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SUR LA VALUATION STRICTE DES HYPERGROUPES POLYSYMETRIQUES CANONIQUES

par

JEAN MITTAS

ABSTRACT. This paper generalizes the theory of valuated and hypervaluated canonical hypergroups in the case of the polysymmetrical canonical hypergroups. We distinguish two types of hypervaluated canonical polysymmetrical hypergroups, the weakly and the strongly hypervaluated ones. The study of the last ones forms the main part of this paper, while the study of the weakly hypervaluated canonical polysymmetrical hypergroups is going to be the subject matter of another paper.

Préliminaires

L' hypergroupe polysymétrique canonique (H.P.C.) est, comme on le sait [26], un cas particulier de l' h y p e r g r o u p e p o l y s y m é t r i q u e (H.P.), qui a été introduit par la considération des matrices à éléments dans un hyperanneau ou dans un hypercorps[1] (ou, autrement dit, des hypermatrices) [3],

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[1] Indépendamment de cette théorie la définition des H.P.C. est la suivante:

On appelle H.P.C. un ensemble H muni d' une hyperopération, qui, notée additivement, vérifie, quels que soient $x,y,z \in H$, les axioms:

[5], [9], [10], [13], [20], [28] moyennant une multiplication spécifique [3], [27]. La théorie des H.P.C. généralise celle des hypergroupes canoniques (H.C.) [12], [19] - qui, comme il est connu, constitue la base pour l' étude d' autres structures hypercompositionnelles, comme les hyperanneaux et les hypercorps (dont leurs parties additives sont des H.C.), les hypermodules et les hyperespaces vectoriels et, encore plus généralement, les hyperalgèbres linéaires [1], [3], [11], [20], [21], [23], [28] - D' autre part l' étude des valuations et hypervaluations des H.C. [14], [15], [16], [17], [18], [25] a enrichi la théorie générale des valuations - en liaison avec celle des structures hypercompositionnelles - par plusieurs résultats intéressants. Cette étude a été réalisée par la généralisation de la théorie correspondante des groupes valués

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I. $x + y = y + x$

II. $(x + y) + z = x + (y + z)$

III. $(\exists 0 \in H)(\forall x \in H)[0 + x = x]$ (L' élément 0, évidemment unique, est le zéro de H).

On note que, comme d' habitude, on identifie, quand rien ne s' y oppose, l' élément x avec le singleton correspondant $\{x\}$. Donc $0 + x = x$ au lieu de $0 + x = \{x\}$.

IV. $(\forall x \in H)(\exists x' \in H)[0 \in x + x']$ (Tout tel $x' \in H$ est un opposé ou symétrique de x et l' ensemble $S(x) = \{x' \in H : 0 \in x + x'\}$ est le symétrique de x)

V. $z \in x + y \Longrightarrow (\exists x' \in S(x))[y \in z + x']$

Un H.P.C. diffère, comme il est visible, d' un hypergroupe canonique [12], [19] du fait que dans ce dernier, pour tout $x \in H$, $S(x)$ est un singleton, $\{x'\}$, auquel cas, si on note $-x$ pour x', on aura pour l' axiome V

$$z \in x + y \Longrightarrow y \in z - x$$

D' autre part de l' axiome III il résulte facilement que, pour tout $x \in H$, $x + H = H$, ce qui avec II, justifie la caractérisation de cette structure comme hypergroupe [7], [8].

(et, pour de cas plus général, hypervalués), comme cette dernière a été formée par M. Krasner par l' introduction de l' ultramétricité et, en particulier, par la considération d' une distance ultramétrique c o m p a t i b l e avec la structure du groupe [4],[5],[6]. Ainsi, si $(G,.)$ est un groupe et $d: G \times G \longrightarrow R_+$ (où R_+ est l' ensemble des nombres réels non négatifs) est une ultramétrique sur G, la compatibilité de d se donne par la condition:

$$d(x,y) = d(ax, ay) = d(xa, ya)$$

quels que soient x,y,a dans G. Mais si on a que $(G,.) \equiv (H,.)$ est un hypergroupe, les "hypercomposés" ax, ay, xa, ya ne sont pas, en général, des singletons et les distances $d(ax, ay)$, $d(xa, ya)$ n' ont pas, généralement, de sens. Mais si on suppose que pour tout $x, y \in H$ les hypercomposés xy sont des cercles de l' espace ultramétrique (H,d), ces distances ont de sens, si $ax \cap ya = \emptyset$, $xa \cap ya = \emptyset$, ce qui donne la possibilité de construire une théorie analogue des valuations (et puis, des hypervaluations) pour les hypergroupes.

En particulier, pour exprimer cette compatibilité dans le cas des H.C. et ayant en vue la notion de l' hypercorps valué introduite par M. Krasner à partir d' un corps valué [5] (mais pour qui Krasner n' a pas occupé avec son étude), on a posé les deux conditions suivantes[1] [17], [25]:

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1 Ces conditions peut être encore utilisées, comme il est clair, pour exprimer, la compatibilité de la distance ultramétrique au cas des hypergroupes complètement réguliers (au sens de Marty [8]) possedant un seul élément unité et qui cas est évidemment encore plus général que celui des H.C.

Si $(H,.)$ est un H.C. et $d: H \times H \longrightarrow R+$ est une (distance) ultramétrique sur H, alors

h_1. Pour tout $x,y \in H$ la somme $x + y$ est un cercle de l' espace ultramétrique (H,d) de rayon proportionnel au $\max \{ d(0,x), d(0,y) \}$, où 0 est le zéro de H. C' est-à-dire il existe un nombre semi-réel $p \geq 0$ d' espèce 0 ou - [4], tel que

$$x + y = C\ (z,\ p\max\{|x|,\ |y|\}),$$

où $z \in x + y$ est quelconque et où, pour tout $x \in H$, on met $|x| = d(0,x)$ en l' appelant valuation de l' élément x (la fonction $|.|: H \longrightarrow R+$ ainsi définie étant la valuation de H associée à l' ultramétrique d).

h_2. Pour tout $x,y,a \in H$ tels que $(x+a) \cap (y+a) = \emptyset$ on a

$$d(x,y) = d(x+a,\ y+a)$$

Un H.C. muni d' une telle ultramétrique compatible avec la structure de l' hypergroupe a été appelé *hypergroupe canonique ultramétrique* et cette définition, plutôt géométrique, équivaut, comme celui-ci a été démontré, avec l' autre, purement algébrique, des *hypergroupes canoniques valués* [17], [25]. Pour le cas des H.C. *hypervalués* ou, de manière équivalente, *hyperultramétriques* on a les mêmes conditions comme ci-dessus, mais, maintenant, l' ensemble où l' hypervaluation $|..|$ et l' hyperultramétrique d prennent leurs valeurs peut être, généralement, au lieu de R+, un ensemble quelconque Ω totalement ordonné et possédant un plus petit élément, noté 0, et encore, au lieu du nombre semi-réel p, on a une fonction $p: \Omega \longrightarrow \Omega \cup \hat{\Omega}$- croissante et telle que $p.0 = 0$ et ou $\hat{\Omega}$ est le complété de Kurepa de Ω et $\hat{\Omega}$-

est l' ensemble des éléments d' espèce - de $\hat{\Omega}$ [14], [15], [17], [18], [25].

La valuation (respec. l' hypervaluation) ainsi définie sur H peut être caractérisée comme s t r i c t e et les H.C. munis de telle valuation, s t r i c t e m e n t v a l u é s (H.C.SV) [respec. h y p e r v a l u é s (H.C.SH-V)] ou, de manière équivalente, s t r i c t e m e n t u l t r a m é t r i q u e s (H.C.SU) [respec. h y p e r u l t r a m é t r i q u e s (H.C.SH-U.)]. Car sauf cette manière de valuation (respec d' hypervaluation) des H.C. il y a encore une autre, selon laqelle l' ultramétrique (respec. l' hyperultramétrique) satisfait aux conditions plus faibles:

h_1'. Pour tout $x,y \in H$ la somme $x + y$ est un cercle de l' espace ultramétrique (H,d).

$h_2' \equiv h_2$

h_3'. Pour tout $x,y \in H$, si $x \in x + y$, alors $x + y = x$.

(h_3' à l' autre cas est une conséquence de h_1 et h_2 [14], [16], [17]). Les H.C. munis de telle ultramétrique (respec. hyperultramétrique) peut être caracterisés, contrairement au premier cas, f a i b l e m e n t v a l u é s (H.C.FV) [respec. h y p e r v a l u é s (H.C.FH-V)] ou f a i b l e m e n t u l t r a m é t r i q u e s (H.C.FU) [resp. h y p e r - u l t r a m é t r i q u e s (H.C.FH-U)]. Maintenant, en ce qui concerne les H.P.C., il est évidemment naturel que pour exprimer la compatibilité d' une ultramétrique définie sur eux avec leur structure d' hypergroupe d' accepter tout abord les conditions h_1' et $h_2' \equiv h_2$ et puis d' étudier les deux cas séparément. Le présent travail est ainsi consacré à l' étude du premier cas, c' est-à-dire aux hypergroupes polysymétriques canoniques strictement ultramétriques (H.P.C.SU), qui par définition satisfont aux conditions h_1 et h_2 [respec. hyperultramétriques (H.P.C.SH-U)], tandis que l' étude des H.P.C., satisfaisant aux conditions h_1', h_2' et a une généralisation convenable de h_3' et avec l' adjonction encore quelques autres axiomes et qui seront appelés hypergroupes polysymétriques canoniques faiblement ultramétriques (H.P.C.FU) (respec. hyperultramétriques (H.P.C.FH-U)) fera l' object d' un autre travail. Quant au premier cas, en étudiant les conséquences de la définition-ayant en vue mon travail [25], où la théorie des H.C. valués et hypervalués (strictement) est exposée et où on peut trouver encore les éléments nécessaires de la théorie des espaces ultramétriques et des nombres semi-réels[1] - on a abouti à la conclusion que tout H.P.C.SU se réduit en H.C.SU. Mais bien que l' on a abouti ainsi, toutefois l' exposé detaillé de ce cas est absolument nécessaire pour l' étude de l' autre. D' autre part on se limite à l'étude seulement des H.P.C.SU, car l' étude des H.P.C.SH-U est pareille (avec de petites modifications évidentes) et, par coséquent, on a le même résultat final. C' est-à-dire que tout H.P.C.SH-U se réduit en H.C.SH-U.

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1 Relativement voir encore [2], [4], [5], [6], [17]

Étude du premier cas: Les hypergroupes polysymétriques canoniques strictement ultra-métriques (H.P.C.SU).

Soit $(H,+,d)$ un H.P.C.SU. Tout abord et d'après les propriétés des cercles des espaces ultramétriques, on déduit la propriété purement algébrique (c'est-a-dire qui s'exprime sans l'intervention de l'ultramétrique) suivante:

Proposition 1. *Quels que soient $x, y, z, w \in H$, si $(x+y) \cap (z+w) \neq \emptyset$, alors ou bien $x+y \subseteq z+w$, ou bien $z+w \subseteq x+y$*

D'autre part, comme aux H.C.SU [25], on a encore:

Proposition 2. *Pour tout $x,y \in H$ on a*

$$d(x,y) \leq p|x| \implies x = y$$

d'où il résulte que, si $H \neq \{0\}$, on a $p < 1$.

[Car $x + 0 = C(x,p|x|) = x$ et $y \in C(x,p|x|)$. Donc, si $0 = y \neq x$, alors $0 \notin C(x,p|x|)$ et $d(0,x) = |x| > p|x|$].

Remarques 1. a) Si on suppose que p est le plus petit des nombres semi-réels pour lesquels la condition h_1 est satisfaite, on aura toujours $0 \leq p < 1$ [c'est-à-dire soit $H \neq \{0\}$, soit $H = \{0\}$].

b) Évidemment, si $p = 0$, H est un groupe abélien.

En particulier, pour les opposés d' un $x \in H$ on a les propositions:

Proposition 3. *Pour tout* $x \in H$, $x' \in S(x)$, *on a*

$$|x| = |x'|$$

D é m o n s t r a t i o n. Il est clair pour $x = 0$. Soit $x \neq 0$ et qu' il existe un $x' \in S(x)$ tel que $|x| \neq |x'|$. Si $|x'| < |x|$, alors $x + x' = C(0,p|x|)$ et $x \notin x + x'$. Car $x \in x + x' \Longrightarrow d(0,x) = |x| \leq p|x| \Longrightarrow p \geq 1$, ce qui est inexact d' après la proposition précédente. Donc, par l' axiome h_2 et en vertu des propriétés des cercles (distance de cercles disjoints)

$$d(0,x') = d(x+0,x+x') = d(x,0)$$

Si $|x| < |x'|$, on a de même $x' \notin x + x'$ et on aboutit par le même raisonnement à la même conclusion $|x| = |x'|$.

Corollaire 1. *Pour tout* $x', x'' \in S(x)$ *on a*

$$|x'| = |x''|$$

Corollaire 2. *Pour tout* $x' \in S(x)$, $x^* \in S(x')$ *on a*

$$|x^*| = |x|$$

Proposition 4. *Pour tout* $x \in H$; $x', x'' \in S(x)$ *on a*

i) $x + x' = x + x''$ *ii)* $x \notin x + x'$

D é m o n s t r a t i o n. i) Évident, car $(x+x') \cap (x+x'') \neq \emptyset$ et $|x| = |x'| = |x''|$.

ii) Il résulte de la démonstration de la proposition précedente.

La proposition qui suit accomplit la proposition 1 et elle joue un rôle important pour la suite.

Proposition 5. *Quels que soient $x, y, a \in H$, si $(x+a) \cap (y+a) \neq \emptyset$, alors $x+a = y+a$.*

Démonstration. On sait [26] que

$$(a + x) \cap (a + y) \neq \emptyset \Longrightarrow$$
$$\Longrightarrow (\exists a' \in S(a))(\forall x' \in S(x))(\exists x'' \in S(x'))(\exists y' \in S(y))$$
$$[(a+a') \cap (x''+y') \neq \emptyset]$$

Alors, si $0 \notin x'' + y'$, on a $x'' + y' \subset a + a'$, d' où (pour les rayons des cercles $x''+y'$, $a+a'$) $p\max\{|x''|, |y'|\} < p\max\{|a|, |a'|$ et, d' après les précédents, $\max\{|x|, |y|\} < |a|$, donc $x + a =$ $= y + a$, ces deux cercles non disjoints ayant des rayons semi-réels égaux. Si $0 \in x'' + y'$, alors $|x''| = |y'|$ (donc $|x| = |y|$) et, encore, ou bien $x'' + y' \subseteq a + a'$, ou bien $a + a' \subset x'' + y'$, d' où il résulte respectivement, ou bien $|x| = |y| < |a|$, ou bien $|a| < |x| = |y|$ et on a, comme auparavant, $x + a = y + a$.

Corollaire 3. *Quels que soient $x, y, a \in H$, les ensembles $x + a$, $y + a$ sont ou bien disjoints, ou bien coïncidents.*

Proposition 6. *Pour tout $x, y \in H$, $x' \in S(x)$ on a*

i) Si $y \notin S(x)$, alors $(x + y) \cap (x + x') = \emptyset$

ii) Si $x \in x + y$, alors $x + y = x$

D é m o n s t r a t i o n. i) En effet, si $(x+y) \cap (x+x') \neq \emptyset$, on aurait $x+y = x+x'$, donc $y \in S(x)$, ce qui est contradictoire.

ii) $x \in x+y \Longrightarrow (x+0) \cap (x+y) \neq \emptyset \Longrightarrow x+y = x+0 = x$

Propositon 7. *Pour tout* $x, y \in H$, $x' \in S(x)$, *si* $y \in x + x'$, *alors* $S(y) \subseteq x + x'$.

D é m o s t r a t i o n. En effet $y \in x+x' \Longrightarrow y \in C(0,p|x|) \Longrightarrow d(0,y) = |y| \leq p|x| \Longrightarrow y' \in C(0,p|x|) = x + x'$, pour tout $y' \in S(y)$.

Proposition 8. *Pour tout* $x \in H$, $x', x'' \in S(x)$ *on a:*

$$S(x') = S(x'')$$

D é m o s t r a t i o n. En effet, $x^* \in S(x') \Longrightarrow 0 \in x' + x^* \Longrightarrow x + x'' \subseteq (x + x'') + (x' + x^*) \Longrightarrow 0 \in (x' + x) + (x'' + x^*) \Longrightarrow 0 \in (x' + x^*) + (x'' + x^*)$, car $x' + x = x' + x^*$ d' après la proposition 4i. Donc il existe $y \in x'' + x^*$ et $y' \in S(y)$ tel que $y' \in x' + x^*$. Mais $y' \in x' + x^* \Longrightarrow S(y') \subseteq x' + x^* \Longrightarrow y \in x' + x^*$, donc $(x' + x^*) \cap (x'' + x^*) \neq \emptyset$, et, en vetru de la proposition 5, $x' + x^* = x'' + x^*$, donc $0 \in x'' + x^*$, c' est-à-dire $x^* \in S(x'')$ et, comme $x^* \in S(x')$ est n' importe quel, alors $S(x') \subseteq S(x'')$. Symétriquement on trouve $S(x'') \subseteq S(x')$, d' où la conclusion.

Corollaire 4. *Pour tout* $x \in H$, $x^* \in S(S(x))$ *on a*

$$S(x) = S(x^*)$$

Corollaire 5. *Pour tout* $x, y \in H$ *on a*

$$S(x) \cap S(y) \neq \emptyset \Longrightarrow S(x) = S(y)$$

[Car il existe $z \in S(x) \cap S(y)$, donc $x \in S(z)$ et $y \in S(z)$, d'où $S(x) = S(y)$].

Remarques 2. a) Pour tout $x \in H$ il est évident que

$$S(S(x)) = \cup_{x' \in S(x)} S(x') = S(x')$$

quelque soit $x' \in S(x)$.

b) Désormais et à cause des propositions 4i et 8 - et seulement quand il n' y a pas le risque de confusion - on va utiliser pour exprimer un n' importe quel élément $x' \in S(x)$ la notation $-x$. Ainsi on aura

$x + x' = x + x'' = \ldots = x - x$, $S(S(x)) = S(-x)$

Proposition 9. *Pour tout* $x, y \in H$, $x' \in S(x)$, $y' \in S(y)$, *si* $S(x) \cap S(y) = \emptyset$ *(donc[1]* $S(x) \neq S(y)$ *) on a*

$$d(x,y) = |x + y'| = |y + x'|$$

où, évidemment, si $A \subseteq H$, $|A| = \{ |z| \in R_+ : z \in A\}$.

D é m o n s t r a t i o n. Il est évident que, si $S(x) \cap S(y) = \emptyset$, alors $x' \notin S(y)$ et $y' \notin S(x)$ et, d' après la proposition 6i on a $(x + y') \cap (x + x') = \emptyset$ et $(y + x') \cap (y + y') = \emptyset$. Par conséquent, vu h_2 et les propriétés des cercles,

$$d(x,y) = d(x+x', y+x') = d(0, y + x') = |y + x'|$$

$$d(x,y) = d(x+y', y+y') = d(0, x + y') = |x + y'|$$

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1 Il est possible que l'on ait $x \neq y$, mais $S(x) = S(y)$, comme on a aux certains exemples en [26] et aux divers cas des H.P.C.

Il en résulte que les ensembles $|x + y'|$ et $|y + x'|$ sont des singletons (même égaux).

La proposition suivante généralise la proposition ci-dessus 6ii

Proposition 10. *Pour tout* $x, y \in H$, $x' \in S(x)$, $x^* \in S(x')$, *si* $x^* \in x + y$, *alors* $x + y = x^*$. *Donc, si* $x \in x^* + y$, *on a* $x^* + y = x$.

D é m o n s t r a t i o n. En effet $x^* \in x + y \Longrightarrow (\exists x'' \in S(x))$ $[y \in x^* + x'']$ donc, en vertu du corollaire 4, et de la proposition 4i, $y \in x + x' = C(0, p|x'|)$, d' où $|y| \leq p|x| < |x|$ Par conséquent $x + y = C(x^*, p|x|) = C(x^*, p|x^*|)$, d' après le corollaire 2. Donc, $x + y = x^*$, car $C(x^*, p|x^*|) = x^* + 0 = x^*$.

