F^*_\star(0) \subseteq S\}$. It is easy to prove that $\mathcal{C}$ is a filter and the set $\mathcal{B} = \{F^*_\star(0) : F \in \mathcal{F}\}$ is a base for it. To prove that $\mathcal{C}$ is a Q^*-cauchy filter, let $q \in Q$. There is a $F \in \mathcal{F}$ such that $F_\star \subseteq q$. If $x, y \in F^*_\star(0)$, then $x \equiv^F y$ and so $(x, y) \in F^*_\star \subseteq F_\star \subseteq q$. This prove that $F^*_\star(0) \times F^*_\star(0) \subseteq q$. Hence $\mathcal{C}$ is a Q^*-cauchy filter. By Proposition 3.7, the set $\{F^*_\star(F^*_\star(0)) : F \in \mathcal{F}\}$ is a base for the uniqe minimal Q^*-cauchy filter $\mathcal{I}$ coarser than $\mathcal{C}$. But it is easy to pove that fo each $F \in \mathcal{F}$, $F^*_\star(F^*_\star(0)) = F^*_\star(0)$. Therefore, $\mathcal{B}$ is a base for $\mathcal{I}$. □

Lemma 3.10. *Let $\mathcal{G}$ and $\mathcal{H}$ be Q^*-cauchy filters on quasi-uniform BL-algebra (A, Q). If $\bullet \in \{\wedge, \vee, \odot, \rightarrow\}$, then $\mathcal{G} \bullet \mathcal{H} = \{G \bullet H : G \in \mathcal{G},\ H \in \mathcal{H}\}$ is a Q^*-cauchy filter base on quasi-uniform BL-algebra (A, Q).*

Proof. Let $\mathcal{C} = \{S \subseteq A : \exists G,\ H\ s.t\ G \in \mathcal{G},\ H \in \mathcal{H},\ G \bullet H \subseteq S\}$. It is easy to prove that $\mathcal{C}$ is a filter and the set $\mathcal{B} = \{G \bullet H : G \in \mathcal{G},\ H \in \mathcal{H}\}$ is a base for it. We prove that $\mathcal{C}$ is a Q^*-cauchy filter. For this, let $q \in Q$. Then for some a $F \in \mathcal{F}$, $F_\star \subseteq q$. Since $\mathcal{G}, \mathcal{H}$ are Q^*-cauchy filters, there are $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $G \times G \subseteq F_\star$ and $H \times H \subseteq F_\star$. We show that $G \bullet H \times G \bullet H \subseteq F_\star \subseteq q$. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$. Then $(g_1, g_2), (g_2, g_1), (h_1, h_2), (h_2, h_1)$ are in $F_\star$. So $g_1 \equiv^F g_2$ and $h_1 \equiv^F h_2$. By Proposition 2.9, $g_1 \bullet h_1 \equiv^F g_2 \bullet h_2$, which implies that $(g_1 \bullet h_1, g_2 \bullet h_2) \in F_\star$. □

Theorem 3.11. *There is a quasi-uniform space $(\widetilde{A}, \widetilde{Q})$ of minimal Q^*-cauchy filters of quasi-uniform BL-algebra (A, Q) that admits a BL-algebra structure.*

Proof. Let $\widetilde{A}$ be the family of all minimal Q^*-cauchy filters on (A, Q). Let for each $q \in Q$,

$$\widetilde{q} = \{(\mathcal{G}, \mathcal{H}) \in \widetilde{A} \times \widetilde{A} : \exists G \in \mathcal{G}, H \in \mathcal{H} \ s.t \ G \times H \subseteq q\}.$$

If $\widetilde{Q} = fil\{\widetilde{q} : q \in Q\}$, then $(\widetilde{A}, \widetilde{Q})$ is a quasi-uniform space of minimal Q^*-cauchy filters of (A, Q). Let $\mathcal{G}, \mathcal{H} \in \widetilde{A}$. Since $\mathcal{G}, \mathcal{H}$ are minimal Q^*-cauchy filters on A, then by Lemma 3.10, $\mathcal{G} \wedge \mathcal{H}$, $\mathcal{G} \vee \mathcal{H}$, $\mathcal{G} \odot \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ are Q^*-cauchy filter bases on A. Now, we define $\mathcal{G} \curlywedge \mathcal{H}$, $\mathcal{G} \curlyvee \mathcal{H}$, $\mathcal{G} \circledcirc \mathcal{H}$ and $\mathcal{G} \hookrightarrow \mathcal{H}$ as the minimal Q^*-cauchy filters contained $\mathcal{G} \wedge \mathcal{H}$, $\mathcal{G} \vee \mathcal{H}$, $\mathcal{G} \odot \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$, respectively. Thus, $\mathcal{G} \curlywedge \mathcal{H}$, $\mathcal{G} \curlyvee \mathcal{H}$, $\mathcal{G} \circledcirc \mathcal{H}$ and $\mathcal{G} \hookrightarrow \mathcal{H}$ are in $\widetilde{A}$. Now, we will prove that $(\widetilde{A}, \curlywedge, \curlyvee, \circledcirc, \hookrightarrow, \mathcal{I}, \mathcal{F}_0)$ is a BL-algebra, where $\mathcal{I}$ is minimal Q^*-cauchy filter in Proposition 3.9 and $\mathcal{F}_0$ is minimal Q^*-cauchy filter in Proposition 3.8. For this, we consider the following steps:

(1) $(\widetilde{A}, \curlywedge, \curlyvee)$ **is a bounded lattice**.

Let $\mathcal{G}, \mathcal{H}, \mathcal{K} \in \widetilde{A}$. We consider the following cases:

Case 1.1: $\mathcal{G} \curlywedge \mathcal{G} = \mathcal{G}$, $\mathcal{G} \curlyvee \mathcal{G} = \mathcal{G}$

By Proposition 3.7, $S_1 = \{F^*_\star(G) : G \in \mathcal{G}, F \in \mathcal{F}\}$ and $S_2 = \{F^*_\star(G_1 \wedge G_2) : G_1, G_2 \in \mathcal{G}, F \in \mathcal{F}\}$ are bases of the minimal Q^*-cauchy filters $\mathcal{G}$ and $\mathcal{G} \curlywedge \mathcal{G}$, respectively. First, we show that $S_2 \subseteq S_1$. Let $F^*_\star(G_1 \wedge G_2) \in S_2$. Put $G = G_1 \cap G_2$, then $G \in \mathcal{G}$. Let $y \in F^*_\star(G)$. Then there is a $x \in G$ such that $(x, y) \in F^*_\star$. Since $x \wedge x = x$, it follows that $(x \wedge x, y) \in F^*_\star$ and so $y \in F^*_\star(G_1 \wedge G_2)$. Hence $S_2 \subseteq S_1$. Therefore, $\mathcal{G} \curlywedge \mathcal{G} \subseteq \mathcal{G}$. By the minimality of $\mathcal{G}$, $\mathcal{G} \curlywedge \mathcal{G} = \mathcal{G}$. The proof of the other case is similar.

Case 1.2: $\mathcal{G} \curlywedge \mathcal{H} = \mathcal{H} \curlywedge \mathcal{G}$, $\mathcal{G} \curlyvee \mathcal{H} = \mathcal{H} \curlyvee \mathcal{G}$

By Proposition 3.7, $S_1 = \{F^*_\star(G \wedge H) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\}$ and $S_2 = \{F^*_\star(H \wedge G) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\}$ are bases of $\mathcal{G} \curlywedge \mathcal{H}$ and $\mathcal{H} \curlywedge \mathcal{G}$, respectively. For each $G \in \mathcal{G}$ and $H \in \mathcal{H}$, since $G \wedge H = H \wedge G$, for each $F \in \mathcal{F}$, $F^*_\star(G \wedge H) = F^*_\star(H \wedge G)$. Hence $\mathcal{G} \curlywedge \mathcal{H} = \mathcal{H} \curlywedge \mathcal{G}$. The proof of the other case is similar.

Case 1.3: $\mathcal{G} \curlywedge (\mathcal{H} \curlywedge \mathcal{K}) = (\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$, $\mathcal{G} \curlyvee (\mathcal{H} \curlyvee \mathcal{K}) = (\mathcal{G} \curlyvee \mathcal{H}) \curlyvee \mathcal{K}$

By Proposition 3.7, the families

$$S_1 = \{F^*_{1\star}(F^*_{2\star}(G \wedge H) \wedge K) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\},$$

$$S_2 = \{F^*_{1\star}(G \wedge F^*_{2\star}(H \wedge K) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\}$$

are bases for the minimal Q^*-cauchy filters $(\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$ and $\mathcal{G} \curlywedge (\mathcal{H} \curlywedge \mathcal{K})$, respectively. Let $F^*_{1\star}(F^*_{2\star}G \wedge (H \wedge K) \in S_2$ and $F = F_1 \cap F_2$. Then $F \in \mathcal{F}$.

Now, we show that $F^*_\star(F^*_\star(G \wedge H) \wedge K) \subseteq F^*_{1\star}(G \wedge F^*_{2\star}(H \wedge K)$. Let $y \in F^*_\star(F^*_\star(G \wedge H) \wedge K)$. Then there are $x \in F^*_\star(G \wedge H)$, $k \in K$, $g \in G$ and $h \in H$ such that $y \equiv^F x \wedge k$ and $x \equiv^F g \wedge h$. Hence $y \equiv^F (g \wedge h) \wedge k = g \wedge (h \wedge k)$, which implies that $y \in F^*_\star(G \wedge F^*_\star(H \wedge K) \subseteq F^*_{1\star}(G \wedge F^*_{2\star}(H \wedge K)$. Therefore, $\mathcal{G} \curlywedge (\mathcal{H} \curlywedge \mathcal{K}) \subseteq (\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$. By the minimality of $(\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$, $\mathcal{G} \curlywedge (\mathcal{H} \curlywedge \mathcal{K}) = (\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$. The proof of the other case is similar.
Case 1.4: $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H}) = \mathcal{G}$, $\mathcal{G} \curlyvee (\mathcal{G} \curlywedge \mathcal{H}) = \mathcal{G}$
It is enough to prove that $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H}) = \mathcal{G}$. The proof of the other case is similar. By Proposition 3.7, the families $S_1 = \{F^*_\star(G) : G \in \mathcal{G}, F \in \mathcal{F}\}$ and $S_2 = \{F^*_{1\star}(G_1 \wedge F^*_{2\star}(G_2 \vee H) : G_1, G_2 \in \mathcal{G}, H \in \mathcal{H}, F_1, F_2 \in \mathcal{F}\}$ are bases for the minimal Q^*-cauchy filters $\mathcal{G}$ and $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H})$, respectively. Let $F^*_{1\star}(G_1 \wedge F^*_{2\star}(G_2 \vee H) \in S_2$. Put $G = G_1 \cap G_2$ and $F = F_1 \cap F_2$. We prove that $F^*_\star(G) \subseteq F^*_{1\star}(G_1 \wedge F^*_{2\star}(G_2 \vee H)$. Let $y \in F^*_\star(G)$. Then there is a $g \in G$ such that $y \equiv^F g$. If $h \in H$, since $g = g \wedge (g \vee h)$, then $y \equiv^F g \wedge (g \vee h)$ and so $y \in F^*_{1\star}(G_1 \wedge F^*_{2\star}(G_2 \vee H)$. Hence $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H}) \subseteq \mathcal{G}$. By the minimality of $\mathcal{G}$, we conclude that $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H}) = \mathcal{G}$.
Now the cases 1.1,1.2,1.3,1.4 imply that $(\widetilde{A}, \curlywedge, \curlyvee)$ is a lattice.
Case 1.5: The lattice $(\widetilde{A}, \curlywedge, \curlyvee)$ is bounded.
For this, for each $\mathcal{G}, \mathcal{H} \in \widetilde{A}$, define $\mathcal{G} \leq \mathcal{H} \Leftrightarrow \mathcal{G} \curlywedge \mathcal{H} = \mathcal{G}$. It is clear that $(\widetilde{A}, \leq)$ is a partial ordered. Now, we prove that for each $\mathcal{G} \in \widetilde{A}$, $\mathcal{I} \leq \mathcal{G} \leq \mathcal{F}_0$. First, we show that $\mathcal{I} \leq \mathcal{G}$. Let $S \in \mathcal{I}$. Then for some a $F \in \mathcal{F}$, $F^*_\star(0) \subseteq S$. Since $\mathcal{G}$ is a minimal Q^*-cauchy filter, there is a $G \in \mathcal{G}$ such that $G \times G \subseteq F_\star$. We show that $F^*_\star(G \wedge F^*_\star(0)) \subseteq S$. Let $y \in F^*_\star(G \wedge F^*_\star(0))$. Then there are $g \in G$ and $x \in F^*_\star(0)$ such that $y \equiv^F g \wedge x$. On the other hand, since $x \equiv^F 0$, we get $g \wedge x \equiv^F 0$. Hence $y \equiv^F 0$ which implies that $y \in F^*_\star(0) \subseteq S$. Since $F^*_\star(G \wedge F^*_\star(0)) \in \mathcal{G} \curlywedge \mathcal{I}$, then $S \in \mathcal{G} \curlywedge \mathcal{I}$. By the minimality of $\mathcal{G} \curlywedge \mathcal{I}$, $\mathcal{G} \curlywedge \mathcal{I} = \mathcal{I}$. Now, we prove that $\mathcal{G} \leq \mathcal{F}_0$. By Proposition 3.7, the set $S_1 = \{F^*_\star(G \wedge F_1) : G \in \mathcal{G},\ F, F_1 \in \mathcal{F}\}$ is a base for $\mathcal{G} \curlywedge \mathcal{F}_0$. Let $F^*_\star(G \wedge F_1) \in S_1$. We prove that $F^*_\star(G) \subseteq F^*_\star(G \wedge F_1)$. Let $y \in F^*_\star(G)$. Then, there is a $g \in G$ such that $y \equiv^F g = g \wedge 1$. Hence $y \in F^*_\star(G \wedge F_1)$. By the minimality of $\mathcal{G}$, $\mathcal{G} \curlywedge \mathcal{F}_0 = \mathcal{G}$.
(2) $(\widetilde{A}, \circledcirc)$ **is a commutative monoid**
Case 2.1: $(\widetilde{A}, \circledcirc)$ is a commutative semigroup.
We will prove that $\mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K}) = (\mathcal{G} \circledcirc \mathcal{H}) \circledcirc \mathcal{K}$. By Proposition 3.7, the sets

$$S_1 = \{F^*_{1\star}(G \odot F^*_{2\star}(H \odot K)) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\},$$

$$S_2 = \{F^*_{1\star}(F^*_{2\star}(G \odot H) \odot K)) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\}$$

are bases from $\mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K})$ and $(\mathcal{G} \circledcirc \mathcal{H}) \circledcirc \mathcal{K}$, respectively. Let $F^*_{1\star}(F^*_{2\star}(G \odot H) \odot K)) \in S_2$, $F = F_1 \cap F_2$ and $y \in F^*_\star(G \odot F^*_\star(H \odot K)$. Then there are $g \in G$, $x \in F^*_\star(H \odot K)$, $h \in H$ and $k \in K$ such that $y \overset{F}{\equiv} g \odot x$ and $x \overset{F}{\equiv} h \odot k$. Hence

$y \overset{F}{\equiv} g\odot(h\odot k) = (g\odot h)\odot k$ and so $y \in F_\star^*(F_\star^*(G\odot H)\odot K) \subseteq F_{1\star}^*(F_{2\star}^*(G\odot H)\odot K))$. Therefore, $S_2 \subseteq S_1$ which implies that $(\mathcal{G} \circledcirc \mathcal{H}) \circledcirc \mathcal{K} \subseteq \mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K})$. Now, by the minimality of $\mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K})$, $\mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K}) = (\mathcal{G} \circledcirc \mathcal{H}) \circledcirc \mathcal{K}$. Finally, it is easy to prove that $\mathcal{G} \circledcirc \mathcal{H} = \mathcal{H} \circledcirc \mathcal{G}$.
Case 2.2: $(\widetilde{A}, \circledcirc)$ is a monoid
We prove that $\mathcal{G}\circledcirc\mathcal{F}_0 = \mathcal{G}$. By Proposition 3.7, the set $S_2 = \{F_\star^*(G\odot F_1) : G \in \mathcal{G}, F, F_1 \in \mathcal{F}\}$ is a base for $\mathcal{G} \circledcirc \mathcal{F}_0$. It is clear that for each $F_\star^*(G \odot F_1) \in S_2$, $F_\star^*(G) \subseteq F_\star^*(G \odot F_1)$ and this implies that $\mathcal{G} \circledcirc \mathcal{F}_0 \subseteq \mathcal{G}$. By the minimality of $\mathcal{G}$, $\mathcal{G} \circledcirc \mathcal{F}_0 = \mathcal{G}$.
(3) $\mathcal{G} \circledcirc (\mathcal{G} \hookrightarrow \mathcal{H}) = \mathcal{G} \curlywedge \mathcal{H}$
By Proposition 3.7, the families

$$S_1 = \{F_\star^*(G \wedge H) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\},$$

$$S_2 = \{F_{1\star}^*(G_1 \odot F_{2\star}^*(G_2 \to H)) : G_1, G_2 \in \mathcal{G}, H \in \mathcal{H}, F_1, F_2 \in \mathcal{F}\}$$

are bases for $\mathcal{G} \curlywedge \mathcal{H}$ and $\mathcal{G} \circledcirc (\mathcal{G} \hookrightarrow \mathcal{H})$, respectively. Let $F_{1\star}^*(G_1 \odot F_{2\star}^*(G_2 \to H)) \in S_2$, $G = G_1 \cap G_2$ and $F = F_1 \cap F_2$. We will prove that $F_\star^*(G \wedge H) \subseteq F_{1\star}^*(G_1 \odot F_{2\star}^*(G_2 \to H))$. Let $y \in F_\star^*(G \wedge H)$. Then there are $g \in G$ and $h \in H$ such that $y \equiv^F g \wedge h$. It follows from $g \wedge h = g \odot (g \to h)$ which $y \in F_{1\star}^*(G_1 \odot F_{2\star}^*(G_2 \to H))$. Hence $F_\star^*(G \wedge H) \subseteq F_{1\star}^*(G_1 \odot F_{2\star}^*(G_2 \to H))$ which implies that $\mathcal{G}\circledcirc(\mathcal{G} \hookrightarrow \mathcal{H}) \subseteq \mathcal{G} \curlywedge \mathcal{H}$. Now, by the minimality of $\mathcal{G} \curlywedge \mathcal{H}$, we get $\mathcal{G} \circledcirc (\mathcal{G} \hookrightarrow \mathcal{H}) = \mathcal{G} \curlywedge \mathcal{H}$.
(4) $\mathcal{G} \leq \mathcal{H} \hookrightarrow \mathcal{K} \Leftrightarrow \mathcal{G} \circledcirc \mathcal{H} \leq \mathcal{K}$
First, we prove the following statements:
(a) $\mathcal{G} \leq \mathcal{H} \Leftrightarrow \mathcal{G} \hookrightarrow \mathcal{H} = \mathcal{F}_0$
(b) $\mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K}) = \mathcal{G} \circledcirc \mathcal{H} \hookrightarrow \mathcal{K}$.
(a) To prove it, let $\mathcal{G} \hookrightarrow \mathcal{H} = \mathcal{F}_0$. Then $\mathcal{G} \circledcirc (\mathcal{G} \hookrightarrow \mathcal{H}) = \mathcal{G} \circledcirc \mathcal{F}_0 = \mathcal{G}$. By (3), $\mathcal{G} \curlywedge \mathcal{H} = \mathcal{G}$ and so $\mathcal{G} \leq \mathcal{H}$.
Conversely, let $\mathcal{G} \leq \mathcal{H}$. By Proposition 3.7, the set $S = \{F_\star^*(G \to H) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\}$ is a base for $\mathcal{G} \hookrightarrow \mathcal{H}$. Let $F_\star^*(G \to H) \in S$. We prove that $1 \in F_\star^*(G \to H)$. Since by Lemma 3.10, $G \to H$ is a Q^*-cauchy filter base, there are $G_1 \in \mathcal{G}$ and $H_1 \in \mathcal{H}$ such that $(G_1 \to H_1) \times (G_1 \to H_1) \subseteq F_\star$. Put $G_2 = G_1 \cap G$ and $H_2 = H_1 \cap H$. It is easy to see that $G_2 \wedge H_2 \subseteq F_\star^*(G_2 \wedge H_2) \in \mathcal{G} \curlywedge \mathcal{H}$. Since $\mathcal{G} \curlywedge \mathcal{H} = \mathcal{G}$, there is a $G_3 \in \mathcal{G}$ such that $G_3 \subseteq G_1$ and $G_3 \subseteq G_2 \wedge H_2$. Since $G_3 \neq \phi$, there are $g_3 \in G_3$, $g \in G_2$ and $h \in H_2$ such that $g_3 = g \wedge h$. Since $(g_3 \to h, g \to h)$ and $(g \to h, g_3 \to h)$ both are in $(G_1 \to H_1) \times (G_1 \to H_1) \subseteq F_\star$, we get $g \to h \equiv^F g_3 \to h = 1$ and so $1 \in F_\star^*(G \to H)$. Hence $F_\star^*(1) \subseteq F_\star^*(G \to H)$. This implies that $\mathcal{G} \hookrightarrow \mathcal{H} \subseteq \mathcal{F}_0$. By the minimality of $\mathcal{F}_0$, $\mathcal{G} \hookrightarrow \mathcal{H} = \mathcal{F}_0$. Therefore, we have (a).

(b) By Proposition 3.7, the families

$$S_1 = \{F^*_{1\star}(G \to F^*_{2\star}(H \to K)) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\},$$

$$S_2 = \{F^*_{1\star}(F^*_{2\star}(G \odot H) \to K) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\}$$

are bases of $\mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K})$ and $(\mathcal{G} \circledcirc \mathcal{H}) \hookrightarrow \mathcal{K}$, respectively. Let $F^*_{1\star}(F^*_{2\star}(G \odot H) \to K) \in S_2$, $F = F_1 \cap F_2$ and $y \in F^*_\star(G \to F^*_\star(H \to K))$. Then there are $g \in G$ and $x \in F^*_\star(H \to K)$ such that $y \equiv^F g \to x$. Also there are $h \in H$ and $k \in K$ such that $x \equiv^F h \to k$. Hence $y \equiv^F g \to x \equiv^F g \to (h \to k) = (g \odot h) \to k$. Therefore, $y \in F^*_{1\star}(F^*_{2\star}(G \odot H) \to K)$. This implies that $(\mathcal{G} \circledcirc \mathcal{H}) \hookrightarrow \mathcal{K} \subseteq \mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K})$. By the minimality of $\mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K})$, we get $\mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K}) = \mathcal{G} \circledcirc \mathcal{H} \hookrightarrow \mathcal{K}$. Hence we have (b).
Now, by (a) and (b), we have

$$\mathcal{G} \leq \mathcal{H} \hookrightarrow \mathcal{K} \Leftrightarrow \mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K}) = \mathcal{F}_0 \Leftrightarrow (\mathcal{G} \circledcirc \mathcal{H}) \hookrightarrow \mathcal{K} = \mathcal{F}_0 \Leftrightarrow \mathcal{G} \circledcirc \mathcal{H} \leq \mathcal{K}.$$

So $\mathcal{G} \leq \mathcal{H} \hookrightarrow \mathcal{K} \Leftrightarrow \mathcal{G} \circledcirc \mathcal{H} \leq \mathcal{K}$.
(5) $(\mathcal{G} \hookrightarrow \mathcal{H}) \curlyvee (\mathcal{H} \hookrightarrow \mathcal{G}) = \mathcal{F}_0$
By Proposition 3.7, the set

$$S = \{F^*_{1\star}(F^*_{2\star}(G_1 \to H_1) \vee F^*_{3\star}(H_2 \to G_2)) : G_1, G_2 \in \mathcal{G}, H_1, H_2 \in \mathcal{H}, F_1, F_2, F_3 \in \mathcal{F}\}$$

is a base for $(\mathcal{G} \hookrightarrow \mathcal{H}) \curlyvee (\mathcal{H} \hookrightarrow \mathcal{G})$. Let $F^*_{1\star}(F^*_{2\star}(G_1 \to H_1) \vee F^*_{3\star}(H_2 \to G_2)) \in S$, $G = G_1 \cap G_2$, $H = H_1 \cap H_2$ and $F = F_1 \cap F_2 \cap F_3$. We show that $1 \in F^*_\star(F^*_\star(G \to H) \vee F^*_\star(H \to G))$. Let $g \in G$ and $h \in H$. Since A is a BL-algebra, we have $(g \to h) \vee (h \to g) = 1$. Since $g \to h \in F^*_\star(G \to H)$ and $h \to g \in F^*_\star(H \to G)$, we have $(g \to h) \vee (h \to g) \in F^*_\star(F^*_\star(G \to H) \vee F^*_\star(H \to G))$ and so $1 \in F^*_\star(F^*_\star(G \to H) \vee F^*_\star(H \to G))$. Hence $F^*_\star(1) \subseteq F^*_\star(F^*_\star(G \to H) \vee F^*_\star(H \to G))$ which implies that $(\mathcal{G} \hookrightarrow \mathcal{H}) \curlyvee (\mathcal{H} \hookrightarrow \mathcal{G}) \subseteq \mathcal{F}_0$. By the minimality of $\mathcal{F}_0$, $(\mathcal{G} \hookrightarrow \mathcal{H}) \curlyvee (\mathcal{H} \hookrightarrow \mathcal{G}) = \mathcal{F}_0$.

$\square$

4 Some topological properties on quasi-unifom BL-algebra (A, Q)

Let $T(Q)$ and $T(Q^*)$ be topologies induced by Q and Q^*, respectively. Our goal in this section is to study (semi)topological BL-algebras $(A, T(Q))$ and $(A, T(Q^*))$. We prove that $(A, \wedge, \vee, \odot, T(Q))$ is a compact connected topological BL-algebra and $(A, T(Q^*))$ is a regular topological BL-algebra. We study separation axioms on $(A, T(Q))$ and $(A, T(Q^*))$. Also we stay conditions under which (A, Q) becomes totally bounded. Finally, we show that if

(A, Q) is a T_0 quasi-uniform space, then the BL-algebra $(\widetilde{A}, \widetilde{Q})$ in Theorem 3.11 is the bicomplition topological BL-algebra of (A, Q).