Proposition 11. *Pour tout* $x \in H$, $y \in x - x$ *il existe* $x^* \in S(-x)$ *tel que* $x + y = x^*$ *et* $x^* - y = x$.

D e m o n s t r a t i o n. En effet $y \in x + x' \Longrightarrow (\exists x'' \in S(x))$ $[x' \in y + x''] \Longrightarrow x + x' \subseteq x + y + x'' \Longrightarrow 0 \in x + y + x'' \Longrightarrow$ $(\exists x^* \in S(x''))[x^* \in x + y]$ et, d' après la proposition précédente, $x + y = x^*$ [Évidemment, si on change x' dans $S(x)$, x^* ne change pas, la somme $x+y$ étant toujours la même]. D' autre part $x + y = x^* \Longrightarrow x + y + y' = x^* + y' \Longrightarrow x \in x^* + y'$ $\Longrightarrow x^* + y' = x$, pour tout $y' \in S(y)$, donc $x^* - y = x$.

Corollaire 6. *Pour tout* $x \in H$, $x' \in S(x)$, $y, y' \in x - x$ [*car, d' après la proposition 7,* $S(y) \subseteq x - x$] *il existe* $x'' \in S(x)$ *tel que* $x' + y' = x''$.

Proposition 12. *Pour tout* $x \in H$ *on a*

$$x + (x - x) = S(-x)$$

D é m o s t r a t i o n. Évidemment, d' après les précédents, $x + (x - x) \subseteq S(-x)$. D' autre part pour tout $x^* \in S(-x)$ on a $x^* \in x^* + (x + x') = x + (x^* + x') = x + (x + x')$ (par Prop. 4i et Cor. 4), donc $S(-x) \subseteq x + (x - x)$, d' où l' égalité.

Remarque 3. Il est visible que, si pour $x, y, a \in H$ on a $x + a =$ $= y + a$, alors il résulte que

$$x + (a - a) = y + (a - a)$$

Mais le reciproque est aussi vrai, c' est-à-dire on a encore la proposition importante suivante:

Proposition 13. *Quels que soient* $x, y, a \in H$, *si* $x + (a - a) = y + (a - a)$, *alors*

$$x + a = y + a$$

D é m o s t r a t i o n. On distingue deux cas: $y \in S(a)$ et $y \notin S(a)$.

i) Soit $y \in S(a)$. Alors: $(x + a) + a' = y + (a + a') \Longrightarrow$ $a' \in (x + a) + a' \Longrightarrow (\exists\, t \in x + a)\ [a' \in t + a'] \Longrightarrow t + a' =$ $= a'$, par la proposition 6ii. Donc on a $t + a' + a = a + a'$, d' où $t \in a + a'$, et, par conséquent, $t \in (x + a) \cap (a + a')$, mais qui est absurde, si $x \notin S(a)$, vu la proposition 6i. Il en résulte que $x \in S(a)$, donc $(x + a) \cap (y + a) \neq \emptyset$ et, en vertu de la proposition 5 (ou, de même, par la prop. 4i), $x + a = y + a$.

ii) Soit $y \notin S(a)$. Alors: $x + (a - a) = y + (a - a) \Longrightarrow$
$\Longrightarrow (x + a) + (a - a) = (y + a) + (a - a)$, dont il s' ensuit

$$(\forall u \in x + a)(\exists v \in y + a)(\exists t \in a - a)[u \in v + t]$$

D' autre part

$$v + t = C(u, p\max\{|v|, |t|\})$$

et, comme $(y + a) \cap (a - a) = \emptyset$, on a $|t| = d(0,t) < d(0,v) =$
$= |v|$ (d' après les propriétés des cercles), donc

$$v + t = C(u, p|v|)$$

Encore on a $S(v) \cap S(t) = \emptyset$, car autrement $S(v) = S(t) \subseteq a - a$ selon le corollaire 5 et la proposition 7. Donc, il vient que $v \in a - a$ et, par suite, $(y + a) \cap (a - a) \neq \emptyset$, mais ce qui est contradictoire, vu la proposition 6i. En appliquant, donc, la proposition 9, on a

$$|v| = d(0,v) = d(a-a,\ y+a) = d(t',v) = d(t'+t,\ v+t) = |v+t|$$

[car $t' \in a-a$ et $(a - a) \cap (y + a) = \emptyset$, $(t'+ t) \cap (v + t) = \emptyset$] et, comme $u \in v+t$, on a $|u| \in |v+t|$, donc $|u| = |v|$. Il en résulte que

$$v + t = C(u, p|u|) = u + 0 = u$$

Mais $v + t = u \Longrightarrow (u + t) \cap (v + t) = (u + t) \cap \{u\}$ et on distingue deux cas: $u \in u + t$ et $u \notin u + t$.

Si $u \in u + t$, alors $u + t = u$, qui entraîne $u + t + t' = u + t'$ donc $u \in u + t'$, d' où, de même, $u + t' = u$ [toujours par la même proposition 6ii et pour tout $t' \in S(t)$]. Mais $v + t = u$ implique $v + t + t' = u + t'$, donc $v \in u + t'$ pour tout $t' \in$ $\in S(t)$. Il en résulte que $u = v$ et par conséquent $(x + a) \cap$ $\cap (y + a) \neq \emptyset$, d' où l' égalité $x + a = y + a$.

Si $u \notin u + t$, on aura $(u + t) \cap (v + t) = \emptyset$, donc

$$d(u,v) = d(u+t,\ v+t) = d(u+t,\ u) = d(u+t,\ u+0) = d(t,0) = |t|$$

Si on suppose $(x+a) \cap (y+a) = \emptyset$, on aura $d(u,v) > p\max\{|x|,|a|\}$

[et $d(u,v) > p\max\{|y|,|a|\}$] donc $d(u,v) > p|a|$. Mais, d' autre côté, $t \in a - a \Longrightarrow |t| \leq p\max\{|a|,|-a|\} = p|a|$, c' est-à-dire $d(u,v) \leq p|a|$, qui contredit au précédent. Par conséquent $(x + a) \cap (y + a) = \emptyset$ est impossible, donc $(x+a) \cap (y+a) \neq \emptyset$ et $x + a = y + a$.

Corollaire 7. *Si $x + (a - a) = y + (a - a)$, alors*

$$x + a^* = y + a^* \quad et \quad x + a' = y + a'$$

quels que soient $a' \in S(a)$, $a^* \in S(-a)$.

Relativement aux sommes $x + x' = x - x$ on a la proposition:

Proposition 14. *i) Pour tout $x \in H$ le sous- -ensemble $x - x$ est un sous-hypergroupe polysymétrique canonique (S-H.P.C) de H.*

ii) Pour tout $x,y \in H$ on a

$$(x-x)+(y-y) = \max\{x-x,\ y-y\} = (x-x)\cup(y-y) = \bigcup_{z\in(x-x)+(y-y)}(z-z)$$

D é m o n s t r a t i o n. i) Étant donné que $y \in x - x \Longrightarrow S(y) \subseteq x - x$, il suffit de montrer que $y_1, y_2 \in x - x \Longrightarrow y_1 + y_2' \subseteq x - x$ pour tout $y_2' \in S(y_2)$ [26]. En effet $y_1,y_2 \in$ $\in x - x = x + x' \Longrightarrow y_1 + y_2' \subseteq (x + x') + y_2' = x + (x' + y_2')$ et, puisque d' après le corollaire 6 il existe $x'' \in S(x)$ tel que $x' + y_2' = x''$, $y_1 + y_2' \subseteq x + x'' = x - x$, pour tout $y_2' \in$ $\in S(y_2)$.

ii) Évident, contenu que $x - x \subseteq y - y$ ou $y - y \subseteq x - x$ et que $x - x$ et $y - y$ sont des S-H.P.C. de H [donc $z \in x - x \Longrightarrow z - z \subseteq -z + (x - x) = x - x$].

Corollaire 8. *Pour tout* $x \in H$ *le sous-ensemble* $x - x = h_x$, *étant S-H.P.C. de H, définit la partition* (mod h_x) *de H, dont les classes sont* $C(z) = z + (x - x)$ [25].

Ensuite et relativement aux partitions de H on voit que l' on a (du corollaire 3) la proposition cosiderable suivante:

Proposition 15. *Si* $x \in H$ *est fixé, les sommes* $x + y$ *quand y parcourt H forment une partiton de H, notée mod x, pour laquelle on a évidemment:*

$$z \equiv w \ (\mathrm{mod}\ x) \iff (\exists y \in H)[(z \in x + y) \wedge (w \in x + y)]$$

D' autre part il est évident que l' on a encore que la relation binaire R_x dans H tel que

$$z \equiv w \ (R_x) \iff x + z = x + w$$

est une relation d' équivalence. Mais de la remarque 3 et la proposition 13 on a que

$$x + z = x + w \iff z + (x - x) = w + (x - x)$$

donc, vu encore le corollaire 8,

$$(R_x) \equiv (\mathrm{mod}\ h_x)$$

Soit maintenant $z \equiv w$ (mod x). Alors, il existe $y \in H$ tel que $z, w \in x + y$ et on a $y \in z + x'$ et $y \in w + x''$, pour $x', x'' \in$ $\in S(x)$ convenables, d' où il vient $x + y \subseteq z + (x + x')$ et $x + y \subseteq w + (x + x'')$, donc $[z + (x - x)] \cap [w + (x - x)] \neq \emptyset$ et, par conséquent, $z \equiv w$ (mod x-x) et, encore, vu les precédents, $z = w$ (R_x).

Inversement, soit $z \equiv w$ (mod x-x). Alors, $z+(x+x') = x+(z+x')$ $= w+(x+x') = x+(w+x')$ et il existe $y_1 \in z+x'$ et $y_2 \in w+x'$ tels que $z \in x+y_1$ et $w \in x+y_2$. Mais, d' après le corollaire 7,

$z+(x+x') = w+(x+x')$ implique $z+x' = w+x'$, donc $y_1, y_2 \in z+x'$, c' est-à-dire on a $y_1 \equiv y_2 \pmod{x'}$ et, en vertu des ci-dessus, $y_1 \equiv y_2 \ (R_{x'})$. Il s' ensuit, donc, que $y_1 + x' = y_2 + x'$, puis $y_1 + (x + x') = y_2 + (x + x')$ et (par la proposition 13) $y_1 + x = y_2 + x$, qui implique que $z, w \in x + y_1$, c' est-à-dire que $z \equiv w \pmod{x}$.

On est arrivé ainsi a l' énoncé suivant, consideré comme lemme pour le théorème considerable qui suit

Lemme. *Les relations d' équivalences dans H mod x, mod$(x-x)$ et R_x sont coïncidantes.*

Par coséquent les classes contenant un $z \in H$ pour chacune d' elles coincident. Ainsi la classe mod$(x - x)$ contenant l' élément $x' \in S(x)$ est $C_{x-x}(x') = x' + (x - x) = x' + (x' + x) = S(x)$, d' après la proposition 12, tandis que pour la classe $C_{x'}(x')$ mod x' pour le même $x' \in H$ on a $C_{x'}(x') = x' + y$, car, d' après la proposition 15, il existe un $y \in H$ tel que $x' \in x' + y$. Mais $x' \in x' + y$ implique $x' + y = x'$ (Prop. 6ii). Donc, on a $S(x) = x'$ et évidemment on a la même chose pour tout $x \in H$ et pour tout $x' \in S(x)$. On a aboutit, ainsi, que pour tout $x \in H$, $S(x)$ est un singleton, c' est-à-dire au théorème, que l' on a mentionné au commencement:

Théorème. *Tout H.P.C.U. est un H.C.U.*

Des précédents on conclût encore la remarque suivante concernant aux hypergroupes fortement canoniques (H.FC)

Remarque 4. Un tel H.C. vérifie de plus par définition les axiomes

$f_1 \equiv$ Proposition 1 et $f_2 \equiv$ Proposition 6ii.

De ces axiomes (comme il est connu de la théorie des H.FC. [24], [25]) il découle la proposition 5. Mais on a vu que très facilement et de manière purement algébrique la proposition 5 implique la proposition 6ii. On déduit, donc, qu' un H.FC. peut être défini de manière équivalente par les axiomes f_1 et $f_2' \equiv$ Proposition 5 [22].

On achève cet exposé en citant un exemple montrant qu il existe des H.P.C.FV et que, par conséquent, le chemin pour leur étude est ouvert. L' ultramétrique d' un tel hypergroupe satisfait aux conditions h_1' et h_2' des H.C.FV citées à l' introduction, mais non à h_3', au lieu de laquelle vérifie d' autres conditions, dont la recherche pour le cas général fait d' objet d' autre travail. Le sujet est ouvert.

<u>**Exemple**</u> Si en partant d' un corps totalement ordonné, p.e. le corps $(\mathbb{R},+,\cdot,\leq)$ des nombres réels, on definit une hyperaddition $x \dotplus y$ comme suit

$$x \dotplus y = y \dotplus x = \begin{cases} y, \text{ si } |x| < |y| \\ \\ [-|x|, |x|], \text{ si } |x| = |y| \end{cases}$$

on obtient une hyperstructure $(\mathbb{R},\dotplus)$, qui, comme on le voit facilement, est un H.P.C. avec $S(x) = \{-x, x\}$, pour tout $x \in \mathbb{R}$. On voit encore que la fonction $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_+$ telle que

$$d(x,y) = d(y,x) = \begin{cases} |y|, & \text{si } |x| < |y| \\ 0, & \text{si } y = x \\ |x|, & \text{si } y = -x \end{cases}$$

est une ultramétrique sur R, qui, comme on le costante aprés une investigation pour les différents cas de $x, y, a \in H$ (en particulier pour $|a| \leq |x| < |y|$, $|x| < |a| < |y|$, $|x| < |y| \leq |a|$ et pour $y = x$, $y = -x$) vérifie l' axiome h_2'. Quant à l' axiome h_1' on voit que l' on a tout abord, si $||x||$ est la valuation associée à l' ultrametrique d, alors

$$d(0,x) = ||x|| = |x| \quad \text{et} \quad d(0,-x) = ||-x|| = |x|$$

donc $||x|| = ||-x||$. Par conséquent

$$\text{si } |x| < |y|, \text{ on a } x \dotplus y = C(y, |y|^-) = C(y, 0),$$

tandis que

$$\text{si } |x| = |y|, \text{ alors } x \dotplus y = [-|y|, |y|] = C(0, |y|).$$

C' est-à-dire, en utilisant le nombre semireél ρ, on a généralement

$$x \dotplus y = C\,(z, \rho\max\{|x|,|y|\})$$

avec

$$\rho = \begin{cases} 1^-, & \text{si } |x| < |y| \\ 1, & \text{si } |x| = |y| \end{cases}$$

Autrement dit, les hypersommes $x \dotplus y$ sont des cercles de l' espace ultramétrique (R, d) dont les rayons sont dépendants du $\max\{|x|, |y|\}$, mais sans un même coefficient de proportionnalité. L' axiome donc h_1' est aussi vérifié.

Enfin, en ce qui concerne l' axiome h_3', celui-ci, évidemment, ne marche pas en général. Mais on voit que l' on a des propriétés qui auraient pu jouer le rôle de h_3' à la définition des H.P.C-FV, comme p.e.

Si $x \in x + y$, alors ou bien $x + y = x$, ou bien $y \in S(x)$ [donc, ici, $S(x) \cap S(y) \neq \emptyset$]

et

Si $S(x) \cap S(y) = \emptyset$, alors pour tout $x', x'' \in S(x)$, $y', y'' \in S(y)$, ou bien $(x \dotplus x') \cap (y \dotplus x'') = \emptyset$, ou bien $(y \dotplus x') \cap (x \dotplus y'') = \emptyset$ [tandis que dans un H.C.-FV, donc dans un H.FC (Remarque 4 et [22]), on a $x \neq y \Longrightarrow -x \neq -y \Longrightarrow$ $\Longrightarrow S(x) \cap S(y) = \emptyset \Longrightarrow (x - y) \cap (x - x) = \emptyset$ et $(y - x) \cap (y - y) = \emptyset$, vu que $S(x) = \{-x\}$, $S(y) = \{-y\}$].

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Ideals in a semihypergroup and Green's relations

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Abstract:

The concept of ideal in a right (left) semihypergroup is defined. Then some connections between ideals and the hyper versions of Green's relations are discussed.

1.Introduction

Marty in 1934[2] Introduced the notion of hypergroup.

We begin by recalling some definitions from [1].

A hyperopration of a non-empty set H, is a function from $H \times H$ into $P^*(H) = P(H)\backslash\{\emptyset\}$.

If "$*$" is a hyperoperation on H, then $(H, *)$ is called a hypergroupoid.

Let $(H, *)$ be a hypergroupoid and A, B two non-empty subsets of H, then $A * B$ is defined by

$$A * B = \bigcup_{a \in A, b \in B} a * b$$

By $x * A$, and $A * x$ we mean $\{x\} * A$ and $A * \{x\}$ respectively, for all $x \in H$, $\emptyset \neq A \subseteq H$.

2.Main results

Definition 2.1. Let $(H, *)$ be a hypergroupoid. Then H is said to be a right (left) semihypergroup (or r.s (l.s)) if

$$(x * y) * z \subseteq x * (y * z), \forall x, y, z \in H$$

$$(x * (y * z) \subseteq (x * y) * z, \forall x, y, z \in H).$$

An hypergroupoid is called a semihypergroup if it is both a left and a right semihypergroup.

Definition 2.2[2]. Let $(H, *)$be a semihyprgroup. Then H is called a hypergroup if $x * H = H * x = H$, for all $x \in H$.

Definition 2.3. Left $(H*)$ be a hypergroupoid and $A \in P^*(H)$. Then A is called

(i) a right ideal in H if

$$x \in A \Longrightarrow x * y \subseteq A, \forall y \in H$$

(ii) a left ideal in H if

$$x \in A \Longrightarrow y * x \subseteq A, \forall y \in H$$

(iii) an ideal in H if it is both a left and a right ideal in H.

Example 2.4. If H is a totally ordered set and the hyperoperation "$*$" on H is defined by

$$x * y = y * x = \begin{cases} \{z \in H : x \leq z\} & \text{if } \ y \leq x \\ \{z \in H : y \leq z\} & \text{if } \ x \leq y, \end{cases}$$

for all $x, y \in H$. Then we can show that $(H, *)$ is a semihypergroup. Infact if $x, y, z \in H$, and $w \in (x * y) * z$ are arbitrary, then we have $w \in a * z$, for some $a \in x * y$. If $x \leq y$, then $y \leq a$. Now we have two cases.

Cases 1: Let $z \geq a$. Then since $w \in a * z$, we have $w \geq z$. On the other hand, since $y \leq a$, we obtain that $x \leq y \leq z$. In other words $z \in y * z$ and $w \in x * z$. Now we get that $w \in x * z \subseteq x * (y * z)$.

Case 2: Let $a > z$. Then, since $w \in a * z$, we conclude that $w \geq a > z$. Now if $z \geq y$, then we have

$$w \geq a > z \geq y \geq x.$$

Hence $w \in x * z \subseteq x * z$ and $z \in y * z$. Thus

$$w \in x * z \subseteq x * (y * z).$$

If $y > z$, then we have

$$z < y \leq a \leq w, \text{ since } w \in a * z.$$

consequently

$w \in x * y$, since $y \leq w$ and $x \leq y \subseteq x * (y * z)$, since $y \in y * z$. Therefore $(x*y)*z \subseteq x*(y*z)$, if $x \leq y$. Now since $x*y = y*x$, we have $(x*y)*z \subseteq x*(y*z)$.

Note that, since $x * B = B * x$. For all $x, y, z \in H$. Thus $(H, *)$ is a semihypergroup. Now Let $A = \{x \in H : a \leq x\}$, where $a \in H$. Then we shall show that A is an ideal of H. To do this let $x \in A, y \in H$ and $z \in x * y$. Then if $x \leq y$, we have

$$z \geq y \geq x \geq a.$$

Hence $z \in A$. If $y \leq x$, we have

$$z \geq x \geq a.$$

That is $z \in A$. Consequently $x * y \subseteq A$.

Definition 2.5. Let $(H, *)$ be a hypergroupoid. For every $a \in H$ we define

$$\begin{aligned} aH &= (a * H) \cup \{a\}; \\ Ha &= (H * a) \cup \{a\}; \\ HaH &= ((H * a) * H) \cup Ha \cup aH. \end{aligned}$$

The hyper versions of Creen's relations are the equivalence relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and $\mathcal{K}$ defined for all $a, b \in H$ by

$$a\mathcal{R}b \Leftrightarrow aH = bH;$$

$$a\mathcal{L}b \Leftrightarrow Ha = Hb;$$

$$a\mathcal{I}b \Leftrightarrow HaH = HbH;$$

$$\mathcal{K} = \mathcal{L} \cap \mathcal{R}.$$

We shall also consider the relations $\leq (\mathcal{R})$, $\leq (\mathcal{L})$ and $\leq (\mathcal{I})$ defined for all $a, b \in H$ by

$$a \leq (\mathcal{R}) b \Leftrightarrow aH \subseteq bH;$$

$$a \leq (\mathcal{L}) b \Leftrightarrow Ha \subseteq Hb;$$

$$a \leq (\mathcal{I}) b \Leftrightarrow HaH \subseteq HbH;$$

(See[1,page 29]).

Teorem 2.6. Let H be a r.s and $\emptyset \neq A \subseteq H$. Then A is a right ideal iff for every $x, y \in H$.

$$x \leq (\mathcal{R}) y \text{ and } y \in A \Rightarrow x \in A. \tag{1}$$

Proof. Let H be a r.s, $A \in P^{*}(H)$ and (1) hold. Then for all $x \in A$ and $y \in H$, we will prove that $x * y \subseteq A$. To do this let $z \in x * y$, and $w \in zH$ are abitrary.Then $w = z$ or $w \in z * t$, for some $t \in H$. If $w = z$, then since $z \in x * y$ we have $w \in x * y$ and hence $w \in xH$. If $w \in z * t$, then since $z \in x * y$ we get that $w \in (x * y) * t \subseteq x * (y * t) \subseteq xH$. Therefore $zH \subseteq xH$. In other words $z \leq (\mathcal{R}) x$.Hence $z \in A$. That is $x * y \subseteq A$.

Conversly, let A be a right ideal in H, $x \leq (\mathcal{R}) y$ and $y \in A$. Then $yH \subseteq A$,since A is a right ideal. Hence

$$x \in xH \subseteq yH \subseteq A.$$

In other words $x \in A$.

Theorem 2.7. Let H be a r.s $a \in H$. Then aH is the smallest right ideals containing a. Right ideals of this form are called principal right ideals.

Proof. The proof is easy.

Theorem 2.8. If H is a r.s and $a, b \in H$, then the following are equivalent:

(1) $a \leq (\mathcal{R})b$

(2) $a \in bH$

(3) $b \in J \Rightarrow a \in J$ for all principal right ideals J in H,

(4) $b \in J \Rightarrow a \in J$ for all right ideals J in H.

Proof. Clearly (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (2). By Theoem 2.7 we have (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (4).

Corollary 2.9. If H is a r.s and $a, b \in H$, Then the following are equivalent:

(1) $a\mathcal{R}b$,

(2) $b \in aH$ and $a \in bH$,

(3) $a \in J \Leftrightarrow b \in J$ for all principal right ideals J in H,

(4) $a \in J \Leftrightarrow b \in J$ for all right ideals J in H.

Proof. The proof follows from Definition 2.5 and Theorem 2.8.

Definition 2.10. Let H be a hypergroupoid. A set F of functions on H is separating if, for all distinct x and y in H, there is an $f \in F$ with $f(x) \neq f(y)$.

Notation. Let H be a hypergroupoid $x \in H$. Then $\mathcal{R}$-class, $\mathcal{L}$ - class, $\mathcal{I}$-class and $\mathcal{K}$-class of x are denoted by $x_{\mathcal{R}}$, $x_{\mathcal{L}}$, $x_{\mathcal{I}}$, and $x_{\mathcal{K}}$ respectively.

Corollary 2.11. Let H be *a* r.s. Then the following are equivalent:

(1) the relation $\leq (\mathcal{R})$ is an order on H.

(2) $x_{\mathcal{R}} = x, \forall x \in H$,

(3) The set of all characteristic function of principal right ideals in H is separating,

(4) The set of all charactcristic functions of right ideals in H is separating .

Proof. Obviously $(1) \Rightarrow (2)$ and $(3) \Rightarrow (4)$. Firstly we shall prove $(2) \Rightarrow (3)$.Let x and y be two distinct elements in H. Then by Corollary 2.9,we have $y \notin xH$ or $x \notin$.Hence $\chi_{xH}(x) \neq \chi_{xH}(y)$ or $\chi_{yH}(y) \neq \chi_{yH}(x)$.

It is now sufficient to show that $(4) \Rightarrow (2)$.Let $x \leq (\mathcal{R})y$ and $y \leq (\mathcal{R})x$. Then $x \in yH$ and $y \in xH$. From Corollary 2.9 we have

$$x \in J \Leftrightarrow y \in J \text{ for all right ideals } J.$$

Hence $x = y$, by (4). Clearly $\leq (\mathcal{R})$ is reflexive and transitive.

Remark 2.12. For a l.s H there are corresponding theorems and corollaries connecting the relation $\mathcal{L}$ with left ideals.Moreover for a semihypergroup H there are corresponding theorems and corollaries connecting the relation $\mathcal{I}$ with ideals, we shall summarise a few of these results in the next theorem.

Theorem 2.13 Let H be a semihypergroup and $a, b \in H$.Then

(1) $a\mathcal{I}b$ iff

$a \in J \Leftrightarrow b \in J$ for all ideals J in H,

(2) $a\mathcal{K}b$ iff

$a \in J \Leftrightarrow b \in J$

whenever J is a left ideal or a right ideal in H.

Theorem 2.14. Let H be a semihypergroup. Then H is a hypergroup iff $x \in x * H$ and $x_{\mathcal{K}} = H, \forall x \in H$.

Proof. The proof is easy.

Definition 2.15. Let H,H' be two hypergroupoid and $f : H \longrightarrow H'$ a function. Then F is called a homomorphism if

$$f(x * y) = f(x) * f(y).$$

Theoerem 2.16. Let H,H' be two hypergroupoids and $f : H \longrightarrow H'$ an onto homomorphism. Then f preserves the relations $\leq (\mathcal{R})$,$\leq (\mathcal{L})$ and $\leq (\mathcal{I})$.Moreover $f(x_{\mathcal{R}}) = (f(x))_{\mathcal{R}}, f(x_{\mathcal{L}}) = (f(x))_{\mathcal{L}}, f(x_{\mathcal{I}}) = (f(x))_{\mathcal{I}}$. and $f(x_{\mathcal{K}}) = (f(x))_{\mathcal{K}}$.

Proof. The proof is easy.

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HYPERCOMPOSITIONAL STRUCTURES FROM THE COMPUTER THEORY

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Abstract This paper presents the several types of hypercompositional structures that have been introduced and used for the approach and solution of problems in the theory of languages and automata.

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Certain properties of the Automata, as well as some essential elements of the structure of the formal languages gave the initiative of introducing the theory of hypercompositional structures into the above theories [5]. This paper will present the structures that have been used for this purpose along with some of the characteristic properties that they have. The use of those structures can be found in the papers that appear as references in the following text.

In [5], it has been proved that the set of the words A^* over an alphabet A, can be organized into a join hypergroup, which was named **B-hypergroup**, through the introduction of the hypercomposition:

$$a+b = \{a, b\} \text{ for every } a, b \in A^*$$

It is worth mentioning that this hypercomposition can be found in a paper by L. Konguetsof, written as early as the 60's. We have introduced again this hypercomposition though, motivated by the theory of Languages and we have named it **B-hypercomposition,** after the Binary result it gives. The deriving structures (**B-hypergroup, Dilated B-hypergroup, B-hyperringoid** etc.) which have already been studied in depth, have produced many and interesting results in both theories.

The join hypergroup is a commutative hypergroup (H,+), which also satisfies the axiom:

(J) $(a : b) \cap (c : d) \neq \varnothing \Rightarrow (a + d) \cap (c + b) \neq \varnothing$ for every $a, b, c, d \in H$

where $a : b = \{x \in H \mid a \in x+b\}$, is the induced hypercomposition from + [2]. We remind that the axiom (J) as well as the join space, i.e. a commutative hypergroup enriched with (J) and certain additional axioms, have been introduced by W. Prenowitz for the study of Geometry with methods and tools from the Hypercompositional Algebra (e.g. see [17]).

Moreover, the set of the words A^* is a semigroup with composition the concatenation of the words. It has been proved though that the concatenation is bilateral distributive to the B-hypercomposition [5]. Thus, there appeared a new hypercompositional structure, the **hyperringoid**.

Definition 1. A triplet (Y,+,.) is called **hyperringoid**, if

i. (Y,+) is a hypergroup

ii. (Y,.) is a semigroup

iii. the composition is bilateral distributive to the hypercomposition

If (Y,+) is a join hypergroup, then the hyperringoid is named **join**.

An important join hyperringoid for the theory of languages is the B-hyperringoid, in which the (Y,+) is a B-hypergroup.

The study of the theory of languages and automata through the theory of the hypercompositional structures has also led to the introduction of a new hypergroup, the **fortified join hypergroup (FJH).** This new hypergroup has been introduced in order to satisfy the need of considering a non scalar neutral element in the join hypergroup and therefore being able to describe the "null word", the use of the symbol <SOS> (Start Of String) in the realization of the automaton etc.

Definition 2. **Fortified Join Hypergroup** is a join hypergroup which also satisfies the axioms:

FJ_1 There exists a unique neutral element, denoted by 0, the zero element of H, such

that for every $x \in H$ holds: $x \in x+0$ and $0+0 = 0$

and

FJ_2 For every $x \in H \setminus \{0\}$, there exists one and only one element $x' \in H \setminus \{0\}$, the opposite

or symmetrical of x, denoted by -x, such that: $0 \in x+x'$ Also $-0 = 0$.

From the above axioms, it is obvious that the FJH places itself between the canonical [13] and the join hypergroup, since, as it is known, if a join hypergroup has a scalar neutral element, it is a canonical one [1], [3].

The relevant analysis of the properties of the FJHs [6] has revealed that they consist of two kinds of elements, the **canonical (c-elements)** and the **attractive** ones **(a-**

elements). This distinction appears due to the different behavior of the elements in their hypersum with the zero element. So, for the canonical elements x, x+0 equals to {x}, i.e. they act like the elements of the canonical hypergroup, while, for the attractive ones, x+0 = {x, 0} i.e. they attract the neutral element in the result of the hypersum x+0. Moreover, since 0+0 is not a biset, we have included 0 among the canonical elements. Furthermore the property -(x-x) = x-x is not always valid in a FJH. Thus, there derives another distinction of the elements, namely the **normal** ones, i.e. the elements that satisfy the above equality, and the **abnormal** ones, i.e. the elements that do not satisfy it. The following Proposition gives a list of the fundamental properties of the elements of this new hypergroup.

Proposition. *In a FJH the following are valid:*

i. *if x is a c-element, then -x is also a c-element.*
ii. *if x is an a-element, then -x is also an a-element.*
iii. *The sum of two a-elements consists only of a-elements (and the 0, if they are opposite) and also it always contains the two addends.*
iv. *The sum of two non opposite c-elements consists of c-elements, while the sum of two opposite c-elements contains all the a-elements.*
v. *The sum of an a-element with a non zero c-element is the c-element.*
vi. *All the c-elements are normal elements.*
vii. *If y is normal, or if* $x \notin y-y$, *then* $-(x : y) = (-x) : (-y)$

For the relevant proofs of the above see [6].

Moreover, it has been proved that in the FJHs the reversibility holds under conditions. More precisely, $z \in x+y \Rightarrow y \in z-x$, except if $z = x \neq y$ where $x \in x+y \Rightarrow x \in x-y$, while generally $y \notin x-x$. This gives as a result that for every $x \neq y$ holds $x-y = (x{:}y) \cup (-y){:}(-x)$, while $x-x \subseteq x{:}x$. If one of the x, y is a c-element then $x-y = x{:}y = (-y){:}(-x)$ [6].

Apart from the different kinds of elements, as described above, the FJH has a variety of subhypergroups [11]. Initially, very significant in every join hypergroup are the intersections $x{:}y \cap z{:}w$ which appear in the first part of the join axiom. Thus, if h is a subhypergroup of a join hypergroup H and if $x, y, z, w \in h$, then the following can happen:

i> $[(x{:}y) \cap (z{:}w)] \cap (H \backslash h) \neq \varnothing$

ii> $[(x{:}y) \cap (z{:}w)] \subseteq h$

If <ii> is valid for every $x, y, z, w \in h$, then h is called **join subhypergroup** of H. Since it has been proved that a subhypergroup h of a commutative hypergroup H is closed in H if and only if $x{:}y \subseteq h$ for every $x, y \in h$ [4], one can easily see that the join subhypergroups are the closed ones. Now if H is a FJH for which $-x \in h$ for every $x \in h$, then h is called **symmetrical subhypergroup** of H. It can be proved that every join subhypergroup of a FJH is a symmetrical one.

A hyperringoid now, with additive hypergroup a FJH, is called **join hyperring.** The join hyperring has properties, some of which are very different from the properties of Krasner's hyperring [12], i.e. hyperringoid in which the additive hypergroup is a canonical one. So, for instance, in a join hyperring, the property $(-x)(-y) = xy$ is not always valid. In [10], example 3.1. one can see a join hyperring for which $(-x)(-y) = -xy$. Also, its canonical and attractive elements give special properties in the multiplication. Thus, while the product of two c-elements is always a c-element, the result of the product of a c-element with an a-element is always the zero element. Furthermore the product of two a-elements is also the zero element, if the join hyperring contains a nonzero c-element.

Another structure which has been used for the approach of the theory of automata through the theory of the hypercompositional structures is the **join polysymmetrical hypergroup.**

Definition 3. Join Polysymmetrical Hypergroup is a join hypergroup which also satisfies the axioms:

JP_1 There exists a unique neutral element, denoted by 0, the zero element of H, such that for every $x \in H$ holds: $x \in x+0$ and $0+0 = 0$

and

JP_2 For every $x \in H \setminus \{0\}$, there exists at least one element $x' \in H \setminus \{0\}$, the opposite or symmetrical of x, such that: $0 \in x+x'$ The set of the opposite elements of x is denoted by S(x) and named **symmetrical** (set) of x. Also $S(0) = 0$.

Those hypergroups have appeared and used for the minimization of the automaton [5] and their study has revealed the significant properties that they have [6]. For instance join polysymmetrical hypergroups are the P-hypergroups i.e. hypergroups that are defined from an abelian group (G,+), a subset P which contains the neutral element of G and hypercomposition "$:^P$" defined as follows:

$$x :^P y = x+y+P, \text{ for every } x, y \in G$$

In addition to the above hypergroups that have been introduced from the study of the theory of automata and languages, notable in the above study, is the role of the canonical hypergroup, with its different types (superiorly canonical and strongly canonical [14], [15]) as well as the canonical polysymmetrical hypergroup [16]. Those hypergroups have been used in order to describe the structure of an automaton. More precisely, through the introduction of the notion of the **order** of a state there appear the **attached order hypergroups** of an automaton, which belong to he above categories [9].

Furthermore, in the study of the theory of languages and automata, except from the special types of hypergroups, we have also used the hypergroup itself. The introduction of the following hypercomposition into the set S of the states of an automaton, has made S a hypergroup:

$$s_1 + s_2 = \begin{cases} \{s \in S \mid s = s_1 w \text{ and } s_2 = sy, \text{ with } w,y \in A^*\}, \\ \qquad \text{if there exist } z \in A^*, \text{ such that } s_2 = s_1 z \\ \{s_1, s_2\}, \quad \text{if there does not exist } z \in A^*, \text{ such that } s_2 = s_1 z \end{cases}$$

This hypergroup has been named **attached hypergroup of the paths.** With a proper generalization of this hypergroup in the case of the operation of the automaton, we have obtained the **attached hypergroup of the operation.** Using this last hypergroup, an algorithm has been developed, which, among other information, gives all the possible states that an automaton can be found at any clock pulse, during its operation, as well as all the possible paths that it may have passed through up to any clock pulse [8].

Lastly, in an automaton, the word (of the language it accepts) causes the system to move from state to state. Therefore, this is an action from a set of operators, which is a subset of A^* on the set of the states of the automaton. This has led to the introduction of two more hypercompositional structures, the **hypermoduloid** and the **supermoduloid.**

Definition 4. If M is a hypergroup and Y is a hyperringoid of operators over M, such that for every $\kappa, \lambda \in Y$ and $s, t \in M$, the axioms:

i. $(s\kappa)\lambda = s(\kappa\lambda)$
ii. $(s+t)\lambda = s\lambda + t\lambda$
iii. $s(\lambda+\kappa) \subseteq s\lambda + s\kappa$

hold, then M is called **hypermoduloid** over Y. If Y is a set of hyperoperators, that is, if there exists an external hyperoperation from MxY to P(M) satisfying axiom <i>, then M is called **supermoduloid** over Y.

The hypermoduloids are being used in the study of the deterministic automata, while the supermoduloids are being used in the case of the non deterministic automata [7].

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CLOSURE SYSTEMS AND CLOSURE HYPERGROUPS

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SUNTO - Dato un sistema di chiusura $(S,\mathfrak{C})$, sull'insieme S si può definire un'iperoperazione $\cdot$ ponendo, per ogni $a,b \in S$, $a\cdot b = <a,b>_{\mathfrak{C}}$, dove $<a,b>_{\mathfrak{C}}$ è il minimo elemento di $\mathfrak{C}$ a cui appartengono sia a che b.
In questo lavoro noi studiamo questo tipo di iperoperazione, evidenziando diverse proprietà significative. Nei casi in cui l'iperoperazione in questione è associativa essa attribuisce ad S una struttura di ipergruppo, che noi chiamiamo *ipergruppo chiusura*.

ABSTRACT - If $(S,\mathfrak{C})$ is a closure system, then one can define on the set S a hyperoperation $\cdot$ by setting, for any $a,b \in S$, $a\cdot b := <a,b>_{\mathfrak{C}}$, where $<a,b>_{\mathfrak{C}}$ is the minimum element of $\mathfrak{C}$ containig a and b.
In this paper we study such a type of hyperoperations and prove several interesting properties. Whenever the above hyperoperation is associative, then it gives S a hypergroup structure that we shall call *closure hypergroup*.

1. PRELIMINARIES AND RECALLS

A function $\cdot : S\text{x}S \to \mathcal{P}(S)$ is said a partial (binary) hyperoperation on S. If $x\cdot y \neq \varnothing$ ($x\cdot y := \cdot(x,y)$) for any $x,y \in S$, then one speaks of hyperoperation.
For any $X,Y \subseteq S$ one can set $X\cdot Y := \bigcup_{x\in X, y\in Y} x\cdot y$ (hence $\varnothing\cdot Y = \varnothing = X\cdot\varnothing$).
Thus one has also a binary operation on $\mathcal{P}(S)$.
If $a \in S$ and $B \subseteq S$, then one usually writes respectively $a\cdot B$ and $B\cdot a$ instead of $\{a\}\cdot B$ and $B\cdot\{a\}$. It is obvious that $\bigcup_{x\in X} x\cdot Y = X\cdot Y = \bigcup_{y\in Y} X\cdot y$.
It is easy to verify that a partial hyperoperation on S is associative or commutative - with an obvious meaning of these terms - if and only if the corresponding operation on $\mathcal{P}(S)$ is associative or commutative.

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Now we recall that a closure system on a set S is a subset $\mathfrak{C}$ of the power set $\mathcal{P}(S)$ which is closed under the arbitrary set intersection (in particular, $S = \cap\varnothing \in \mathfrak{C}$). One says also that $(S,\mathfrak{C})$ is a closure system.
For any $X \subseteq S$ one can consider the so called closure of X under $\mathfrak{C}$, given by the intersection of the elements of $\mathfrak{C}$ including X, and represented by $<X>_{\mathfrak{C}}$. If $x_1, x_2, \dots, x_n$ are elements of S, then one writes $<x_1, x_2, \dots, x_n>_{\mathfrak{C}}$ instead of $<\{x_1, x_2, \dots, x_n\}>_{\mathfrak{C}}$ and says that $<x_1, x_2, \dots, x_n>_{\mathfrak{C}}$ is finitely generated. The elements of $\mathfrak{C}$ of the type $<x>_{\mathfrak{C}}$ are said principal; moreover, if every element of $\mathfrak{C}$ is principal, then $(S,\mathfrak{C})$ is said principal.