Theorem 4.1. *The set* $T(Q) = \{G \subseteq A : \forall x \in G\ \exists F \in \mathcal{F}$ *s.t* $F_\star(x) \subseteq G\}$ *is the topology induced by* Q *on* A *such that* $(A, \{\wedge, \vee, \odot\}, T(Q))$ *is a topological BL-algebras. Also* $(A, \rightarrow, T(Q))$ *is a left topological BL-algebra. Furthermore, if the negation map* $c(x) = x'$ *is one to one, then* $(A, T(Q))$ *is a topological BL-algebra.*

Proof. First we prove that $T(Q)$ is a nonempty set. For this, we prove that for each $F \in \mathcal{F}$ and each $x \in A$, $F_\star(x) \in T(Q)$. Let $F \in \mathcal{F}$, $x \in A$ and $y \in F_\star(x)$. If z is an arbitrary element of $F_\star(y)$, then $z \rightarrow y \in F$. Since $y \rightarrow x \in F$, by (B_{15}), we get $z \rightarrow x \in F$. Hence $F_\star(y) \subseteq F_\star(x)$ which implies that $F_\star(x) \in T(Q)$. Now we prove that $T(Q)$ is a topology on A. Clearly, $\phi, A \in T(Q)$. Also it is easy to prove that the arbitrary union of members of $T(Q)$ is in $T(Q)$. Let $G_1, ..., G_n$ be in $T(Q)$ and $x \in \bigcap_{i=1}^{i=n} G_i$. There are $F_1, ..., F_n \in \mathcal{F}$ such that $F_{i\star}(x) \subseteq G_i$, for $1 \leq i \leq n$. Let $F = F_1 \cap ... \cap F_n$. Then $F \in \mathcal{F}$ and $F_\star(x) \subseteq F_{1\star}(x) \cap ... \cap F_{n\star}(x) \subseteq \bigcap_{i=1}^{i=n} G_i$. Hence $T(Q)$ is a topology. Since for each $F \in \mathcal{F}$, $F_\star$ belongs to Q, then $T(Q)$ is the topology induced by Q. Now, by Lemmas 3.1, it is clear that $(A, \{\wedge, \vee, \odot\}, T(Q))$ is a topological BL-algebra. In continue, we prove that $(A, \rightarrow, T(Q))$ is a left topological BL-algebra. Let $x, y, z \in A$, and $z \in F_\star(y)$. By (B_9), $(x \rightarrow z) \rightarrow (x \rightarrow y) \geq z \rightarrow y$ which implies that $(x \rightarrow z) \rightarrow (x \rightarrow y) \in F$. So $x \rightarrow z \in F_\star(x \rightarrow y)$. Hence $x \rightarrow F_\star(y) \subseteq F_\star(x \rightarrow y)$ and so $(A, \rightarrow, T(Q))$ is a left topological BL-algebra.
To complete the proof, suppose that the negation map c is one to one. Since $(A, \rightarrow, T(Q))$ is a topological BL-algebra, c is continuous. Now by [[2], Theorem(3.15)], $(A, T(Q))$ is a topological BL-algebra. □

Theorem 4.2. *BL-algebra* $(A, T(Q))$ *is a connected and compact space and each* $F \in \mathcal{F}$, *is a closed compact set in* $(A, T(Q))$.

Proof. First we prove that if $\{G_i : i \in I\}$ is an open cover of A in $T(Q)$, then for some $i \in I$, $A = G_i$. Let $A = \bigcup_{i \in I} G_i$, where $G_i \in T(Q)$. Then, there are $i \in I$ and $F \in \mathcal{F}$ such that $1 \in G_i$ and $F_\star(1) \subseteq G_i$. By Lemma 3.1 (vi), $A = F_\star(1)$. Hence $A = G_i$. Now, it is easy to show that $(A, T(Q))$ is connected and compact. In continue we prove that each $F \in \mathcal{F}$, is a closed, compact set in $(A, T(Q))$. For this, let $F \in \mathcal{F}$ and $x \in \overline{F}$. Then, there is a $y \in F_\star(x) \cap F$. Since $y \in F$ and $y \rightarrow x \in F$, we get $x \in F$. Hence $\overline{F} = F$. Now, Since $(A, T(Q))$ is compact, F is compact. □

Theorem 4.3. (i) *BL-algebra* $(A, T(Q))$ *is not a* T_1 *and* T_2 *topological space.*
(ii) *BL-algebra* $(A, T(Q))$ *is a* T_0 *topological space iff, for each* $1 \neq x \in A$, *there is a* $F \in \mathcal{F}$ *such that* $x \notin F$.

Proof. (i) $(A, T(Q))$ is not a T_1 and T_2 topological space because for each $G \in T(Q)$, $1 \in G$ if and only if $G = A$.
(ii) Suppose for each $1 \neq x \in A$, there is a $F \in \mathcal{F}$ such that $x \notin F$. We prove that $(A, T(Q))$ is a T_0 topological space. For this, let $1 \neq x \in A$. Then for some $F \in \mathcal{F}$, $x \notin F$. Since $1 \to x = x$, then $1 \notin F_\star(x)$. Moreover, since $(A, \to, T(Q))$ is a left topological BL-algebra, by [[2], Proposition(4.2)], $(A, T(Q))$ is a T_0 topological space. Conversely, let $(A, T(Q))$ is a T_0 topological space and $1 \neq x \in A$. Then for some $F \in \mathcal{F}$, $1 \notin F_\star(x)$. Hence $x = 1 \to x \notin F$. □

Theorem 4.4. *The set* $T(Q^*) = \{G \subseteq A : \forall x \in G\ \exists F \in \mathcal{F}\ s.t\ F^*_\star(x) \subseteq G\}$ *is the topology induced by* Q^* *on BL-algebra* A *such that* $(A, T(Q^*))$ *is a topological BL-algebras.*

Proof. By the similar argument as Theorem 4.1, we can prove that $T(Q^*)$ is the topology induced by Q^* on A. By Lemma 3.2(v), $(A, T(Q^*))$ is a topological BL-algebra. □

Theorem 4.5. (i) *BL-algebra* $(A, T(Q^*))$ *is connected iff,* $\mathcal{F} = \{A\}$,
(ii) $\mathcal{F}$ *has only a proper filter iff, each* $F \in \mathcal{F}$ *is a component.*

Proof. (i) Let $\mathcal{F} = \{A\}$. Then it is easy to prove that $T(Q^*) = \{\phi, A\}$. Hence $(A, T(Q^*))$ is connected.
Conversely, let $\mathcal{F} \neq \{A\}$. Then, there is a filter $F \in \mathcal{F}$ such that $F \neq A$. Since for each $x \in F$, $F^*_\star(x) \subseteq F$, we conclude that $F \in T(Q^*)$. Let $y \in \overline{F}$. Then there is a $z \in F^*_\star(y) \cap F$. This proves that $y \in F$. Hence F is closed. Now, since F is a closed and open subset of A, then A is not connected.
(ii) Let $\mathcal{F}$ has a proper filter F. By the similar argument as (i), we get that F is closed and open. We show that F is connected. Let G_1 and G_2 be in $T(Q^*)$ and $F = (F \cap G_1) \cup (F \cap G_2)$. Without loss of generality, Suppose that $1 \in F \cap G_1$, then $F \subseteq F^*_\star(1) \subseteq G_1$. Hence $F \cap G_1 = F$, which implies that F is connected. Therefore, F is a component.
Conversely, suppose each $F \in \mathcal{F}$ is a component. If F_1 and F_2 are in $\mathcal{F}$, then $F_1 \cap F_2$ is in $\mathcal{F}$ and is component. Hence $F_1 = F_1 \cap F_2 = F_2$. □

Recall that a topological space $(X, \mathcal{U})$ is regular if for each $x \in G \in \mathcal{U}$ there is a $U \in \mathcal{U}$ such that $x \in U \subseteq \overline{U} \subseteq G$.

Theorem 4.6. *BL-algebra* $(A, T(Q^*))$ *is a regular space.*

Proof. First we prove that for each $F \in \mathcal{F}$ and $x \in A$, $\overline{F^*_\star(x)} = F^*_\star(x)$. Let $y \in \overline{F^*_\star(x)}$. Then there is a $z \in F^*_\star(y) \cap F^*_\star(x)$. Hence $y \equiv^F z \equiv^F x$ which implies that $y \in F^*_\star(x)$. Therefore, $\overline{F^*_\star(x)} = F^*_\star(x)$. Now if $x \in G \in T(Q^*)$, then for some a $F \in \mathcal{F}$, $x \in \overline{F^*_\star(x)} = F^*_\star(x) \subseteq G$. Hence $(A, T(Q^*))$ is a regular space. □

Theorem 4.7. *On BL-algebra $(A, T(Q^*))$ the follwing statements are equivalent.*
(i) $(A, T(Q^*))$ *is a* T_0 *space,*
(ii) $\bigcap_{F\in\mathcal{F}} F^*_\star(1) = \{1\}$,
(iii) $(A, T(Q^*))$ *is a* T_1 *space,*
(iv) $(A, T(Q^*))$ *is a* T_2 *space.*

Proof. $(i \Rightarrow ii)$ Let $(A, T(Q^*))$ be a T_0 space and $1 \neq x \in A$. By [[2], Proposition(4.2)], there is a $F \in \mathcal{F}$ such that $1 \notin F^*_\star(x)$. Hence $x \notin F$. This implies that $x \notin F^*_\star(1)$. Therefore, $x \notin \bigcap_{F\in\mathcal{F}} F^*_\star(1)$.
$(ii \Rightarrow i)$ Let $\bigcap_{F\in\mathcal{F}} F^*_\star(1) = \{1\}$ and $1 \neq x \in A$. Then for some a $F \in \mathcal{F}$, $x \notin F$. Hence $1 \notin F^*_\star(x)$. Now by [[2], Proposition(4.2)], $(A, T(Q^*))$ is a T_0 space.
By Theorems 4.4 and 4.6, $(A, T(Q^*))$ is a regular topological BL-algebra. Hence by [[2], Theorem(4.7)], the statements (ii), (iii) and (iv) are equivalent. □

Example 4.8. *In Example 3.4, For each $a \in [0, 1)$ and $x \in [0, 1]$*

$$F_{a*}(x) = \begin{cases} [0,x] & , x \leq a, \\ [0,1] & , x > a. \end{cases} \quad F^{-1}_{a*}(x) = \begin{cases} [x,1] & , x \leq a, \\ (a,1] & , x > a. \end{cases}$$

$$F^*_{a*}(x) = \begin{cases} x & , x < a, \\ a & , x = a \\ (a,1] & , x > a. \end{cases}$$

If $T(Q)$ is the induced topology by Q and $G \in T(Q)$, then for each $x \in G$, there is a $a \in [0, 1)$ such that $F^*_{a\star}(x) \subseteq G$. Hence $[0, x] \subseteq G$ or $G = [0, 1]$. If $G \in T(Q)$ and $G \neq [0, 1]$, then for each $x \in G$, $[0, x] \subseteq G$. If $g = supG$, then $G = [0, g]$ or $[0, g)$. Therefore $T(Q) = \{[0, x] : x \in [0, 1]\} \cup \{[0, x) : x \in [0, 1]\}$. Also if $T(Q^*)$ is topology induced by Q^* and $G \in T(Q^*)$, then for each $x \in G$, there is a $a \in [0, 1)$ such that $F^*_{a\star}(x) \subseteq G$. Hence if $G \in T(Q^*)$, then for some $a \in [0, 1)$, $a \in G$ or $(a, 1] \subseteq G$.
Now since for each $a \in [0, 1)$, $F^*_{a\star}(1) = (a, 1]$, we get that $\bigcap_{a\in[0,1)} F^*_{a\star}(1) = \{1\}$. Hence by Theorems 4.4, 4.6 and 4.7, $(A, T(Q^*))$ is a T_i regular topological BL-algebra, when $0 \leq i \leq 2$.

Theorem 4.9. *Let $(A, \rightarrow, \mathcal{U})$ be a semitopological BL-algebra and F_0 be an open proper BL-filter in A. Then, there exists a nontrivial topology $\mathcal{V}$ on A such that $\mathcal{V} \subseteq \mathcal{U}$ and $(A, \mathcal{V})$ is a topological BL-algebra.*

Proof. Let $\mathcal{F}$ be a collection of BL-open filters in A which closed under finite intersection and $F_0 \in \mathcal{F}$. Let Q be the quasi-uniformity induced by $\mathcal{F}$. Since

$F_0 \neq A$, by Lemma 3.1(vi), there is a $x \in A$ such that $F^*_{0\star}(x) \neq A$. So $T(Q^*)$ is a nontrivial topology. We prove that $T(Q^*) \subseteq \mathcal{U}$. Let $x \in G \in T(Q^*)$. Then, there is a $F \in \mathcal{F}$ such that $F^*_\star(x) \subseteq G$. Since $x \to x = 1 \in F \in \mathcal{U}$, there is a $U \in \mathcal{U}$ such that $x \in U$ and $U \to x \subseteq F$ and $x \to U \subseteq F$. If $z \in U$, then $z \to x, x \to z \in F$ and so $z \in F^*_\star(x)$. Hence $x \in U \subseteq G$. Therefore, $T(Q^*)$ is a nontrivial topology coaser than $\mathcal{U}$ and so by Theorem 4.4, $(A, T(Q^*))$ is a topological BL-algebra. $\square$

Example 4.10. *Let $\mathcal{I}$ be the BL-algebra in Example 2.5(ii), and $\mathcal{U}$ be a topology on $\mathcal{I}$ with the base $S = \{(a, b] \cap \mathcal{I} : a, b \in \mathbb{R}\}$. We prove that $(\mathcal{I}, \to, \mathcal{U})$ is a semitopological BL-algebra. Let $x, y \in I$, and $x \to y \in (a, b]$. If $x \leq y$, then $[0, x]$ and $(ax, y]$ are two open neighborhoods of x and y, respectively, such that $(0, x] \to y \subseteq (a, 1]$ and $x \to (ax, y] \subseteq (a, 1]$. If $x > y$ and $y = 0$, then $(0, x]$ and $\{0\}$ are two open neighborhoods of x and 0, respectively, such that $(0, x] \to 0 \subseteq [0, b]$ and $x \to \{0\} \subseteq [0, b]$. If $x > y$ and $y \neq 0$, then $(y/b, y/a]$ and $(ax, bx]$ are two open sets of x, y, respectively, such that $(y/b, y/a] \to y \subseteq (a, b]$ and $x \to (ax, bx] \subseteq (a, b]$. It is easy to prove that $\mathcal{F} = \{(0, 1], A\}$ is a collection of BL-filters which is closed under intersection. Now since for each $x \in A$, $A^*_\star(x) = A$ and $(0, 1]^*_\star(x) = (0, 1]$, we conclude $T(Q^*) = \{\phi, (0, 1], A\}$. By Theorem 4.9, $(A, T(Q^*))$ is a topological BL-algebra.*

Recall a quasi-uniform space (X, Q) is *totally-bounded* if for each $q \in Q$, there exist sets $A_1, ..., A_n$ such that $X = \bigcup_{i=1}^{i=n} A_i$ and for each $1 \leq i \leq n$, $A_i \times A_i \subseteq q$.(See [10])

Theorem 4.11. *The following conditions on BL-algebra $(A, T(Q^*))$ are equivalent.*
(i) For each $F \in \mathcal{F}$, A/F is finite,
(ii) (A, Q) is totally bounded,
(iii) $(A, T(Q^*))$ is compact.

Proof. ($i \Rightarrow ii$) Let for each $F \in \mathcal{F}$, A/F be finite. We prove that (A, Q) is totally bounded. For this it is enough to prove that, for each $F \in \mathcal{F}$, there are $a_1, ..., a_n \in A$, such that for each $1 \leq i \leq n$, $a_i/F \times a_i/F \subseteq F_\star$. Let $F \in \mathcal{F}$. Since A/F is finite, there are $a_1, ..., a_n \in A$, such that $A = \cup_{i=1}^n a_i/F$. For each $1 \leq i \leq n$, $a_i/F \times a_i/F \subseteq F_\star$ because if $(x, y) \in a_i/F \times a_i/F$, then $x \equiv^F a_i \equiv^F y$ and so $(x, y) \in F_\star$. This proves that (A, Q) is totally bounded.
($ii \Rightarrow iii$) Let (A, Q) be totally bounded and $F \in \mathcal{F}$. There exist sets $A_1, ..., A_n$, such that $\bigcup_{i=1}^{i=n} A_i = A$ and for each $1 \leq i \leq n$, $A_i \times A_i \subseteq F_\star$. Let $1 \leq i \leq n$ and $x, y \in A_i$. Since (x, y) and (y, x) are in $F_\star$, we get $x \equiv^F y$. This proves that $A_i = a_i/F$, for some $a_i \in A_i$.

Now to prove that $(A, T(Q^*))$ is compact let $A = \bigcup_{i \in I} G_i$, where each G_i is in $T(Q^*)$. Then there are $H_1, ..., H_n \in \{G_i : i \in I\}$, such that $a_i \in H_i$, for each $1 \leq i \leq n$. Now suppose $x \in A$, then $x \in a_i/F$, for some $1 \leq i \leq n$, and so $x \in F^*_\star(a_i) \subseteq H_i$. Therefore, $A \subseteq \bigcup_{i=1}^n H_i$, which shows that $(A, T(Q^*))$ is compact.
$(iii \Rightarrow i)$ Let $F \in \mathcal{F}$. Since $\{F^*_\star(x) : x \in A\}$ is an open cover of A in $T(Q^*)$, then there are $a_1, ..., a_n \in A$, such that $A \subseteq \bigcup_{i=1}^n F^*_\star(a_i)$. Now, it is easy to see that $A/F = \{a_1/F, ..., a_n/F\}$.

□

In the end, we prove that the quasi-uniform Bl-algeba $(\widetilde{A}, \widetilde{Q})$ in Theorem 3.11, is T_0 bicomplition quasi-uniform of BL-algebra (A, Q).

Theorem 4.12. *If quasi-uniform BL-algebra (A, Q) is T_0, then*
(i) $(\widetilde{A}, \widetilde{Q})$ is the bicompletion of (A, Q).
(ii) $(\widetilde{A}, T(\widetilde{Q}))$ is a topological BL-algebra.
(iii) A is a sub BL-algebra of $\widetilde{A}$.
(iv) $(\widetilde{A}, T(\widetilde{Q^*}))$ is a topological BL-algebra.

Proof. (i) By Theorem 3.11 and Lemma 2.18, $(\widetilde{A}, \widetilde{Q})$ is an unique T_0-bicompletion quasi-uniform of (A, Q) and the mapping $i : A \to \widetilde{A}$ by $i(x) = \{W \subseteq A : W \textit{ is a } T(Q^*) - \textit{neighborhood of } x\}$ is a quasi-uniform embedded and $cl_{T(Q^*)} i(A) = \widetilde{A}$.
(ii) It is clear that

$$T(\widetilde{Q}) = \{S \subseteq \widetilde{A} : \forall \mathcal{G} \in S \ \exists F \in \mathcal{F} \ s.t \ \widetilde{F_\star}(\mathcal{G}) \subseteq S\}.$$

Let $\bullet \in \{\wedge, \vee, \odot\}$ and $\widetilde{\bullet} \in \{\curlywedge, \curlyvee, \circledcirc\}$. We have to prove that for each $\mathcal{G}, \mathcal{H} \in \widetilde{A}$, $\widetilde{F_\star}(\mathcal{G}) \widetilde{\bullet} \widetilde{F_\star}(\mathcal{H}) \subseteq \widetilde{F_\star}(\mathcal{G} \widetilde{\bullet} \mathcal{H})$. Let $\mathcal{G}_1 \in \widetilde{F_\star}(\mathcal{G})$ and $\mathcal{H}_1 \in \widetilde{F_\star}(\mathcal{H})$. Then, there are $G \in \mathcal{G}$, $G_1 \in \mathcal{G}_1$, $H \in \mathcal{H}$ and $H_1 \in \mathcal{H}_1$ such that $G \times G_1 \subseteq F_\star$, $H \times H_1 \subseteq F_\star$. By Proposition 3.7, $S_1 = \{F^*_\star(G \bullet H) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\}$ and $S_2 = \{F^*_\star(G_1 \bullet H_1) : G_1 \in \mathcal{G}_1, H_1 \in \mathcal{H}_1, F \in \mathcal{F}\}$ are bases of $\mathcal{G} \widetilde{\bullet} \mathcal{H}$ and $\mathcal{G}_1 \widetilde{\bullet} \mathcal{H}_1$, respectively. We show that $\mathcal{G}_1 \widetilde{\bullet} \mathcal{H}_1 \in \widetilde{F_\star}(\mathcal{G} \widetilde{\bullet} \mathcal{H})$. For this, it is enough to show that $F^*_\star(G \bullet H) \times F^*_\star(G_1 \bullet H_1) \subseteq F_\star$. Let $(y, y_1) \in F^*_\star(G \bullet H) \times F^*_\star(G_1 \bullet H_1) \subseteq F_\star$. Then, there are $g \in G$, $g_1 \in G_1$, $h \in H$ and $h_1 \in H_1$ such that $y \equiv^F g \bullet h$ and $y_1 \equiv^F g_1 \bullet h_1$. By $(B_{17}), (B_{18})$ and (B_{19}), we have $(g_1 \to g) \odot (h_1 \to h) \leq (g_1 \bullet h_1) \to (g \bullet h)$. It follows from $(g, g_1) \in G \times G_1 \subseteq F_\star$ and $(h, h_1) \in H \times H_1 \subseteq F_\star$ that $g_1 \to g$ and $h_1 \to h$ are in F. Hence $g_1 \bullet h_1 \to g \bullet h \in F$. Therefore, $y_1 \to y \in F$ and so $(y, y_1) \in F_\star$. Thus we proved that $\widetilde{F_\star}(\mathcal{G}) \widetilde{\bullet} \widetilde{F_\star}(\mathcal{H}) \subseteq \widetilde{F_\star}(\mathcal{G} \widetilde{\bullet} \mathcal{H})$.
(iii) Let $\bullet \in \{\wedge, \vee, \odot, \to\}$, $\widetilde{\bullet} \in \{\curlywedge, \curlyvee, \circledcirc, \hookrightarrow\}$ and $a, b \in A$. We shall prove

that $i(a)\widetilde{\bullet}i(b) = i(a \bullet b)$. By Proposition 3.7, the set $S = \{F^*_\star(W_a \bullet W_b) : F \in \mathcal{F},\ W_a, W_b\ are\ T(Q^*) - neighborhoods\ of\ a, b\}$ is a base for $i(a)\widetilde{\bullet}i(b)$. Since $F^*_\star(a \bullet b) \subseteq F^*_\star(W_a \bullet W_b)$ and $F^*_\star(a \bullet b) \in i(a \bullet b)$, we deduce that filter $i(a)\widetilde{\bullet}i(b)$ is contained in the filter $i(a \bullet b)$. Since they are minimal Q^*-cauchy filters, $i(a)\widetilde{\bullet}i(b) = i(a \bullet b)$. Hence A is a sub-BL-algebra of $\widetilde{A}$.
(iv) By Lemma 2.18, $\widetilde{Q^*} = (\widetilde{Q})^*$. Hence

$$T(\widetilde{Q^*}) = \{S \subseteq \widetilde{A} : \forall \mathcal{G} \in S\ \exists F \in \mathcal{F}\ s.t\ \widetilde{F^*_\star}(\mathcal{G}) \subseteq S\}.$$

We prove that $(\widetilde{A}, T(\widetilde{Q^*}))$ is a topological BL-algebra. Let $\bullet \in \{\wedge, \vee, \odot, \rightarrow\}$ and $\widetilde{\bullet} \in \{\curlywedge, \curlyvee, \circledcirc, \hookrightarrow\}$ and let $\mathcal{G}\widetilde{\bullet}\mathcal{H} \in \widetilde{F^*_\star}(\mathcal{G}\widetilde{\bullet}\mathcal{H})$. We show that $\widetilde{F^*_\star}(\mathcal{G})\widetilde{\bullet}\widetilde{F^*_\star}(\mathcal{H}) \subseteq \widetilde{F^*_\star}(\mathcal{G}\widetilde{\bullet}\mathcal{H})$. Let $\mathcal{G}_1 \in \widetilde{F^*_\star}(\mathcal{G})$ and $\mathcal{H}_1 \in \widetilde{F^*_\star}(\mathcal{H})$. Then, there are $G \in \mathcal{G}$, $G_1 \in \mathcal{G}_1$, $H \in \mathcal{H}$ and $H_1 \in \mathcal{H}_1$ such that $G \times G_1 \subseteq F^*_\star$ and $H \times H_1 \subseteq F^*_\star$. By Proposition 3.7, $F^*_\star(G_1 \bullet H_1) \in \mathcal{G}_1\widetilde{\bullet}\mathcal{H}_1$ and $F^*_\star(G \bullet H) \in \mathcal{G}\widetilde{\bullet}\mathcal{H}$. We have to prove that $\mathcal{G}_1\widetilde{\bullet}\mathcal{H}_1 \in \widetilde{F^*_\star}(\mathcal{G}\widetilde{\bullet}\mathcal{H})$. For this, it is enough to show that $F^*_\star(G \bullet H) \times F^*_\star(G_1 \bullet H_1) \subseteq F^*_\star$. Let $y \in F^*_\star(G \bullet H)$ and $y_1 \in F^*_\star(G_1 \bullet H_1)$. Then $y \equiv^F g \bullet h$ and $y_1 \equiv^F g_1 \bullet h_1$ for some $g \in G$, $g_1 \in G_1$, $h \in H$ and $h_1 \in H_1$. Since $(g, g_1), (h, h_1)$ are in $F^*_\star$, we get $g \bullet h \equiv^F g_1 \bullet h_1$. Hence $(y, y_1) \in F^*_\star$. $\square$

5 Conclusions

The aim of this paper is to study In [2] and [4] we study (semi)topological BL-algebras and metrizability on BL-algebras. We showed that continuity the operations $\odot$ and $\rightarrow$ imply continuity $\wedge$ and $\vee$. Also, we found some conditions under which a locally compact topological BL-algebra become metrizable. But in there we can not answer some questions, for example:
(i) Is there a topology $\mathcal{U}$ on BL-algera A such that $(A, \mathcal{U})$ be a (semi)topological BL-algebra?
(ii) Is there a topology $\mathcal{U}$ on a BL-algebra A such that $(A, \mathcal{U})$ be a compact connected topological BL-algebra?
(iii) Is there a topological BL-algebra $(A, \mathcal{U})$ such that T_0, T_1 and T_2 spaces be equivalent?
(iv) If $(A, \rightarrow, \mathcal{U})$ is a semitopological BL-algebra, is there a topology $\mathcal{V}$ coarsere than $\mathcal{U}$ or finer than $\mathcal{U}$ such that $(A, \mathcal{V})$ be a (semi)topological BL-algebra?

Now in this paper, we answered to some above questions and got some interesting results as mentioned in abstract.

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Case Studies on the Application of Fuzzy Linear Programming in Decision-Making

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Abstract

This study demonstrated the effectiveness of fuzzy method in decision-making and recommends the integration of fuzzy methods in decision-making in production, transportation, power production and distribution and utility maintenance in Nigeria companies.
Keywords: Fuzzy set, Fuzzy linear programming, Fuzzy constraints, Fuzzy optimization.

1 Introduction

Linear Programming (LP), an important tool in operations research, has developed over the years in solving management problems [13]. It is in two forms: classical and fuzzy linear programming. It takes various linear inequalities relating to the situation being considered and finds the best value obtainable under that situation.

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Let $A_i's$ be the constraint functions and $b_i's$ the available resources. Generally, a linear programming problem can be written as

$$Min(Max)\ \ z = cx \tag{1}$$

subject to $A_i(x) \leq b_i,$ where $x \geq 0$. In practice, all of the needed information such as c, $A_i's$, b_is are not completely available or determined; these parameters are uncertain and are said to be fuzzy variables [10].

A typical mathematical programming problem is to optimise an objective function subject to some constraints. Usually, the classes of objects encountered in the real world do not have clearly defined criteria of membership. Hence, constraints and objective functions could be fuzzy [25].

In production processes, hardly does the firm utilize the exact resources available to meet a proposed target. This may be due to waste in the cause of production and/or machine wear and tear over time or some other factors due to exigency. Thus, a firm is required to optimally plan around its available resources.

Having recognized the shortcomings of traditional mathematical models in some areas of real life application, Zadeh (1965) proposed the notion of a fuzzy set. It began as an effort to use mathematics to define such concepts as "slightly" or "tall" or "fast" or "beautiful" or any other concept that has ambiguous boundaries. The fuzzy set theory was developed to improve the over simplified model, thereby developing a more robust and flexible model in order to solve real-world complex systems involving human aspects.

Fuzziness was modeled by membership functions which might be described as an extension of the usual characteristic function in the setting of mathematical sets [16]. Fuzzification offers superior expressive power, greater generality and an improved capability to model complete problems at a low solution cost. The application of fuzzy set theory is claimed to be effective in decision making and coordinating multiple system requirements [18],[11]. Thus, it is an excellent method for planning and making decision under uncertainty.

2 Preliminaries

Definition 2.1. [23] *A fuzzy set A in* X *is a set of ordered pairs* $A = \{(x, \mu_A(x)) : x \in X\}$*, where* $\mu_A(x)$ *is the grade of membership of* $x \in A$ *and* $\mu_A : X \longrightarrow [0, 1]$.

Example 2.1. [25] *Let* $X = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110\}$ *be possible speeds(mph) at which cars can cruise over long distances. Then the fuzzy set A of "uncomfortable speeds for long distances" may be defined by a certain individual as:*

$$A = \{(30, 0.7), (40, 0.75), (50, 0.8), (60, 0.8), (70, 1.0), (80, 1.0), (90, 1.0)\},$$

where 0.7, 0.75, 0.8 and 1.0 are the degree of uncomfortability, attaining "certainly uncomfortable" at ≥ 70 *mph.*

Definition 2.2. [14] *The support of a fuzzy set* A, $S(A) = \{x \in A : \mu_A(x) > 0\}$.

Definition 2.3. [23] *A fuzzy set* A *is empty if and only if* $\mu_A(x) = 0, \forall x \in X$.

Definition 2.4. [23] *Two fuzzy sets* A *and* B *are equal if and only if* $\mu_A(x) = \mu_B(x)$, $\forall x \in X$.

Definition 2.5. [23] *A fuzzy set* A *is contained in a fuzzy set* B*, written as* $A \subseteq B$*, if and only if* $\mu_A(x) \leq \mu_B(x)$.

Definition 2.6. [23] *The intersection of two fuzzy sets* A *and* B *is denoted by* $A \cap B$ *and is defined as the largest fuzzy set contained in both* A *and* B*. The membership function of* $A \cap B$ *is given by* $\mu_A(x) \wedge \mu_B(x) = min\{\mu_A(x), \mu_B(x), \forall x \in X\}$.

Example 2.2. *Consider the following set of cars,*

$$X = \{Mercedez,\ Camry,\ Chevrolet,\ Accord\}.$$

Suppose A *is the fuzzy subset of "durable cars" and* B *is the fuzzy subset of "fast cars".*

$$A = \{0.8/Me,\ 0.6/Ac,\ 0.5/Ca,\ 0.3/Ch\}$$

and

$$B = \{0.3/Me,\ 0.8/Ac,\ 0.6/Ca,\ 1.0/Ch\}.$$

The intersection of A *and* B,

$$\mu_A(x) \wedge \mu_B(x) = \{0.3/Me,\ 0.6/Ac,\ 0.5/Ca,\ 0.3/Ch\},$$

is the fuzzy subset of the degree of compatibility of the quality of the cars being "durable and fast".

Definition 2.7. [23] *The union of* A *and* B*, denoted as* $A \cup B$*, is defined as the smallest fuzzy set containing both* A *and* B*. The membership function of* $A \cup B$ *is given by* $\mu_A(x) \vee \mu_B(x) = max\{\mu_A(x), \mu_B(x), \forall x \in X\}$.

Example 2.3. *Consider the following set of cars,*

$$X = \{Mercedez,\ Camry,\ Chevrolet,\ Accord\}.$$

Suppose A *is the fuzzy subset of "durable cars" and* B *is the fuzzy subset of "fast cars". Consider* A *and* B *as in Example 2.2. The union of* A *and* B,

$$\mu_A(x) \vee \mu_B(x) = \{0.8/Me,\ 0.8/Ac,\ 0.6/Ca,\ 1.0/Ch\},$$

is the fuzzy subset of the degree of the quality of either "durable or fast or both".

Definition 2.8. [23] *If A is a fuzzy subset of X, then an α-level set of A is a non-fuzzy set A_α which comprises all elements of X whose grade of membership is greater than or equal to α. It is denoted by $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha \; \forall \; x \in X\}$.*

Example 2.4. *The intelligence quotient of students were tested and some were discovered to possess high intelligence quotient while some very low. Let $FSIQ$ be fuzzy set of intelligence quotient.*

$$FSIQ = \{(C, 0.9), (M, 0.7), (B, 0.5), (S, 0.4), (P, 0.3)\}.$$

Then, $A_{0.5} = (B, M, C)$.

3 Methodology

The data were collected from two places: the production data for two products from a Water Venture and the value-added services provided by an Institute of Economic and Law, both in Oyo State, Nigeria.

3.1 Fuzzy Linear Programming Models

In fuzzy linear programming, the fuzziness of available resources is characterised by the membership function over the tolerance range. The general model of linear programming with fuzzy resources is:

$$Max(Min)z = cx, \tag{2}$$

subject to $(s.t.)\ A_i(x) \leq \tilde{b_i},\ i = 1, 2, ..., m, x \geq 0$, where, for each i, $A_i(x)'s$ are the m constraints, $\underset{\sim}{b_i} \in [b_i, b_i + p_i]$ are the real numbers representing the quantities of each fuzzy resources and $p_i's$ are the tolerance levels of the decision-maker for each of the resources.
The fuzzy linear programming may also be considered as:

$$Max(Min)\ z = cx, \tag{3}$$

subject to $(s.t.)\ A_i(x) \lessapprox b_i,\ i = 1, 2, ..., m, x \geq 0$, where $\lessapprox$ is called "fuzzy less than or equal to". If the tolerance p_i is known for each fuzzy constraint, $A_i(x) \lessapprox b_i$ could be seen as $A_i(x) \leq (b_i + \theta p_i)$, for all i, where $\theta \in [0, 1]$.

3.2 Verdegay's Approach- A Nonsymmetric Model

Verdegay [21] considered that if the membership functions of the fuzzy constraints.

$$\mu_i(x) = \begin{cases} 1, \text{ if } A_i(x) < b_i \\ \\ 1 - \frac{A_i(x) - b_i}{p_i}, \; b_i \leq A_i(x) \leq b_i + p_i, i = 1, ..., m+1 \\ \\ 0, \; A_i(x) > b_i + p_i \end{cases} \tag{4}$$

are continuous and monotonic functions, and trade-off between those fuzzy constraints are allowed, the general model of linear programming with fuzzy resources will be equivalent to:

$$Max\ cx, \quad s.t\ x \in X_\alpha, \tag{5}$$

where $X_\alpha = \{x : \mu(x) \geq \alpha, x \geq 0, \text{ for each } \alpha \in [0,1]\}$. The α-level concept is based on the work of [20]. It is indicated in the membership function that if $A_i(x) \leq b_i$ then the $i-th$ constraint is satisfied and $\mu_i(x)$ = 1. But, on the other hand, if $A_i(x) \geq b_i + p_i$, where p_i is the maximum tolerance from b_i, (which is always determined by the decision-maker), then the $i-th$ constraint is violated at this point and $\mu_i(x) = 0$. Finally, if $A_i(x) \in (b_i, b_i + p_i)$, then the membership function is monotonically decreasing and, the less satisfied the decision-maker becomes. Using parametric programming, where $\alpha = 1 - \theta$, we can substitute membership function of Equation (4) into (5) and the problem below is obtained:

$$Max\ cx, \quad s.t\ (Ax)_i \leq b_i + (1-\alpha)p_i, \ \forall i, \tag{6}$$

for $x \geq 0$ and $\alpha \in [0,1]$.

4 Result Analysis and Discussions

In this section, fuzzy linear programming method is applied to some cases to optimize the decisions. These are the cases of a Water Venture and an Institute.

4.1 The Water Ventures

The study was based on two different bottles of water which the Venture produces : $75cl$ and $50cl$. It makes $134.62NGN$ per carton of $50cl$ and $150.26NGN$ per carton of $75cl$ as profits. The firm employs machine for 7 hours in a day, with

Basic Variables	x_1	x_2	g_1	g_2	b
x_1	1	1.189	23.681	0	166.573
g_2	0	0.032	-1.003	1	1.003
p	0	10.002	3,204.03	0	22,437.129

Table 1: Final Solution to Equation (7) by Simplex Method

tolerance level of 2 hours and labor for 8 hours with tolerance level of 1 hour. The classical linear programming problem is constructed thus:

$$Max\ p = 134.62x_1 + 150.26x_2, \tag{7}$$

$s.t.\ g_1(x) = 0.042x_1 + 0.05x_2 \leq 7, g_2(x) = 0.042x_1 + 0.082x_2 \leq 8.$

where g_1 is machine time, g_2 is labour time, x_1 is the 50cl bottle water and x_2 is the 75cl bottle water. The final result of the simplex method is in Table 1.

The fuzzy membership function of the machine time:

$$\mu_1(x) = \begin{cases} 1, \text{ if } g_1(x) \leq 7 \\ \\ 1 - \frac{g_1(x)-7}{2}, \ 7 < g_1(x) < 9 \\ \\ 0, \ g_1(x) \geq 9 \end{cases} \tag{8}$$

The membership function of the labour time:

$$\mu_2(x) = \begin{cases} 1, \text{ if } g_2(x) \leq 8 \\ \\ 1 - \frac{g_2(x)-8}{1}, \ 8 < g_2(x) < 9 \\ \\ 0, \ g_2(x) \geq 9 \end{cases} \tag{9}$$

The fuzzy linear programming problem associated with Equation (7) is

$$Max\ p = 134.62x_1 + 150.26x_2, \tag{10}$$

$s.t.\ \mu_1(x) \geq \alpha,\ \mu_2(x) \geq \alpha,$

where $\alpha \in [0, 1]$ and $x_1, x_2 \geq 0$. The fuzzy linear programming problem is expanded thus:

$$Max\ p = 134.62x_1 + 150.26x_2, \tag{11}$$

$s.t.\ g_1 = 0.042x_1 + 0.05x_2 \leq 7 + 2(1 - \alpha),$ and $g_2(x) = 0.042x_1 + 0.082x_2 \leq 8 + (1 - \alpha)$, where $x_1, x_2 \geq 0$ and $\alpha \in [0, 1]$.

Basic Variables	x_1	x_2	g_1	g_2	b
x_1	1	1.189	23.681	0	166.573 + 47.362θ
g_2	0	0.032	-1.003	1	1.003 - 1.006θ
p	0	10.002	3,204.03	0	22,437.129 + 6,408.06θ

Table 2: Solution to the fuzzy linear programming Equation (12)

θ	p^*	x_1^*	g_1	g_2
0.0	22,437.13	166.573	6.996	6.663
0.1	22,077.94	171.309	7.195	6.852
0.2	23,718.74	176.045	7.394	7.042
0.3	24,359.55	180.782	7.593	7.231
0.4	25,000.35	185.518	7.792	7.421
0.5	25,641.16	190.254	7.991	7.610
0.6	26,281.97	194.990	8.189	7.799
0.7	26,922.77	199.726	8.389	7.989
0.8	27,563.58	204.463	8.587	8.178
0.9	28,204.38	209.199	8.786	8.368
1.0	28,845.19	213.935	8.925	8.557

Table 3: Result of the Parametric Problem

Setting $\theta = 1 - \alpha$, the programming problem above becomes

$$Max\ p = 134.62x_1 + 150.26x_2 \tag{12}$$

s.t. $g_1 = 0.042x_1 + 0.05x_2 \leq 7 + 2\theta$, $g_2(x) = 0.042x_1 + 0.082x_2 \leq 8 + \theta$, $x_1, x_2 \geq 0$, where $\theta \in [0, 1]$ is a parameter determining the tolerance level. Using the parametric technique and final result of simplex method, Table 2 was obtained.

The optimal solution is

$$(x_1^*, x_2^*) = (166.573 + 47.362\theta, 0)$$

and $p^* = 22,424.06 + 6,375.87\theta$. Therefore, the solution of the parametric programming problem is in Table 3.

From the analysis above, it is observed that the Water Venture could make more profit by producing more of 50cl bottles than producing 75cl bottles. In essence, it will be more profitable for the firm to scale up its production of 75cl bottle water and cut down the production of 50cl.

$Basis$	E	L	g_1	g_2	g_3	b
g_1	0	$\frac{2}{3}$	1	0	$\frac{-1}{1,440}$	$\frac{710}{3}$
g_2	0	$\frac{2}{3}$	0	1	$\frac{-1}{1,440}$	$\frac{4,496}{3}$
E	1	$\frac{1}{3}$	0	0	$\frac{1}{1,440}$	$\frac{4}{3}$
p	0	$\frac{259}{3}$	0	0	$\frac{235}{1,440}$	$\frac{940}{3}$

Table 4: Final Result of the Simplex Method

4.2 The Institute of Energy Law and Energy Economics

This section seeks to maximise profit and minimise cost in the sessional operation of the institute based on tuition alone. Annually, the institute admits Law and Energy Studies students.

On each Law student, the institute makes a loss of approximately 8,000NGN and on each Energy Studies student, a profit of approximately 235,000NGN. For both Energy Law and Energy Economics, if the institute is willing to spend 238,000NGN with tolerance of 70,000NGN on internet, 1,500,000NGN with tolerance of 500,000NGN on conference support, and 3 graduate assistant for Energy Study, 1 graduate assistant for Energy Law, with tolerance of 2 additional graduate assistants, the following will be the linear programming problem.

$$Max\ p(E, L) = 235E - 8L, \tag{13}$$

$s.t.\ g_1(E, L) = E+L \leq 238(Internet),\ g_2(E, L) = E+L \leq 1,500(Coference\ Support)$ and $g_3(E, L) = 1,440E + 480L \leq 1,920(GraduateAssistants)$.
where E is Energy Studies, L is Energy Law, g_1 is Internet, g_2 is Conference Support and g_3 is Graduate Assistants.
Using the Simplex method, Table 4 was obtained.

The membership function of Internet

$$\mu_1(E, L) = \begin{cases} 1,\ \text{if } g_1(E, L) \leq 238 \\ 1 - \frac{g_1(E,L)-238}{70},\ 238 < g_1(E, L) < 308 \\ 0,\ g_1(E, L) \geq 308 \end{cases} \tag{14}$$

$Basis$	E	L	g_1	g_2	g_3	b
g_1	0	$\frac{2}{3}$	1	0	$\frac{-1}{1{,}440}$	$\frac{710}{3} + \frac{208\theta}{3}$
g_2	0	$\frac{2}{3}$	0	1	$\frac{-1}{1{,}440}$	$\frac{4{,}496}{3} + \frac{1{,}498\theta}{3}$
E	1	$\frac{1}{3}$	0	0	$\frac{1}{1{,}440}$	$\frac{4}{3} + \frac{2\theta}{3}$
p	0	$\frac{259}{3}$	0	0	$\frac{235}{1{,}440}$	$\frac{940}{3} + \frac{470\theta}{3}$

Table 5: Matrix Multiplication of the Simplex Method Solution and the Tolerance Level

The membership function of Conference Support

$$\mu_2(E, L) = \begin{cases} 1, \text{ if } g_2(E, L) \leq 1,500 \\ \\ 1 - \frac{g_2(E,L) - 1{,}500}{500}, \ 1,500 < g_2(E, L) < 2,000 \\ \\ 0, \ g_2(E, L) \geq 2,000 \end{cases} \tag{15}$$

The membership function of Graduate Assistants

$$\mu_3(E, L) = \begin{cases} 1, \text{ if } g_3(E, L) \leq 1,920 \\ \\ 1 - \frac{g_3(E,L) - 1{,}440}{960}, \ 1,920 < g_3(E, L) < 2,880 \\ \\ 0, \ g_3(E, L) \geq 2,880 \end{cases} \tag{16}$$

The fuzzy linear programming is

$$Max\ p(E, L) = 235E - 8L, \tag{17}$$

$s.t.$ $g_1(E, L) = E + L \leq 238 + 70(1 - \alpha)$ $g_2(E, L) = E + L \leq 1,500 + 500(1 - \alpha)$ and $g_3(E, L) = 1,440E + 480L \leq 1,920 + 960(1 - \alpha)$.
Setting $\theta = 1 - \alpha$, the following is the parametric problem:

$$Max\ p = 235E - 8L, \tag{18}$$

$s.t.$ $g_1(E, L) = E + L \leq 238 + 70\theta,\ g_2(E, L) = E + L \leq 1,500 + 500\theta$ and $g_3(E, L) = 1,440E + 480L \leq 1,920 + 960\theta$, where $\theta \in [0, 1]$ is a parameter given the tolerance level.
Using the parametric technique and final result of simplex method, Table 5 was obtained.

θ	E^*	p^*	Internet	Conf. Supp.	G.A
0.0	1.33	313.33	1.33	1.33	1,920.00
0.1	1.40	329.00	1.40	1.40	2,016.00
0.2	1.47	344.67	1.47	1.47	2,112.00
0.3	1.53	360.33	1.53	1.53	2,208.00
0.4	1.60	376.00	1.60	1.60	2,304.00
0.5	1.67	391.67	1.67	1.67	2,400.00
0.6	1.73	407.33	1.73	1.73	2,496.00
0.7	1.80	423.00	1.80	1.80	2,592.00
0.8	1.87	438.67	1.87	1.87	2,688.00
0.9	1.93	454.33	1.93	1.93	2,784.00
1.00	2.00	470.00	2.00	2.00	2,880.00

Table 6: Result of the Parametric Problem

The optimal solution is

$$p^* = (\frac{940}{3} + \frac{470\theta}{3})NGN$$

and $x^* = (E^*, L^*) = (\frac{4}{3} + \frac{2\theta}{3}, 0)$. Therefore, the final result for the parametric problem is in Table 6.

From the above analysis, it is observed that (under varying resources) the profit gotten by the institute comes from the Energy Study program. It is observed that the Energy Law program is not adding to the institute, instead they run at loss to keep the program. The researcher also observed that the random allocation of conference support to both program is not profiting the institute, but will rather jeopardise its continuity.

4.3 Minimisation of Cost

Minimising the cost of operation of the institute, the classical linear programming problem becomes

$$Min\ c = 125E + 368L, \tag{19}$$

$s.t.\ g_1(E, L) = E + L \leq 238,\ g_2(E, L) = E + L \leq 1,500$ and $g_3(E, L) = 1,440E + 480L \leq 1,920.$
Using the Simplex method, Table 7 was obtained.

$Basis$	E	L	g_1	g_2	g_3	b
g_1	-2	0	1	0	$\frac{-1}{480}$	234
g_2	-2	0	0	1	$\frac{-1}{480}$	1,496
L	3	1	0	0	$\frac{1}{480}$	4
p	979	0	0	0	$\frac{368}{480}$	1,472

Table 7: Final Result of the Simplex Method

The membership functions of the constraints, respectively Internet, conference support and Graduate Assistants are:

$$\mu_1(E,L) = \begin{cases} 1, \text{ if } g_1(E,L) \leq 238 \\ 1 - \frac{g_1(E,L)-238}{70}, \ 238 < g_1(E,L) < 308 \\ 0, \ g_1(E,L) \geq 308 \end{cases} \tag{20}$$

$$\mu_2(E,L) = \begin{cases} 1, \text{ if } (g_2(E,L)) \leq 1,500 \\ 1 - \frac{g_2(E,L)-1,500}{500}, \ 1,500 < g_2(E,L) < 2,000 \\ 0, \ g_2(E,L) \geq 2,000 \end{cases} \tag{21}$$

$$\mu_3(E,L) = \begin{cases} 1, \text{ if } g_3(E,L) \leq 1,920 \\ 1 - \frac{g_3(E,L)-1,920}{960}, \ 1,920 < g_3(E,L) < 2,880 \\ 0, \ g_3(E,L) \geq 2,880 \end{cases} \tag{22}$$

The required fuzzy linear programming is

$$Min\ c = 125E + 368L, \tag{23}$$

$s.t.\ g_1(E,L) = E+L \leq 238+70(1-\alpha),\ g_2(E,L) = E+L \leq 1,500+500(1-\alpha)$ and $g_3(E,L) = 1,440E + 480L \leq 1,920 + 960(1-\alpha)$.
Setting $\theta = 1-\alpha$, the following is the parametric programming problem:

$$Min\ c = 125E + 368L, \tag{24}$$

$s.t.\ g_1(E,L) = E+L \leq 238 + 70\theta\ g_2(E,L) = E+L \leq 1,500 + 500\theta$ and $g_3(E,L) = 1,440E + 480L \leq 1,920 + 960\theta$, where $\theta \in [0,1]$ is a parameter.
Using the parametric technique and final result of simplex method, Table 8 was obtained.

$Basis$	E	L	g_1	g_2	g_3	b
g_1	-2	0	1	0	$\frac{-1}{480}$	$234 + 68\theta$
g_2	-2	0	0	1	$\frac{-1}{480}$	$1,496 + 498\theta$
E	3	1	0	0	$\frac{1}{480}$	$4 + 2\theta$
C	979	0	0	0	$\frac{368}{480}$	$1,472 + 736\theta$

Table 8: Matrix Multiplication of the Simplex Method and the Tolerance Level

θ	C^*	Internet	Conf. Supp.	G.A	Energy Law
0.0	1,472.00	4.00	4.00	1,920.00	4.00
0.1	1,545.60	4.20	4.20	2,016.00	4.20
0.2	1,619.20	4.40	4.40	2,112.00	4.40
0.3	1,692.80	4.60	4.60	2,208.00	4.60
0.4	1,766.40	4.80	4.80	2,304.00	4.80
0.5	1,840.00	5.00	5.00	2,400.00	5.00
0.6	1,913.60	5.20	5.20	2,496.00	5.20
0.7	1,987.20	5.40	5.40	2,592.00	5.40
0.8	2,060.80	5.60	5.60	2,688.00	5.60
0.9	2,134.40	5.80	5.80	2,784.00	5.80
1.0	2,208.00	6.00	6.00	2,880.00	6.00

Table 9: Result of the Parametric Problem

The optimal solution is

$$C^* = (1,472 + 736\theta)NGN$$

and $x^* = (E^*, L^*) = (0, 4 + 2\theta)$. Therefore, the final result for the parametric problem is in Table 9.

From the analysis above on cost minimisation, the Energy Law program viably increases the cost of running the institute.

4.4 Proposed Model

From the results above, the institute is discovered not to be making optimal profit running both Energy Law and Energy Studies' program. Therefore, the researcher proposes that the fees of the Energy Law should be increased in such a way that it contributes meaningfully to the institute. Suppose the Law Student and Energy Student contribute 230,000NGN and 235,000NGN respectively, and the conference support is given in the ratio 742 to 758 (from contribution made by

$Basis$	E	L	g_1	g_2	g_3	b
g_1	0	0	1	$\frac{-238}{1{,}468}$	$\frac{90202}{1{,}056{,}960}$	0
L	0	1	0	$\frac{3}{1{,}468}$	$\frac{-758}{704{,}640}$	1
E	1	0	0	$\frac{-1}{1{,}468}$	$\frac{2{,}226}{2{,}113{,}920}$	1
p	0	0	0	$\frac{455}{1{,}468}$	$\frac{1}{23{,}488}$	$\frac{1{,}395}{3}$

Table 10: Final Result of the Simplex Method

each program), then the new linear programming problem becomes:

$$Max\ p(E,L) = 235E + 230L, \tag{25}$$

$s.t.\ g_1(E,L) = 119E + 119L \leq 238\ g_2(E,L) = 758E + 742L \leq 1,500$ and $g_3(E,L) = 1,440E + 480L \leq 1,920$, where E is Energy Studies and L is Energy Law.
Using the Simplex method, Table 10 was obtained.

The membership functions of the constraints, respectively Internet, conference support and Graduate Assistants are:

$$\mu_1(E,L) = \begin{cases} 1, \text{ if } g_1(E,L) \leq 238 \\ 1 - \frac{g_1(E,L)-238}{70}, \ 238 < g_1(E,L) < 308 \\ 0, \ g_1(E,L) \geq 308 \end{cases} \tag{26}$$

$$\mu_2(E,L) = \begin{cases} 1, \text{ if } g_2(E,L) \leq 1,500 \\ 1 - \frac{g_2(E,L)-1{,}500}{500}, \ 1,500 < g_2(E,L) < 2,000 \\ 0, \ g_2(E,L) \geq 2,000 \end{cases} \tag{27}$$

$$\mu_3(E,L) = \begin{cases} 1, \text{ if } g_3(E,L) \leq 1,920 \\ 1 - \frac{g_3(E,L)-1{,}920}{960}, \ 1,920 < g_3(E,L) < 2,880 \\ 0, \ g_3(E,L) \geq 2,880 \end{cases} \tag{28}$$

$Basis$	E	L	g_1	g_2	g_3	b
g_1	0	0	1	$\frac{-230}{1,468}$	$\frac{90202}{1,056,960}$	$\frac{74,901,120\theta}{1,056,960}$
L	0	1	0	$\frac{3}{1,468}$	$\frac{-758}{704,640}$	1 - $\frac{11,520\theta}{1,056,960}$
E	1	0	0	$\frac{-1}{1,468}$	$\frac{2,226}{2,113,920}$	1 + $\frac{708,480\theta}{1,056,960}$
p	0	0	0	$\frac{455}{1,468}$	$\frac{1}{23,488}$	$\frac{1,395}{3} + \frac{163,939,200\theta}{1,056,960}$

Table 11: Matrix Multiplication of the Simplex Method and the Tolerance Level

Hence,

$$Max\ p = 235E + 230L, \tag{29}$$

$s.t.\ g_1(E, L) = E + L \leq 238 + 70(1 - \alpha)$ $g_2(E, L) = 758E + 742L \leq 1,500 + 500(1 - \alpha)$ and $g_3(E, L) = 1,440E + 480L \leq 1,920 + 960(1 - \alpha)$.
Setting $\theta = 1 - \alpha$, the following is the parametric problem:

$$Max\ p = 235E + 230L, \tag{30}$$

$s.t.\ g_1(E, L) = E + L \leq 238 + 70\theta$ $g_2(E, L) = 758E + 742L \leq 1,500 + 500\theta$ and $g_3(E, L) = 1,440E + 480L \leq 1,920 + 960\theta$,

where $\theta \in [0, 1]$ is a parameter.
Using the parametric technique and final result of simplex method, Table 11 was obtained. The optimal solution is

$$p^* = (\frac{1,395}{3} + \frac{163,939,200\theta}{1,056,960})NGN = 465 + 155.10445NGN$$

and $x^* = (E^*, L^*) = (1 + \frac{708,480\theta}{1,056,960}, 1 - \frac{11,520\theta}{1,056,960})$.
Therefore, the final result for the parametric problem is given in Table 12.

From the analysis above, the profit of the institute increased greatly as a result of the viable contribution from both programs.