2. BINARY CLOSURE SYSTEMS

Through a closure system $(S,\mathfrak{C})$ one can define a commutative (binary) hyperoperation $\cdot$ on S by setting, for any $a,b \in S$, $a \cdot b := <a,b>_{\mathfrak{C}}$. We shall say that $\cdot$ is a *(binary) closure hyperoperation*.

Remark 1. If $\cdot$ is the above hyperoperation, since $a \cdot b = <a,b>_{\mathfrak{C}}$ and $a \cdot b \subseteq <a>_{\mathfrak{C}} \cdot b \subseteq <a>_{\mathfrak{C}} \cdot <b>_{\mathfrak{C}} \subseteq <a,b>_{\mathfrak{C}}$ for any $a, b \in S$, then one has:

1) $a \cdot b = <a>_{\mathfrak{C}} \cdot b = a \cdot <b>_{\mathfrak{C}} = <a>_{\mathfrak{C}} \cdot <b>_{\mathfrak{C}}$.

Consequently, if $a \in S$ and $B \subseteq S$, then one has:

2) $a \cdot B = \bigcup_{b \in B} a \cdot b = \bigcup_{b \in B} <a>_{\mathfrak{C}} \cdot b = <a>_{\mathfrak{C}} \cdot B$ $(= B \cdot <a>_{\mathfrak{C}})$. In particular, if $\cdot$ is an associative hyperoperation and $x_1, x_2, \dots, x_n \in S$, with $n > 2$, then $x_1 \cdot x_2 \cdot \dots \cdot x_n = <x_1>_{\mathfrak{C}} \cdot <x_2>_{\mathfrak{C}} \cdot \dots \cdot <x_n>_{\mathfrak{C}}$.

Now, if $\mathfrak{C}$ is a closure system on S, let $\mathfrak{C}_2$ be the set of the parts X of S which are closed under the hyperoperation $\cdot$ associated to $\mathfrak{C}$ (i.e.: $X \cdot X \subseteq X$). Thus we shall say that the set $\mathfrak{C}_2$ is a *binary* (or *linear*) *closure system* on S and call *binary subspace* (or *pseudo-linear subspace*) of $(S,\mathfrak{C})$ every element of $\mathfrak{C}_2$. Obviously, $\varnothing$ is a binary subspace of $(S,\mathfrak{C})$; moreover $\mathfrak{C} \subseteq \mathfrak{C}_2$.
Now we recall that a closure system $(S,\mathfrak{C})$ is said algebraic if, for any subset X of S and for any $x \in <X>_{\mathfrak{C}}$, there is a finite subset F of X such that $x \in <F>_{\mathfrak{C}}$. It is known that $(S,\mathfrak{C})$ is algebraic if and only if $\mathfrak{C}$ is closed under the set union of the elements of any subset of $\mathfrak{C}$, which is upper directed (in particular, which is a chain) with respect to $\subseteq$. Therefore one can easily verify that $\mathfrak{C}_2$ is an algebraic closure system.
If $+$ is a partial hyperoperation on S, then it is clear that the set of the parts of S which are closed under $+$ is a binary closure system on S.

Remark 2. Let $(S,\mathfrak{C})$ and $(S,\mathfrak{C}')$ be closure systems. If $<x, y>_{\mathfrak{C}'} \subseteq <x, y>_{\mathfrak{C}}$ for any $x, y \in S$, then $\mathfrak{C}_2 \subseteq \mathfrak{C}'_2$. Consequently, if $\mathfrak{C} \subseteq \mathfrak{C}'$, then $\mathfrak{C}_2 \subseteq \mathfrak{C}'_2$.

Remark 3. 1) If $x,y \in S$, then $<x,y>_{\mathfrak{C}_2} \subseteq <x,y>_{\mathfrak{C}}$. Moreover by definition of $\mathfrak{C}_2$, since $x,y \in <x,y>_{\mathfrak{C}_2}$, one has $<x,y>_{\mathfrak{C}} \subseteq <x,y>_{\mathfrak{C}_2}$. As a consequence one gets:

(a) $<x, y>_{\mathfrak{C}} = <x, y>_{\mathfrak{C}_2}$ (in particular, $<x>_{\mathfrak{C}} = <x>_{\mathfrak{C}_2}$). Therefore $\mathfrak{C}$ and $\mathfrak{C}_2$ define the same binary closure hyperoperation, and hence $\mathfrak{C}_2 = (\mathfrak{C}_2)_2$.

(b) Since $\mathfrak{C}_2 = (\mathfrak{C}_2)_2$, $\mathfrak{C}$ is a binary closure system if and only if $\mathfrak{C} = \mathfrak{C}_2$.

2) $\mathfrak{C}_2$ is the lower binary closure system on S including $\mathfrak{C}$. Indeed, if $\mathfrak{C}'$ is a binary closure system and $\mathfrak{C} \subseteq \mathfrak{C}'$, then $\mathfrak{C}_2 \subseteq \mathfrak{C}'_2 = \mathfrak{C}'$.

3) If X is an element of $\mathfrak{C}_2$, then $X \subseteq X{\cdot}X$, and hence $X = X{\cdot}X$.

Theorem 4. If $(S,\mathfrak{C})$ is a binary closure system, let C be a fixed element of $\mathfrak{C}$, and let $\mathfrak{C}' = \{Y \in \mathfrak{C} \mid Y = \varnothing \text{ or } C \subseteq Y\}$. Then $(S,\mathfrak{C}')$ is a binary closure system.

Proof. It is obvious that $(S,\mathfrak{C}')$ is a closure system. Thus, by Remark 3 (see (b) of property 1)), it is sufficient to prove that $\mathfrak{C}'_2 \subseteq \mathfrak{C}'$.

To this end, let X be a non empty element of $\mathfrak{C}'_2$. Thus, for every $x, y \in X$, one has $C \subseteq <x, y>_{\mathfrak{C}'} \subseteq X$ and $<x, y>_{\mathfrak{C}} \subseteq <x, y>_{\mathfrak{C}'} \subseteq X$. Therefore $X \in \mathfrak{C}'$. ∎

2. PARA-NORMAL CLOSURE SYSTEMS

Now, in order to extend some interesting properties of the normal subgroups of a group, in this paragraph let us assume that $\mathfrak{N}$ is a <u>binary</u> closure system on S, $+$ is a hyperoperation on S and $\mathfrak{C}$ is the closure system of the subsets of S closed under $+$. Thus we shall indicate respectively with $\cdot_{\mathfrak{N}}$ and $\cdot_{\mathfrak{C}}$ the closure hyperoperations associated to $\mathfrak{N}$ and to $\mathfrak{C}$.

We shall say that $(S,\mathfrak{N})$ is *para-normal* with respect to $+$ whenever the following condition holds:

$$(°) \qquad \forall\, x,y \in S\colon <x>_{\mathfrak{N}} + <y>_{\mathfrak{N}} = <x, y>_{\mathfrak{N}} .$$

Hence, since if $x,y \in S$ we have $x+y \subseteq <x>_{\mathfrak{N}} + <y>_{\mathfrak{N}} = <x, y>_{\mathfrak{N}} =$ $= x \cdot_{\mathfrak{N}} y$, then the following property holds:

$$(^{\circ\circ}) \qquad \forall\, x,y \in S\colon\ x+y \subseteq x \cdot_{\mathfrak{N}} y.$$

And now let us assume that the binary closure system $(S, \mathfrak{N})$ is para-normal with respect to the hyperoperation $+$. Then we have the following theorems.

Theorem 5. for any $A,B \in \mathfrak{N}$ one has:

$$(*) \qquad A+B = A \cdot_{\mathfrak{C}} B = A \cdot_{\mathfrak{N}} B.$$

Furthermore $\mathfrak{N}$ is included in $\mathfrak{C}$.

Proof. Preliminarily let us remark that $A+B = A \cdot_{\mathfrak{N}} B$. In fact the following equalities are trivial:

$$A+B = \bigcup_{a\in A, b\in B} <a>_{\mathfrak{N}} + <b>_{\mathfrak{N}} = \bigcup_{a\in A, b\in B} <a, b>_{\mathfrak{N}} =$$

$$= \bigcup_{a\in A, b\in B} a \cdot_{\mathfrak{N}} b = A \cdot_{\mathfrak{N}} B.$$

In particular, one has $A+A = A \cdot_{\mathfrak{N}} A = A$. Thus $\mathfrak{N}$ is included in $\mathfrak{C}$ and hence $A \cdot_{\mathfrak{C}} B \subseteq A \cdot_{\mathfrak{N}} B$. As a consequence - since it is obvious that $A+B \subseteq A \cdot_{\mathfrak{C}} B$ - we get $A+B = A \cdot_{\mathfrak{C}} B = A \cdot_{\mathfrak{N}} B$. ∎

Theorem 6. Let $A \in \mathfrak{C}$ be a union of elements of $\mathfrak{N}$. Then $A \in \mathfrak{N}$ [1].

Proof. It is sufficient to prove that $A \cdot_{\mathfrak{N}} A \subseteq A$. Indeed one has:

$$A \cdot_{\mathfrak{N}} A \subseteq \bigcup_{a,a' \in A} <a>_{\mathfrak{N}} \cdot_{\mathfrak{N}} <a'>_{\mathfrak{N}} = \bigcup_{a,a' \in A} <a, a'>_{\mathfrak{N}} =$$

$$= \bigcup_{a,a' \in A} <a>_{\mathfrak{N}} + <a'>_{\mathfrak{N}} \subseteq A+A \subseteq A. \quad ∎$$

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[1] See the case of a subgroup which is a union of normal subgroups.

Remark 7. Let us point out that if the binary closure system $(S, \mathcal{N})$ is para-normal with respect to an associative hyperoperation $+$ then, as in the case of normal subgroups of a group, also $\cdot_{\mathcal{N}}$ is associative. Indeed, by Remark 1 and by Theorem 5, for any $a, b, c \in S$ we have:

$$a \cdot_{\mathcal{N}} (b \cdot_{\mathcal{N}} c) = <a>_{\mathcal{N}} \cdot_{\mathcal{N}} (<b, c>_{\mathcal{N}}) = <a>_{\mathcal{N}} + (<b>_{\mathcal{N}} + <c>_{\mathcal{N}}) =$$

$$= (<a>_{\mathcal{N}} + <b>_{\mathcal{N}}) + <c>_{\mathcal{N}} = (<a, b>_{\mathcal{N}}) \cdot_{\mathcal{N}} <c>_{\mathcal{N}} = (a \cdot_{\mathcal{N}} b) \cdot_{\mathcal{N}} c.$$

In the meantime $\cdot_{\mathcal{C}}$ can be not associative, as in most groups $(S, +)$ in which $+$ is a non commutative operation.

3. ASSOCIATIVE CLOSURE SYSTEMS

Now let $\cdot$ be the hyperoperation associated to a given closure system $(S, \mathcal{C})$; hence, for any $x, y \in S$, x and y belong to the hyperproduct $x \cdot y$. As a consequence, if $\cdot$ is associative, then $\cdot$ gives S a structure of commutative hypergroup (in the sense of [1], p. 8). Therefore we shall say that $(S, \mathcal{C})$ is *associative* and $(S, \cdot)$ is a *closure hypergroup*.
Furthermore, we shall say that $(S, \mathcal{C})$ is 3-*strong associative* if, for any $x, y, z \in S$, $x \cdot (y \cdot z) = <x, y, z>_{\mathcal{C}}$ $(= z \cdot (x \cdot y) = (x \cdot y) \cdot z)$. In such a case, since $x \cdot (y \cdot z) \subseteq <x, y, z>_{\mathcal{C}_2} \subseteq <x, y, z>_{\mathcal{C}}$, then one has $<x, y, z>_{\mathcal{C}} = <x, y, z>_{\mathcal{C}_2}$.
More generally, given a natural numbers $n \geq 2$, we shall say that $(S, \mathcal{C})$ is n-*strong associative* if it is associative and $x_1 \cdot x_2 \cdot \ldots \cdot x_n = <x_1, x_2, \ldots, x_n>_{\mathcal{C}}$ for any $x_1, x_2, \ldots, x_n \in S$. Furthermore, we shall say that $(S, \mathcal{C})$ is *finitely strong associative* if it is n-strong associative for any natural numbers $n \geq 2$.
Obviously, a closure system $(S, \mathcal{C})$ is 2-strong associative if and only if it is associative. Furthermore, if m and n are natural numbers such that $2 \leq m < n$ and $(S, \mathcal{C})$ is n-strong associative, then $(S, \mathcal{C})$ is m-strong associative. In fact, if one set $x_m = x_{m+1} = \ldots = x_n$, then (see 1) of Remark 1) one gets [2]:

$$x_1 \cdot x_2 \cdot \ldots \cdot x_{m-1} \cdot x_m = x_1 \cdot x_2 \cdot \ldots \cdot x_{m-1} \cdot <x_m>_{\mathcal{C}} =$$

[2] We recall that $<x>_{\mathcal{C}} \cdot <x>_{\mathcal{C}} = <x>_{\mathcal{C}}$ for any $x \in S$.

$$= x_1 \cdot x_2 \cdot \ldots \cdot x_{m-1} \cdot <x_m>_{\mathfrak{C}} \cdot <x_{m+1}>_{\mathfrak{C}} \cdot \ldots \cdot <x_n>_{\mathfrak{C}} =$$

$$= x_1 \cdot x_2 \cdot \ldots \cdot x_m \cdot x_{m+1} \cdot \ldots \cdot x_n = <x_1, x_2, \ldots, x_n>_{\mathfrak{C}} =$$

$$= <x_1, x_2, \ldots, x_m>_{\mathfrak{C}} .$$

Theorem 8. Let n be a natural number greather than 1. Then a closure system $(S,\mathfrak{C})$ is n-strong associative if and only if the following property holds [3]:

(*) $\forall\, x_1, x_2, \ldots, x_n \in S$: $<x_1, x_2, \ldots, x_n>_{\mathfrak{C}} = x_1 \cdot <x_2, \ldots, x_n>_{\mathfrak{C}}$.

Proof. The assertion is obvious if $n = 2$ or $n = 3$. Thus let $n > 3$. If $(S,\mathfrak{C})$ is n-strong associative then, since it is also $(n-1)$-strong associative, we have:

$$<x_1, x_2, \ldots, x_n>_{\mathfrak{C}} = x_1 \cdot x_2 \cdot \ldots \cdot x_n = x_1 \cdot <x_2, \ldots, x_n>_{\mathfrak{C}} .$$

Conversely, let the condition (*) hold. Then it holds also with n replaced by a natural number m such that $2 < m < n$. In fact we can set $x_m = x_{m+1} = \ldots = x_n$. Thus, by setting $m = 3$, we have that $(S,\mathfrak{C})$ is associative.
As an immediate consequence, by induction, we get $<x_1, x_2, \ldots, x_n>_{\mathfrak{C}} = x_1 \cdot x_2 \cdot \ldots \cdot x_n$. ∎

Theorem 9. Let $(S,\mathfrak{C})$ be a closure system and let $a_1, a_2, \ldots, a_n, b \in S$. If $<a_2, \ldots, a_n>_{\mathfrak{C}} = <b>_{\mathfrak{C}}$, then $a_1 \cdot <a_2, \ldots, a_n>_{\mathfrak{C}} = <a_1, a_2, \ldots, a_n>_{\mathfrak{C}}$.

Proof. Indeed (see 1) of Remark 1), $a_1 \cdot <a_2, \ldots, a_n>_{\mathfrak{C}} = a_1 \cdot <b>_{\mathfrak{C}} = a_1 \cdot b = <a_1, b>_{\mathfrak{C}} = <a_1, a_2, \ldots, a_n>_{\mathfrak{C}}$; whence the thesis. ∎

Remark 10. If the finitely generated and non empty elements of a closure system $\mathfrak{C}$ are principal, then (by Theorem 6) the condition (*) of Theorem 8 is true for any natural number n, and hence $(S,\mathfrak{C})$ is finitely strong associative. In particular, the closure system $\mathfrak{I}$ of the ideals of a semilattice $(S,\cup)$ [4] is a finitely strong associative closure system. In fact one can immediately verify that, for any $x_1, x_2, \ldots, x_n \in S$, the ideal generated by $x_1, x_2, \ldots, x_n$ is equal to $<x_1+x_2+ \ldots +x_n>_{\mathfrak{I}}$.

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[3] If $n = 2$ then, by 1) of Remark 1, property (*) is true even if $(S,\mathfrak{C})$ is non associative.

[4] A semilattice is a structure $(S,\cup)$, where $\cup$ is an idempotent, commutative and associative binary operation; an ideal is a subset B of S closed under $\cup$ such that, for any $x \in S$ and $x' \in B$, if $x \leq x'$ (i.e.: $x \cup x' = x'$), then $x \in B$.

Theorem 11. Let $(S,\mathfrak{C})$ be an algebraic and associative closure system. Then $\mathfrak{C}$ is finitely strong associative if and only if $\mathfrak{C} \cup \{\varnothing\} = \mathfrak{C}_2$.

Proof. Let $(S,\mathfrak{C})$ be finitely strong associative. Thus, since $\mathfrak{C} \cup \{\varnothing\} \subseteq \mathfrak{C}_2$, in order to prove that $\mathfrak{C} \cup \{\varnothing\} = \mathfrak{C}_2$ it is sufficient to verify that if X is a non empty element of $\mathfrak{C}_2$, then $<X>_{\mathfrak{C}} = X$ (hence X is also an element of $\mathfrak{C}$).

Since $X \subseteq <X>_{\mathfrak{C}}$, we only have to verify that $<X>_{\mathfrak{C}} \subseteq X$. Thus let $x' \in$ $\in <X>_{\mathfrak{C}}$ and (by the hypothesis that $(S,\mathfrak{C})$ is algebraic) let us consider $x_1, \ldots, x_n \in X$ such that $x' \in <x_1, \ldots, x_n>_{\mathfrak{C}}$. As $\mathfrak{C}$ is finitely strong associative, $<x_1, \ldots, x_n>_{\mathfrak{C}} = x_1 \cdot \ldots \cdot x_n \subseteq X$, and hence $x' \in X$.

On the contrary, let $\mathfrak{C} \cup \{\varnothing\} = \mathfrak{C}_2$. Hence $\mathfrak{C}$ and $\mathfrak{C}_2$ determine the same hyperoperation $\cdot$; moreover $<X>_{\mathfrak{C}_2} = <X>_{\mathfrak{C}}$ for any non empty subset X of S, hence $\mathfrak{C}$ is finitely strong associative if and only if $\mathfrak{C}_2$ is finitely strong associative. Thus let us verify that if $x_1, x_2, \ldots, x_n \in S$, then $x_1 \cdot x_2 \cdot \ldots \cdot$ $\cdot x_n = <x_1, x_2, \ldots, x_n>_{\mathfrak{C}_2}$.

Indeed, since $x_1, x_2, \ldots, x_n \in x_1 \cdot x_2 \cdot \ldots \cdot x_n \subseteq <x_1, x_2, \ldots, x_n>_{\mathfrak{C}_2}$ and $<x_1, x_2, \ldots, x_n>_{\mathfrak{C}_2}$ is the minimum element of $\mathfrak{C}_2$ containing $\{x_1, x_2, \ldots, x_n\}$, it is sufficient to point out that, by associativity and by commutativity, one has (cf. Remark 1):

$$(x_1 \cdot x_2 \cdot \ldots \cdot x_n) \cdot (x_1 \cdot x_2 \cdot \ldots \cdot x_n) = x_1 \cdot x_1 \cdot x_2 \cdot x_2 \cdot \ldots \cdot x_n \cdot x_n =$$

$$= <x_1>_{\mathfrak{C}_2} \cdot <x_2>_{\mathfrak{C}_2} \cdot \ldots \cdot <x_n>_{\mathfrak{C}_2} = x_1 \cdot x_2 \cdot \ldots \cdot x_n. \quad \blacksquare$$

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ON SOME APPLICATIONS OF FUZZY SETS AND COMMUTATIVE HYPERGROUPS TO EVALUATION IN ARCHITECTURE AND TOWN-PLANNING[1]

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SUNTO: In alcune recenti leggi sulla determinazione dei canoni di affitto e dei valori catastali degli alloggi è richiesta una "divisione in zone omogenee" della città in base ad assegnati criteri. In molte città, come risulta anche dai quotidiani, è stata a tale scopo effettuata una classificazione crisp dell'insieme degli alloggi. In alcuni nostri lavori abbiamo osservato che, dato il modo "non brusco" in cui variano le caratteristiche degli alloggi, sembra più opportuna una classificazione fuzzy.
In questo lavoro, partendo dal concetto di join space associato ad un insieme fuzzy, indaghiamo sulle relazioni fra partizioni fuzzy ed ipergruppi commutativi. Più in generale, introduciamo i concetti di "insieme fuzzy qualitativo lineare" e mostriamo le relazioni fra le famiglie di tali insiemi, le partizioni fuzzy e gli ipergruppi commutativi. Lo scopo del lavoro è di mostrare come la teoria degli ipergruppi commutativi possa essere un utile strumento di lavoro per affrontare problemi di valutazione in urbanistica. In particolare, per mezzo dei blocchi associati ad un opportuno ipergruppo commutativo si individuano aree "quasi omogenee", per le quali si possono determinare le oscillazioni di affitti e valori catastali.

ABSTRACT: In some recent laws about the determination of the rents and of the estimated income of properties, it is required a subdivision of the municipal area in homogeneous zones on the basis of assigned criteria. For this reason in many cities, as it also appeared in some daily newspapers, it has been made a crisp classification of the set of buildings. In some our papers we have observed that the peculiarities of the buildings change in a "not sharp" way and so a fuzzy classification seems more suitable.
In this paper, starting from the concept of join space associated to a fuzzy set, we study the relations between fuzzy partitions and commutative hypergroups. More in

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general, we introduce the concepts of "qualitative linear fuzzy set" and we show the relations among the families of these sets, fuzzy partitions and commutative hypergroups. The aim of the study is to show that the commutative hypergroups are a useful tool to study problems on the evaluation in town-planning. In particular, from the blocks associated to a suitable commutative hypergroup, we single out "almost homogeneous" areas and we can determine, in such areas, the fluctuation of the rents and of the values of the buildings.

KEYWORDS: Commutative hypergroups, Fuzzy classifications, Evaluation in Architecture and in Town-planning, Qualitative fuzzy sets

1. CLASSIFICATION FOR THE EVALUATION IN TOWN-PLANNING

Many problems about the evaluation in town-planning lead to a classification of a city or of a territory: for example the organization of the taxation of the building, the evaluation of the soil or the distribution of the mail and of the shops.

In recent times, many Italian newspapers show the classifications of some cities made by municipal governments for the definition of the rents. Every city is considered as a set Ω with elements the buildings and it is divided in a fixed number of subsets, called microzones, that are a partition of Ω. Such classifications depend on a set of criteria fixed by the national law.

The classifications considered by the municipal authorities are all of crisp type, that is any element of Ω belongs totally only to a class. Besides, by reading of the laws and of the reportages of the newspapers it would seem that such classifications are not obtained with precise statistical methods but rather in some empirical ways.

In some our recent papers, [6], [7], we note that for the characteristics of a city a fuzzy classification seems to be more appropriate than a crisp one. In fact the variations of the characteristics of the buildings are not crisp but they are always variable almost with continuity and so it is not possible to consider "walls" that divide the microzones from one to other. In [7] we propose some algorithms of fuzzy classification that we think suitable for the formation of the microzones.

In this paper we investigate about the relations between fuzzy classifications and commutative hypergroups. We think that the hypergroups are a very useful tool to individuate, both by an algebraic and a geometric point of view, homogeneous zones in the city. We consider also a generalization of the concept of fuzzy set, the "qualitative linear fuzzy set" that is a more natural function than fuzzy sets in problems of Architecture in which we can give judgments but not precise measures about the degree in which a building belongs to a given class or has a particular characteristic.

2. FUZZY CLASSIFICATION AND HYPERGROUPS

We recall some fundamental definition

Definition 2.1 A *fuzzy set* with universe Ω is a function $\varphi:\Omega\to[0,1]$. If $\varphi(\Omega) = \{h\}$, $h\in[0,1]$ φ is called *constant fuzzy set*. In particular for h=1 is called *null fuzzy set* and for h=1 *unitary fuzzy set*. A *fuzzy partition* is a finite o countable family of non null fuzzy sets $\{\varphi_i, i\in I\}$ such that, $\forall x\in\Omega$, $\Sigma_{i\in I}\ \varphi_i(x) = 1$.
A fuzzy set $\varphi:\Omega\to[0,1]$ with $\varphi(\Omega)\subseteq\{0, 1\}$ is called *crisp set* and a finite o countable family of non null crisp sets $\{\varphi_i, i\in I\}$ such that, $\forall x\in\Omega$, $\Sigma_{i\in I}\ \varphi_i(x) = 1$ is said to be a *crisp partition.*

If we consider the bijection Ψ: $S\in \wp(\Omega)\to (\varphi_S: \Omega\to\{0, 1\} \ / \ \varphi_S^{-1}(1) = S)$ we can identify every subset S of Ω with the crisp set φ_S with universe Ω. Then a crisp partition is a usual partition of Ω.

Definition 2.2 A hypergroupoid $H = (\Omega, \sigma)$ is said to be a *hypergroup* if we have the following properties

1. *associative* $\forall a,b,c\in\Omega$, $(a\sigma b)\sigma c = a\sigma(b\sigma c)$;
2. *reproducibility* $\forall a,b\in\Omega$, $\exists x,y\in\Omega$: $b \in a\sigma x \cap y\sigma a$.

A hypergroup is said to be *commutative* if $\forall a,b\in\Omega$, $a\sigma b = b\sigma a$. In a commutative hypergroup we define "in a natural way" the division "/" if we put

$$\forall a,b\in\Omega,\ b/a = \{x\in\Omega\text{: } b\in a\sigma x\}.$$

Definition 2.3 Let $H = (\Omega, \sigma)$ be a commutative hypergroup. It is

- *open* if $\forall a,b\in\Omega$, $a\neq b$, $a\bullet b\cap\{a,b\}=\varnothing$;
- *closed* if $\forall a,b\in\Omega$, $a\bullet b\supseteq\{a,b\}$;
- *geometric* if $\forall a\in\Omega$, $a\bullet a = \{a\} = a/a$;
- *join space* if we have the following incidence property:

$$\forall a, b, c, d \in\Omega,\ a/b\cap c/d\neq\varnothing \Rightarrow a\bullet d\cap b\bullet c\neq\varnothing.$$

Let $\varphi:\Omega\to[0,1]$ be a fuzzy set with universe Ω. By a result of Corsini, [3], if we put

$$\forall a, b\in\Omega,\ a\sigma b = \{z\in\Omega\text{: } \min\{\varphi(a), \varphi(b)\}\leq \varphi(z)\leq \max\{\varphi(a), \varphi(b)\}\}$$

then $H = (\Omega, \sigma)$ is a *closed join space*, called *associated* to φ.

Let I be a finite or countable set and let $\forall i\in I$, $\varphi_i:\Omega\to[0,1]$ be a fuzzy set with universe set Ω. If we put,

$$\forall i\in I,\ a\sigma_i b = \{z\in\Omega\text{: } \min\{\varphi_i(a), \varphi_i(b)\} \leq \varphi_i(z)\leq \max\{\varphi_i(a), \varphi_i(b)\},$$

we have the family of the closed join spaces $\{H_i = (\Omega, \varphi_i), i\in I\}$ associated to the family of fuzzy sets $\Phi = \{\varphi_i, i\in I\}$.

Now we introduce some hypergroups associated to the fuzzy partitions. We begin by considering the particular case of the crisp partitions.
Let $\Phi = \{\varphi_i, i\in I\}$ be a crisp partition with universe Ω and, $\forall a\in\Omega$, let C(a) the class of a. If we put $\forall a, b\in\Omega$, $a*b = C(a)\cup C(b)$ we can consider the hypergroupoid $H=(\Omega, *)$, that we call *hypergroupoid associated* to Φ.

We have the following
Theorem 2.4 Let $\Phi = \{\varphi_i, i\in I\}$ be a crisp partition with universe Ω and let $H=(\Omega, *)$ be the hypergroupoid associated to Φ. Then H is a commutative closed hypergroup. Moreover we have $\forall a, b\in\Omega$, $a*b = \cap_{i\in I} a\sigma_i b$.
Proof The associative and commutative properties are consequences of the ones of the union. Since $\{a, b\} \subseteq a*b$ we have the closure and the reproducibility and so H is a commutative hypergroup.
Let x an element Ω. We have that
$(x\in a*b) \Leftrightarrow (x\in C(a)\cup C(b)) \Leftrightarrow (\forall i\in I, \varphi_i(x) = \varphi_i(a) \text{ or } \varphi_i(x) = \varphi_i(b)) \Leftrightarrow$
$\Leftrightarrow (\forall i\in I, x\in a\sigma_i b) \Leftrightarrow (x\in\cap_{i\in I} a\sigma_i b)$.

In general, let $\Phi = \{\varphi_i, i\in I\}$ be a fuzzy partition with universe Ω. If we wish to extend the results achieved for the crisp partitions to the fuzzy ones, we have to consider the hyperoperation $*: (x, y)\in\Omega\to\cap_{i\in I} a\sigma_i b$. But, for the applications to the architecture and town-planning it is convenient to examine a concept more general than the one of fuzzy set, that we call *qualitative linear fuzzy set*.
In fact, if Ω is a set of objects (e. g. the buildings of a city) to evaluate as to a criterion K, in general we do not have a numeric function $\varphi_K: \Omega\to[0,1]$ such that, $\forall x\in\Omega$, $\varphi_K(x)$ gives the *measure* in which x satisfies the criterion K. We are happy if we can find a set totally ordered S of the "possible judgements" about the truth or the falsity of the proposition "x satisfies K". For example S may be "false, not all false, partially false, partially true, not all true, true". S is called *set of qualitative values*. We suppose that S has a minimum F and a maximum T, that are, respectively, the qualitative values "false" and "true". For any judgement g of S we call *the opposite of g* the judgement g^c obtained by g with the change between the words "true" and "false". We call *the opposite of S* the set S^c of the opposites of the elements of S.

Definition 2.5 Let S be a set of qualitative values. An application $\alpha: \Omega\to S$ is called *qualitative linear fuzzy set (qlfs)*. A qlfs $\alpha^c: \Omega\to S^c$ that to any $x\in\Omega$ associates $(\alpha(x))^c$ is called *the opposite* of α.

Definition 2.6 Let S be a set of qualitative values. We call *numerical evaluation* on S any function $v: S\to[0,1]$ such that $v(F) = 0$, $v(V)=1$, $\forall z, t\in S$, $z\leq t \Rightarrow v(z)\leq v(t)$.

We say that v: $S\to[0,1]$ is a *strong numerical evaluation* on S if it is a numerical evaluation on S and $\forall z, t\in S$, $v(z)\le v(t) \Rightarrow z\le t$.

We can note that, if v: $S\to[0,1]$ is a numerical evaluation on S then v^c: $S^c\to[0,1]$ such that, $\forall z\in S$, $v^c(z^c) = 1-v(z)$ is a numerical evaluation on S^c such that $\forall z, t\in S$, $z\le t \Rightarrow v^c(z^c)\ge v^c(t^c)$. If v is strong then also v^c is strong. We call v^c *the opposite of v*. We have the relations $(S^c)^c=S$, $(\alpha^c)^c=\alpha$ and $(v^c)^c=v$.

If α: $\Omega\to S$ is a qualitative linear fuzzy set and v: $S\to[0,1]$ is a numerical evaluation then the function $\varphi=v\circ\alpha$ is a fuzzy set with universe Ω. Then, from a qlfs on Ω, if we give a suitable numerical evaluation, we can have also a fuzzy set on Ω. Let $\varphi^c=v^c\circ\alpha^c$. It is easy to prove that $\varphi^c=1-\varphi$.

We can prove that the theorem of Corsini can be extended also to the qualitative linear fuzzy set. In fact the proof of this theorem utilizes only the properties of the total order relation on [0, 1]. So, if α: $\Omega\to S$ is a qualitative linear fuzzy set, if we put

$$\forall a, b\in\Omega,\ a\sigma b = \{z\in\Omega: \min\{\alpha(a), \alpha(b)\}\le \alpha(z)\le \max\{\alpha(a), \alpha(b)\}\},$$

with $\le$ order relation on S, we have that $H = (\Omega, \sigma)$ is a closed join space, that we call *join space associated* to α.

We note that, if we consider the opposite α^c of α, the order relation $\le^c$ on S^c is the opposite of $\le$ on S. Then we have $\{z\in\Omega: \min\{\alpha(a),\alpha(b)\} \le\alpha(z)\le\max\{\alpha(a),\alpha(b)\}\} = =\{z\in\Omega:\max\{\alpha^c(a),\alpha^c(b)\}\ge\alpha^c(z)\ge\min\{\alpha^c(a),\alpha^c(b)\}\}$. So the *join space associated* to α^c is equal to the one associated to α.

Let $\Psi = \{\alpha_i, i\in I\}$ a family of qualitative linear fuzzy sets and let, $\forall i\in I$, $H_i = (\Omega, \sigma_i)$ the join space associated to α_i. Let "*" the hyperoperation such that to any pair (a, b) of elements of Ω associates $a*b = \cap_{i\in I}\, a\sigma_i b$. We have the following

Theorem 2.7 The pair $H=(\Omega, *)$ is a closed commutative hypergroup, called *associated to the family of the qualitative linear fuzzy sets* $\Psi = \{\alpha_i, i\in I\}$.

Proof Since $a*b = \cap_{i\in I}\, a\sigma_i b$, by definitions it follows that H is a closed and commutative hypergroupoid and since $b\in a*b$, $\forall a, b\in\Omega$, the reproducibility holds. Therefore, it remain to prove that * is associative. We have, $\forall a,b,c\in\Omega$, $(a*b)*c = =\cap_{i\in I} (a*b)\sigma_i c$. Let $m_i= \min\{\alpha_i(a), \alpha_i(b), \alpha_i(c)\}$, $M_i= \max\{\alpha_i(a), \alpha_i(b), \alpha_i(c)\}$. Since $a*b\subseteq a\sigma_i b$, $\forall x\in a*b$, $\alpha_i(x)$ belongs to the closed interval with extremes $\alpha_i(a)$ and $\alpha_i(b)$. Then, we have $(a*b)\sigma_i c = \{z\in\Omega: m_i\le \alpha_i(z)\le M_i\}$ and so $(a*b)*c = \cap_{i\in I} \{z\in\Omega: m_i \le \alpha_i(z) \le M_i\}$. Similarly, we prove that $a*(b*c) = \cap_{i\in I} \{z\in\Omega: m_i \le \alpha_i(z) \le M_i\}$ and so the associative property holds.

Since a family of fuzzy sets $\Phi = \{\varphi_i, i\in I\}$ with universe Ω is also a family of qualitative linear fuzzy sets with universe Ω and with [0, 1] as set of judgements, the previous theorem is valid also if Ψ is a family of fuzzy sets. In particular, if Ψ is a fuzzy partition, the commutative hypergroup $H=(\Omega, *)$ generalizes the one considered in theorem 2.4 for the crisp partitions.

Now we consider the following problem: given a closed commutative hypergroup $H=(\Omega,*)$, we wish to find, if it exists, a fuzzy partition $\Phi = \{\varphi_i, i\in I\}$ such that $H=(\Omega, *)$ is the commutative hypergroup associated. For this aim, we introduce the following

Definition 2.8 We say that a hypergroup $H=(\Omega,*)$ is *fuzzy decomposable* if it is a closed commutative hypergroup and there exists a finite or countable family $\{H_i=(\Omega, \sigma_i)\}_{i\in I}$ of closed commutative hypergroups, called *fuzzy decomposition* of H, such that

(FD1) $\forall i\in I$, H_i is associated to a qualitative non-constant linear fuzzy set α_i;
(FD2) $\forall$ a, $b\in\Omega$, $a*b = \cap_{i\in I} a\sigma_i b$.

If $H=(\Omega, *)$ is a *fuzzy decomposable* hypergroup and $\{H_i=(\Omega, \sigma_i)\}_{i\in I}$ is a fuzzy decomposition of H, then, for any $i\in I$, there exists a qualitative linear non-constant fuzzy set $\alpha_i:\Omega\rightarrow S_i$ such that H_i is associated to α_i and so H_i is associated also to α_i^c.
Let v_i be a strong numerical evaluation of S_i. Then $\varphi_i = v_i\circ\alpha_i$ and $\varphi_i^c = v_i^c\circ\alpha_i^c$ are two non-constant fuzzy sets associated to H_i, and such that $\varphi_i^c = 1 - \varphi_i$. Moreover, $\forall i\in I$ and $\forall\lambda_i\in(0, 1]$ also $\lambda_i\, \varphi_i$ and $\lambda_i\, \varphi_i^c$ are non-constant fuzzy sets associated to H_i.
We say that a family of fuzzy sets $\Phi = \{\varphi_i, i\in I\}$ is associated to H if, $\forall i\in I$, φ_i is a non-constant fuzzy set associated to H_i.
We say that two fuzzy sets φ and ψ are similar if there exists a $\lambda\in(0, 1]$ such that $\psi=\lambda\varphi$ or $\psi=\lambda\varphi^c$. Moreover, we say that two family of fuzzy sets $\Phi = \{\varphi_i, i\in I\}$ and $\Psi = \{\psi_i, i\in I\}$ are similar if, $\forall i\in I$, φ_i and ψ_i are similar.
Now we consider the following problem: given a family of fuzzy sets $\Phi = \{\varphi_i, i\in I\}$ associated to a fuzzy decomposable hypergroup H to find a family of fuzzy sets $\Psi = \{\psi_i, i\in I\}$ similar to Φ and fuzzy partition of Ω. We consider the case in which I is finite. We prove the following

Theorem 2.9 Let $H=(\Omega, *)$ be a fuzzy decomposable hypergroup, and let $\Phi = \{\varphi_i, i\in I\}$ be a family of fuzzy sets associated to (H, v). Then there exists a fuzzy partition of Ω $\Psi=\{\psi_i, i\in I\}$ similar to Φ if and only if, $\exists k\in R$ and $\forall i\in I$ $\exists\beta_i\neq 0$ such that $\Sigma_{i\in I}\, \beta_i\varphi_i = k$.
Proof Let $\Phi = \{\varphi_i, i\in I\}$ be a family of fuzzy sets associated to H and suppose $\Psi = \{\psi_i, i\in I\}$ is a fuzzy partition similar to Φ. Then, $\forall i\in I$, there exists a $\lambda_i\in(0, 1]$ such that $\psi_i=\lambda_i\varphi_i$ or $\psi_i=\lambda_i\varphi_i^c$ and $\Sigma_{i\in I}\, \psi_i=1$.
Let $P=\{i\in I: \psi_i=\lambda_i\varphi_i\}$ and let $Q==\{i\in I: \psi_i=\lambda_i\varphi_i^c\}$.
We have $\Sigma_{i\in P}\, \lambda_i\varphi_i +\Sigma_{i\in Q}\, \lambda_i(1-\varphi_i) = 1$ and so $\Sigma_{i\in P}\, \lambda_i\varphi_i +\Sigma_{i\in Q}\, (-\lambda_i)\varphi_i = 1-\Sigma_{i\in Q}\, \lambda_i$. If we put $\beta_i = \lambda_i$ for $i\in P$, $\beta_i = -\lambda_i$ for $i\in Q$ and $1-\Sigma_{i\in Q}\, \lambda_i = k$, the first part of theorem is proved.