5 Conclusions

The potency of fuzzy set theory, fuzzy logic and so on in decision-making cannot be over-emphasized. Its use has proved very efficient from the above analysis, and gives the decision-maker the opportunity to make decision in a robust and flexible environment.

θ	E	L	Internet	Conf. Supp.	G.A.	P
0.0	1.000	1.000	238.00	1500.00	1920.00	465.00
0.1	1.070	1.001	246.09	1551.53	2016.96	480.51
0.2	1.130	1.002	254.18	1603.06	2113.92	496.02
0.3	1.200	1.003	262.28	1654.58	2210.88	511.53
0.4	1.270	1.004	270.37	1706.11	2307.84	527.04
0.5	1.340	1.005	278.46	1757.64	2404.80	542.55
0.6	1.400	1.006	286.55	1809.17	2501.76	558.06
0.7	1.470	1.007	294.64	1860.70	2598.72	573.57
0.8	1.540	1.008	302.74	1912.22	2695.68	589.08
0.9	1.600	1.009	310.83	1963.75	2792.64	604.59
1.0	1.670	1.010	318.92	2015.28	2889.60	620.10

Table 12: Result of the Parametric Problem

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A Proof of Descartes' Rule of Signs

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Abstract

In 1637 Descartes, in his famous Géométrie, gave the rule of the signs without a proof. Later many different proofs appeared of algebraic and analytic nature. Among them in 1828 the algebraic proof of Gauss. In this note, we present a proof of Descartes' rule of signs that use the roots of the first derivative of a polynomial and that can be presented to the students of the last year of a secondary school.

Keywords: roots of a polynomial; derivative of a polynomial.
2010 AMS subject classification: 12D10; 26C10.

1 Introduction

Descartes' Rule of signs first appeared in 1637 in Descartes' famous *Géométrie* [1], where also analytic geometry was given for the first time. Descartes gave the rule without a proof. Later several discussions appear trying to understand which one was the first proof of the Rule. It seems that a first proof of the Rule was given in Segner's degree thesis in 1728 and it is contained in a letter that Segner sent to Hamberger [3]. In 1828 Gauss [2] gave a purely algebraic and very simple proof. Many other proofs, both

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algebraic and analytic in nature have been given later. One of the possible statements giving Descartes' Rule of signs is the following:

Theorem 1.1 *If a polynomial with real coefficients in one unknown has all of its roots being real number, then the number of positive roots, counted with their multiplicity, equals the number of variations of signs among the ordered sequence of his coefficients.*

A more general statement covers the case when one does not know if all roots are real and it is given in the following:

Theorem 1.2 *The number of variations of sign is the maximum number of positive roots of a polynomial with real coefficients. The number of positive roots equals either the maximum or the maximum minus an even number.*

The previous theorems do not give results on the number of negative roots. The negative roots of $p(x)$ are in number equal to the number of positive roots of $p(-x)$ and hence in order to count the number of negative roots of $p(x)$ one can count the number of positive roots of $p(-x)$ by applying Descartes' Rule of signs.

Let $p(x)$ be a polynomial whose monomials are given either in increasing or in decreasing order. Consider the sequence of its coefficients in the same order. One says that there is a "change of sign" if two consecutive terms have opposite sign.

For example if $p(x)=x^6-3x^5+4x^3+x^2-5x+9$, then the sequence of its coefficients is:

$1,-3,4,1,-5,9$ and the number of variations is 4. Hence the number of positive roots of $p(x)$ is either 4 or 2 or 0.

Observe that if the number of variations is even, then the Rule cannot say that the polynomial has a positive root. If the number of variations is odd, the Rule says that there is at least a positive root.

2 The Proof

2.1 The Derivative of a Polynomial

We first start with some results giving information between the roots of a polynomial $p(x)$ and the roots of its derivative $p'(x)$.

Lemma 2.1 *A roots of $p(x)$ is also a root of $p'(x)$ with multiplicity one less.*

Proof. Let $p(x) = (x - a)^k q(x)$ with $(x - a)$ that does not divide $q(x)$. Hence $q(a) ¹ 0$. Then k is the multiplicity of a as root of $p(x)$. It is

$$p'(x) = k(x - a)^{k-1} q(x) + (x - a)^k q'(x) =$$

$$= (x - a)^{k-1}[kq(x) + (x - a)q'(x)]$$

$$= (x - a)^{k-1} F(x).$$

Since $F(a) = kq(a) \neq 0$, $(x - a)$ does not divide the polynomial $F(x)$. Hence $k - 1$ is the multiplicity of a as root of $p'(x)$.

Next results follows from Rolle's theorem.

Lemma 2.2 *If all roots of a polynomial $p(x)$ are real numbers, then also all roots of $p'(x)$ are real numbers. Moreover between to consecutive roots of $p(x)$ there is a simple (multiplicity 1) root of $p'(x)$.*

Proof. Let $x_1 < x_2 < \cdots\cdots < x_k$ be the roots of $p(x)$ with multiplicity $m_1, m_2, \dots\dots\dots, m_k$, respectively. Since all roots are real numbers we have that

$$m_1 + m_2 + \cdots\cdots + m_k = n = \deg(p(x)).$$

From the previous lemma $p'(x)$ has roots $x_1 < x_2 < \cdots\cdots < x_k$ with multiplicity $m_1 - 1, m_2 - 1, \dots\dots\dots, m_k - 1$. Moreover, from Rolle's theorem between two real roots of $p(x)$ there is at least a real root of $p'(x)$. Hence $p'(x)$ has at least other $k - 1$ real roots. From

$$(m_1 - 1) + (m_2 - 1) + \cdots + (m_k - 1) + k - 1 = n - 1 = \deg(p'(x)),$$

it follows that $p'(x)$ cannot have other roots. The assertion follows.

Lemma 2.3 *If all roots of a polynomial* $p(x)$ *are real numbers and* k *of them are positive numbers, then* $p'(x)$ *has either* k *or* $k-1$ *positive roots.*

Proof. Let $x_1 < x_2 < \cdots\cdots < x_s$ be the positive roots of $p(x)$ with multiplicity $m_1, m_2, \ldots\ldots\ldots, m_s$, respectively. From the hypothesis we have

$$m_1 + m_2 + \cdots\cdots + m_s = k.$$

The derivate $p'(x)$ will have as positive roots $x_1 < x_2 < \cdots\cdots < x_s$ with multiplicity

$$m_1 - 1, m_2 - 1, \ldots\ldots\ldots, m_s - 1,$$

the simple roots $y_1, y_2, \ldots\ldots\ldots, y_{s-1}$ between consecutive positive roots and, possibly, another simple root y_0 between the maximum negative root and x_1.

So the total number of positive roots of $p'(x)$ is either

$$(m_1 - 1) + (m_2 - 1) + \cdots\cdots + (m_s - 1) + s - 1 = k - 1$$

if y_0 is not a positive number or

$$(m_1 - 1) + (m_2 - 1) + \cdots\cdots + (m_s - 1) + s - 1 + 1 = k$$

if y_0 a positive number.

2.2 Proof of Theorem 1.1

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_0$ be a degree n polynomial. Hence $a_n \neq 0$. We may assume that $a_n > 0$. In what follows we assume that all roots of $p(x)$ are real numbers.

Lemma 2.4 *If* $p(x)$ *has* k *positive roots, then the sign of the last non zero coefficient of* $p(x)$ *is* $(-1)^k$.

Proof. Let a_h be the last non zero coefficient. Since all roots of $p(x)$ are real numbers we can factorize the polynomial as

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_h x^h =$$

$$= a_n x^h (x - x_1) \cdots\cdots (x - x_k)(x - x_{k+1}) \cdots\cdots (x - x_{n-h})$$

where $x_1, \ldots\ldots\ldots, x_k$ are the positive roots and $x_{k+1}, \ldots\ldots\ldots, x_{n-k}$ are the negative roots.

It follows that $a_h = a_n \cdot (-1)^k \cdot x_1 x_2 \cdots\cdot x_k \cdot (-x_{k+1}) \cdots (-x_{n-h})$ and since all numbers are positive the sign is $(-1)^k$.

We will now give the proof of the Theorem 1.1 by induction on $n = \deg(p(x))$.

If $n = 1$, the assertion holds. Indeed $p(x) = a_1 x + a_0$ has a unique root $x_1 = -a_0/a_1$. It is a positive root if and only if a_1 and a_0 have opposite sign, that is there is a variation.

Suppose the assertion holds for all polynomials of degree $n-1$ with all real roots. Let

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots\cdots + a_0$$

be a polynomial of degree n.

If $a_0 = 0$, then $p(x) = xq(x)$ and the polynomials $p(x)$ and $q(x)$ have the same number of positive roots and the same number of variations of sign. Since $\deg\big(q(x)\big) = n-1$ and the assertion holds for $q(x)$, then it also holds for $p(x)$.

If $a_0 \neq 0$, then

$$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots\cdots\cdots + a_1.$$

The last non zero coefficient in $p'(x)$ is the non zero coefficient consecutive to a_0 in $p(x)$. If the sign of a_0 and the last non zero coefficient in $p'(x)$ coincide, then $p(x)$ and $p'(x)$ have the same number of variations of sign, otherwise $p'(x)$ has one less variations of sign compared with $p(x)$. Since the sign of the last non zero coefficient determines the parity of the number of positive roots (Lemma 2.4) in the first case the parity of the number of roots of $p(x)$ and $p'(x)$ is the same in the second case it is different.

On the other hand from Lemma 2.3 the number of roots of $p(x)$ and $p'(x)$ is different for at most 1. Hence either $p(x)$ and $p'(x)$ have the same number of positive roots or this number is different for 1.

From the inductive hypothesis $p'(x)$ has the same number of positive roots as the number of variations of sign. Since from $p(x)$ to $p'(x)$ the number of positive roots and the number of variations of sign either remains the same for both or it is 1 more for both, the assertion follows also for $p(x)$.

3 Conclusions

The proof of Descartes rule of signs is a good example of math reasoning and it should be taught to the students of last year of secondary schools. Contrary to this in many schools it is given the Rule without a proof. In particular it is a good example for understanding the relation between the roots of a polynomials and its first derivative. It also uses Rolle's theorem, that is one of the most important result shown to the students of last year of secondary schools. Moreover Descartes' rule of signs is one of the Math results that puts together analysis and algebra and it doesn't happen so often in curricula of secondary school. In Math, except for axioms, everything should be demonstrated.

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The Proof of the Fermat's Conjecture in the Correct Domain

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Abstract

The distinction between the Domain of Natural Numbers and the Domain of Line gets highlighted. This division provides the new perception to the Fermat's Conjecture, where to place it and how to prove it. The reasons why the Fermat's Conjecture remained unproven for 382 years are examined. The Fermat's Conjecture receives the proof in the Domain of Natural Numbers only. The equation $a^n + b^n = c^n$ with positive integers *a, b, c, n* is not the Fermat's Conjecture in the Domain of Line.
Keywords: Fermat's Conjecture; Fermat's Last Theorem; Domain of Natural Numbers; Domain of Line
2010 AMS subject classification: 11D41

1 Introduction

There are two fundamental domains in mathematics: The Domain of Natural Numbers (positive whole numbers or positive integers) and the Domain of Line. They appear in Table 1.

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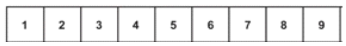
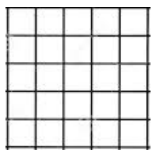
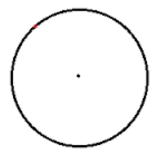
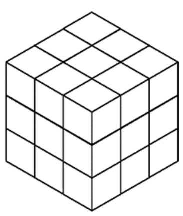
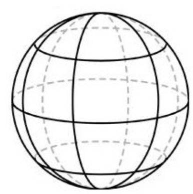
<table>
<tr><th></th><th>The Domain of Natural Numbers</th><th>The Domain of Line</th></tr>
<tr><td>One-dimensional filled space</td><td>Numbered Unit Squares (Squarits)

</td><td>All Kinds of Line

Euclidian, Hyperbolic, Elliptic, dashed, etc.</td></tr>
<tr><td>Two-dimensional filled space</td><td>The Squared Circle

From the centre, it is the same distance equal to a specific integer or number of squarits (unit squares). A rotation brings the position to the same beginning squarit.</td><td>The Lined Circle

From the centre, it is the same distance equal to a specific decimal or integer number. A rotation brings the position to the same beginning point.</td></tr>
<tr><td>Three-dimensional filled space</td><td>The Cube

From the centre, it is the same distance equal to a specific integer number of cubits (unit cubes). A rotation brings the position to the same beginning cubit.</td><td>The Sphere

From the centre, it is the same distance equal to a specific decimal or integer number. A rotation brings the position to the same beginning point.</td></tr>
<tr><td>Zero</td><td>It is the impassable wall at the bottom. Zero refers to none, nothing, no-one. Zero gets used for counting the natural numbers to mark the new set of 9. Thus, the numbers that contain zeros can be viewed as multiples of nine plus one</td><td>In all accepted combi-nations and expressions.</td></tr>
</table>

	digit from 1 to 9, e.g. 10 = 9 + 1, 103 = 9×9 + 2×9 + 4.	
Numbers	The whole positive numbers and their ratios (rational numbers) only. The array of even numbers *2n* starts at zero: 0, 2, 4, 6, 8, … The collection of odd numbers *2n-1* begins at one: 1, 3, 5, 7, 9... On the graph, the positive integers constitute a straight collection of dots, at the same pace, stretching at geometrically 45°, and numbering n-dots.	The real and complex numbers, whole and rational numbers, positive and negative numbers, logarithmic and decimal numbers, irrational and transcendental numbers. On the graph, the function $y = x$ is a continuous line stretching at geometrically 45° and containing an uncounted number of dots.
Relations	The relations with and for positive integers only.	The algebraic relations and mathematical analyses. Functions and equations of all possible lines, groups, rings, and fields. Euclidian and non-Euclidian geometries. Diophantine and algebraic geometry. Calculus and analytical geometry.

Table 1. The Domain of Natural Numbers versus the Domain of Line

In the Domain of Line, zero gets assigned to the origin or the beginning point. In one-dimensional space, the geometry determines the distance within two points or one point on the coordinative axis and zero (the origin) and what kinds of lines are passing through those points: parallel (Euclidian), hyperbolic (Lobachevskian) or elliptic. In two-dimensional space, the geometry determines the inner area between three points or one point on each of the two coordinative axes and zero (the origin). In three-dimensional space, the geometry determines the inner volume between four points or one point on each of the two coordinative axes and zero (the origin).

The Domain of Line makes use of the conclusions that come from the Domain of Natural Numbers, but not the opposite. Our existence starts at one. Below zero, $n < 0$, the meaning of life and existence loses. Things, living and self-thinking entities get numbered positively. We exist as numbers and get shaped by lines. In the Domain of Natural Numbers, the one- or two- or three-dimensional entities are geometrically unconnected objects. Numbers connect them, because of numbers bond numbers. After squarits or cubits get packed in their respective spaces, there are no void spaces left in between. In one-dimensional space, for example, three connects one and two, because $1 + 2 = 3$.

In two-dimensional space, 5^2 connects 3^2 and 4^2, because $9 + 16 = 25$. This heavenly-existed set 3-4-5 is the first square set in the unique sequence commonly called the Pythagorean Triples. These can, for example, be generated by the Fibonacci's method (since the year 1225), by the Michael Stifel's method (since the year 1544) and Jacques Ozanam's technique (since the year 1694) of the progressions of whole and fractional numbers. The Pythagorean Triples get also produced using either the Leonard Eugene Dickson's method (since the year 1920) or Euclid's algebraic quadratic equations or the matrices and linear transformations, etc. The first set of positive integers 3-4-5 is followed by 6-8-10, 5-12-13, 9-12-15, 8-15-17, 12-16-20, 15-20-25, 7-24-25, 10-24-26, 20-21-29, 18-24-30, and so on. [1] Any relationship in the Pythagorean Triples can be proved using squared circles. Only for the Pythagorean Triples, the three squared circles form between the geometrical shape of the right-angled triangle with sides taking integer numbers. Otherwise, the right-angled triangles are geometrical lines and have the length of at least one of their sides taking a non-integer number.

In three-dimensional space, 6^3 connects 3^3, 4^3 and 5^3, because $27 + 64 + 125 = 216$. This essentially natural cubic set is the first in the unique cubic sequence 3-4-5-6, 6-8-01-9, 6-8-10-12, 12-16-02-18, 9-12-15-18, 12-16-20-24, 18-24-03-27, and so on.

A lined circle cannot take positive integers and get converted to a lined (geometrical) square with positive integers. Because, a lined square consists of four equal sides with either an odd or an even integer number of steps, which so produce either an odd or an even integer number of squarits. Thus, a lined square fundamentally falls into the Domain of Natural Numbers at a time when the lined circle divided into an irrational number $\pi = 3.14159265358\ldots$ of steps remains in the Domain of Line. The geometric irrational number $\pi = 3.14159265358\ldots$ mirrors the ratio 22/7 in the Domain of Natural Numbers. A lined circle and a lined square bond only when they have an equal geometrical inner area or by inscribing the lined circle inside the lined square and *vice versa*.

2 The Fermat's Last Theorem

Around the year of 1636, Pierre de Fermat (1607-1665) wrote a comment on the margin of a page in a copy of 1621 edition of the book *Arithmetica*, that translations since the third century A.D. had brought as written by Diophantus of Alexandria. The first part of the comment stated that four positive integers or natural numbers *a, b, c, n* when $n > 2$ cannot be a solution to the following equation:

$$a^n + b^n = c^n \qquad 1)$$

The second part of the comment stated that he, Pierre de Fermat, had the proof for Eq. (1) but he could not write it because the page margin did not have enough space for it. Likely, Pierre de Fermat had a flash that could prove Eq. (1), because he did not write anytime later a general proof of Eq. (1). What he communicated in detail was the use of an original logic known as "The Infinite Descent" to derive a contradiction to an invented counterexample from himself. [1-10] He stated that if the area of a right-angled triangle were equal to the square of an integer, e.g., r^2, then there would exist two numbers *p, q* in the fourth power the difference of which equals r^2. [3, 10, 11] And without his assertion what the numbers *p* and *q* were, the following was his equation:

$$p^4 - q^4 = r^2 \qquad 2)$$

In the Domain of Line, if by wish r^2 is chosen equal to s, then Eq. (2) appears as $p^4 - q^4 = s$. If by wish $r = t^2$, then $p^4 - q^4 = t^4$ which is a form of Eq. (1) for $n = 4$. If by wish $t = u^2$ then $p^4 - q^4 = u^8$, and so on.

Eq. (2) is inaccurately taken as the specific case of $n = 4$ for Eq. (1), because by command it puts the condition of $r = t^2$. Also, the counterexample built by Pierre de Fermat or his Eq. (2) falls in the Domain of Line, while the mathematical relationship bodied in Eq. (1) is in the Domain of Natural Numbers.

As a sort of indirect proof, the technique of Infinite Descent is more a wording logic looking for a contradiction to its start than a mathematical method of proof. Though it relies on geometry and numbers, the purpose of this technique is to decide by language. The contradiction emerges since the start is either non-existent or untrue or unproven. The Infinite Descent by Pierre de Fermat trailed the logic of *reductio ad absurdum* (reduction to absurdity) by ancient Aristotle. Though *reductio ad absurdum* has full power in philosophical perception, it is not enough in the mathematics of numbers. It is so because reasoning is subjective (coming or accepted from the thinking) and numbers are objective (existing independently of thought).

He activated his proving approach using the formula of the Pythagorean Triples, where the sides of the right-angle triangles are sets of specific positive integers and belong to the Domain of Natural Numbers. Also, he guessed that the edges of such triangles were relatively prime numbers. Then through further calculations and assumptions, e.g., any time the difference of two integers in fourth power was assumed a squared integer, a descending spiral of infinite smaller and smaller such right-angled triangles emerged. The only way to stop the descending loop or the

Infinite Descent was by the wording, as Pierre de Fermat wrote, "…this is impossible since there is not an infinitude of positive integers than a given one". Thus, in accord with Pierre de Fermat, the Infinite Descent was in contradiction to the original counterexample, and so it proved that a right triangle could not have an area equal to a squared integer. [3, 5, 11]

The proof for a problem that stays within the Domain of Natural Numbers is not enough or valid to become credible proof for the Domain of Line. Reversibly, a general proof extracted in the Domain of Line is larger than the gate of the Domain of Natural Numbers, and thus unacceptable there.

The Infinite Descent or the descending spiral did not produce anything new, except the need to stop it verbally on purpose. The Infinite Descent generated right-angled triangles with decreasing size and headed to infinitely small such triangles. It is equivalent to the direction of the Infinite Ascent, which creates right-angled triangles with increasing size and leading to unbelievably big such triangles. Geometrically, as Pierre de Fermat created his counterexample and the procedure for finding its contradiction, there are not any contradictions going down to infinitely small or up to infinitely big right-angled triangles. As such, both the Infinite Descent and Infinite Ascent cannot be stopped verbally except than on purpose.

In the Domain of Line, the area of a right-angled triangle equal to a squared integer is possible and can be only when the lengths of the adjacent sides to the right angle relates in the ratio 2:1. In which case, the length of the side opposite the right angle equals the unit number multiplying $\sqrt{5}$. It means that such a right-angled triangle is not one of the Pythagorean Triples and precisely it appears within the Domain of Line.

In the Domain of Natural Numbers, the sequence of natural numbers begins at one and has zero its bottom limit. The chain of natural numbers has no top boundary and increases infinitely by an increment of one. The existence of the bottom base cannot constitute a contradiction in the process of the Infinite Descent for the invented counterexample because it is just an arrival at the lower limit. It is just a trial in the engineering optimization.

After the death of Pierre de Fermat, his son Clément-Samuel examined his father's papers, letters, and notes and published them as a book in 1670. [8] Then, Eq. (1) came into sight for other mathematicians who began a pursuit to prove it. Equation (1) is known as the Fermat's Last Theorem or the Fermat's Conjecture, because since then in century XVII it has not been proved in a general form.

3 The Endeavours for Proving the Fermat's Last Theorem

The first effort for the specific case $n = 4$ to prove the relation embodied in Eq. (1) appeared in 1676 and accelerated in century XIX and early century XX. Due to its outward ease, Eq. (1) attracted all mathematicians and leaders in mathematics. [1, 2, 6, 8, 12-15] The diving efforts of brilliant minds into the ocean of mathematics for solving the Fermat's Conjecture advanced the science of mathematics in new directions. [10, 16, 17]

There have been many publications related to the efforts for proving the Fermat's Conjecture. They cover a range of peer-reviewed top mathematical journals to the simplest personal trials and progress reports posted on the Internet. Such relevant publications keep coming into the scientific view. [7-9, 13, 14, 18-29] It is impossible to cite for reference all of them. However, it is possible to praise all researchers for the time spent for searching to prove the Fermat's Last Theorem.

The shared characteristics of the efforts exerted to prove the Fermat's Conjecture and the root reasons why not a final general proof has been reached unfold below.

FIRST – The proofs have been searched geometrically (e.g., using elliptic curves) or algebraically (e.g., using Bernoulli or complex numbers) in the Domain of Line at a time that Eq. (1) is inside the Domain of Natural Numbers; please refer to Table 1 and associated elucidations. Likewise, the proof of Eq. (1) has been examined on algebraic equations, abstract functions, and conditions noticeable other than Eq. (1). [2, 4, 6, 8, 18, 19, 23-26, 29-41]

SECOND – The logic of conclusion has been the logic of contradiction to the one assumed either counterexample or new starting conditions; please refer to Figure 1.

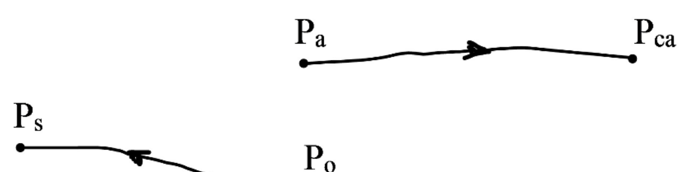


Figure 1. The two paths of the solution, where: P_o is the original point of conditions of the problem. P_s is the solution point of the problem. P_a is the point of the assumed to-be-original-conditions of the problem. P_{ca} is the contradicting point to P_a.

The path or vector of solution P_oP_s is the path that preserves the original conditions of the problem. While the imaginary route P_aP_{ca} starts with an assumed identity of conditions at point P_a that is detached from the point P_o, the original status of conditions. And sometimes, one counterexample or invented supposition is planted at point P_a. Then, a solution is accepted if a contradiction to point P_a comes across in the path P_aP_{ca}. The rejection by contradiction at point P_{ca} proves only that the assumed to-be-original-conditions or the counterexample at point P_a were not accurate or could not exist. That is, an encounter at a point P_{ca} will undoubtedly contradict its self-non-existence that rooted at point P_a. The emerged contradiction relates to the false assumption made at point P_a and ruins only the characteristics of position P_a, which stays detached from the point P_o. Thus, the emerged contradiction at point P_{ca} has no connection with path P_oP_s and conditions of the solution at the position P_s. Also, a counterexample is specific, and there is not any general counterexample.

THIRD – The proofs have progressed on steps that incorporated the assumption or supposition of specific conditions or properties for variables, equations, and functions. [1, 2, 4-8, 10, 12, 14, 16-18, 22, 23, 25, 27, 29, 31-35, 37-48, 50] That is, the conditions or properties or counterexamples have been created on purpose, taken for granted, personally accepted or assigned, thought or imagined to be that way. The examination of the natural Eq. (1) in imaginary systems or the endeavours to reach its proof with tools of the imaginative mathematics beget misleading results. It reaffirms Figure 1.

FOURTH – The proofs of the Fermat's Conjecture have been researched for isolated power numbers, for example, n = 3, 4, 5, 7, 6, 10, 14 or ideal numbers, and especially for prime numbers. [1, 2, 4, 6, 8, 12, 14, 18, 24-27, 30-39, 41, 42, 46-49]

The trail of attempts to prove the Fermat's Conjecture by selecting prime numbers for the exponent in Eq. (1) started by Sophie Germain in 1823. Sophie Germain grouped in Case One the prime numbers p that cannot divide a, b, c in Eq. (1) and in Case Two those that do. Moreover, she reformulated Eq. (1) into the following equation, which both had different conditions from Eq. (1) and it was not the Fermat's Last Theorem anymore [18, 24, 31-36, 47]:

$$a^p + b^p + c^p = 0 \qquad 3)$$

In 1847, Gabriel Lamé tried unsuccessfully to factorize the Fermat's Last Theorem in the cyclotomic field of complex numbers. Based on that experience, Ernst E. Kummer developed the theory of ideal numbers in 1849. Within that theory, and using the compound Bernoulli numbers, Ernst E. Kummer defined the set of regular prime numbers. He used them to prove the first case of Fermat's Last Theorem. [1, 2,4, 6, 8, 18, 31, 33, 38, 39, 47-49]

The ideal numbers are algebraic integers, which means they are complex numbers. They are part of the ring theory studied in the Abstract Algebra. They represent the ideals (subsets) in the rings of integers of algebraic number fields, which have finite dimensions. As such, the Bernoulli, complex and ideal numbers differ totally from natural numbers and do not reside in the Domain of Natural Numbers. Their incorporation in the form of regular prime numbers for proving Eq. (1) cannot give the proof or at least a general solution. Above all, the past and modern researchers that try to find a proof for Sophie Germain's First Case embodied in Eq. (3) have tried to find a proof of a relationship which is not the Fermat's Conjecture embodied in Eq. (1).