On the converse, suppose that $\exists k\in R$, $\forall i\in I$ $\exists\beta_i\neq 0$ such that $\Sigma_{i\in I}\,\beta_i\varphi_i = k$. Let $P=\{i\in I: \beta_i>0\}$ and let $Q=\{i\in I: \beta_i<0\}$. We have $\Sigma_{i\in P}\,\beta_i\varphi_i + \Sigma_{i\in Q}\,(-\beta_i)(1-\varphi_i) = k-\Sigma_{i\in Q}\,(-\beta_i)$. Let $h = k-\Sigma_{i\in Q}\,(-\beta_i)$. We have that $h>0$. If we put $\psi_i = \beta_i\varphi_i/h$ for $i\in P$, $\psi_i = -\beta_i(1-\varphi_i)/h$ for $i\in Q$ then we have $\Sigma_{i\in I}\,\psi_i=1$ and so $\{\psi_i, i\in I\}$ is a fuzzy partition similar to Φ.

If I and $\Omega=\{O_j\}_{j\in J}$ are finite, $|I|=c$, $|J|=n$, a family of fuzzy sets $\Phi = \{\varphi_i, i\in I\}$ is represented by a matrix A_Φ of type [c, n], with generic element $a_{ij}=\varphi_i(O_j)$, called *first matrix* of Φ. Denote by A_Φ^*, called *second matrix* of Φ, the matrix obtained by A_Φ by adding a vector row, denoted u, with n columns and with any element equal to 1. By previous theorem we have the following

Corollary 2.10 Let $\Phi = \{\varphi_i, i\in I\}$ be a family of non-constant fuzzy sets associated to H and let A_Φ^* be the second matrix of Φ. If there exists a fuzzy partition $\Psi = \{\psi_i, i\in I\}$ similar to Φ then the rank of A_Φ^* is not superior to $c=|I|$. On the converse, if the rank of A_Φ^* is not superior to c then there exists a $I^*\subseteq I$ and a fuzzy partition $\Psi^*=\{\psi_i^*, i\in I^*\subseteq I\}$ that is similar to the subfamily, of Φ, $\Phi^* = \{\varphi_i, i\in I^*\}$.

Proof In fact, if there exists a fuzzy partition of Ω $\Psi = \{\psi_i, i\in I\}$ similar to Φ, for the previous theorem $\exists k\in R$, $\forall i\in I$ $\exists\beta_i\neq 0$ such that $\Sigma_{i\in I}\,\beta_i\varphi_i = k$. If $k=0$ the φ_i, $i\in I$ are linearly dependent and, if $k\neq 0$, u is linearly dependent on the φ_i, $i\in I$. In both the cases the rank of A_Φ^* is not superior to c.

On the converse, if the the rank of A_Φ^* is not superior to c we have that $\forall i\in I$, $\exists\beta_i\in R$ such that $\Sigma_{i\in I}\,\beta_i\varphi_i = 0$ if the φ_i, $i\in I$, are linearly dependent and $\forall i\in I$, $\exists\beta_i\in R$ such that $\Sigma_{i\in I}\,\beta_i\varphi_i = u$ if the φ_i, $i\in I$, are linearly independent. In both the cases, let $I^*=\{i\in I: \beta_i\neq 0\}$. By theorem 2.9 we have that from the subfamily of fuzzy sets $\Phi^* =\{\varphi_i, i\in I^*\}$ we can find a family of fuzzy sets $\Psi^* =\{\psi_i, i\in I^*\}$ that is a fuzzy partition of Ω.

We can obtain, in the previous theorem, a fuzzy partition $\Psi^*=\{\psi_i^*, i\in I^*\subseteq I\}$ with I^* maximal. In fact, the rank of A_Φ^* is not superior to c if and only if the homogeneous system $\Sigma_{i\in I}\,x_i\varphi_i - x_{c+1}u$ has at least a not trivial solution. The set S of solutions is a vector space. If d is the dimension of S and $\{v_1, v_2,\ldots,v_d\}$ is a base of S, we can find a vector $v=\{\beta_1, \beta_2,\ldots,\beta_c, \beta_{c+1}\}\in S$ such that, $\forall j\in\{1, 2, \ldots,n+1\}$, β_j is null if and only if the component j of v_s is null for all $s\in\{1, 2,\ldots,d\}$. Since, $\forall w=\{w_1, w_2,\ldots,w_c, w_{c+1}\}\in\in S$ and $\forall j\in\{1, 2, \ldots,n+1\}$ if β_j is null then also w_j is null, the set $I_w^*=\{j\in I: w_j\neq 0\}$ is maximal for v. By theorem 2.9 by the subfamily of fuzzy sets $\Phi^* =\{\varphi_i, i\in I_v^*\}$ of Φ we obtain a maximal fuzzy partition $\Psi^* =\{\psi_i, i\in I_v^*\}$.

In the application to evaluation in Architecture and Town-Planning, we have a set Ω of objects to evaluate and a family $\Gamma=\{C_i, i\in I\}$ of classes. We suppose that, for any class C_i, we can find a total preorder relation ρ_i on Ω such that, $\forall x, y\in\Omega$, we have $x\rho_i y$ if and only if we think that the measure in which x belongs to C_i is not superior the one that y belongs to C_i. We suppose ρ_i is not trivial, that is that there are at least two elements a, b of Ω such that $a\rho_i b$ but not $b\rho_i a$. If for any $i\in I$, we put, $\forall a, b\in\Omega$, $a\sigma_i b = \{z\in\Omega$: ($a\rho_i z$ and $z\rho_i b$) or ($b\rho_i z$ and $z\rho_i a$)$\}$ and $a*b = \cap_{i\in I} a\sigma_i b$, we can prove that $H = (\Omega, *)$ is a commutative closed hypergroup and $\{H_i=(\Omega, \sigma_i), i\in I\}$ is a fuzzy decomposition of H. In fact, $\forall i\in I$, if we put, $\forall a, b\in\Omega$, $a \sim_i b$ if and only if ($a\rho_i b$ and $b\rho_i a$), $\sim_i$ is an equivalence relation on Ω. Let $E_i = \Omega/\sim_i$. Denote by $[a]_i$ the equivalence class of $a\in\Omega$. If we put, $\forall a, b\in\Omega$, $[a]_i<[b]_i \Leftrightarrow$ ($a\rho_i b$ and not $b\rho_i a$) we have a total order on E_i. Any element of E_i is a judgment about the truth of the proposition P(x)="the element $x\in\Omega$ belongs to C_i". If there is not an element of E_i that means "true" we add as "true" the symbol "T" and we assume $[a]_i<T$, $\forall a\in\Omega$. In an analogous way, if there is not an element of E_i that means "false" we add as "false" the symbol "F" and we assume $F<[a]_i<T$, $\forall a\in\Omega$. Let $S_i=E_i\cup\{V, F\}$. The function α_i: $x\in\Omega\rightarrow[a]_i\in S_i$ is a qualitative linear fuzzy set and $H_i=(\Omega,\sigma_i)$ is the associate closed commutative hypergroup. Then $\{H_i, i\in I\}$ is a fuzzy decomposition of H.

By theorem 2.9 and corollary 2.10 we can find, under suitable conditions, a fuzzy partition of Ω.

3. A REAL CASE OF STUDY

In this paragraph we describe the methodology we propose to classify an urban territory in homogeneous areas. This is a "multicriteria" procedure, since the clustering is made by considering a set of *criteria*.

We start our study by the law 431/98 that has recently pointed out the problem of a fairer regulation of the rents: it fixes new criteria to define the prices on extended metropolis and on towns with high density of population.

This act prescribes to establish the reference values of the rents in a municipal area on the basis of the quality of real estates, considering both the conditions of buildings and the conditions of services of the zones in which they are placed; sectors of fluctuation of the prices have to be singled out by dividing municipal areas in homogeneous zones, said *microzones*.

According to the aforesaid law the microzones, as sectors with similar peculiarities, are individuated by the municipality on the basis of the following elements:

1. *market price* of the area;
2. *infrastructures (transport, public parks and gardens, schools, health services, commercial equipments)*;
3. *kind of housing,* considering cadastral categories and classes;
4. *artistic quality* of the area;
5. presence of *urban decay*.

These parameters, or *criteria*, are useful also to define precise maximum and minimum values of prices within each sector. In order to assign the actual rent

between this two aforesaid extreme values in each microzone the law prescribes to consider also the following elements: *typology of the house*; *maintenance state of the house and of the whole building*; *pertinences of the house* (box, car place, cellar, etc.); *presence of common spaces* (courtyard, open spaces and gardens, sport facilities, etc.); *technical fixtures and fittings* (lift, independent or centralized heating system, air conditioner, etc.); *possible equipment of forniture*.
The cities of Milan and Pescara conformed to the law, defining their microzones by observing peculiarities of the settlements. The first one has been divided in nine zones; each zone is subdivided in three strips: an economical sector, a middle sector and a luxury sector and each of these is characterized by a minimum value and a maximum value of the rent [11]. In the city of Pescara the present three cadastral zones, to which three different sectors of classification and cadastral rent correspond, are now replaced with ten new microzones [12].
As we have underlined in the paragraph 1, there are some perplexities about the trait of homogeneousness of the microzones: is it suitable the empirical clustering made in Milano and Pescara? Is it possible a so "hard" difference among peculiarities of near zones? We think the reality is more complex and that there is a gradual, soft passing from the traits of a certain zone to the traits of another zone. Really it seems that the microzones include elements with characteristics not to distinguish so sharply and that their frontiers have a fuzzy connotation: a building could belong even to more zones simultaneously. For this reason we deem it opportune the recourse to methods of statistical clustering that permit to obtain a subdivision of an urban territory in clusters having traits of major homogeneousness than clusters recently singled out in an empirical way. Moreover we think that the fuzzy classification is more suitable than the crisp one.

The aim of our study is the formulation of algorithms to define the grade in which a building, having defined peculiarities deduced by analyzing the context in which the building is located, belongs to various zones.
Our mathematical model of clustering is based on the attribution of *qualitative judgments* representing the grade of achievement of the set of the predefined criteria: this step corresponds to the formulation of a *qualitative matrix* in which the rows are the criteria, the columns are the objects and the element in the row i and column j is the judgment on the grade in which the object j achieves the criterion i.
Successively, through a suitable strong numerical evaluation v: $S \rightarrow [0,1]$, it is possible to parametrize the model, by substituting the qualitative judgments with numerical values deduced by analysing the urban context: this step corresponds to the formulation of a *quantitative matrix*.
At this step, for the formulation of the *clustering*, it is necessary a suitable algorithm, by fixing the number of classes and a *distance* among the objects.
In [7] we have formulated for the case of Pescara an algorithm of fuzzy classification obtained by considering a *distance* among the buildings based both on the geographical position (as urban metric) and on the difference among the grades of achievement of criteria fixed by the law.

This classification is represented by a matrix A = $[a_{ij}]$ such that $a_{ij} \in [0,1]$ represents the grade in which the building O_j belongs to the class C_i.
Each row of the matrix is a fuzzy set and it is possible to associate to this one a hypergroup of Corsini H_i. Moreover we can associate to the set of the rows of the matrix the commutative hypergroup considered in the theorem 2.7. Each block $a_1*a_2*\dots*a_n$ of this hypergroup is the set of buildings having peculiarities, in particular rents and estimated incomes, constrained by the ones of $a_1, a_2, \dots, a_n$.
By utilizing the theorem 2.9, the corollary 2.10 and the definition 2.8 we notice that an alternative criterion of clustering is obtained by fixing "a priori" the classes C_1, $C_2, \dots, C_p$. For any class we assign a qualitative linear fuzzy set with universe the set Ω of the buildings. Finally with the methods considered in the previous paragraph we have a fuzzy classification on Ω.

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HOMOMORPHIC RELATIONS ON HYPERRINGOIDS AND JOIN HYPERRINGS

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Abstract This paper is a study of the Join Hyperringoid, which is a hypercompositional structure that has appeared recently. Here appear the homomorphic relations and a special type of such relations, the congruence ones. Moreover, the homomorphisms of the join hyperringoids are being studied, along with the homomorphisms of the Fortified Join Hyperringoids.

1. INTRODUCTION

The **hyperringoid** is a hypercompositional structure that has been introduced by G. Massouros and J. Mittas for the study of the theory of Automata and Languages [4]. The hyperringoid is a triplet (Y,+, .), for which the following axioms are valid:

I. (Y,+) is a hypergroup
ii. (Y,.) is a semigroup
iii. the composition "." is bilaterally distributive with regard to the hypercomposition "+"

If the hypergroup (Y,+) is a join one, then the hyperringoid is called **join**. The join hypergroup is a commutative hypergroup in which the Prenowitz's join axiom is also valid, i.e., it holds that:

(J) $(a:b) \cap (c:d) \neq \varnothing \Rightarrow (a+d) \cap (b+c) \neq \varnothing$, for every $a,b,c,d \in H$

where $a:b = \{x \in H \mid a \in x+b\}$ is the induced from "+" hypercomposition [1].

An important join hyperringoid for the theory of Languages, is the **B-hyperringoid** [6], in which the hypercomposition is defined as follows:

$$a + b = \{a, b\}$$

We shall begin the study of the congruence relations starting with certain Propositions which hold in more general hypercompositional structures, the hypergroupoids. So let (H,+) and (H′,+) be two hypergroupoids with hyperoperations defined in the entire sets and always giving non void result, i.e., $a+b \neq \varnothing$, for every two of their elements a, b. Then:

Definition 1.1. A binary relation $R \subseteq H \times H'$ is called **homomorphic** if for every $(a_1,b_1) \in R$ and $(a_2,b_2) \in R$ holds:
$(\forall x \in a_1+a_2)(\exists y \in b_1+b_2)\ [(x,y) \in R]$ and $(\forall y' \in b_1+b_2)(\exists x' \in a_1+a_2)\ [(x',y') \in R]$ (D_1)
or equivalently for every $x \in a_1+a_2$ and for every $y \in b_1+b_2$ holds:
$[\{x\} \times (b_1+b_2)] \cap R \neq \varnothing$ and $[(a_1+a_2) \times \{y\}] \cap R \neq \varnothing$ (D_1')

From the definition it derives that the inverse binary relation R^{-1} is also a homomorphic one. Moreover, when R defines a mapping $\varphi : H \to H'$, then, if $a,b \in H$ for every $x \in a+b$, we have $\varphi(x) \in \varphi(a)+\varphi(b)$, and therefore $\varphi(a+b) \subseteq \varphi(a)+\varphi(b)$. Also, for every $y \in \varphi(a)+\varphi(b)$ there exists $x \in a+b$ such that $\varphi(x) = y$, thus $\varphi(a)+\varphi(b) \subseteq \varphi(a+b)$. Consequently the condition $\varphi(a_1+a_2) = \varphi(a_1)+\varphi(a_2)$ is being verified and so the Proposition:

Proposition 1.1. *If a homomorphic relation between two hypergroupoids defines a mapping, then it is a normal homomorphism.*

We remind that, according to the terminology which has been introduced by M. Krasner, a mapping φ from the hypergroupoid H to the power-set $\wp(H')$ of the hypergroupoid Ç′ is called **homomorphism** if $\varphi(x+y) \subseteq \varphi(x)+\varphi(y)$ for every $x,y \in H$. A homomorphism is called **strong** if the above relation holds as an equality. Moreover, if φ is a mapping from H to H′ for which $\varphi(x+y) \subseteq \varphi(x)+\varphi(y)$, then φ is a **strict homomorphism**. Lastly, if for a strict homomorphism holds $\varphi(x+y) = \varphi(x)+\varphi(y)$, then we have a **normal** or **good homomorphism.**

For the homomorphic relations and the normal homomorphisms we give the Propositions:

Proposition 1.2. *If R, S are homomorphic relations between the hypergroupoids H′, H and H, H″ respectively, then their composition SR is a homomorphic relation between H′ and H″.*

Next let R be a homomorphic relation between the hypergroupoids (H,+) and (H′,+). If $h' \subseteq H'$ is a subhypergroupoid of H′ and h the image of h′ under R^{-1}, then:

Proposition 1.3. *If h′ is stable under the hypercomposition, then h is stable as well.*
P r o o f. Let $x,y \in h$. It will be proved that $x + y \subseteq h$. Indeed, since $x,y \in h$ then there exist t_1, t_2 from h′, such that (x,t_1), (y,t_2) belong to R. But since R is a homomorphic relation, it derives that for every $w \in x + y$ holds:

$$[\{w\} \times (t_1 + t_2)] \cap R \neq \varnothing.$$

But h′ is stable with regard to the hypercomposition and therefore $t_1 + t_2$ is a subset of h′. Thus, for every w from x + y, there exists t from h′ such that $(w,t) \in R$. So $w \in h$ and therefore, for every y from h, x + y is a subset of h. Consequently h is stable.

Corollary 1.1 *The inverse image of a semi-subhypergroup through a homomorphism between two hypergroups is a semi-subhypergroup.*

2. Homomorphic Relations in the Join Hypergroups

As it is known, from the general theory of the hypergroups, a subhypergroup h of a hypergroup H is closed in H if $a{:}b \subseteq h$ for every $a,b \in h$ [2]. Thus, in a closed subhypergroup h of a join hypergroup H, the axiom (J) is being verified in h. Moreover if a subhypergroup h of H is a join hypergroup itself, then it is called **join subhypergroup** of H. Therefore the closed subhypergroups of H are its join subhypergroups. For the following, let H be a join hypergroup, h a join subhypergroup of H and E a hypergroupoid with hyperoperation defined for every two elements of E and always giving non void result. If R is a homomorphic relation from H to E with the property: $y = y'$, when (x,y), (x,y') belong to R, then:

Proposition 2.1. *The image h′ of h through R is a subhypergroup of E. Also if all the elements of h′: h′ are images through R of elements of H, then the elements of h′ satisfy the join axiom inside E, but not necessarily inside h′.*

P r o o f. Let $(x,y) \in R$ with $x \in h$ and $y \in h'$ and let's consider the hypersum $y + t$, $t \in h'$. For $t \in h'$ there exists $v \in h$ such that $(v,t) \in R$. Consequently for every $b \in y + t$ there exists $a \in x + v$ such that $(a,b) \in R$, and therefore $y + t \subseteq h'$. Thus $y + h' \subseteq h'$. Next let $t \in h'$. Then $(v,t) \in R$ for some $v \in h$. Now, for v, there exists $a \in h$ such that $v \in x + a$. Let b be an element of h′ such that $(a,b) \in R$. Then:

$$[\{v\} \times (y + b)] \cap R \neq \varnothing$$

so $t \in y + b$ and therefore $h' \subseteq y + h'$. Thus $h' = y + h'$. Also it can be proved that for every three elements a′, b′ and c′ from h′ the associativity holds and so h′ is a hypergroup. Next let's assume that all the elements of h′:h′ are images, through R, of elements of H. Suppose that for the elements a′,b′,c′,d′ of h′ holds:

$$(a'{:}b') \cap (c'{:}d') \neq \varnothing$$

We remark that the a′:b′ and c′:d′ are not necessarily subsets of h′. If $t \in a'{:}b'$ and $t \in c'{:}d'$, then $a' \in b' + t$ and $c' \in d' + t$. Next we choose the elements $v \in H$, and $b, d \in h$ in such a way that the pairs (b,b'), (d,d') and (v,t) belong to R. Then for every $x \in b + v$ and $y \in d + v$ holds:

$$[\{x\} \times (b' + t)] \cap R \neq \varnothing \text{ and } [\{y\} \times (d' + t)] \cap R \neq \varnothing.$$

Therefore, for the a', c' which belong to $b' + t$ and $d' + t$ respectively, there exist a, c, such that the pairs (a,a') and (c,c') belong to R and also $a \in b + v$ and $c \in d + v$. Then $v \in (a:b) \cap (c:d)$ and thus $(a:b) \cap (c:d) \neq \varnothing$. From the last relation, and since H is a join hypergroup, it derives that $(a + d) \cap (c + b) \neq \varnothing$. Now let w be an element of the intersection $(a + d) \cap (c + b)$. Then:

$$[\{w\} \times (a' + d')] \cap R \neq \varnothing \text{ and } [\{w\} \times (c' + b')] \cap R \neq \varnothing$$

So there exists w' which belongs to the hypersums $(a'+ d')$ and $(c'+ b')$ such that $(w,w') \in R$. Thus

$$(a' + d') \cap (c' + b')] \neq \varnothing$$

Therefore it has been proved that the join axiom is being verified for the elements of h', not necessarily inside it, but inside E.

Corollary 2.1. *Let φ be a normal epimorphism from the join hypergroup H on the hypergroupoid E. Then E is a join hypergroup and the image through φ of every join subhypergroup of H is a subhypergroup of E.*

A homomorphic relation which is also an equivalence relation will be named **congruence relation**.

Proposition 2.2. *Every congruence relation R on a hypergroup H is a normal equivalence relation and therefore the set H/R is a hypergroup if we define the hypercomposition:*

$$C_x \bullet C_y = \{ C_z \mid z \in x + y \}$$

where C_a is the class of an arbitrary element $a \in H$.

Proposition 2.3. *If the hypergroup H is join, then H/R is also a join hypergroup.*

3. Homomorphic Relations and Homomorphisms in the Join Hyperringoids

Let Y and Y' be two hyperringoids and let $R \subseteq Y \times Y'$ be a binary relation from Y to Y'.

Definition 3.1. R will be called **homomorphic relation**, if it satisfies the axioms of the Definition 1.1. and moreover if for every $(a_1,b_1) \in R$ and $(a_2,b_2) \in R$ holds:

$$(a_1a_2,\ b_1b_2) \in R \qquad (D_2)$$

The notion of the homomorphism, as well as the different special types of homomorphisms that exist in the hypergroups are also being defined in the hyperringoids, with the use of the additional axiom:

$$\varphi(x.y) = \varphi(x) . \varphi(y)$$

for every x, y from the domain of φ. For the following, let K and K′ be two join hyperringoids and E a hyperringoid. Then:

Proposition 3.1. *If φ is a strict homomorphism between K and K′, then the inverse image through φ, of a join subhyperringoid of K′, is a join subhyperringoid of K.*

Proposition 3.2. *If R is a homomorphic relation between K and E, then the image of every join subhyperringoid of K is a subhyperringoid of E.*

Corollary 3.1. *Let φ be a strong epimorphism from K to E, then E is a join hyperringoid and the image, through φ of every join subhyperringoid of K is a subhyperringoid of E.*

Proposition 3.3. *Every congruence relation R over E is a normal equivalence relation and therefore the set E/R is a hyperringoid with the following hypercomposition and composition:*

$$C_x \bullet C_y = \{ C_z \in E/R \;/\; z \in x + y \}$$
$$C_x . C_y = C_{zy}$$

Proposition 3.4. *If E is a join hyperringoid, then E/R is also a join hyperringoid.*

Proposition 3.5. *Let A be a bilateral hyperidealoid of K. If we define in K a relation R as follows:*

$$(\kappa,\lambda) \in R \;\; if \;\; (\kappa : \lambda) \cap A \neq \varnothing \;\; and \;\; (\kappa : \lambda) \cap A \neq \varnothing$$

Then R is a homomorphic relation.

P r o o f. Let $(\kappa_1, \lambda_1) \in R$ and $(\kappa_2, \lambda_2) \in R$. Then from the definition of R we have: $(\kappa_1 : \lambda_1) \cap A \neq \varnothing$, $(\lambda_1 : \kappa_1) \cap A \neq \varnothing$ and $(\kappa_2 : \lambda_2) \cap A \neq \varnothing$, $(\lambda_2 : \kappa_2) \cap A \neq \varnothing$
So there exist x, x′ belonging to A and such that $x \in \kappa_1 : \lambda_1$ and $x' \in \lambda_1 : \kappa_1$. From here it derives that $\kappa_1 \in x + \lambda_1$ and $\lambda_1 \in x' + \kappa_1$, from where $\kappa_1 \in \lambda_1 + A$ (1) and $\lambda_1 \in \kappa_1 + A$ (2). Similarly, $\kappa_2 \in \lambda_2 + A$ (3) and $\lambda_2 \in \kappa_2 + A$ (4). From (1) and (3) we have $\kappa_1 + \kappa_2 \subseteq (\lambda_1 + \lambda_2) + A$ i.e. for every $a \in \kappa_1 + \kappa_2$ there exists $b \in \lambda_1 + \lambda_2$ such that $a \in b + A$, or equivalently $(a:b) \cap A \neq \varnothing$, from where, due to the definition of R, $(a,b) \in R$. So $[\, \{ a \} \times (\lambda_1 + \lambda_2) \,] \cap R \neq \varnothing$ for every $a \in \kappa_1 + \kappa_2$. Similarly, from (2) and (4) it derives that for every $b \in \lambda_1 + \lambda_2$ there exists $a \in \kappa_1 + \kappa_2$ such that $(a,b) \in R$. Thus $[\, (\kappa_1 + \kappa_2) \times \{ b \} \,] \cap R \neq \varnothing$.

Moreover, from the relations $\kappa_1 \in x + \lambda_1$ and $\kappa_2 \in y + \lambda_2$ it derives that $\kappa_1.\kappa_2 \in (x + \lambda_1) . (y + \lambda_2)$ and due to the Properties III.1.1 of [5]

$$(x + \lambda_1) . (y + \lambda_2) \subseteq x.y + x.\lambda_2 + \lambda_1.y + \lambda_1.\lambda_2$$

Therefore $\kappa_1.\kappa_2 \in x.y + x.\lambda_2 + \lambda_1.y + \lambda_1.\lambda_2$ But, because of the multiplicative property of A, we have $x.\lambda_2, \lambda_1.y, x.y \in A$, so $\kappa_1.\kappa_2 \in \lambda_1.\lambda_2 + A$ or $(\kappa_1.\kappa_2 : \lambda_1.\lambda_2) \cap A \neq \varnothing$ thus $(\kappa_1.\kappa_2 , \lambda_1.\lambda_2) \in R$ and so R is a homomorphic relation.

Proposition 3.6. *Let R be a congruence relation over K. Then the mapping φ from K to K/R which is defined as follows:*

$$\varphi(x) = C_x \text{ for every } x \in K$$

is a normal homomorphism from K on K/R.

Proposition 3.7. *Let φ be a normal epimorphism of K on K΄. We define in K a relation R as follows:*

$$(x,y) \in R \text{ if and only if } \varphi(x) = \varphi(y)$$

Then R is a congruence relation in K and K/R is isomorphic to K΄.

P r o o f. Obviously the relation R is an equivalence relation and with not much difficulty it can be proved that it is also homomorphic. Next let C_a be the equivalence class of R which is defined from a. If σ is the mapping from K/R to K΄ which is defined by $\sigma(C_a) = \varphi(a)$, then σ is well defined, 1-1 and it maps K/R to K΄. Also

$$\sigma(C_a . C_b) = \sigma(C_{ab}) = \varphi(ab) = \varphi(a).\varphi(b) = \sigma(C_a).\sigma(C_b)$$

and
$$\sigma(C_a + C_b) = \sigma\{C_x \mid x \in a + b \} = \{ \varphi(x) \mid x \in a + b \}$$
$$= \varphi(a + b) = \varphi(a) + \varphi(b) = \sigma(C_a) + \sigma(C_b)$$

Therefore σ is indeed an isomorphism.

Corollary 3.2. *Let φ be a normal homomorphism from K to K΄. Then there exists a congruence relation R in K, a natural epimorphism $\pi : K \to K/R$ and a monomorphism $\psi : K/R \to K'$ such that $\varphi = \psi\bullet\pi$.*

Next we observe that if an equivalence relation R in a hyperringoid E satisfies the property:

$$xRy \text{ and } w \in E \Rightarrow xwRyw \text{ and } wxRwy \qquad [D_2']$$

then it satisfies the axiom $[D_2]$ of the Definition 3.1. If a relation satisfies $[D_2']$, then it is called **compatible** to the composition. It is possible though that an equivalence relation satisfies only one of the conditions of the second part of $[D_2']$. We will call such a relation **right** or **left compatible** to the composition respectively.

Theorem 3.1. *Let L be a subset of E. We define in E a relation R_L as follows:*
$x\, R_L\, y \Leftrightarrow (\forall a,b \in E)[x.a \in L \Leftrightarrow y.a \in L \quad (i) \quad \text{and} \quad b.x \in L \Leftrightarrow b.y \in L \quad (ii)]$

Then R_L is an equivalence relation in E compatible to the composition. If R_L satisfies only (i), [symb. R_L'] or only (ii) [symb. $'R_L$] then it is right or left compatible respectively. If E is a B-hyperringoid, then R_L is a congruence relation.

P r o o f. Obviously this relation is reflexive and symmetric, and it is not very difficult to prove that it is transitive as well. Next let $x_1R_Ly_1$ and $x_2R_Ly_2$. Suppose that for some $b \in E$ holds $b(x_1x_2) \in L$ or equivalently $(bx_1)x_2 \in L$ Then, since $x_2R_Ly_2$ we have $(bx_1)y_2 \in L$ or equivalently $b(x_1 y_2) \in L$ and so $x_1x_2R_Lx_1y_2$. Similarly $x_1y_2R_Ly_1y_2$, and thus $x_1x_2R_Ly_1y_2$, that is the axiom [D_2]. Next let E be a B-hyperringoid. If $w \in x_1 + x_2$, then $w \in \{x_1, x_2\}$. Thus $[\{w\} \times (y_1 + y_2)] \cap R_L = [\{w\} \times \{y_1, y_2\}] \cap R_L$ and therefore this intersection is non void. Similarly, for $z \in y_1 + y_2$ we have $[(x_1 + x_2) \times \{z\}] \cap R_L \neq \varnothing$. Thus the axiom [$D_1$] of Definition 1.1. is being satisfied and so the Theorem.

Corollary 3.3. *If E is a B-hyperringoid, then the quotient E/R_L is a B-hyperringoid as well.*

Now, let's suppose that the subset L of E is a union of classes with regard to an equivalence relation R. Then, if R is right compatible with regard to the multiplication, from xRy, it derives that xaRya for every $a \in E$. Therefore the classes $(xa)_R$ and $(ya)_R$ are equal for every $a \in E$ and since L is a union of classes, it derives that:

$$xa \in L \Leftrightarrow ya \in L \quad \text{for every } a \in E$$

So, according to Theorem 3.1, the above relation defines an equivalence relation R_L' in E, for which $xRy \Rightarrow x R_L' y$, and consequently every class of R is contained in a class of R_L'. Therefore every class of R_L' is a union of one or more classes of R and so $rk(R_L') \leq rk(R)$. Respective results we get when R is a right compatible or a compatible relation. Thus:

Theorem 3.2. *If there exists an equivalence relation R in E compatible to the multiplication, with regard to which L is a union of classes, then $rk(R_L) \leq rk(R)$ and therefore, if $rk(R) < \infty$ then $rk(R_L) < \infty$. Respective properties hold for R_L' and $'R_L$, if R is right or left compatible with regard to the multiplication.*

A special case of join hyperringoid is the **Join Hyperring** [7], in which the additive hypergroup is a **Fortified Join Hypergroup** [7], i.e. a join hypergroup (H,+) that also satisfies the axioms:

FJ_1 There exists a unique neutral element, denoted by 0, the zero element of H, such

that for every $x \in H$ holds: $x \in x+0$ and $0+0 = 0$

and

FJ_2 For every $x \in H \backslash \{0\}$, there exists one and only one element $x' \in H \backslash \{0\}$, the opposite
or symmetrical of x, denoted by -x, such that: $0 \in x+x'$ Also $-0 = 0$.

In the following we will present a few Propositions which refer to the homomorphisms of the join hyperrings. If Y, Y′ are two join hyperrings and φ is a normal homomorphism from Y to Y′, then, as usual [8], we define the kernel of ö, denoted by kerφ, to be the subset $\varphi^{-1}(\varphi(0))$ of Y and we denote the homomorphic image φ(Y) of Y, with Imφ. In accordance now to what holds in the case of the normal homomorphisms of the fortified join hypergroups [3], in the join hyperrings holds:

Proposition 3.8.

i. kerφ is a subhyperringoid of Y

ii. Imφ is a subhyperringoid of Y′, which generally does not contain the element $0' \in Y'$, but φ(0) is neutral element in Imφ

iii. If T is a join subhyperring of Y which contains the kernel of φ, and if φ is an epimorphism, then φ(T) is a join subhyperring of Y′.

Proposition 3.9. *If Y is an integral join hyperring, then kerφ is a symmetrical hyperideal of Y.*

P r o o f. It has been proved (see Proposition 2.5 of [3]) that the set $[\text{kerö}] = -\varphi^{-1}(\varphi(0)) \cup \varphi^{-1}(\varphi(0))$ is a symmetrical subhypergroup. And since Y is an integral join hyperring, if $\varphi(x) = \varphi(0)$, then for $\varphi(-x)$ we have:

$$\varphi(-x) = \varphi(x)\varphi(-1) = \varphi(0)\varphi(-1) = \varphi(0)$$

Thus $\varphi(-x) \in \ker\varphi$ and therefore $[\ker\varphi] = \ker\varphi$. Now if $x \in \ker\varphi$ and w is an arbitrary element of Y, then $\varphi(xw) = \varphi(0)$. Consequently $xw \in \ker\varphi$ and so kerφ is a symmetrical hyperideal.

The study of the homomorphisms in the case of the fortified join hypergroups [3] has shown that if φ is a normal homomorphism (and much more a homomorphism), then its kernel does not necessarily contain the opposite of every element it consists of. Thus a new type of homomorphism, the **complete homomorphism** was introduced, for which $-x \in \ker\varphi$ for every $x \in \ker\varphi$. As it has been proved in the previous Proposition, this relation holds when Y is an integral join hyperring and so:

Proposition 3.10. *Every normal homomorphism with domain an integral join hyperring, is complete.*

Also we have the Proposition:

Proposition 3.11. *If φ is a complete and normal homomorphism from Y to Y′ with the property φ(0) = 0, then:*

i. $ker\varphi$ is a symmetrical hyperideal of Y
ii. $Im\varphi$ is a symmetrical subhyperring of Y'
iii. if T is a symmetrical subhyperring of Y, then $\varphi(T)$ is a symmetrical subhyperring of Y'

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Lower and Upper Approximations in H_v-groups

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Abstract

The purpose of this paper is to introduce and discuss the concept of lower and upper approximations. A simple and straightforward way for interpreting rough sets is to use membership functions. We investigate the similarity between rough membership function and conditional probability. We also consider the fundamental relation β^* defined on an H_v-group H and interprete the lower and upper approximations as subsets of the group H/β^* and give some properties of such subsets.

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1 Introduction

The notion of rough sets has been introduced by Pawlak [11] in 1982 and subsequently the algebraic approach of rough sets has been studied by some authors, for example, Bonikowaski [2], Iwinski [8], Pomykala and Pomykala [12], Gehrke and Walker [7]. Recently, Biswas and Nanda [1] introduced the notion of rough subgroups. Kuroki and Wang gave some properties of the lower and upper approximations with respect to the normal subgroups in [9].

The concept of hypergroup was introduced in 1934 by Marty [10] and has been studied in the following decades and nowadays by many mathematicians among whom, Krasner, Prenowitz, Mittas, Corsini, Sureou, Comer, Jantosciak, Vougiouklis.

The last of these, at the fourth A.H.A congress, Xanthi (1990), introduced the definitions of H_v-group.

The principal notions of hypergroup theory can be found in [3]. The basic results of H_v-groups are in [13].

In this paper we apply the concept of rough sets theory in the theory of algebraic hyperstructures. We consider the fundamental relation β^* defined on an H_v-group H and interprete the lower and upper approximations as subsets of the fundamental group H/β^* and obtain some results in this connection. In particular, we show that if X is an H_v-subgroup of H then upper approximation of X is a subgroup of H/β^*.

2 Interval sets

Given two subsets $A_1, A_2 \subseteq U$ with $A_1 \subseteq A_2$, we define the following closed interval set:

$$[A_1, A_2] = \{X \in \mathcal{P}(U) |\ A_1 \subseteq X \subseteq A_2\}$$

which is a subset of $\mathcal{P}(U)$. The set A_1 is called the lower bound, and A_2 the upper bound. That is, members of an interval set are subsets of the universe U. An interval set consists of all those subsets that are bounded by two particular elements of the Boolean algebra $\mathcal{P}(U)$. Let $I(\mathcal{P}(U))$ denote the set of all closed interval sets.

Set-theoretic operators on interval sets can be defined based on set operators on their members. For two interval sets $\mathcal{A} = [A_1, A_2]$ and $\mathcal{B} = [B_1, B_2]$, interval set intersection, union, and difference are defined by

$$\mathcal{A} \sqcap \mathcal{B} = \{X \cap Y | X \in \mathcal{A}, Y \in \mathcal{B}\},$$

$$\mathcal{A} \sqcup \mathcal{B} = \{X \cup Y | X \in \mathcal{A}, Y \in \mathcal{B}\},$$

$$\mathcal{A} \backslash \mathcal{B} = \{X - Y | X \in \mathcal{A}, Y \in \mathcal{B}\}.$$

The above defined operators are closed on $I(\mathcal{P}(U))$, namely, $\mathcal{A} \sqcap \mathcal{B}, \mathcal{A} \sqcup \mathcal{B}$, and $\mathcal{A} \backslash \mathcal{B}$ are interval sets. They can be explicitly computed by

$$\mathcal{A} \sqcap \mathcal{B} = [A_1 \cap B_1, A_2 \cap B_2],$$

$$\mathcal{A} \sqcup \mathcal{B} = [A_1 \cup B_1, A_2 \cup B_2],$$

$$\mathcal{A} \backslash \mathcal{B} = [A_1 - B_2, A_2 - B_1].$$

The interval set complement $\neg$ is defined by $[U, U] \backslash [A_1, A_2]$. This is equivalent to $[U - A_2, U - A_1] = [\sim A_2, \sim A_1]$. Clearly, we have $\neg[\emptyset, \emptyset] = [U, U]$ and $\neg[U, U] = [\emptyset, \emptyset]$.

Degenerate interval sets of the form $[A, A]$ are equivalent to ordinary sets. For degenerate interval sets, the proposed operators $\sqcap, \sqcup, \backslash$, and $\neg$ reduce to set operators. Interval set operators obey most properties of set operators. For example, idempotence, commutativity, associativity, and distributivity laws hold for $\sqcap$ and $\sqcup$; De Morgan's and double negation laws hold for $\neg$. Thus, the system $(I(\mathcal{P}(U)), \sqcap, \sqcup)$ is a complete distributive lattice, with zero element

$[\emptyset, \emptyset]$ and unit element $[U, U]$. The associated order relation is called interval set inclusion. It can be defined using the set inclusion relation:

$$\mathcal{A} \sqsubseteq \mathcal{B} \Longleftrightarrow A_1 \subseteq B_1 \text{ and } A_2 \subseteq B_2.$$

The system $(I(\mathcal{P}(U)), \sqcap, \sqcup, \neg, [\emptyset, \emptyset], [U, U])$ is called an interval set algebra.

3 Lower and upper approximations

Let ρ be an equivalence relation defined on the set U and $[x]_\rho$ equivalence class of the relation ρ generated by an element $x \in U$.

Any finite union of equivalence classes of U is called a definable set in U. Let A be any subset of U. In general, A is not a definable set in U. However, the set A may be approximated by two definable set in U. The first one is called a ρ-lower approximation of A in U, denoted by $\underline{\rho}(A)$ and defined as follows:

$$\underline{\rho}(A) = \{x \in U | \ [x]_\rho \subseteq A\}.$$

The second set is called a ρ-upper approximation of A in U, denoted by $\overline{\rho}(A)$ and defined as follows:

$$\overline{\rho}(A) = \{x \in U | \ [x]_\rho \cap A \neq \emptyset\}.$$

The ρ-lower approximation of A in U is the greatest definable set in U contained in A. The ρ-upper approximation of A in U is the least definable set in U containing A. The difference $\widehat{\rho(A)} = \overline{\rho}(A) - \underline{\rho}(A)$ is called the ρ-boundary region of A. In the case when $\widehat{\rho(A)} = \emptyset$ the set A is said to be ρ-exact.