FIFTH – A wording instrument linked to integer numbers, known as *modulus operandi*, has been used in algebraic or number formulas. [2, 4, 7, 18, 24-26, 30, 31, 33-35,37, 38, 42-44, 47, 49]

The modulo operation depicts the integer remaining after another integer number divides one integer. Thus, for two integers *x, y* that give the same remainder *R* after divided by another shared integer *z*, it gets worded that both *x* and *y* are congruent modulo *z* and *x – y* is a multiple of *z*. It becomes mathematically visible with the following wordy phrase:

$$x \equiv y \pmod{z} \tag*{4)}$$

Arithmetically, the relations among the integers *x, y, z* are generalized as the following:

$$\frac{x}{z} = v + R \tag*{5)}$$

$$\frac{y}{z} = w + R \tag*{6)}$$

$$\frac{(x - y)}{z} = v - w \tag*{7)}$$

The wording phrase (4) is not a numeral operator, a *numeralis operandi*, and only describes the ratio *(x – y)/z* in Eq. (7) by implying that it is equal to an integer number. As just a notation, the wording phrase (4) does not display the values of *v – w* and *R*. It is not a mathematical formula or a line equation or a numerical function. The wording phrase (4) is a *verbum operandi* and does not bring anything new mathematically. The Eqs. (5-7) give the complete explicit information. In the Domain of Natural Numbers, mathematics gets explicitly expressed through numeral operators of plus, minus, multiplication, division (ratio), power, equal and sum.

The use of the *verbum operandi* (4) in *numeralis operandi* for proving the Fermat's Conjecture does not fit. It does not offer any specific sets of natural numbers that can be examples for Eq. (1). [19, 31, 34, 35] The Arithmetic is an explicit and exact science, while modulo operation is both a wording phrase and an implying operator. The modulo itself deals with cyclic numbers and all integers, while the natural numbers *a, b, c, n* in Eq. (1) are only positive integers and not cyclic. A modulo solution used for proving Eq. (1) must be congruent with a proof using arithmetic operators and mathematical formulas. It just complicates a mathematical expression, e.g., Eq. (7), by making invisible and undetermined the integers *v-w* and *R* in Eqs. (5-7).

Even when Eq. (1) is arranged in the following rational-number form,

$$\left(\frac{a}{c}\right)^n + \left(\frac{b}{c}\right)^n = 1 \qquad 8)$$

there is not any condition in the Fermat's Conjecture that the first term is congruent to the second term or *a* is congruent to *b* modulo *c* in Eq. (8). Anyway, a solution must keep or provide the variables *a, b, c, n* as positive integers.

SIXTH – The effort to use the elliptic curves and imaginary Galois representations to prove the Fermat's Conjecture gets separately examined here. Between 1955 and 1967, Goro Shimura, Yutaka Taniyama, and André Weil set forth the modularity theorem, also known as the Taniyama-Shimura-Weil conjecture. It claimed that all elliptic curves in the field of rational numbers (at rational number coordinates) associated with the modular forms; that is, they were modular. [2, 4, 6, 12, 16, 42, 50]

Yves Hellegouarch in 1974 and Gerhard Frey in 1982 claimed that the following algebraic equation of the geometrical semi-stable elliptic curves, where *p* is an odd prime number, is correlated with Fermat's Last Theorem or Eq. (1). [2, 6, 12, 42, 51]

$$y^2 = x(x - a^p)(x + b^p) \qquad 9)$$

Gerhard Frey proposed that if a solution for *a, b, c, p* exists from Eq. (1) then *a, b* of it would give a semi-stable elliptic curve from Eq. (9), referred to as the Frey-Hellegouarch curve, which would not be modular. Thus, referring to point P_a in Figure 1, Gerhard Frey established a counterexample to Fermat's Conjecture. In 1985, Gerhard Frey deepened the mathematical abstraction by articulating that the Taniyama-Shimura-Weil conjecture implied Fermat's Last Theorem. [2, 4, 6, 7, 12] So, referring to point P_{ca} in Figure 1, Gerhard Frey laid down the imaginary path of solution P_aP_{ca}. On it, someone could investigate for a proof of the Taniyama-Shimura-Weil conjecture that would contradict the counterexample flagged at point Pa, thus proving the Fermat's Last Theorem. [46]

In 1985, Jean-Pierre Serre wrote that a Frey-Hellegouarch curve could not be modular and since he did not offer a solid proof for his proposition it turned to be known as the Epsilon Conjecture. In the summer of 1986, Kenneth A. Ribet proved the Epsilon conjecture for a semi-stable elliptic curve, which meant that the Taniyama-Shimura-Weil conjecture implied the Fermat's Last Theorem. [2, 4, 6, 13, 22, 39, 41, 46]

A highlighted effort for proving Eq. (1) emerged when Andrew J. Wiles published a final article 108-page-long in parallel with a supportive article co-authored with Richard Taylor 19-page-long in the Annals of Mathematics in 1995. [43, 44] Using those two pieces, Andrew J. Wiles confirmed the modularity theorem for semistable elliptic curves to be adequate for contradicting the Gerhard Frey's proposition and thus implying the truth of Fermat's Last Theorem. Very a few mathematicians seem to understand the depths of abstract mathematics contained in those two published papers and the connection to the proof of Fermat's Last Theorem. [2, 7, 13] The whole approach summarizes in the following Figure 2:

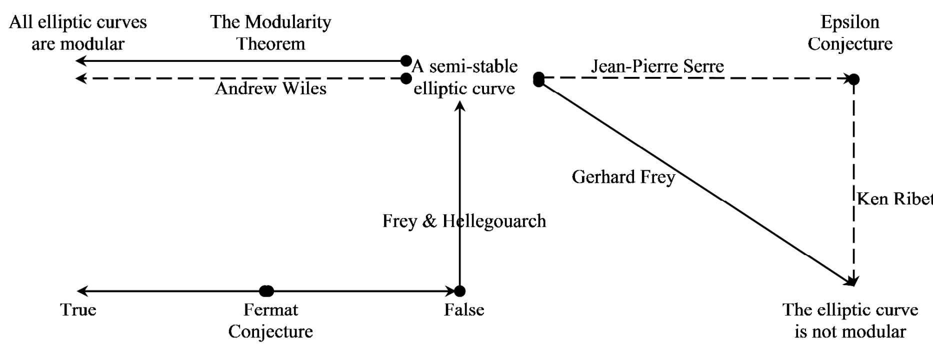


Figure 2. The paths associated with the efforts to prove the Fermat's Conjecture using geometric elliptic curves.

As a preface, the proposed solution first guessed by Gerhard Frey and later laid out by Andrew J. Wiles did not provide a general proof because they treated prime numbers instead of the natural numbers for the exponent in Eq. (1). Also, the elliptic curves, modular forms or Galois representations incorporated by them are tools for inside the Domain of Line while the Fermat's Conjecture is inside the Domain of Natural Numbers.

The counterexample proposed by Yves Hellegouarch and Gerhard Frey was a false assumption because the solution to Fermat's Conjecture never existed. Something cannot both exist and not to be at the same time, place, and under the same conditions. Ancient Aristotle had summarized this in his principle of non-contradiction, as well. That is, a solution cannot be both known and unknown at the same time, place, and conditions. That is, it was and is impossible to find a set of four natural numbers *a, b, c, n* that can prove Eq. (1).

Figure 2 confirms the Figure 1 and both Figures endorse the principle of explosion *ex contradictione sequitur quodlibet* (from a contradiction, anything follows). Since both the right and left paths started from a false point or non-existing key, their time-shifted final points had neither any connection with nor an authority on the precise spot of the Fermat's Conjecture. Even if both branches are opposite, their disagreement is dual and not general. Both right and left routes did not comply with the Gottfried W. Leibniz's principle of the Truth of Reasoning, in which an object is resolved into its simplest ideas and truths, into its primitives, to prove it.

As brilliant mathematicians, Yves Hellegouarch, Gerhard Frey, Jean-Pierre Serre and Kenneth A. Ribet on the right route and Yutaka Taniyama, Goro Shimura, André Weil, Andrew J. Wiles and Richard Taylor on the left path were correct in their conclusions about the modularity of geometrical semis-stable elliptic curves. They built their conjectures on detached assumptions, conditions, and tools, independently. Therefore, they produced various products (conclusions). Otherwise, they should have reached the same conclusions. Their right and left approaches to exploration were not even contradicting. Their findings in conceptual mathematics were only different in seeing the geometrical semi-stable elliptic curves from diverse viewpoints. Their research brought highlighted advancements in theoretical mathematics.

As a natural science, mathematics is an explicitly exact science that makes unfit the implying proposition that the Modularity Theorem can imply the Fermat's Last Theorem. Both routes do not end at the precise point of the Fermat's Conjecture. The course for going to the correct spot of the Fermat's Conjecture is explicitly apparent. Eq. (1) was not born from Eq. (9) or some modular forms, or *vice versa*. There is no genetic connection between Eq. (1) and Eq. (9), independently that the two pairs a^n, b^n and a^p, b^p seem of the same gender. Whatever solution that the values a^p, b^p can for elliptic curves in the field of rational numbers, the pair a^p, b^p does not deliver the duo a^n, b^n. And this, at a time that c^n is not known, and so even the sum $a^p + b^p$ cannot be evaluated. Along with Eq. (9), a solution to any other elliptic or non-elliptic equation $y = f(x)$ that combines a^n, b^n, c^n is not a condition of eligibility for giving any hint how to prove Eq. (1). Also, a Galois Field is a theoretic finite-field enclosing a limited number of elements, while the array of natural numbers is a chain without end. Therefore, any discovery on Eq. (9) has no sway on Eq. (1).

The elliptic Eq. (9) is a specific equation and the other elliptic curves are two-dimensional geometric functions $y^2 = f(x^3)$ that give continuous geometric lines, which contain an incalculable amount of numbers of all kinds. The properties that the elliptic curves might have at rational number coordinates have no link to Eq. (1), which contains only four arrays of positive integers. While Eq. (1) has as variables the natural number *a, b, c, n*, Eq. (9) has geometrical variables *x, y, a, b* and prime number variable *p*. [4, 6, 22, 46] A solution for Eq. (9) is an optimum solution that incorporates and belongs to the set of the geometrical variables *x, y, a, b* and prime number variable *p*. That is, even when *a* and *b* in Eq. (9) are positive integers, they get processed and so lose their originality and individuality as positive integers. Therefore, such a solution has no authority over the solution of Eq. (1).

Also, by definition, a modular form is a complex analytic function (a holomorphic function) on the upper half-plane, which itself is a set of complex numbers with the positive imaginary part. Furthermore, a meromorphic function, expressed as a ratio between two holomorphic functions, is a complex-valued function and unlinked to the chain of natural numbers. A modular form is a function that has superior symmetries and complexity on a unit disk. [7, 22, 42, 46, 51] Which means that a modular form is not an array of natural numbers. A function can be symmetric. On the other side, the collection of natural numbers has no symmetries because it is a chain of increasing positive integers. The modular forms are absolutely part of the Domain of Line and not part of the Domain of Natural Numbers.

In the article by Andrew J. Wiles, there is no conclusive formula where any substitution with concrete natural numbers *a, b, c, n* would confirm the Fermat's Conjecture. Except mentioning the Fermat's Last Theorem by name six times in the title and introduction, Eq. (1) was not engaged in the article. It was so because Andrew J. Wiles theoretically proved using related Galois representations only that the semi-stable elliptic curves were modular. [4, 12, 39, 43, 46, 51] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor advanced the path laid down by Andrew J. Wiles and proved the modularity theorem for all elliptic curves in 2001 [45]. Both right and left pathways in Figure 2 constitute a non-constructive proving endeavour for the Fermat's Conjecture because they provide no numeral examples for Eq. (1).

4 The Proof of the Fermat's Last Theorem

4.1 The Initial Cases for n < 3

To prove the Fermat's Conjecture expressed in Eq. (1), initially, means to assume (to bear the error) that Eq. (1) will remain the same for all $n > 2$: two terms on the left and one term on the right. But the fact of the unique cubic sequence 3-4-5-6, 6-8-01-9, 6-8-10-12… is the example just at the beginning for $n > 2$ that Eq. (1) does not exist with two terms on the left and one term on the right when *a, b, c* are positive integers. It means that the efforts for proving the Fermat's Conjecture have conveyed the untruth that Eq. (1) with the natural numbers *a, b, c, n* has only two terms on the left and one term on the right. With the knowledge of this error, summing up Eq. (1) side by side for all *n* gives the following:

$$\sum_{n=1}^{n} a^n + \sum_{n=1}^{n} b^n = \sum_{n=1}^{n} c^n \qquad 10)$$

$$\frac{a^{n+1} - a}{a - 1} + \frac{b^{n+1} - b}{b - 1} = \frac{c^{n+1} - c}{c - 1} \qquad 11)$$

The naturalness and conditions of natural numbers *a, b, c, n* of Eq. (1) are kept undisturbed in Eq. (11). However, Eq. (11) cannot be used for proving the Fermat's Conjecture because it is untrue that Eq. (1) will remain with only two terms on the left and one term on the right for all *n*.

For $n = 1$, Eq. (1) or Eq. (11) becomes $a + b = c$ that is true for unlimited cases in which the numbers *a, b*, and *c* form the bond $a + b = c$. This. This relation also tells that always $c > \{a, b\}$.

For $n = 1$ and $a = b$ Eq. (1) becomes $2a = c$, which is true for all cases when $c = 2a$. Both cases for $n = 1$ refers to the situation of one-dimensional array of unit squares (squarits) in Table 1.

For $n = 2$, $a \neq b$, $a + b \neq c$ and $c > \{a, b\}$, Eq. (1) is true only for the Pythagorean Triples. These are generated when *a, b,* and *c* relate through, for example, Euclid's algebraic quadratic equations with $a = k(p^2 - q^2)$, $b = k(2pq)$, $c = k(p^2 + q^2)$ and where *p, q* are coprime and not both odd, and *k* is an additional positive integer. It refers to the situation of a two-dimensional collection of squarits in Table 1 that comply with the rule in Figure 3.

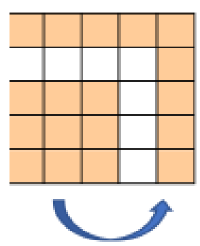


Figure 3. The Pythagorean Rule in the Domain of Natural Numbers, e.g., $3^2 + 4^2 = 5^2$.

When $n = 2$ and $a = b$ then Eq. (1) becomes $2a^2 = c^2$, which cannot make *c* be a natural number (positive integer) because $2^{1/2}$ cannot be a positive integer. Since $2^{1/2}$ is an irrational number, $2^{1/2}$ cannot be constructed as a ratio of two integer numbers. As such, $2^{1/}a$ cannot give an integer number of squarits for *c,* which would make the dimension of the squared field $2^{1/}a$ an integer number of squarits. In other words, $2a^2$ squarits cannot be arranged in a square field.

For $n \geq 3$, the situation in the Domain of Natural Numbers belong to a n-dimensional space. For instance, for $n = 3$, the space is cubic and filled by cubits (unit cubes). The three dimensions of the cube are equal to an integer number of cubits. Finding the value $(a^3)^{1/3} = a$ means finding the dimension an of the cube that contains a3 cubits. In general, the value $(a^n)^{1/n} = a$ means finding the dimension an of the n-dimensional body that contains a^n space units.

Therefore, for $n \geq 3$ and $a = b$, Eq. (1) becomes $2a^n = c^n$. As such, c cannot have an integer value because $2^{1/n}$ is an irrational number and cannot be either a positive integer or expressed as a ratio of two integers. That is, $2^{1/n}a$ cannot give an integer number of space units for c. Which means that the dimension of the equally-shaped spatial body $2^{1/n}a$ cannot bear an integer number of space units. In other words, $2a^n$ space units cannot be arranged in an equally-shaped spatial body.

So, the Fermat's Conjecture is proved for these initial scenarios. The remaining general set, which is the epical quest of mathematicians to prove the Fermat's Conjecture, is the case for $n \geq 3$, $a \neq b$, and $a < b < c$.

4.2 Eq. (1) Arranged in a Fractional Form

The Fermat's Last Theorem provides only one equation, the Eq. (1), with four variables and no specific link between them. As such and since there are no fixed pair distances in the set $\{a, b, c, n\}$, the Eq. (1) does not get measured. The use of *modulus operandi* does not help either because the bonds among a, b, c, n are undefined and unconditioned. Staying in the Domain of Natural Numbers and without disturbing the identity of natural numbers, the only equations that can be used to prove the Fermat's Conjecture are Eq. (1) or those like Eq. (8). Let's arrange Eq. (1) as follows:

$$1 + \left(\frac{b}{a}\right)^n = \left(\frac{c}{a}\right)^n \qquad 12)$$

Summing up both side from $n = 1$ to $n = n$ as follows:

$$\sum_{n=1}^{n} 1 + \sum_{n=1}^{n} \left(\frac{b}{a}\right)^n = \sum_{n=1}^{n} \left(\frac{c}{a}\right)^n \qquad 13)$$

It results to:

$$n + \frac{\left(\frac{b}{a}\right)^{n+1} - \frac{b}{a}}{\frac{b}{a} - 1} = \frac{\left(\frac{c}{a}\right)^{n+1} - \frac{c}{a}}{\frac{c}{a} - 1} \qquad 14)$$

After some arrangements, it becomes the following:

$$\left(\frac{b}{a}\right)^n = \frac{(c-a)(na-bn+b)}{a(c-b)} \tag{15}$$

or,

$$b = a\left[\frac{(c-a)(na-bn+b)}{a(c-b)}\right]^{1/n} \tag{16}$$

The left side of Eq. (15) is a positive rational number. Since $0 < a < b < c$ and $n \geq 3$, the right side of Eq. (15) will be positive only if $b/a \leq 3/2$. In this event, it will be a rational number too. The nth root of a rational number with at least either its nominator or denominator being not at nth power gives an irrational number. Which means that, the nth root of the expression inside the square bracket in Eq. (16) is an irrational number. The multiplication of an irrational number with an integer produces an irrational number as well. Thus, b is an irrational number in Eq. (16), meaning not an integer number. It so proves the Fermat's Conjecture.

The other event is when $b/a > 3/2$ and thus the right side of Eq. (15) will be a negative number. It also proves the Fermat's Conjecture because the path paved by $b/a > 3/2$ meets a contradiction with $\left(b/a\right)^n > 0$.

Besides, both these events border at the value $3/2$ that is the perfect fifth interval or the tone *G* in the diatonic musical scale; or the note *Sol* at the solfeggio system. After the fully-consonant octave interval 1 : 2, the next best harmony ratio is the perfect fifth 2 : 3. The just perfect fifth and octave intervals are the foundation of the Pythagorean musical tuning. The border value $3/2$ holds the number *2* that replicates $n = 2$ in the Pythagorean Triples and the number *3* that replicates $n \geq 3$ in the endeavour to prove the Fermat's Conjecture.

4.3 Eq. (1) Arranged in a Squared Form

With a general setting of $a < b < c$ and $n \geq 3$, another technique to verify the Fermat's Conjecture is to start with the following modified Eq. (1):

$$\left(a^{n/2}\right)^2 + \left(b^{n/2}\right)^2 = \left(c^{n/2}\right)^2 \tag{17}$$

Only when the three squared terms are bonded in the Domain of Natural Numbers in the form of the Pythagorean Triples through Euclid's algebraic quadratic equations, they can contain integer numbers. That is, they relate to the following equations:

$$a = [k(p^2 - q^2)]^{2/n} \quad 18)$$

$$b = [k(2pq)]^{2/n} \quad 19)$$

$$c = [k(p^2 + q^2)]^{2/n} \quad 20)$$

In Eq. (18-19), p and q are coprime, not both odd and $0 < q < p,\ n \geq 3$, while k is an additional positive integer. It is enough for proving the Fermat's Conjecture to look only at Eq. (19). Wherein, no matter what the value of $(2kpq)$ is, there will be no integer value for b because of the power $2/n$ at $[k(2pq)]^{2/n}$. Furthermore, no matter what the value of $[kpq]^{2/n}$ will be, b will not be an integer number because $2^{2/n}$ is an irrational number. This is adequate to affirm that for $n > 2$, the values of a, b, c discovered with Eqs. (17-20) will not simultaneously be all positive integers. Therefore, the Fermat's Conjecture holds true in the Domain of Natural Numbers wherein the Eq. (1) does not have a solution for positive integer values of a, b, c, n when $n > 2$.

4.4 Incorporating a New Integer in Eq. (1)

For $a < b < c$ and $n \geq 3$, another approach is to discover, e.g., whether b will be an integer when $c = a + d$ and a, c, d are the known integers. Then, Eq. (1) becomes:

$$a^n + b^n = (a + d)^n \quad 21)$$

then

$$b = a\left[\left(1 + \frac{d}{a}\right)^n - 1\right]^{1/n} \quad 22)$$

and

$$b = [(a + d)^n - a^n]^{1/n} =$$

$$= \left(na^{n-1}d + \frac{n(n-1)}{2!}a^{n-2}d^2 + \cdots + nad^{n-1} + d^n\right)^{1/n} \quad 23)$$

In the Domain of Natural Numbers, b is the dimension of an equally-shaped spatial body with volume b^n space units and unit subsection having a^n space units. The removal of a unit subsection from an equally-shaped spatial body with volume $(a + d)^n$ space units leaves a number of space units that cannot be finitely divided into an integer number of identical unit subsections needed for the new equally-shaped spatial body.

The multiplication of an integer or rational number with an irrational number gives an irrational number. Saying it differently from Eq. (22), the dimension b cannot be an integer number because $\left[\left(1+\frac{d}{a}\right)^n - 1\right]^{1/n}$ is an irrational number; that is, not an integer number. Therefore, the spatial units in the resulting spatial body cannot be arranged in a way that the spatial body will be equally-shaped, having a dimension b equal to an integer number, and containing an integer number b^n of spatial units.

In addition, whatever is the value of the sum inside the bracket in Eq. (23), it cannot give an integer value for b because a^n has been cancelled out and the power of the big bracket is $1/n$. Which means that b will be an irrational number, so not an integer. Thus, Eq. (1) cannot be true for simultaneous integer values for a, b, c and $n \geq 3$ in the Domain of Natural Numbers. This is proof of the Fermat's Conjecture.

4.5 Incorporating a Multiple in Eq. (1)

Having $a < b < c$ and $n \geq 3,$ any such three numbers in the series of natural numbers may relate in pairs in the forms of $c = ga$ and $b = ha$. The positive coefficients g, h are larger than one. They can be integers (e.g., $a = 3, c = 9$, then $g = 9/3 = 3$; and e.g., $a = 3, b = 6$, thus $h = 6/3 = 2$) or non-integers (e.g., $a = 3, c = 8$, then $g = 8/3 > 1$; and e.g., $a = 3, b = 5$, thus $h = 5/3 > 1$). The search for the proof means to discover, using Eq. (1), whether the third term can be an integer when the two other terms are integers.

Let's take the case of $c = ga$ with $g > 1$. It means that the positive integers a, c are known and the discovery will be whether b can be another natural number. Now, Eq. (1) appears in the following form:

$$a^n + b^n = (ga)^n \qquad 24)$$

then

$$b = a(g^n - 1)^{1/n} \qquad 25)$$

With g being either an integer or a non-integer, since $g^n = ggggggg\ldots$ n-times and $(g^n - 1) < g^n$ by one, then $g^n - 1 = eg^{n-1} = g^n(e/g),$ where $1 < e < g$ or $ae < c$. The quantity e is a non-integer because:

$$e = \frac{g^n - 1}{g^{n-1}} \tag{26}$$

Then, the Eq. (25) becomes:

$$b = a\left(g^n \, \frac{e}{g}\right)^{1/n} = ag\left(\frac{e}{g}\right)^{1/n} = c\,\left(\frac{e}{g}\right)^{1/n} = \tag{27}$$

$$= \,(c^{n-1}ae)^{1/n} = \, a^{1/n}\, c^{(n-1)/n}\, e^{1/n}$$

While the multiplication of an integer with a non-integer can give either an integer or a non-integer number, the Eq. (27) produces only a non-integer value for *b*. Because of no matter whether ($c^{n-1}ae$) will give an integer value or not, its power *1/n* omit the option that *b* will have an integer value. The multiplication of an integer or rational number with an irrational number gives an irrational number. It explicitly means that *b* cannot be an integer because $\left(\frac{e}{g}\right)^{1/n}$ is an irrational number; so, confirming the Fermat's Conjecture in the Domain of Natural Numbers.

The proof of the Fermat's Conjecture that concluded by using Eqs. (12 - 27) make evident that for $a < b < c$ and $n \geq 3$ it stays true in the Domain of Natural Numbers only. Whereas a general Eq. (1) has its field of the degrees of freedom in the Domain of Line where *a, b, c, n* can be real or complex numbers. In the Domain of Line, Eq. (1) can be analysed with all possible mathematical, geometrical, algebraic, analytical, complex and imaginary tools. In the Domain of Line, the Eq. (1) is not the Fermat's Conjecture anymore.

5 Conclusion

A mathematical conjecture or any formula and equation needs be first defined to which Domain it belongs: to the Domain of Natural Numbers or the Domain of Line. Then, this will determine the point of view and tools directed to the analysed conjecture or equation. If a conjecture or equation is entirely on natural numbers (it is inside the Domain of Natural Numbers), then the mathematical tools should be extracted from the Domain of Natural Numbers. If a conjecture or equation gets defined for the Domain of Line, then the precise tools should be derived from the Domain of Line and the Domain of Natural Numbers if they fit.

The Fermat's Last Theorem preserves its original identity if it is proved within the Domain of Natural Numbers and with mathematical tools from this Domain. Pierre de Fermat was correct that Eq. (1) having positive integers *a, b, c, n* cannot be possible for $n > 2$. However, he missed defining both in which Domain he was conjuring the

Eq. (1) and any relationship among numbers *a, b, c, n.* It took 382 years to outline and prove the Fermat Last Theorem correctly.

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The Sum of the Series of Reciprocals of the Quadratic Polynomials with Complex Conjugate Roots

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Abstract

This contribution is a follow-up to author's papers [1], [2], [3], [4], [5], [6], [7], and in particular [8] dealing with the sums of the series of reciprocals of quadratic polynomials with different positive integer roots, with double non-positive integer root, with different negative integer roots, with double positive integer root, with one negative and one positive integer root, with purely imaginary conjugate roots, with integer roots, and with the sum of the finite series of reciprocals of the quadratic polynomials with integer purely imaginary conjugate roots respectively. We deal with the sum of the series of reciprocals of the quadratic polynomials with complex conjugate roots, derive the formula for the sum of these series and verify it by some examples evaluated using the basic programming language of the computer algebra system Maple 16. This contribution can be an inspiration for teachers of mathematics who are teaching the topic Infinite series or as a subject matter for work with talented students.

Keywords: sum of the series, harmonic number, imaginary conjugate roots, hyperbolic cotangent, computer algebra system Maple.
2010 AMS subject classifications: 40A05, 65B10.

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1 Introduction

Let us recall the basic terms. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \ .$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum\limits_{k=1}^{\infty} a_k$ is defined for each n as the sum

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \, .$$

The series $\sum\limits_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence of partial sums $\{s_n\}$ converges to s, i.e. $\lim\limits_{n\to\infty} s_n = s$. We say that the series $\sum\limits_{k=1}^{\infty} a_k$ has a *sum* s and write $\sum\limits_{k=1}^{\infty} a_k = s$.

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. For example the sum of the reciprocals of the square numbers (the *Basel problem*) is $\pi^2/6$:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \doteq 1.644934 \, .$$

The *nth harmonic number* is the sum of the reciprocals of the first n natural numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \, ,$$

H_0 being defined as 0. The *generalized harmonic numbers* of order n in power r is the sum

$$H_{n,r} = \sum_{k=1}^{n} \frac{1}{k^r} \, ,$$

where $H_{n,1} = H_n$ are harmonic numbers.

Generalized harmonic number of order n in power 2 can be written as a function of harmonic numbers using formula (see [9])

$$H_{n,2} = \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n} \, .$$

From formulas for $H_{n,r}$, where $r = 1,2$ and $n = 1,2,\ldots,9$, we get the following table:

n	1	2	3	4	5	6	7	8	9
H_n	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$
$H_{n,2}$	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	$\frac{5269}{3600}$	$\frac{5369}{3600}$	$\frac{266681}{176400}$	$\frac{1077749}{705600}$	$\frac{771817}{352800}$

Table 1: Nine first harmonic numbers H_n and generalized harmonic numbers $H_{n,2}$

The *hyperbolic cotangent* is defined as a ratio of hyperbolic cosine and hyperbolic sine

$$\coth x = \frac{\cosh x}{\sinh x}, \qquad x \neq 0.$$

Because hyperbolic cosine and hyperbolic sine can be defined in terms of the exponential function

$$\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = \frac{\mathrm{e}^{2x} + 1}{2\,\mathrm{e}^x}, \qquad \sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = \frac{\mathrm{e}^{2x} - 1}{2\,\mathrm{e}^x},$$

we get

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} = \frac{\mathrm{e}^{2x} + 1}{\mathrm{e}^{2x} - 1}, \qquad x \neq 0.$$

2 The sum of the series of reciprocals of the quadratic polynomials with integer roots

As regards the sum of the series of reciprocals of the quadratic polynomials with different positive integer roots a and b, $a < b$, i.e. of the series

$$\sum_{\substack{k=1\\k\neq a,b}}^{\infty} \frac{1}{(k-a)(k-b)},$$

in the paper [1] it was derived that the sum $s(a,b)^{++}$ is given by the following formula using the nth harmonic numbers H_n

$$s(a,b)^{++} = \frac{1}{b-a}\left(H_{a-1} - H_{b-1} + 2H_{b-a} - 2H_{b-a-1}\right). \tag{1}$$

In the paper [2] it was shown that the sum $s(a,b)^{--}$ of the series of reciprocals of the quadratic polynomials with different negative integer roots a and b, $a < b$,

i.e. of the series
$$\sum_{k=1}^{\infty} \frac{1}{(k-a)(k-b)},$$
is given by the simple formula
$$s(a,b)^{--} = \frac{1}{b-a}\left(H_{-a} - H_{-b}\right). \tag{2}$$

The sum of the series
$$\sum_{\substack{k=1\\k\neq b}}^{\infty} \frac{1}{(k-a)(k-b)},$$
of reciprocals of the quadratic polynomials with integer roots $a < 0$, $b > 0$ was derived in the paper [4]. This sum $s(a,b)^{-+}$ is given by the formula
$$s(a,b)^{-+} = \frac{(b-a)\left(H_{-a} - H_{b-1}\right) + 1}{(b-a)^2}. \tag{3}$$

In the paper [3] it was derived that the sum $s(a,a)^{--}$ of the series of reciprocals of the quadratic polynomials with double non-positive integer root a, i.e. of the series
$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^2},$$
is given by the following formula using the generalized harmonic number $H_{-a,2}$ of order $-a$ in power 2
$$s(a,a)^{--} = \frac{\pi^2}{6} - H_{-a,2}. \tag{4}$$

The sum of the series
$$\sum_{\substack{k=1\\k\neq a}}^{\infty} \frac{1}{(k-a)^2},$$
of reciprocals of the quadratic polynomials with double positive integer root a, was derived in the paper [5]. This sum $s(a,a)^{++}$ is given by the formula with the generalized harmonic number in power 2
$$s(a,a)^{++} = \frac{\pi}{2} + H_{a-1,2}. \tag{5}$$

The formula for the sum $s(a,0)^{-0}$ of the series
$$\sum_{k=1}^{\infty} \frac{1}{k(k-a)}$$

of reciprocals of the quadratic polynomials with one zero and one negative integer root a and also the formula for the sum $s(0,b)^{0+}$ of the series

$$\sum_{\substack{k=1 \\ k\neq b}}^{\infty} \frac{1}{k(k-b)}$$

of reciprocals of the quadratic polynomials with one zero and one positive integer root b were derived in the paper [7]. These sums are given by the simple formulas

$$s(a,0)^{-0} = \frac{H_{-a}}{-a}, \tag{6}$$

$$s(0,b)^{0+} = \frac{1-bH_{b-1}}{b^2}. \tag{7}$$

3 The sum of the series of reciprocals of the quadratic polynomials with complex conjugate roots

We deal with the problem to determine the sum $S(a,b)$, where a,b are non-zero integers, of the infinite series of reciprocals of the quadratic polynomials with complex conjugate roots $a \pm b\mathrm{i}$, i.e. the series of the form

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^2+b^2}. \tag{8}$$

The quadratic trinomial $(k-a)^2+b^2 = k^2-2ak+a^2+b^2$ can be in the complex domain written in the product form $\big[k-(a+b\mathrm{i})\big]\cdot\big[k-(a-b\mathrm{i})\big]$, so the quadratic trinomial $k^2-2ak+a^2+b^2$ has complex conjugate roots $k_1 = a+b\mathrm{i}$, $k_2 = a-b\mathrm{i}$.

The series (8) is convergent because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k-a)^2+b^2} \leq \sum_{k=1}^{\infty} \frac{1}{(k-a)^2}$. For non-positive integer a we get by (4) an equality $S(a,b) < \pi^2/6 \doteq 1.6449$ and for positive integer a we have by (5) $S(a,b) < \pi/2+\pi^2/6 \doteq 3.2157$ (see [5]).

Because it obviously does not matter the sign of an imaginary part b, let us deal further with two cases of the series (8) – with a positive real part a and with a negative one. If the integer real part $a > 0$, then the sum $S(a,b)$ has the form

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(k-a)^2+b^2} &= \sum_{k=1}^{\infty}\frac{1}{(a-k)^2+b^2} = \\ &= \frac{1}{(a-1)^2+b^2}+\frac{1}{(a-2)^2+b^2}+\cdots+\frac{1}{1^2+b^2}+ \\ &+ \frac{1}{b^2}+\frac{1}{1^2+b^2}+\frac{1}{2^2+b^2}+\frac{1}{3^2+b^2}+\cdots = s(a-1,b)+\frac{1}{b^2}+s(b),\end{aligned}$$

where

$$s(b) = \sum_{k=1}^{\infty} \frac{1}{k^2 + b^2}$$

is the sum that was derived in the paper [6] and which is given by the formula

$$s(b) = \frac{\pi}{2b} \cdot \frac{\mathrm{e}^{2\pi b} + 1}{\mathrm{e}^{2\pi b} - 1} - \frac{1}{2b^2} = \frac{\pi}{2b} \coth \pi b - \frac{1}{2b^2} \tag{9}$$

and where

$$s(a-1, b) = \sum_{k=1}^{a-1} \frac{1}{k^2 + b^2} \tag{10}$$

is the sum of the finite series, which was in the paper [8] derived by means of the trapezoidal rule and which is given by the approximate formula

$$s(a-1, b) \doteq \frac{1}{b} \arctan \frac{a}{b} - \frac{1}{2} \left(\frac{1}{b^2} + \frac{1}{a^2 + b^2} \right). \tag{11}$$

In this paper it was shown that this approximate formula is a suitable approximation of the sum $s(a-1, b)$, because one hundred results obtained by means of this formula when modelling in Maple 16 have very small relative errors (in the range of 0.60% to 0.05%). In total, we get

$$\begin{aligned} S(a, b) &= s(a-1, b) + \frac{1}{b^2} + s(b) \doteq \\ &\doteq \frac{1}{b} \arctan \frac{a}{b} - \frac{1}{2} \left(\frac{1}{b^2} + \frac{1}{a^2 + b^2} \right) + \frac{1}{b^2} + \frac{\pi}{2b} \coth \pi b - \frac{1}{2b^2}, \end{aligned}$$

so after simple arrangement we have the following result:

$$S(a, b) \doteq \frac{1}{b} \arctan \frac{a}{b} - \frac{1}{2(a^2 + b^2)} + \frac{\pi}{2b} \coth \pi b, \qquad a > 0.$$

If the integer real part $a < 0$, then the sum $S(a, b)$ has for $A = -a > 0$ the form

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{(k-a)^2 + b^2} = \sum_{k=1}^{\infty} \frac{1}{(k+A)^2 + b^2} = \\ &= \frac{1}{(1+A)^2 + b^2} + \frac{1}{(2+A)^2 + b^2} + \frac{1}{(3+A)^2 + b^2} + \cdots = \\ &= \frac{1}{1^2 + b^2} + \frac{1}{2^2 + b^2} + \cdots + \frac{1}{A^2 + b^2} + \frac{1}{(1+A)^2 + b^2} + \frac{1}{(2+A)^2 + b^2} + \cdots \\ &\cdots - \left(\frac{1}{1^2 + b^2} + \frac{1}{2^2 + b^2} + \cdots + \frac{1}{(A-1)^2 + b^2} \right) - \frac{1}{A^2 + b^2} = \\ &\qquad = s(b) - s(A-1, b) - \frac{1}{A^2 + b^2} = s(b) - s(-a-1, b) - \frac{1}{a^2 + b^2}. \end{aligned}$$

So we have

$$S(a,b) = s(b) - s(-a-1,b) - \frac{1}{a^2+b^2} \doteq$$
$$\doteq \frac{\pi}{2b}\coth\pi b - \frac{1}{2b^2} - \left[\frac{1}{b}\arctan\frac{-a}{b} - \frac{1}{2}\left(\frac{1}{b^2} + \frac{1}{a^2+b^2}\right)\right] - \frac{1}{a^2+b^2}$$

and after simple arrangement we get the same result as above for $a > 0$:

$$S(a,b) \doteq \frac{1}{b}\arctan\frac{a}{b} - \frac{1}{2(a^2+b^2)} + \frac{\pi}{2b}\coth\pi b, \qquad a < 0.$$

Therefore for every integer a including zero we get the main result

$$S(a,b) \doteq \frac{1}{b}\arctan\frac{a}{b} - \frac{1}{2(a^2+b^2)} + \frac{\pi}{2b}\coth\pi b. \tag{12}$$

4 Numerical verification

We solve the problem to determine the values of the sum $S(a,b)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^2+b^2}$$

for $a = -5, -4, \ldots, 5$ and for $b = 1, 2, \ldots, 10$. We use on the one hand an approximative direct evaluation of the sum

$$s(a,b,t) = \sum_{k=1}^{t} \frac{1}{(k-a)^2+b^2},$$

where $t = 10^5$, using the basic programming language of Maple 16, and on the other hand the formula (12) for evaluation the sum $S(a,b)$. We compare 110 pairs of these two ways obtained sums $s(a,b,10^5)$ and $S(a,b)$ to verify the formula (12). We use following procedure `sumsab` and succeeding for-loop statement:

```
sumsab=proc(a,b,t)
   local k,s,S; s:=0;
   for k from 1 to t do
       s:=s+1/((k-a)*(k-a)+b*b);
   end do;
   print("s(",a,b,t,")=",evalf[6](s);
   S:=evalf[6]((1/b)*arctan(a/b)-1/(2*(a*a+b*b))
       +(Pi/(2*b))*coth(Pi*b));
   print("S(",a,b,")=",evalf[6](S));
   print("relerr(S)=",evalf[10](100*abs(s-S)/s),"%");
end proc:
```

```
for a from -5 to 5 do
    for b from 1 to 10 do
        sumsab(100000,a,b);
    end do;
end do;
```

Forty of these one hundred and ten approximative values of the sums $s(a,b,10^5)$ and $S(a,b)$ rounded to four decimals obtained by these procedure and the relative quantification accuracies

$$r(a,b) = \frac{|s(a,b,10^5) - S(a,b)|}{s(a,b,10^5)}$$

of the sums $s(a,b,10^5)$ (expressed as a percentage) are written into Table 2 below. Let us note that the computation of 110 values $s(a,b,10^5)$ (abbreviated in Table 2 as $s(a,b)$) and $S(a,b)$ took over 5 hours 24 minutes. The relative quantification accuracies are approximately between 6.14% and 0.0006%, 96 of these 110 approximative values have the relative quantification accuracy smaller than 0.5%.

$s\,\|\,S\,\|\,r$	$a=-3$	$a=-2$	$a=-1$	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$
$s(a,1)$	0.2766	0.3767	0.5767	1.0767	2.0767	2.5767	2.7767	2.8767
$S(a,1)$	0.2776	0.3695	0.5413	1.0767	2.1121	2.5838	2.7757	2.8731
$r(a,1)$	0.34	1.90	6.14	0.0006	1.70	0.28	0.04	0.12
$s(a,2)$	0.2585	0.3354	0.4604	0.6604	0.9104	1.1104	1.2354	1.3123
$S(a,2)$	0.2555	0.3302	0.4536	0.6604	0.9172	1.1156	1.2383	1.3140
$r(a,2)$	1.13	1.55	1.48	0.002	0.75	0.47	0.24	0.13
$s(a,3)$	0.2356	0.2911	0.3680	0.4680	0.5791	0.6791	0.7561	0.8116
$S(a,3)$	0.2340	0.2891	0.3663	0.4680	0.5808	0.6811	0.7576	0.8127
$r(a,3)$	0.65	0.38	0.46	0.002	0.29	0.29	0.21	0.13
$s(a,4)$	0.2126	0.2526	0.3026	0.3614	0.4239	0.4828	0.5328	0.5728
$S(a,4)$	0.2118	0.2518	0.3020	0.3614	0.4245	0.4836	0.5335	0.5734
$r(a,4)$	0.37	0.33	0.19	0.003	0.14	0.18	0.15	0.12
$s(a,5)$	0.1918	0.2212	0.2557	0.2941	0.3341	0.3726	0.4071	0.4365
$S(a,5)$	0.1914	0.2208	0.2554	0.2942	0.3344	0.3730	0.4075	0.4369
$r(a,5)$	0.22	0.18	0.09	0.003	0.08	0.11	0.11	0.09

Table 2: Some approximate values of the sums $s(a,b,10^5)$, $S(a,b)$ and the relative quantification accuracies $r(a,b)$ of the sums $s(a,b,10^5)$ for some values of a and b

5 Conclusions

We dealt with the problem to determine the sum $S(a, b)$, where a, b are non-zero integers, of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^2+b^2}$$

of reciprocals of the quadratic polynomials with complex conjugate roots $a \pm b\mathrm{i}$.

We derived that the sum $S(a, b)$ is given by the approximate formula

$$S(a, b) \doteq \frac{1}{b} \arctan \frac{a}{b} - \frac{1}{2(a^2+b^2)} + \frac{\pi}{2b} \coth \pi b.$$

We verified this result by computing 110 various sums by using the computer algebra system Maple 16. This result also includes a special case, when $b = a$. In this case we get the approximate formula

$$S(a, a) \doteq \frac{\pi}{4a} - \frac{1}{4a^2} + \frac{\pi}{2a} \coth \pi a.$$

Because for integer $a \geq 1$ it holds $\coth \pi a \to 1$ (e.g. $\coth \pi \doteq 1.004$, $\coth 2\pi \doteq 1.000007$, $\coth 3\pi \doteq 1.00000001$), we have the simple aproximate formula

$$S(a, a) \doteq \frac{3\pi a - 1}{4a^2}.$$

Let us note that this consequence of the main result corresponds to the numeric values in Table 2: $S(1, 1) \doteq (3\pi - 1)/4 \doteq 2.1062$, $S(2, 2) \doteq (6\pi - 1)/16 \doteq 1.1156$, $S(3, 3) \doteq (9\pi - 1)/36 \doteq 0.7576$, $S(4, 4) \doteq (12\pi - 1)/64 \doteq 0.5734$.

The series of the quadratic polynomials with complex conjugate roots $a \pm b\mathrm{i}$ so belong to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically and also presented by means of a simple numerical expression.

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Preparation and Application of Mind Maps in Mathematics Teaching and Analysis of their Advantages in Relation to Classical Teaching Methods

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Abstract

In this article, we are dealing with mind maps and describing the experiment with the application of mind maps in teaching mathematics at secondary schools. The experiment is aiming at comparing classical teaching and learning with mind maps. In the past, we created two groups of students (25 students per group), an experimental and a control group. We have set up a pre-test consisting of tasks not related to the subject that will be taken through mind maps. By the end of the experiment, we apply a post-test with tasks directly focused on the subject that we will teach through mind maps. We will then evaluate the individual tests and then we will evaluate the effectiveness of the mind maps in the teaching process compared to the traditional methods.

Keywords: mind maps, experiment, mathematics, application, pre-test, post-test, comparing

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1. Mind Maps

Mind maps are diagrams that express essential relationships between terms in the form of claims. The statements are represented by briefly characterized combinations of terms that describe relationship information and describe the interrelation of concepts. The mind map illustrates the structure, hierarchy, and the relationship between the terms. It enhances the learning process efficiency and promotes creativity. They are very economical in expressing a very complex content, helpful to memorize, allowing a view of the same thing from multiple angles, allowing see the relationships between ideas in a complex way [8]. They also help to see paradoxes and opposites, which motivates students to ask new questions. It is very important to determine the central idea, from which lead the main and the secondary branches, which gradually form certain relations. We use different colors, shortcuts, diagrams, symbols, equations, and images in the map.

Mind maps as a schematic expression of thoughts, ideas or notes are not inventions of the 21st century. In addition, the use of learning methods built on the creation and presentation, which are logically arranged and the links of the conjugated terms are not the discovery of current educational trends. In the past, teachers structured the learned curriculum by using key concepts placed on a magnetic board, notice board, or complemented the prepared schemes with cutout images and characters. Many important artists and geniuses, such as Leonardo da Vinci, Michelangelo, Isaac Newton, Pablo Picasso, Thomas Edison, Galileo, Marie Curie and others, brought certain schemes into their own ideas in the past. They tried to highlight their ideas, not just linear, using lines and words, but also with a strong language of images, drawings, schemes, codes, symbols, and graphs [1].
In current professional, as well as popular-learning literature, we can encounter several names of the linear and nonlinear layout of concepts, data and main themes in graphically integrated structures. The authors present conceptual maps, mental maps, thought maps, cognitive maps, semantic maps, knowledge maps, webs, mind maps, and so on. Some of these terms do not distinguish individually and call their collectively mental maps or cognitive schemes [2].

J. D. Novak [3] considered as a founder of conceptual map theories and their construction speaks of mind maps as a hierarchically arranged, graphical representation of relations between selected concepts. There are general terms at the top of the map that are associated with terms that are more specific in the lower tree level. From the central concept, the "branches" are connecting with the concept in the lower parts of the map, from which the "branches" are connecting again with the concept at the lower levels of the map.

Psychologists Veselský [4] and Stewart [5] talk about conceptual maps as graphical imaging systems, whose basic building unit are concepts. They are represented by frames with inscribed notional names and the relationships are expressed by marked orientated lines linking the respective conceptual expressions. Focus on the non-linear abstract representation of the structure of the subject and notes an opposite to the written, printed, projected, or otherwise presented text followed by the sentences one after the other [6], stresses Mareš. According to him, it is based on the idea of organizing the best and the most transparent key concepts and relationships by "visualizing" them and creating a sketch, a schematic of an easily accessible abstract "outer" memory. Although a learner learns to organize the key elements of the curriculum on paper first. He has to begin with organizing them in the head. Thus, he is forced to consciously construct and reconstruct a network of concepts and relationships in his "mental space" [7].

It follows from the above that there is no terminological unity and consensus among the experts in understanding the different concepts of capturing concept ideas in the graphical structure of related concepts with the designation of relationships and links between them. The Fisher’s definition of mind maps is probably the most precise according to them. A conceptual, thought-based, or otherwise called mental map is a diagram that illustrates the context and relationship between knowledge, serves to organize them. We understand the conceptual, mental or idea map as synonyms. We do this in particular because some of the used conceptual maps do not have a typical structure of mental maps, they consist of several levels, and there are significant links – relationships between some terms [2].

2. Preparation, Application and Evaluation of the Pre-Test

If we wanted to compare the two different teaching methods, we needed to have the experimental and control groups at the same level of knowledge before the comparisons began. To test knowledge of these groups, we created a pre-test that tested both groups before applying the mind maps. Students in both groups wrote this pre-test on the same day. The pre-test included three tasks from the previous non-geometry related lesson. Tasks aimed at adjusting the fractions, creating, and solving the equation. In each assignment, we identified several characters, which we took into consideration during the evaluation and allowed us to compare the two groups of students more objectively.

<table>
<tr><td colspan="2">Pre-test, Task 1: Determine when the expression is meaningful and adjust it to the simplest form.</td></tr>
<tr><td>$$\frac{\frac{a+b}{a-b}-\frac{a-b}{a+b}}{1-\frac{a^2+b^2}{a^2-b^2}}:\frac{\frac{1}{b^2}-\frac{2}{b}+1}{2-\frac{1+b^2}{b}}$$</td><td>Rated characters in task # 1:
$a \neq b$
$b \neq 0{,}1$
transcription for multiplication
common denominator
modifying a fraction
cutting fractions
excluding -1 from the second fraction
result</td></tr>
<tr><td colspan="2">Pre-test, Task 2: If we enlarge one side of the square by 4 units and at the same time reduce the other side by 2 units; we create a rectangle whose content is 12% larger than the square. Specify the square size of the square.</td></tr>
<tr><td></td><td>Rated characters in task # 2:
picture
content of a square
rectangle content
increase content by 12%
equality of contents
edit quadratic equation
result</td></tr>
<tr><td colspan="2">Pre-test, Task 3: If we increase the unknown number by 7 and if we create the square root of this enlarged number, we get a number that is by 5 smaller than the original number. Specify an unknown number.</td></tr>
<tr><td></td><td>Rated characters in task # 3:
enlarged number
root
reduced number
equality
squaring
writing of quadratic equation
modification of quadratic equation
writing results</td></tr>
</table>

The selected characters represented the various conditions within the given task, important for its solvability, mathematical operations, mathematical entries, various comparisons, adjustments of equations, fractions and results of individual tasks. The choice of characters within the assignments helped us evaluate the solution of these tasks objectively without any external influence.

Picture 1: Selected sample of students in pre-test, experimental group, Task 1

a ≠ b	b ≠ 0,1	transcription for multiplication	common denominator	modifying a fraction	cutting fractions	excluding -1 from the second fraction	result
N	N	N	N	N	N	N	N
N	N	N	N	N	N	N	N
N	N	Y	Y	N	N	N	N
N	N	Y	Y	N	N	N	N

Picture 2: Selected sample of students in pre-test, experimental group, Task 2

picture	content of a square	rectangle content	increase content by 12%	equality of contents	edit quadratic equation	result
Y	N	Y	N	N	N	N
N	N	N	N	N	N	N
Y	Y	Y	N	N	N	N
Y	Y	Y	N	N	N	N

Picture 3: Selected sample of students in pre-test, experimental group, Task 3

enlarged number	root	reuced number	equality	squaring	writting of quadratic equation	modification of quadratic equation	writting results
Y	Y	Y	Y	N	N	N	N
Y	Y	Y	Y	N	N	N	N
Y	Y	Y	Y	N	N	N	N
Y	Y	N	Y	N	N	N	N

Picture 4: Selected sample of students in pre-test, control group, Task 1

a ≠ b	b ≠ 0,1	transcription for multiplication	common denominator	modifying a fraction	cutting fractions	excluding -1 from the second fraction	result
N	Y	N	N	N	N	N	N
N	N	Y	Y	N	N	N	N
N	N	Y	Y	N	N	N	N
N	N	N	N	N	N	N	N

Picture 5: Selected sample of students in pre-test, control group, Task 2

picture	content of a square	rectangle content	increase content by 12%	equality of contents	edit quadratic equation	result
N	Y	Y	N	N	Y	N
Y	N	N	N	N	N	N
N	N	N	N	N	N	N
N	N	N	N	N	N	N

Picture 6: Selected sample of students in pre-test, control group, Task 3

enlarged number	root	reuced number	equality	squaring	writting of quadratic equation	modification of quadratic equation	writting results
Y	Y	Y	Y	N	N	N	N
Y	Y	Y	Y	N	N	N	N
Y	Y	Y	Y	N	N	N	N
N	N	N	N	N	N	N	N

We evaluated the pre-test as the ratio of the characters in the resolution (Y means, that character was in resolution, N means, that character was missing in solution) from all students within the given task to the total number of characters within the given task. In Table 1 there is shown percentage of students' success rate in each group in a particular task.

Table 1: Comparison of pre-test results in experimental and control groups (%)

Task / Group	Experimental	Control
Task 1	10.2	5.7
Task 2	14.3	15.6
Task 3	33	34

Even though the students did not properly calculate the tasks, it was clear from the pre-test that both groups of students were about the same level of knowledge, what was essential for our experiment and we could move to the next stage, the application of mind maps in the teaching process.

3. Application of Conceptual Maps in the Teaching Process in the Subject of Mathematics

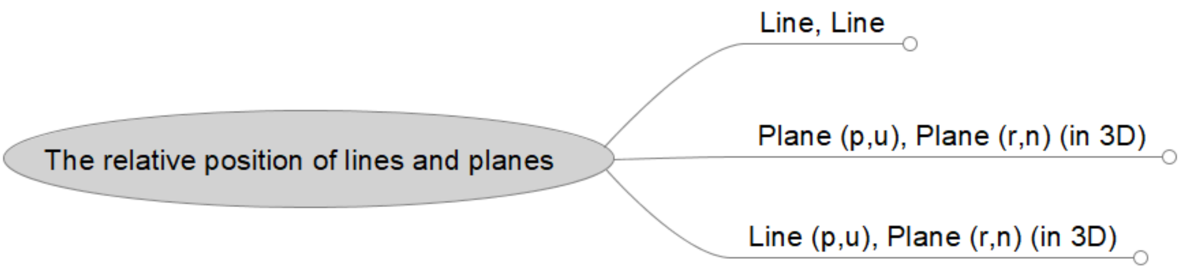
After agreement with the mathematics teacher in the experimental group, we had three lessons available, during which we presented the curriculum of geometry dealing with the mutual positions of lines and planes.

We used a computer and a projector for this activity. The curriculum was processed using conceptual maps and inserted into the presentation.

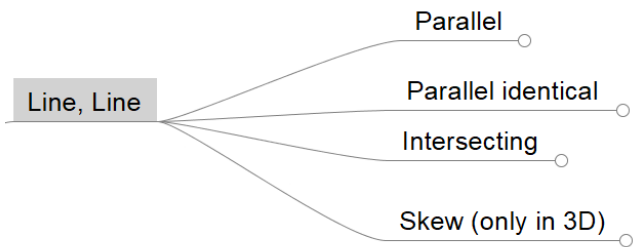
We divided the curriculum for lessons into individual groups as follows:
1st lesson: Mutual position of lines (parallel, parallel identical, concurrent),
2nd lesson: Mutual position of lines (skew), mutual position of lines and planes (parallel, parallel identical),
3rd lesson: Mutual position of lines and planes (concurrent), mutual position of planes (parallel, parallel identical, concurrent).

We informed students about the content for next three lessons.

Picture 7: Introductory division of the curriculum

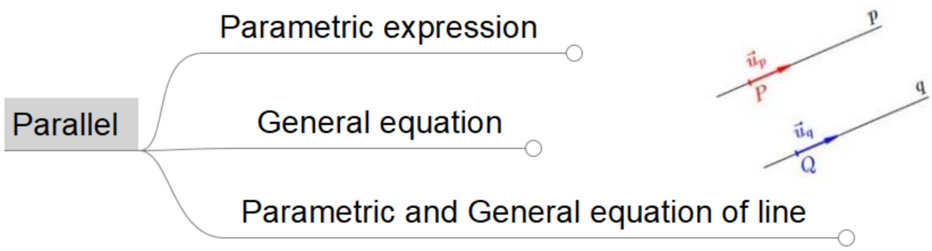


Picture 8: Detailed division of the mutual positions of the two lines



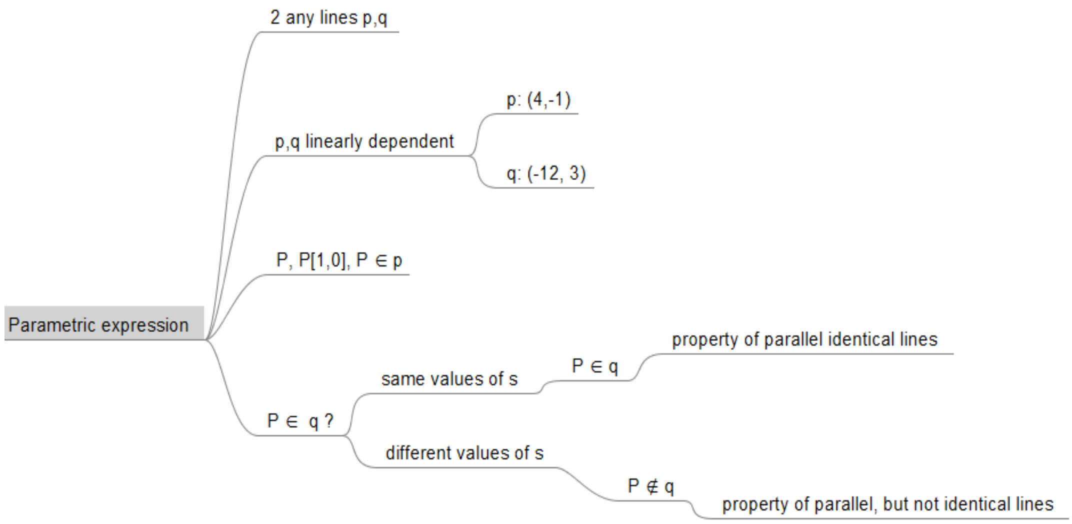
In Picture 8, we have explored in more detail the possible mutual positions of the two lines.

Picture 9: Detailed division of the parallel positions of the two lines



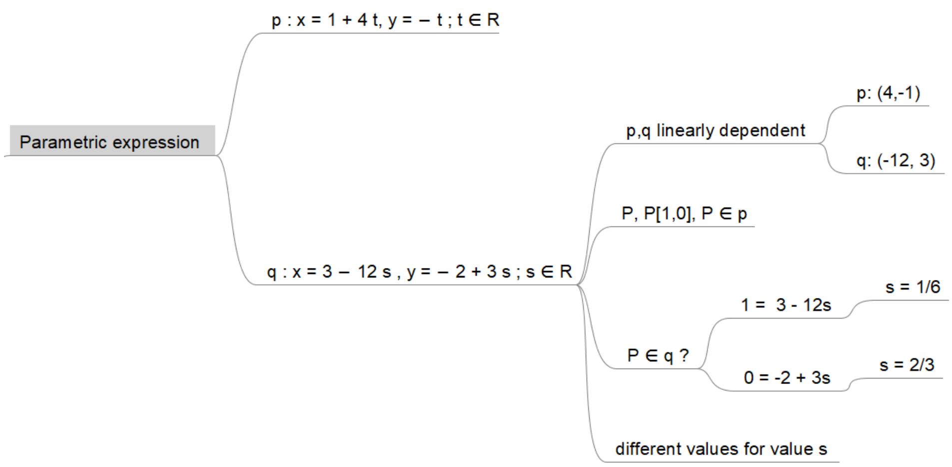
Picture 9 focused on the case of two parallel lines and the possible representation of these lines.

Picture 10: Detailed parametric representation of mutual parallel positions of two lines



In Picture 10 there is an illustrative and detailed description of a branch of parametric representation.

Picture 11: An example of the mutual position of two parallel lines in parametric representation



In Picture 11 there is task, which the students were trying to solve after the theoretical part was completed. On the map there were marked intermediate results that served to students for check.

As can be seen in Picture 7 – 11, the principles of the mind map were retained. The deeper we got into the mind map, the more specific terms were in that part. Our task was to explain these concepts to the students so they can join these concepts together alone and can apply them in solving different problems. After these three lessons, during which we were teaching using mind maps, we moved into the final phase of our experiment.
This part consisted of the post-test we gave to the students. The post-test consisted of tasks that focused directly on the subject discussed at our three lessons.

4. Preparation, Application and Evaluation of the Post-Test

After completing the pre-test, which showed us that the students are about the same level of knowledge, following the use of mind maps in three teaching lessons, we have reached the final stage of our experiment. This final phase consisted of two phases: application of post-test and evaluation of results from post-test. The tasks in the post-test were, this time, directly focused on the mutual positions of planes and lines, in order to compare the effectiveness of this method in the experimental group against the classical way of teaching in the control group. In the given tasks, we have re-selected the characters that represented the key elements in the solving of the task.

Post-test, Task 1: Show that the planes α and β are concurrent and write the parametric representation of the intersection of these planes.	
α: $5x - 3y + 2z - 5 = 0$ β: $2x - y - z - 1 = 0$	Rated characters in task # 1: • normal vector p • normal vector q • vector products • parameter at point P • parametric representation of intersection

Post-test, Task 2: Determine the mutual position of plane β and line p.	
β: x – 5y + 4z – 6 = 0 p: x = 2 – t, y = 3t, z = 3 + 4t, t ∈ R	Rated characters in task # 2: • placing **p** to β, scalar product • place P into equation β • adjusting equation after placing P into β • determine final position
Post-test, Task 3: Determine the mutual positions of p, q. If p = $\overleftrightarrow{(AB)}$, q = $\overleftrightarrow{(CD)}$	
A = [7, 6] B = [6, 8] C = [6, -5] D = [4, -1]	Rated characters in task # 3: • line p • line q • expression of **p** • expression of **q** • vector comparison • computation and comparison of parameters • result

Picture 12: Selected sample of students in post-test, experimental group, Task 1

normal vector p	normal vector q	vector products	parameter at point P	parametric expression of intersection
Y	Y	Y	Y	Y
Y	Y	Y	Y	Y
Y	Y	Y	Y	Y
Y	Y	Y	Y	N

Picture 13: Selected sample of students in post-test, experimental group, Task 2

placing p to β, scalar product	placing p to β, scalar product	adjusting equation after placing P into β	determine final position
Y	Y	Y	Y
Y	Y	Y	Y
Y	Y	Y	Y
Y	Y	Y	Y

Picture 14: Selected sample of students in post-test, experimental group, Task 3

line p	line q	expression of p	expression of q	vector comparison	computation and comparison of parameters	result
Y	Y	N	N	Y	N	N
Y	Y	Y	Y	Y	Y	Y
Y	Y	Y	Y	Y	Y	Y
Y	Y	Y	Y	Y	Y	Y

Picture 15: Selected sample of students in post-test, control group, Task 1

normal vector p	normal vector q	vector products	parameter at point P	parametric expression of intersection
Y	Y	N	N	N
Y	Y	Y	Y	N
Y	Y	Y	Y	N
Y	Y	Y	Y	Y

Picture 16: Selected sample of students in post-test, control group, Task 2

placing p to β, scalar product	placing p to β, scalar product	adjusting equation after placing P into β	determine final position
Y	N	N	N
N	N	N	N
Y	Y	Y	Y
Y	Y	Y	Y

Picture 17: Selected sample of students in post-test, control group, Task 3

line p	line q	expression of p	expression of q	vector comparison	computation and comparison of parameters	result
Y	Y	N	N	Y	N	N
Y	N	N	N	N	N	N
Y	Y	N	N	Y	N	N
Y	N	N	N	Y	N	N

The post-test was evaluated in the same way as the pre-test and the results from both tests were subsequently recorded in Table 2

Table 2: Comparison of the results of the post-test in the experimental and control group (%)

Task / Group	Experimental	Control
Task 1	88.3	78.3
Task 2	91.7	68.8
Task 3	82.1	53.6

As we can see from Table 2, the results compared to the pre-test are much better. The students were able to apply the acquired knowledge in solving of the given tasks. For our experiment is much more important that the results achieved in the experimental group, in the group where we were teaching with the help of mind maps, are obviously better than in the control group where the classic teaching methods were used.

5. Conclusion

This article was focused on the application of mind maps in the teaching process and the comparison of the mind map's effectiveness with the classical way of teaching. This comparison consisted of three important steps:

1st Pre-test and evaluation
2nd Application of conceptual maps in the teaching process
3rd Post-test and evaluation

In both tests, pre-test and post-test, we chose the rated characters which we were looking for during the correction of students' tests. We subsequently evaluated and compared these rated characters. The students wrote the tests on the same day to prevent the possible influence and improvement of the results in one or the other group.

In the first step, we gave the students a pre-test, in which the balance or imbalance of students' knowledge in the experimental and control group should be demonstrated. The results of the pre-test showed that the students were about the same level of knowledge.
In the next step, we had three lessons from the geometry. Subject of these lessons was the mutual positioning of the lines and the planes. During these lessons we were using mind maps.
In the last step, the students wrote a post-test with tasks related to the mutual positions of the lines and planes. The post-test was then evaluated and the results from both groups were compared in Table 2.
According to the values, we can see that success in solving problems is higher in the experimental group. These results are better in the range of 10% to almost 30% compared to the control group, which is not negligible. Only for comparison, the results of the pre-test in both groups varied from 1% to less than 5%.
At the end, we can conclude that the mind maps in our presentation with our teaching are more effective compared to the traditional classical teaching method.

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Note on Heisenberg Characters of Heisenberg Groups

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Abstract

An irreducible character χ of a group G is called a Heisenberg character, if $Ker\chi \supseteq [G, [G, G]]$. In this paper, the Heisenberg characters of the quaternion Heisenberg, generalized Heisenberg, polarised Heisenberg and three other types of infinite Heisenberg groups are computed.

Keywords: Heisenberg character, Heisenberg group.

1 Introduction

Suppose G is a finite group and V is a vector space over the complex field $\mathbb{C}$. A representation of G is a homomorphism $\varphi : G \longrightarrow GL(V)$, where $GL(V)$ denotes the group of all invertible linear transformations $V \longrightarrow V$ equipped with

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composition of functions. The commutator subgroup $[G, G]$ is the subgroup generated by all the commutators $[x, y] = xyx^{-1}y^{-1}$ of the group G.

An irreducible character χ of a group G is called a **Heisenberg character**, if $ker\chi \supseteq [G, [G, G]]$ [1]. Suppose $\varphi : G \longrightarrow GL(V)$ is an irreducible representation with irreducible character χ. Since $[G, G'] \leq Ker\chi$, $\overline{\varphi} : \frac{G}{[G,G']} \longrightarrow GL(V)$ is an irreducible representation of $\frac{G}{[G,G']}$. Conversely, we assume that $\chi \in Irr\left(\frac{G}{[G,G']}\right)$ and $\delta : \frac{G}{[G,G']} \longrightarrow GL(V)$ affords the irreducible character χ. If $\gamma : G \longrightarrow \frac{G}{[G,G']}$ denotes the canonical homomorphism then $\delta o\gamma : G \longrightarrow \frac{G}{[G,G']}$ is an irreducible representation for G and $Ker\ \delta o\gamma \supseteq [G, G']$. This proves that there is a one to one correspondence between Heisenberg characters of G and irreducible characters of $\frac{G}{[G,G']}$, see [2, 8] for details.

Marberg [8] in his interesting paper proved that the number of Heisenberg characters of the group $U_n(q)$ is a polynomial in $q - 1$ with nonnegative integer coefficients, with degree $n - 1$, and whose leading coefficient is the $(n - 1)-$th Fibonacci number. The present authors [1], characterized groups with at most five Heisenberg characters. The aim of this paper is to compute all Heisenberg characters of five classes of infinite Heisenberg groups. These are as follows:

1. Suppose $\mathbf{T}$ denotes the set of all complex numbers of unit modulus and $H = \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{T}$. Define $(y_1, x_1, z_1)(y_2, x_2, z_2) = (y_1 + y_2, x_1 + x_2, e^{-2\pi i y_2.x_1} z_1 z_2)$. It is easy to see that H is a group under this operation. This group is called the Heisenberg group of second type [5].

2. The polarised Heisenberg group H_n^3 is defined as the set of all triples in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ under the multiplication
$$(x, y, z)(a, b, c) = (x + a, y + b, z + c + \frac{1}{2}(x.b - y.a)),$$
see [3] for details.

3. Suppose $a = (a_1, a_2, \ldots, a_n)$ is an $n-$tuple in $\mathbb{R}^n$, where a_i's are positive real constants, $1 \leq i \leq n$. Following Tianwu and Jianxun [10], we define a group operation on $H_n^a = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ given by
$$(x, y, z)(r, s, t) = (x + r, y + s, z + t + \frac{1}{2}\sum_{j=1}^{n} a_j(r_j y_j - s_j x_j)).$$

This group is called the generalized Heisenberg group. In the mentioned paper, the authors proved that the group operation of the generalized Heisenberg group can be simplified in the following way:

Suppose $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Define $x * y = (x_1y_1, x_2y_2, \ldots, x_ny_n)$ and $ab = \sum_{j=1}^{n} a_jb_j$. If $c = (c_1, c_2, \ldots, c_n)$ and $\lambda \in \mathbb{R}$ then we can see that (i) $x * (y + c) = x * y + x * c$; (ii) $(x * y)c = x(y * c)$; and (iii) $(\lambda x) * y = \lambda(x * y)$. Therefore, the group operation of the generalized Heisenberg group can be written as $(x, y, z)(r, s, t) = (x + r, y + s, z + t + \frac{1}{2}((a * r)y - (a * s)x))$.

4. Suppose $\mathbb{H}$ denotes the set of all of quaternion numbers with three imaginary units i, j and k such that $i^2 = j^2 = k^2 = ijk = -1$. Following Liu and Wang [7], we define the quaternion Heisenberg group $\mathcal{N}$ as a nilpotent Lie group with underlying manifold $\mathbb{R}^4 \times \mathbb{R}^3$. The group structure is given by
$$(q, t)(p, s) = (q + p, t + s + \frac{1}{2}Im(\overline{p}q)),$$
where $p, q \in \mathbb{R}^4$ and $t, s \in \mathbb{R}^3$.

5. Following Qingyan and Zunwei [9], the Heisenberg group $\mathbb{H}^n$ of third type is a non-commutative nilpotent Lie group, with the underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$. The group operation can be given as:
$$(x_1, x_2, \ldots, x_{2n}, x_{2n+1})(x_1', x_2', \ldots, x_{2n}', x_{2n+1}') =$$
$$(x_1 + x_1', x_2 + x_2', \ldots, x_{2n} + x_{2n}', x_{2n+1} + x_{2n+1}' + 2\sum_{j=1}^{n}(x_j'x_{n+j} - x_jx_{n+j}').$$

6. Suppose $\mathcal{H}^n = \mathbb{C}^n \times \mathbb{R}$ with group law defined by $(z, t) \cdot (w, s) = (z + w, t + s + 2Im(z.\overline{w}))$. This is our sixth class of Heisenberg groups. Following Chang et al. [4], this group can be realized as the boundary of the Siegel upper half-space $\mathcal{U}_{n+1}$ in $\mathbb{C}^{n+1}$, where the group operation gives a group action on the hypersurface.

Throughout this paper our notation is standard and can be taken from the famous book of Isaacs [6]. Suppose G is a group and $\{\{e\} = A_0, A_1, \ldots, A_n = G\}$ is a set of normal subgroups of G such that
$$A_0 \lhd A_1 \lhd \ldots \lhd A_n = G. \tag{1}$$

The sequence (1) is called a **central series** for G, if $[G, A_{i+1}] \leq A_i$ in which $[G, H]$ denotes the subgroup of G generated by all commutators $ghg^{-1}h^{-1}$, where $g \in G, h \in H$. The group G is called **nilpotent**, if it has a central series. The **nilpotency class** of G, $nc(G)$, is the length of its central series. The set of all irreducible characters of G is denoted by $Irr(G)$ and the trivial character of G is denoted by 1_G.

2 Main Results

The aim of this section is to compute the Heisenberg characters of five different types of Heisenberg groups. To do this, we first note that every linear character of a group G is Heisenberg. This proves that all irreducible characters of abelian groups are Heisenberg.

Lemma 2.1. *All irreducible characters of a group G are Heisenberg if and only if G is nilpotent of class two.*

Proof. Suppose $nc(G) = 2$. Then $[G, G'] = 1$ and so all irreducible characters are Heisenberg. If all irreducible characters are Heisenberg then $[G, G'] \leq \cap_{\chi \in Irr(G)}$ = $\{e\}$, as desired. $\square$

Theorem 2.2. *All irreducible characters of the Heisenberg groups H, H_n^3, H_n^a, $\mathcal{N}$, $\mathbb{H}^n$ and $\mathcal{H}^n$ are Heisenberg.*

Proof. Apply Lemma 2.1. Our main proof will consider five separate cases as follows:

1. *The Heisenberg Group H.* We first compute the derived subgroup H'. We have,

$$\begin{aligned} H' &= \left\langle [(x,y,z),(x^{'},y^{'},z^{'})] \mid (x,y,z),(x^{'},y^{'},z^{'}) \in H, z = e^{i\Theta_1}, z^{'} = e^{i\Theta_2} \right\rangle \\ &= \langle (x+x^{'}, y+y^{'}, e^{i(\Theta_1+\Theta_2-2\pi x^{'}y)})(-x-x^{'}, -y-y^{'}, \\ &\quad e^{i(\Theta_1+\Theta_2+2\pi xy+2\pi x^{'}y^{'}+2\pi x^{'}y)}) \mid (x,y,z),(x^{'},y^{'},z^{'}) \in H, z = e^{i\Theta_1}, z^{'} = e^{i\Theta_2} \rangle \\ &= \left\langle (0,0,e^{2\pi i(x.y^{'}-x^{'}.y)}) \mid x,y,x',y' \in \mathbb{R}^n \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned}[H, H'] &= \big\langle (x, y, e^{i\Theta_1})(0, 0, e^{i\Theta_2})(-x, -y, e^{-i\Theta_1 - 2\pi ixy})(0, 0, e^{-i\Theta_2}) \\ &\qquad \mid (x, y, e^{i\Theta_1}) \in H, (0, 0, e^{i\Theta_2}) \in H' \big\rangle \\ &= \langle (x, y, e^{i\Theta_1 + i\Theta_2})(-x, -y, e^{-i(\Theta_1 + \Theta_2 + 2\pi ixy}) \mid \\ &\qquad (x, y, e^{i\Theta_1}) \in H, (0, 0, e^{i\Theta_2}) \in H' \rangle = \{(0, 0, 1)\}.\end{aligned}$$

So, all irreducible characters of H are Heisenberg.

2. *The Heisenberg group* H_n^3. The commutator subgroup of H_n^3 can be computed as follows:

$$\begin{aligned}(H_n^3)' &= \langle [(x, y, z), (x^{'}, y^{'}, z^{'})] \mid (x, y, z), (x^{'}, y^{'}, z^{'}) \in H_n^3 \rangle \\ &= \langle (x, y, z)(x^{'}, y^{'}, z^{'})(x, y, z)^{-1}(x^{'}, y^{'}, z^{'})^{-1} \mid (x, y, z), (x^{'}, y^{'}, z^{'}) \in H_n^3 \rangle \\ &= \langle (0, 0, (x.y^{'} - y.x^{'}) \mid x, y, x', y' \in \mathbb{R}^n \rangle.\end{aligned}$$

On the other hand, $[H_n^3, (H_n^3)'] = \{(0, 0, 0)\}$ and so all irreducible characters of this group are Heisenberg.

3. *The Heisenberg group* H_n^a. Again, we first compute the commutator subgroup $(H_n^a)'$. We have,

$$\begin{aligned}(H_n^a)' &= \langle [(x, y, z), (x^{'}, y^{'}, z^{'})] \mid (x, y, z), (x^{'}, y^{'}, z^{'}) \in H_n^a, a \in \mathbb{R}_+^n \rangle \\ &= \Big\langle (x + x^{'}, y + y^{'}, \frac{1}{2}((a * x^{'})y - (a * y^{'})x))(-x - x^{'}, -y - y^{'}, \\ &\quad + \frac{1}{2}\big((a * -x^{'})(-y) - (a * -y^{'})(-x)\big) \mid (x, y, z), (x^{'}, y^{'}, z^{'}) \in H_n^a, a \in \mathbb{R}_+^n \rangle \\ &= \Big\langle (0, 0, (a * x^{'})y - (a * y^{'})x) \mid x, y, x', y' \in \mathbb{R}^n, a \in \mathbb{R}_+^n \Big\rangle.\end{aligned}$$

Therefore, $[H_n^a, (H_n^a)'] = \{(0, 0, 0)\}$. This shows that all irreducible characters are Heisenberg.

4. *The Heisenberg group* $\mathcal{N}$. By definition of this group, we have

$$\begin{aligned}\mathcal{N}' &= \langle [(p, t), (q, s)] \mid (p, t), (q, s) \in \mathcal{N} \rangle \\ &= \langle (p + q, t + s + \frac{1}{2}Im(\overline{q}p))(-p - q, -t - s + \frac{1}{2}Im(\overline{q}p)) \mid p, q \in \mathbb{R}^4, t, s \in \mathbb{R}^3 \rangle \\ &= \langle (0, Im(\overline{q}p)) \mid p, q \in \mathbb{R}^4 \rangle.\end{aligned}$$

Therefore, we have again $[\mathcal{N}, \mathcal{N}'] = \{(0, 0)\}$. Now apply Lemma 2.1 to deduce that all irreducible characters of this group are Heisenberg.

5. *The Heisenberg group* $\mathbb{H}^n$. By definition of this group,

$$\begin{aligned}
(\mathbb{H}^n)' &= \big\langle [(x_1,\ldots,x_{2n},x_{2n+1}),(x_1^{'},\ldots,x_{2n}^{'},x_{2n+1}^{'})] \mid \\
&\qquad (x_1,\ldots,x_{2n},x_{2n+1}),(x_1^{'},\ldots,x_{2n}^{'},x_{2n+1}^{'}) \in \mathbb{H}^n \big\rangle \\
&= \langle (x_1+x_1^{'},\ldots,x_{2n}+x_{2n}^{'},x_{2n+1}+x_{2n+1}^{'}+2\big(\sum_{j=1}^{n}(x_j^{'}x_{n+j}-x_jx_{n+j}^{'})\big) \\
&\quad (-x_1-x_1^{'},\ldots,-x_{2n}-x_{2n}^{'},-x_{2n+1}-x_{2n+1}^{'}+2\big(\sum_{j=1}^{n}(x_j^{'}x_{n+j}-x_jx_{n+j}^{'})\big) \\
&\qquad \mid (x_1,\ldots,x_{2n},x_{2n+1}),(x_1^{'},\ldots,x_{2n}^{'},x_{2n+1}^{'}) \in \mathbb{H}^n \rangle \\
&= \big\langle (0,\ldots,4\big(\sum_{j=1}^{n}(x_j^{'}x_{n+j}-x_jx_{n+j}^{'})\big) \\
&\qquad \mid x_1,\ldots,x_{2n},x_{2n+1},x_1^{'},\ldots,x_{2n}^{'},x_{2n+1}^{'} \in \mathbb{R}\big\rangle.
\end{aligned}$$

Therefore, $[\mathbb{H}^n,(\mathbb{H}^n)'] = \{(0,\ldots,0,0)\}$ and by Lemma 2.1 all irreducible characters of this group are Heisenberg.

6. *The Heisenberg group* $\mathcal{H}^n$. The derived subgroup of this group can be computed as follows:

$$\begin{aligned}
(\mathcal{H}^n)' &= \langle [(z,t),(w,s)] \mid (z,t),(w,s) \in \mathcal{H}^n \rangle \\
&= \langle (z,t)(w,s)(-z,-t)(-w,-s) \mid (z,t),(w,s) \in \mathcal{H}^n \rangle \\
&= \langle (z+w,t+s+2Im(z\overline{w}))(-z-w,-t-s+2Im(z\overline{w})) \mid z,w \in \mathbb{C}^n, \\
&\qquad\qquad t,s \in \mathbb{R}\rangle \\
&= \langle (0,2Im(z\overline{w}-w\overline{z})) \mid z,w \in \mathbb{C}^n\rangle.
\end{aligned}$$

Therefore, $[\mathcal{H}^n,(\mathcal{H}^n)'] = \{(0,0,0)\}$ and by Lemma 2.1, all irreducible characters of this group are again Heisenberg.

This completes our argument. $\square$

In the end of this paper we compute the factor groups of six types of Heisenberg groups modulus their centers.

Theorem 2.3. *The factor groups of all Heisenberg groups modulus their centers* *can be computed as:*

$$\begin{aligned}\frac{H}{Z(H)} &\cong \mathbb{R}^n \times \mathbb{R}^n, \frac{H_n^3}{Z(H_n^3)} \cong \mathbb{R}^n \times \mathbb{R}^n, \frac{H_n^a}{Z(H_n^a)} \cong \mathbb{R}^n \times \mathbb{R}^n \\ \frac{\mathcal{N}}{Z(\mathcal{N})} &\cong \mathbb{R}^4, \frac{\mathbb{H}^n}{Z(\mathbb{H}^n)} \cong \mathbb{R}^n \times \mathbb{R}^n, \frac{\mathcal{H}}{Z(\mathcal{H})} \cong \mathbb{C}^n.\end{aligned}$$

Proof. An easy calculations show that $Z(H) = \{(0,0,z)|z \in \mathbf{T}\}$, $Z(H_n^3) = \{(0,0,s)|s \in \mathbb{R}\}$, $Z(H_n^a) = \{(0,0,s)|s \in \mathbb{R}\}$, $Z(\mathbb{H}^n) = \{(0,\dots,0,x_{2n+1})| x_{2n+1} \in \mathbb{R}\}$ and $Z(\mathcal{H}^n) \cong \mathbb{R}$. Therefore, $\frac{H}{Z(H)} \cong \mathbb{R}^n \times \mathbb{R}^n$, $\frac{H_n^3}{Z(H_n^3)} \cong \mathbb{R}^n \times \mathbb{R}^n$, $\frac{H_n^a}{Z(H_n^a)} \cong \mathbb{R}^n \times \mathbb{R}^n$, $\frac{\mathbb{H}^n}{Z(\mathbb{H}^n)} \cong \mathbb{R}^n \times \mathbb{R}^n$ and $\frac{\mathcal{H}}{Z(\mathcal{H})} \cong \mathbb{C}^n$. So, it is enough to compute $\frac{\mathcal{N}}{Z(\mathcal{N})} \cong \mathbb{R}^4$. To do this, we assume that $(p,t) \in Z(\mathcal{N})$ is arbitrary. Hence for each pair (q,s), $(p,t)(q,s) = (q,s)(p,t)$. This proves that $(p+q, s+t+\frac{1}{2}Im(\overline{q}p)) = (q+p, s+t+\frac{1}{2}Im(\overline{p}q))$ and so $Im(\overline{p}q) = 0$. Suppose $p = p_0 + ip_1 + jp_2 + kp_3$. Then by considering three different values $q = (1,0,0,0), (0,1,0,0), (0,0,1,0)$, we will have the following system of equations:

$$\begin{cases} p_1 + p_2 + p_3 = 0, \\ p_0 + p_2 - p_3 = 0, \\ p_0 - p_1 + p_3 = 0. \end{cases}$$

Hence $Z(\mathcal{N}) \cong \mathbb{R}^3$ and $\frac{\mathcal{N}}{Z(\mathcal{N})} \cong \mathbb{R}^4$ that completes the proof. □

3 Concluding Remarks

In this paper the Heisenberg characters of six classes of Heisenberg groups were computed. It is proved that all irreducible characters of these Heisenberg groups are Heisenberg. We also compute all factor groups of these Heisenberg groups which show these factor groups are abelian and so all irreducible characters of these factor groups are again Heisenberg.

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An Analysis of Loan Repayment Plans According to the Bank Customer Profile

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Abstract

It has been demonstrated that there exists a general preference for improvement in loan repayment plans in the way that people prefer decreasing sequences of installments as tested by Hoelzl et al. (2011). Moreover, they also demonstrated that there exists a positive correlation between financial capability and financial literacy when it is given the possibility of having a gain by investing a part of the available money. In these cases, the most financial literate consumers showed a preference for increasing loan plans instead of decreasing ones. In this vein, independently of the level of the borrowers' risk profile, we suggest that an *ad hoc* offer should be made to the customers taking into account these two characteristics by distinguishing three different levels for both personal traits: low, medium and high. Thus, we have analyzed the interest rate which makes both the decreasing and the increasing loan plans indifferent when considering that the option to invest part of the money in savings products is given. Moreover, the analysis has been carried out by considering that the loan repaid principal is variable either in arithmetic progression or in geometric progression. Thus, regarding the main repayment plans offered by banks we have analyzed which one fits better to the defined customer's profile.

Keywords: Loan Plan, Sequence of Payments, Financial Literacy, Financial Capability, Customer's Profile.

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1. Introduction

Traditionally, the offer of loans by banks has been focused on the analysis and description of the main repayment schedules and the calculation of the effective interest rate of the corresponding operation. Thus, given the rate of interest, the client may indistinctly choose a loan with constant installments, constant repaid principal or payments variable in arithmetic progression, among others. From a theoretical point of view, all these repayment methods are equivalent because the interest rate (which is the price of the operation) is the same.

However, in our opinion, neither the loan offer nor the choice of a repayment schedule should be indifferent for banks and costumers, respectively. In effect, it is necessary to previously analyze the costumer profile and the needs of the bank in order to try to satisfy both requirements. Accordingly, we are going to first examine both the main elements defining the costumer profile and the variables of interest for the management of banking institutions.

This paper starts from the conclusions obtained by Hoelzl *et al*. (2011) consisting in the proposal of some suitable advises to different consumption groups based on their preferences on loan repayment plans considered as sequences of installments. To do that, this paper is focused on the results obtained from the three studies implemented by these scholars, especially in the second one. Thus, in order to make an *ad hoc* offer to customers, the financial capability and the level of financial literacy could be the main two inputs which determine if borrowers choose falling, constant or rising installments, according to their preferences. In particular, as pointed out by Hoelzl *et al.* (2011), there is a positive correlation between the rising profile and the financial education level when a part of the monthly available cash can be saved.

In spite of the fact that financial security and consumer self-protection also require to have a good understanding of financial topics (Kozup and Hogarth, 2008), we are going to only focus on the financial literacy. Thus, with respect to the concept of financial literacy, the following definition has been considered: "Financial literacy is a measure of the degree to which one understands key financial concepts and possesses the ability and confidence to manage personal finances through appropriate, short-term decision-making and sound, long-range financial planning, while mindful of life events and changing economic conditions" (Remund, 2010). Therefore, it has been demonstrated that financial literacy is a factor which determines the financial decisions taken by customers when they have to define their credit portfolio (Disney and Gathergood, 2013). In this regard, a low level of financial literacy has direct repercussions in self-control and over-indebtedness (Gathergood, 2012). That is the reason whereby it

has been analyzed how this characteristic could be measured (Huston, 2010). Moreover, there is a relationship between financial satisfaction and financial capability for those consumers who have certain level of financial education and avoid risky decisions (Xiao *et al*., 2014). In this vein, the effects of financial education in financial capability have been analyzed (Xiao and O'Neill, 2016). Thus, considering that financial capability is "the ability to manage and take control of their finance", the age and the unemployment status are the two factors with most impact on the financial capability (Taylor, 2011). Different analyses show that younger adults have more financial difficulties and make worse financial plans (Atkinson *et al*., 2006; Kempson *et al*., 2004).
Therefore, with respect to the customer profile, there are three significant variables of interest for banks:

1. The *payment capability* which can be defined as the financial potential to face the installments derived from the financial operation. Usually, it is stated that the sum of all installments corresponding to the same client must be between 30% and 40% of his/her overall incomes.

2. The *risk* which is the probability of default in the operation. The possibility of failure in payments can be covered by personal or real guarantees.

3. The *financial literacy* which implies a financial knowledge sufficient to adequately invest the extra money and an understanding of the meaning of the effective rate of the operation. On the contrary, a costumer with a low financial training does not have enough financial knowledge to invest and will analyze the total amount of interests instead of the effective rate of the operation.

In our opinion, these three variables are related with the loan schedule which is most suitable for a specific borrower. Table 1 shows the relationship of variables #1 and #3 with the loan repayment plan, where the symbol ↑ means that the principal repaid must be increasing in arithmetic progression, whilst ↓ denotes that the plan must be decreasing. The variable #2 will be incorporated to the analysis in Section 2.

		Financial literacy		
		High	Medium	Low
Payment capability	High	↑	↓	↓
	Medium	↑	↑	↓
	Low	↑	↑	↑

Table 1. Different customer profiles.

Figure 1 helps to understand the existing relationship between variables #1 and #3 and the three identified profiles (decreasing, constant and increasing).

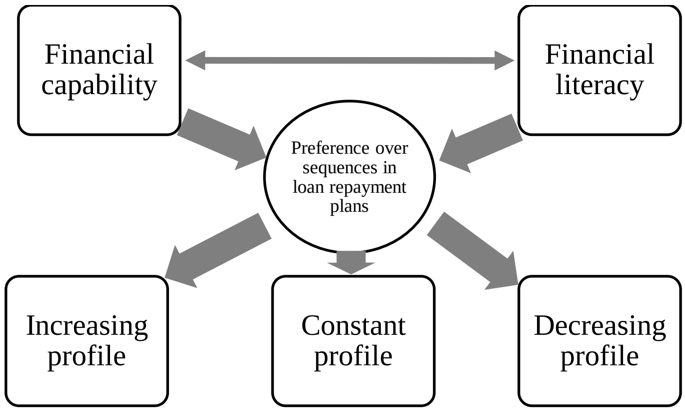

Figure 1. Relationship between variables #1 and #3 and the three considered profiles: decreasing, constant and increasing.

The organization of this paper is as follows. Section 2 shows an analysis of the interest rate according to the loan repayment plan. In Section 3 and Section 4 the main repayment plans offered by banks are analyzed by considering that the principal is repaid in arithmetic progression and geometric progression, respectively. In Section 5, the relationship between variables #1 and #3 and the three profiles is remade by considering the repayment plans obtained in Section 3 and 4. Finally, Section 6 summarizes and concludes.

2. An Analysis of the Interest Rate According to the Loan Repayment Plan

In general, in a loan with a principal C_0 to be repaid in *n* periods, the repayment plan can follow either a constant or a variable repaid principal (increasing or decreasing arithmetic progression). Figure 2 displays these three possible situations for a loan to be repaid in four periods where the principal is €20.

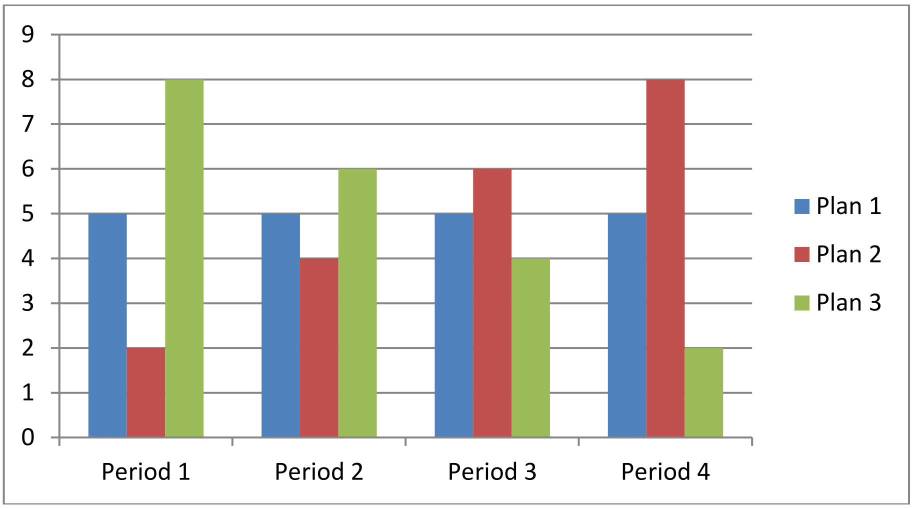

Figure 2. Evolution of the repaid principal variable in arithmetic progression: constant (in blue), increasing (in red) or decreasing (in green).

In general, the following identity holds for the three repayment plans:

$$\sum_{s=1}^{n} A_s = C_0 .$$

Although the financial cost of these three alternatives is the same (more specifically, the interest rate *i*), it has been experimentally shown that people prefer the third option in which the repaid principal is a decreasing arithmetic progression, all terms amounting C_0. Maybe this is because people prefer to leave the best for last (Kahneman *et al*., 1993) or because the total burden in this option is lower than the corresponding one to other plans. This is consistent with a borrower characterized by a low financial training or a high financial capability. Moreover, this type of clients supposes a small risk for banking institutions (see variable #2 in Section 1) since the most part of the principal is repaid at the beginning of the operation. Therefore, a good initiative of banks could be to offer a smaller rate of interest because these clients pay more attention to the total amount of interests instead of the effective rate of the operation. The opposite can be said for clients with a high financial capability or a high financial training because they prefer a repaid principal increasing in arithmetic progression which implies a higher risk for banks and the possibility to invest the extra money. Therefore, this type of clients can support a greater rate of interest.

Therefore, the following question could be addressed: what is the new interest rate, say i' ($i' > i$), such that plan 2 is indifferent to the plan 3? If the difference involved in plan 3 is d ($d < 0$) and the common difference of the second plan is d' ($d' = 0$ or $d' > 0$), then we can require that the sum of payments corresponding to plan 3 must coincide with the sum of payments corresponding to plan 2 minus the amount A earned in the investments. As the aggregated principal repaid in both plans must coincide with C_0, we can propose the following equality between the aggregated interest due in both plans:

$$\sum_{s=1}^{n} I_s = \sum_{s=1}^{n} I'_s - A,$$

or equivalently,

$$i\sum_{s=1}^{n}\left[C_0 - (s-1)A - \frac{(s-1)(s-2)}{2}d\right] = i'\sum_{s=1}^{n}\left[C_0 - (s-1)A' - \frac{(s-1)(s-2)}{2}d'\right] - A.$$

As $A = \frac{C_0}{n} - \frac{n-1}{2}d$ and $A' = \frac{C_0}{n} - \frac{n-1}{2}d'$, then we can write:

$$i\sum_{s=1}^{n}\left[C_0 - (s-1)C_0 + \frac{(s-1)(n-1)}{2}d - \frac{(s-1)(s-2)}{2}d\right]$$

$$= i'\sum_{s=1}^{n}\left[C_0 - (s-1)C_0 + \frac{(s-1)(n-1)}{2}d' - \frac{(s-1)(s-2)}{2}d'\right] - A.$$

Finally, as $n-1 > s-2$, for every s = 1, 2, …, n, we can deduce that $i' > i$.

As indicated in sections 1 and 2, the loan repayment schedule is conditioned by the customer profile and the risk supported by the banking entity. Therefore, Section 3 will be devoted to deduce the different repayment plans of a loan starting from the different variability of the repaid principal.

3. The Principal Repaid is Variable in Arithmetic Progression

In this case, the principal repaid is an arithmetic progression whose first term is A and whose common difference is d:

- $A_1 = A$,
- $A_2 = A + d$,
- $A_3 = A + 2d$,

  $\vdots$
- $A_n = A + (n-1)d$.

Let us analyze the structure of the periodic payments:

- $a_1 = A + C_0 i$,
- $a_2 = (A+d) + (C_0 - A)i$
  $= (A + C_0 i) + (d - Ai)$
  $= a_1 + d - Ai$,
- $a_3 = (A+2d) + (C_0 - 2A - d)i$
  $= a_2 + d - (A+d)i$
  $= a_2 + d - A_2 i$,
- $a_4 = (A+3d) + (C_0 - 3A - 3d)i$
  $= a_3 + d - (A+2d)i$
  $= a_3 + d - A_3 i$,

  $\vdots$
- In general, $a_s = a_{s-1} + d - A_{s-1} i$.

This series is an arithmetic progression of second order. In effect,

- $d_{k-1} := a_k - a_{k-1} = d - A_{k-1} i$, and
- $d_k := a_{k+1} - a_k = d - A_k i$.

Therefore, $d_k - d_{k-1} = -di$. The general solution of an arithmetic progression of second order is:

$$a_s = \frac{r}{2} s^2 + \left(d_1 - \frac{3r}{2} \right) s + (r - d_1 + a_1), \tag{1}$$

where r is the common difference of $d_k - d_{k-1}$, where $d_k - d_{k-1} = -di$. By applying Equation (1) to our concrete case, it would remain:

$$a_s = -\frac{di}{2}s^2 + \left(d - Ai + \frac{3di}{2}\right)s + (-di - d + Ai + A + C_0 i). \qquad (2)$$

The outstanding principal at time s is:

$$C_s = C_0 - sA - \frac{(1+s-1)(s-1)}{2}d = C_0 - sA - \frac{s(s-1)}{2}d\,.$$

Therefore, the structure of the interest due is the following:

$$\begin{aligned} I_s = C_{s-1}i &= \left[C_0 - (s-1)A - \frac{(s-1)(s-2)}{2}d\right]i \\ &= C_0 i - (n-1)Ai - \frac{di}{2}(n^2 - 3n + 2) \\ &= -\frac{di}{2}n^2 + \left(-Ai + \frac{3di}{2}\right)n + (C_0 i + Ai - di)\,. \end{aligned}$$

As expected, observe that $a_s = A_s + I_s$. Finally, Table 2 shows the repayment plan of this loan category.

Period	Payment a_s	Interest due I_s	Principal repaid A_s	Outstanding principal C_s
0	-	-	-	C_0
1	$a_1 = A + C_0 i$	$I_1 = C_0 i_1$	$A_1 = A$	$C_1 = C_0 - A$
2	$a_2 = a_1 + d - Ai$	$I_2 = C_1 i_2$	$A_2 = A + d$	$C_2 = C_0 - A - d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$a_n = a_{n-1} + d - A_{n-1} i$	$I_n = C_{n-1} i_n$	$A_n = A + (n-1)d$	$C_n = 0$

Table 2. Repayment plan of a loan where the principal repaid varies in arithmetic progression.

A special case is when $d = 0$ in whose case a constant repayment plan is obtained (see Table 3).

Period	Payment a_s	Interest due I_s	Principal repaid A_s	Outstanding principal C_s
0	-	-	-	C_0
1	$a_1 = A + C_0 i$	$I_1 = C_0 i_1$	$A_1 = A$	$C_1 = C_0 - A$
2	$a_2 = a_1 - Ai$	$I_2 = C_1 i_2$	$A_2 = A$	$C_2 = C_0 - A$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$a_n = a_{n-1} - A_{n-1} i$	$I_n = C_{n-1} i_n$	$A_n = A$	$C_n = 0$

Table 3. Repayment plan of a loan where the principal repaid varies in arithmetic progression where $d = 0$.

Moreover, if $A = 0$, the American repayment plan can be obtained (see Table 4).

Period	Payment a_s	Interest due I_s	Principal repaid A_s	Outstanding principal C_s
0	-	-	-	C_0
1	$a_1 = I_1$	$I_1 = C_0 i_1$	$A_1 = 0$	$C_1 = C_0$
2	$a_2 = I_2$	$I_2 = C_0 i_2$	$A_2 = 0$	$C_2 = C_0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$a_n = I_n$	$I_n = C_0 i_n$	$A_n = C_0$	$C_n = 0$

Table 4. Repayment plan of a loan where the principal repaid varies in arithmetic progression where $d = 0$ and $A = 0$.

4. The Principal Repaid is Variable in Geometric Progression

In this case, the principal repaid is a geometric progression whose first term is *A* and whose common ratio is *r*:

- $A_1 = A$,
- $A_2 = Ar$,
- $A_3 = Ar^2$,

$\vdots$

- $A_n = Ar^{n-1}$.

Let us analyze the structure of the periodic payments:

- $a_1 = A + C_0 i$,
- $a_2 = Ar + (C_0 - A)i$
  $= Ar + C_0 ir - C_0 ir + (C_0 - A)i$
  $= a_1 r + (C_0 - A - C_0 r)i$,
- $a_3 = Ar^2 + (C_0 - A - Ar)i$
  $= Ar^2 + (C_0 - A)ir - (C_0 - A)ir + (C_0 - A - Ar)i$
  $= a_2 r + (C_0 - A - C_0 r)i$,
- $a_4 = Ar^3 + (C_0 - A - Ar - Ar^2)i$
  $= Ar^3 + (C_0 - A - Ar)ir - (C_0 - A - Ar)ir + (C_0 - A - Ar - Ar^2)i$
  $= a_3 r + (C_0 - A - C_0 r)i$,
  $\vdots$
- In general, $a_s = a_{s-1} r + (C_0 - A - C_0 r)i$.

This series is an arithmetic-geometric sequence whose general solution is:

$$a_s = a_1 r^{s-1} + d\,\frac{1 - r^{n-1}}{1 - r}, \tag{3}$$

where r is the common ratio and d is the difference of the progression. By applying Equation (3) to our concrete case, it would remain:

$$\begin{aligned} a_s &= (A + C_0 i) r^{n-1} + (C_0 - A - C_0 r) i \frac{1 - r^{n-1}}{1 - r} \\ &= (A + C_0 i) r^{n-1} + C_0 i (1 - r^{n-1}) - Ai \frac{1 - r^{n-1}}{1 - r} \\ &= Ar^{n-1} + C_0 i - Ai \frac{1 - r^{n-1}}{1 - r} \end{aligned} \tag{4}$$

The outstanding principal at time s is:

$$C_s = C_0 - \sum_{k=1}^{s} A_k = C_0 - A\frac{1 - r^s}{1 - r} .$$

Therefore, the structure of the interest due is the following:

$$I_s = C_{s-1} i = \left(C_0 - A \frac{1-r^{s-1}}{1-r} \right) i$$
$$= C_0 i - A \frac{1-r^{s-1}}{1-r} i .$$

As expected, observe that $a_s = A_s + I_s$. Finally, Table 5 shows the repayment plan of this loan category.

Period	Payment a_s	Interest due I_s	Principal repaid A_s	Outstanding principal C_s
0	-	-	-	C_0
1	$a_1 = A + C_0 i$	$I_1 = C_0 i_1$	$A_1 = A$	$C_1 = C_0 - A$
2	$a_2 = a_1 r + (C_0 - A - C_0 r) i$	$I_2 = C_1 i_2$	$A_2 = Ar$	$C_2 = C_0 - A - Ar$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$a_n = a_{n-1} r + (C_0 - A - C_0 r) i$	$I_n = C_{n-1} i_n$	$A_n = Ar^{n-1}$	$C_n = 0$

Table 5. Repayment plan of a loan where the principal repaid varies according to a geometric progression.

Thus, when $r = 1 + i$, the French repayment plan can be obtained (see Table 6), where $a_1 = a_2 = \cdots = a_n$ and $i_1 = i_2 = \cdots = i_n$.

Period	Payment a_s	Interest due I_s	Principal repaid A_s	Outstanding principal C_s
0	-	-	-	C_0
1	$a_1 = A + C_0 i = a$	$I_1 = C_0 i_1$	$A_1 = A$	$C_1 = C_0 - A$
2	$a_2 = A(1+i) + (C_0 - A) i = a$	$I_2 = C_1 i_2$	$A_2 = A(1+i)$	$C_2 = C_0 - A - A(1+i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	a	$I_n = C_{n-1} i_n$	$A_n = A(1+i)^{n-1}$	$C_n = 0$

Table 6. Repayment plan of a loan where the principal repaid varies according to a geometric progression and $r = 1 + i$.

5. General Discussion

In sections 3 and 4 we have obtained the main repayment plans offered by banks: the American plan, the constant principal repaid plan and the French plan. Thus, we can observe that, the same characteristics being considered, the amount of interests in the constant principal repaid plan and the French plan is less than in the American plan (Figure 3). Therefore, the American plan will be the best choice for those customers with a high level of financial literacy and a high level of financial capability since they could invest the available money and repay the principal at the end of the loan. Finally, for a low financial capability profile, the choice would be the French plan independently of the financial literacy profile.

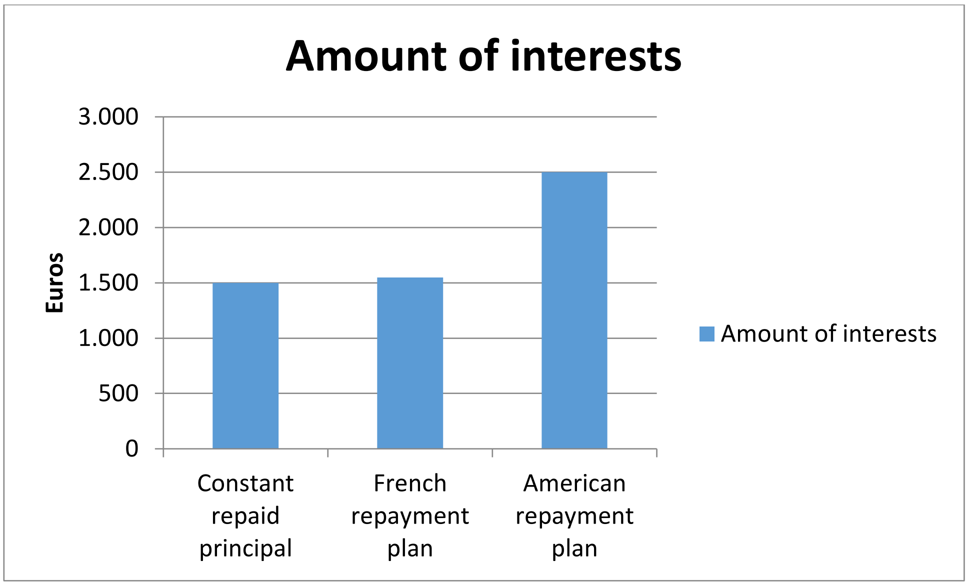

Figure 3. Amount of interests for a loan with a principal of €10,000 to be repaid in 5 years at a 5% interest rate.

Regarding now the relationship between variables #1 and #3 (shown in Table 1), we can remake this relationship by considering, in this case, that the borrowers have to choose one of the aforementioned plans (see Table 7). It is necessary to point out that the choice of the American plan and the constant principal repaid plan will depend on the possibility to invest the available money with a suitable profitability.

		Financial literacy		
		High	Medium	Low
Payment capability	High	American	Constant	Constant
	Medium	American/Constant	Constant	French
	Low	French	French	French

Table 7. Customer profiles considering the three main repayment plans (American, constant principal repaid and French).

We would like also to remark that the use of a repayment plan where the payments are distributed in different years allows combining this yearly schedule with other repayment plans inside each year. In that way, the payment schedule will be adapted according to the borrower's financial availability inside each year.

6. Conclusions

A preference for improvement has been demonstrated for sequences of both incomes (Loewenstein and Sicherman, 1991) and outcomes (Loewenstein and Prelec, 1993). Nevertheless, focusing on loan repayment plans as sequences of installments, we have to take into account the conclusions obtained by Hoelzl *et al*. (2011) about the positive correlation found between the preferences for increasing installments and a high level of financial literacy, by considering the same financial capability. In this regard, we have suggested that banks can evaluate the level of each borrower by considering two main categories: financial literacy and financial capability. In that way, they could make a classification for each of them in three different levels: low, medium and high. Thus, according to the level of risk, we have to consider that it is higher if the loan schedule is based on increasing installments than if it is based on decreasing ones since most of principal is repaid at the beginning when considering the falling profile. Taking into account the aforementioned statement, Table 1 shows the suitability of each plan (falling or rising) according to the borrower's level in each category. It seems that those consumers with a low level of financial literacy and high level of financial capability (low-high profile) prefer a falling plan. In those cases, banks could offer low interest rates since borrowers focus on the total amount of interests and the risk is lower. However, it is likely that customers with a high level of financial literacy and high level of financial capability (high-high profile) prefer rising plans since they are interested in investing part of their available money. Thus, in this case a higher interest rate could be offered by banks. In that way, we have obtained the interest rate which makes equivalent both offers by considering the amount saved by the consumer with a high-high profile.

On the other hand, by considering that the principal is repaid in arithmetic progression, we have analyzed the evolution experienced by each parameter of the repayment plan during the loan lifetime. Thus, when $d = 0$, the constant principal repaid plan is obtained. Moreover, if $d = 0$ and $A = 0$, the American repayment plan is obtained. The same analysis has been done by considering that the principal is repaid in geometric progression. Here, the special case is when $r = 1 + i$, by obtaining the French repayment plan. In that way, we have considered that these three plans are the main ones offered by banks. Thus, taking into account the amount of interests of the three plans for a loan with the same characteristics, we obtained that the corresponding to the American repayment plan is quite higher than for the other plans. This leads us to conclude that this plan will be the best choice for those customers with a high-high profile since they could invest the available money and repay the principal at the end of the loan. Finally, for a low financial capability profile, the choice would be the French plan independently of the financial literacy profile.

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