Using ρ-lower and ρ-upper approximations, we define a binary relation on subsets of U:

$$X \approx Y \Longleftrightarrow \underline{\rho}(X) = \underline{\rho}(Y) \text{ and } \overline{\rho}(X) = \overline{\rho}(Y).$$

It is an equivalence relation which induces a partition $\mathcal{P}(U)/ \approx$ of $\mathcal{P}(U)$. An equivalence class of $\approx$ is called a ρ-rough set. Therefore a ρ-rough set of X is the family of all subsets of U having the same ρ-lower and the same ρ-upper approximations of X. More specifically, a ρ-rough set is the following family of subsets of U:

$$< A_1, A_2 >= \{X \in \mathcal{P}(U) | \ \underline{\rho}(X) = A_1, \ \overline{\rho}(X) = A_2\}.$$

A set $X \in < A_1, A_2 >$ is said to be a member of the ρ-rough set.

Rough set intersection $\sqcap$, union $\sqcup$, and complement $\neg$ are defined by set operators as follows: for two ρ-rough sets $< A_1, A_2 >$ and $< B_1, B_2 >$,

$$\begin{aligned} < A_1, A_2 > \sqcap < B_1, B_2 > &= \{X \in \mathcal{P}(U) | \ \underline{\rho}(X) = A_1 \cap B_1, \ \overline{\rho}(X) = A_2 \cap B_2\} \\ &=< A_1 \cap B_1, A_2 \cap B_2 >, \end{aligned}$$

$$\begin{aligned} < A_1, A_2 > \sqcup < B_1, B_2 > &= \{X \in \mathcal{P}(U) | \ \underline{\rho}(X) = A_1 \cup B_1, \ \overline{\rho}(X) = A_2 \cup B_2\} \\ &=< A_1 \cup B_1, A_2 \cup B_2 >, \end{aligned}$$

$$\begin{aligned} \neg < A_1, A_2 > &= \{X \in \mathcal{P}(U) | \ \underline{\rho}(X) =\sim A_2, \ \overline{\rho}(X) =\sim A_1\} \\ &=<\sim A_2, \sim A_1 > . \end{aligned}$$

The results are also ρ-rough sets. The induced system $(\mathcal{P}(U)/ \approx, \sqcap, \sqcup, \neg, [\emptyset]_\approx, [U]_\approx)$ is called a ρ-rough set algebra.

The corresponding order is called ρ-rough set inclusion and is given by

$$< A_1, A_2 > \sqsubseteq < B_1, B_2 > \Longleftrightarrow A_1 \subseteq B_1 \text{ and } A_2 \subseteq B_2.$$

The proof of the following theorem is similar to the Proposition 2.2 of Pawlak [11] and Theorem 2.1 of Kuroki [9]. We shall give a proof for completeness.

Theorem 1. Let ρ be an equivalence relation on a set U. If A and B are non-empty subsets of U, then the following hold:

1) $\underline{\rho}(A) \subseteq A \subseteq \overline{\rho}(A)$;

2) $\overline{\rho}(A \cup B) = \overline{\rho}(A) \cup \overline{\rho}(B)$;

3) $\underline{\rho}(A \cap B) = \underline{\rho}(A) \cap \underline{\rho}(B)$;

4) $A \subseteq B$ implies $\underline{\rho}(A) \subseteq \underline{\rho}(B)$;

5) $A \subseteq B$ implies $\overline{\rho}(A) \subseteq \overline{\rho}(B)$;

6) $\underline{\rho}(A \cup B) \supseteq \underline{\rho}(A) \cup \underline{\rho}(B)$;

7) $\overline{\rho}(A \cap B) \subseteq \overline{\rho}(A) \cap \overline{\rho}(B)$;

Proof. (1) If $a \in \underline{\rho}(A)$, then $a \in [a]_\rho \subseteq A$. Hence $\underline{\rho}(A) \subseteq A$. Next, if $a \in A$, then, since $a \in [a]_\rho$, we have $[a]_\rho \subseteq A \neq \emptyset$, and $a \in \overline{\rho}(A)$. Thus $A \subseteq \overline{\rho}(A)$.

$$\begin{aligned}(2)\ \ a \in \overline{\rho}(A \cup B) &\iff [a]_\rho \cap (A \cup B) \neq \emptyset \iff ([a]_\rho \cap A) \cup ([a]_\rho \cap B) \neq \emptyset \\ &\iff [a]_\rho \cap A \neq \emptyset \text{ or } [a]_\rho \cap B) \neq \emptyset \iff a \in \overline{\rho}(A) \text{ or } a \in \overline{\rho}(B) \\ &\iff a \in \overline{\rho}(A) \cup \overline{\rho}(B)\end{aligned}$$

Thus $\overline{\rho}(A \cup B) = \overline{\rho}(A) \cup \overline{\rho}(B)$.

$$\begin{aligned}(3)\ \ a \in \underline{\rho}(A \cap B) &\iff [a]_\rho \subseteq A \cap B \iff [a]_\rho \subseteq A \text{ and } [a]_\rho \subseteq B \\ &\iff a \in \underline{\rho}(A) \text{ and } a \in \underline{\rho}(B) \iff a \in \underline{\rho}(A) \cap \underline{\rho}(B).\end{aligned}$$

Thus $\underline{\rho}(A \cap B) = \underline{\rho}(A) \cap \underline{\rho}(B)$.

(4) Since $A \subseteq B$ iff $A \cap B = A$, by (3) we have

$$\underline{\rho}(A) = \underline{\rho}(A \cap B) = \underline{\rho}(A) \cap \underline{\rho}(B).$$

This implies that $\underline{\rho}(A) \subseteq \underline{\rho}(B)$.

(5) Since $A \subseteq B$ iff $A \cup B = B$, by (2) we have

$$\overline{\rho}(B) = \overline{\rho}(A \cup B) = \overline{\rho}(A) \cup \overline{\rho}(B).$$

This implies that $\overline{\rho}(A) \subseteq \overline{\rho}(B)$.

(6) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (4) we have

$$\underline{\rho}(A) \subseteq \underline{\rho}(A \cup B) \ \text{ and } \ \underline{\rho}(B) \subseteq \underline{\rho}(A \cup B),$$

which yields

$$\underline{\rho}(A) \cup \underline{\rho}(B) \subseteq \underline{\rho}(A \cup B).$$

(7) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (5) we have

$$\overline{\rho}(A \cap B) \subseteq \overline{\rho}(A) \text{ and } \overline{\rho}(A \cap B) \subseteq \overline{\rho}(B),$$

which yields

$$\overline{\rho}(A \cap B) \subseteq \overline{\rho}(A) \cap \overline{\rho}(B). \Box$$

4 Probabilistic rough sets

The notion of conditional probability is a basic tool of probability thery, and it is unfortunate that its great simplicity is somewhat obscured by a singularly clumsy terminology.

Let X be an event with positive probability. For an arbitrary event A we shall write

$$P(A|X) = \frac{P(A \cap X)}{P(X)}.$$

The quantity so defined will be called the conditional probability of A on the hypothesis X (or for given X). When all sample points have equal probabilities, $P(A|X)$ is the ratio $\frac{|A \cap X|}{|X|}$ of the number of sample points common to A and X, to the number of points in X. All theorems on probabilities are valid also for conditional probabilities with respect to any particular hypothesis X. For example, the fundamental relation for probability of the occurrence of either A or B or both takes on the form

$$P(A \cup B|X) = P(A|X) + P(B|X) - P(A \cap B|X).$$

For any $A \subseteq U$, a rough membership function is defined by

$$\mu_A(x) = \frac{|A \cap [x]_\rho|}{|[x]_\rho|}.$$

By definition, elements in the same equivalence class have the same degree of membership. One can see the similarity between rough memberghip function and conditional probability. The rough membership value $\mu_A(x)$ may be interpreted as the probability of x belonging to A given that x belongs to an equivalence class. Under this interpretation, one obtains the notion of probabilistic rough sets. By the laws of probability, the intersection and union of probabilistic rough sets are not truth-functional. Nevertheless, we have

1) $\mu_A(x) = 1 \Longleftrightarrow x \in \underline{\rho}(A)$,

2) $\mu_A(x) = 0 \Longleftrightarrow x \in \underline{\rho}(A^c)$,

3) $0 < \mu_A(x) < 1 \Longleftrightarrow x \in \widehat{\rho(A)}$,

4) $\mu_A(x) = 1 - \mu_{A^c}(x)$,

5) $\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_{A\cap B}(x)$,

6) $max\{\mu_A(x), \mu_B(x)\} \leq \mu_{A\cup B}(x) \leq min\{1, \mu_A(x) + \mu_B(x)\}$,

7) $\mu_{A\cap B} \leq min\{\mu_A(x), \mu_B(x)\}$,

8) for any pairwise disjoint collection P of subsets

$$\mu_{\cup P}(x) = \sum\{\mu_Y(x) | \ Y \in P\}.$$

They follow from the properties of probabability.

With the rough membership function, One may view a probabilistic rough set as a special type of fuzzy set. By drawing such a link between these two theories, the non-truth-functionality of the operators on probabilistic rough sets may provide more insights into the definition of fuzzy set operators.

The notion of probabilistic rough sets may be related to ρ-rough set algebra $(\mathcal{P}(U/\approx, \sqcap, \sqcup, \neg, [\emptyset]_\approx, [U]_\approx)$. For two members of the same membership function, i.e., $A \approx B$, they may not be characterized by the same membership function, i.e., $\mu_A \neq \mu_B$. Let $c(\mu_A)$ and $s(\mu_A)$ denote the core and support of μ_A defined by

$$c(\mu_A) = \{x \mid \mu_A(x) = 1\},$$

$$s(\mu_A) = \{x \mid \mu_A(x) > 0\}.$$

By properties (1) and (2), one can verify that if $A \approx B$, then $c(\mu_A) = c(\mu_B)$, and $s(\mu_A) = s(\mu_B)$. In other words, a ρ-rough set is a family of probabilistic rough sets with the same core and support.

5 Algebraic hyperstructures

A hyperstructure is a set H together with a function $\cdot : H \times H \longrightarrow \mathcal{P}^*(H)$ called hyperoperation, where $\mathcal{P}^*(H)$ denotes the set of all the non-empty subsets of H. According to [10] Marty defined a hypergroup as a hyperstructure $(H, \cdot)$ such that the following axioms hold: (i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y, z in H, (ii) $a \cdot H = H \cdot a = H$ for all a in H. The second axiom is called reproduction axiom. In the above definition if $A, B \subseteq H$ and $x \in H$ then we define

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b, \quad x \cdot B = \{x\} \cdot B, \quad A \cdot x = A \cdot \{x\}.$$

An H_v-group (cf. [4,5,13,14,15]) is a hyperstructure $(H, \cdot)$ such that (i) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset$ for all x, y, z in H, (ii) $a \cdot H = H \cdot a = H$ for all a in H. The first axiom is called weak associativity. If $(H, \cdot)$ satisfies only the first axiom, then it is called an H_v-semigroup. A subset K of H is called an H_v-subgroup if $(K, \cdot)$ is itself an H_v-group.

Let $(H, \cdot)$ be an H_v-group. The relation β^* is the smallest equivalence relation on H such that the quotient H/β^*, the set of all equivalence classes, is a group. β^* is called the fundamental equivalence relation on H. This relation is studied by Corsini [3] concerning hypergroups, see also [6], [13] and [16].

According to [13] if $\mathcal{U}$ denotes the set of all the finite products of elements

of H, then a relation β can be defined on H whose transitive closure is the fundamental relation β^*. The relation β is as follows: for x and y in H we write $x\beta y$ if and only if $\{x, y\} \subseteq u$, for some $u \in \mathcal{U}$. We can rewrite the definition of β^* on H as follows:
$a\beta^* b$ iff there exist $z_1, \dots, z_{n+1} \in H$ with $z_1 = a,\ z_{n+1} = b$ and $u_1, \dots, u_n \in \mathcal{U}$ such that

$$\{z_i, z_{i+1}\} \subseteq u_i \quad (i = 1, \dots, n).$$

Suppose $\beta^*(a)$ is the equivalence class containing $a \in H$. Then the product $\odot$ on H/β^* is defined as follows: $\beta^*(a) \odot \beta^*(b) = \{\beta^*(c) |\ c \in \beta^*(a) \cdot \beta^*(b)\}$ for all a, b in H. It is proved in [13] that $\beta^*(a) \odot \beta^*(b)$ is the singleton $\{\beta^*(c)\}$ for all $c \in \beta^*(a) \cdot \beta^*(b)$. In this way H/β^* becomes a hypergroup. If we put $\beta^*(a) \odot \beta^*(b) = \beta^*(c)$, then H/β^* becomes a group.

Let ρ be an equivalence relation on an H_v-group H. If $\{A, B\} \subseteq \mathcal{P}^*(H)$, we write $A\overline{\rho}B$ to denote that for every $a \in A$, there exists $b \in B$ such that $a\rho b$ and for every $b \in B$, there exists $a \in A$ such that $a\rho b$.

We write $A\overline{\overline{\rho}}B$ if for every $a \in A$ and for every $b \in B$, one has $a\rho b$.

Definition 2.(cf. [3]). An equivalence relation ρ on an H_v-group H is called regular to the right if for every $(x, y) \in H \times H$, one has

$$x\rho y \Longrightarrow x \cdot a\overline{\rho}y \cdot a \text{ for all } a \in H.$$

We say that ρ is strongly regular to the right if for every $(x, y) \in H \times H$, the implication

$$x\rho y \Longrightarrow x \cdot a\overline{\overline{\rho}}y \cdot a \quad rm for\ all\ a \in H$$

is valid.

Analogously we define the regularity (strong regularity) of an equivalence relation to the left. A regular equivalence (strongly regular) relation to the right and to the left is called regular (strongly regular).

The following corollary is exactly obtained from above definitions.

Corollary 3. β^* is a strongly regular relation.

Definition 4. Let $(H_1, \cdot)$ and $(H_2, *)$ be H_v-groups. A mapping T from H_1 into H_2 is called a strong homomorphism if

$$\bigcup_{c \in a \cdot b} T(c) = T(a) * T(b)$$

for all $a, b \in H$. The set $K = \{(a, b) \in H_1 \times H_1 | \ T(a) = T(b)\}$ is called the kernel of T.

Proposition 5. Let $T : H_1 \longrightarrow H_2$ be a strong homomorphism of the H_v-groups $(H_1, \cdot)$ and $(H_2, *)$. Then K is a regular relation on H_1.

Proof. The proof is straightforward and omitted.□

6 On the fundamental relation β^*

Throughout this section we let H be an H_v-group.

The lower and upper approximations can be presented in an equivalent form as shown below. Let X be a non-empty subsets of H. Then

$$\underline{\beta^*(X)} = \{\beta^*(x) \in H/\beta^* | \ \beta^*(x) \subseteq X\}$$

and

$$\overline{\beta^*(X)} = \{\beta^*(x) \in H/\beta^* | \ \beta^*(x) \cap X \neq \emptyset\}.$$

Now, we discuss these sets as subsets of the fundamental group H/β^*.

Proposition 6. Let X and Y are non-empty subsets of H, then the following hold:

1) $\overline{\beta^*(X \cup Y)} = \overline{\beta^*(X)} \cup \overline{\beta^*(Y)}$;

2) $\underline{\beta^*(X \cap Y)} = \underline{\beta^*(X)} \cap \underline{\beta^*(Y)}$;

3) $X \subseteq Y$ implies $\underline{\beta^*(X)} \subseteq \underline{\beta^*(Y)}$;

4) $X \subseteq Y$ implies $\overline{\beta^*(X)} \subseteq \overline{\beta^*(Y)}$;

5) $\underline{\beta^*(X \cup Y)} \supseteq \underline{\beta^*(X)} \cup \underline{\beta^*(Y)}$;

6) $\overline{\beta^*(X \cap Y)} \subseteq \overline{\beta^*(X)} \cap \overline{\beta^*(Y)}$.

Proof. (1)

$$\begin{aligned}
\beta^*(x) \in \overline{\beta^*(X \cup Y)} &\iff \beta^*(x) \cap (X \cup Y) \neq \emptyset \\
&\iff (\beta^*(x) \cap X) \cup (\beta^*(x) \cap Y) \neq \emptyset \\
&\iff \beta^*(x) \cap X \neq \emptyset \text{ or } \beta^*(x) \cap Y \neq \emptyset \\
&\iff \beta^*(x) \in \overline{\beta^*(X)} \text{ or } \beta^*(x) \in \overline{\beta^*(Y)} \\
&\iff \beta^*(x) \in \overline{\beta^*(X)} \cup \overline{\beta^*(Y)}.
\end{aligned}$$

Thus $\overline{\beta^*(X \cup Y)} = \overline{\beta^*(X)} \cup \overline{\beta^*(Y)}$.

(2)

$$\begin{aligned}
\beta^*(x) \in \underline{\beta^*(X \cap Y)} &\iff \beta^*(x) \subseteq X \cap Y \\
&\iff \beta^*(x) \subseteq X \text{ and } \beta^*(x) \subseteq Y \\
&\iff \beta^*(x) \in \underline{\beta^*(X)} \text{ and } \beta^*(x) \in \underline{\beta^*(Y)} \\
&\iff \beta^*(x) \in \underline{\beta^*(X)} \cap \underline{\beta^*(Y)}.
\end{aligned}$$

Thus $\underline{\beta^*(X \cap Y)} = \underline{\beta^*(X)} \cap \underline{\beta^*(Y)}$.

(3) Since $X \subseteq Y$ iff $X \cap Y = X$, by (2) we have

$$\underline{\beta^*(X)} = \underline{\beta^*(X \cap Y)} = \underline{\beta^*(X)} \cap \underline{\beta^*(Y)}.$$

This implies that $\underline{\beta^*(X)} \subseteq \underline{\beta^*(Y)}$.

(4) Since $X \subseteq Y$ iff $X \cup Y = Y$, by (1) we have

$$\overline{\beta^*(Y)} = \overline{\beta^*(X \cup Y)} = \overline{\beta^*(X)} \cup \overline{\beta^*(Y)}.$$

This implies that $\overline{\beta^*(X)} \subseteq \overline{\beta^*(Y)}$.

(5) Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, by (3) we have

$$\underline{\beta^*(X)} \subseteq \underline{\beta^*(X \cup Y)} \quad \text{and} \quad \underline{\beta^*(Y)} \subseteq \underline{\beta^*(X \cup Y)},$$

which yields

$$\underline{\beta^*(X)} \cup \underline{\beta^*(Y)} \subseteq \underline{\beta^*(X \cup Y)}.$$

(6) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, by (4) we have

$$\overline{\beta^*(X \cap Y)} \subseteq \overline{\beta^*(X)} \quad \text{and} \quad \overline{\beta^*(X \cap Y)} \subseteq \overline{\beta^*(Y)},$$

which yields

$$\overline{\beta^*(X \cap Y)} \subseteq \overline{\beta^*(X)} \cap \overline{\beta^*(Y)}.\square$$

Theorem 7. If X is an H_v-subgroup of $(H, \cdot)$, then $\overline{\beta^*(X)}$ is a subgroup of $(H/\beta^*, \odot)$.

Proof. The kernel of the canonical map $\varphi : H \longrightarrow H/\beta^*$ is called the core of H and is denoted by ω_H. Here we also denote by ω_H the unit element of H/β^*.

First we show that $\omega_H \in \overline{\beta^*(X)}$. Since X is an H_v-subgroup of $(H, \cdot)$, then for every $a \in X$ we have $a \cdot X = X$. Therefore $a \in a \cdot X$ and so there exists $b \in X$ such that $a \in a \cdot b$ which implies $\beta^*(a) = \beta^*(a \cdot b) = \beta^*(a) \odot \beta^*(b)$. Therefore $\beta^*(b) = \omega_H$ and so $b \in \omega_H \cap X$ which implies $\omega_H \cap X \neq \emptyset$. Therefore $\omega_H \in \overline{\beta^*(X)}$.

Now, suppose $\beta^*(x), \beta^*(y) \in \overline{\beta^*(X)}$, we show that $\beta^*(x) \odot \beta^*(y) \in H/\beta^*$.

We have $\beta^*(x) \cap X \neq \emptyset$ and $\beta^*(y) \cap X \neq \emptyset$ then there exist $a \in \beta^*(x) \cap X$ and $b \in \beta^*(y) \cap X$. Thus $a \in \beta^*(x)$, $a \in X$, $b \in \beta^*(y)$, $b \in X$ and so

$$a \cdot b \subseteq \beta^*(x) \cdot \beta^*(y) \subseteq \beta^*(x \cdot y) = \beta^*(x) \odot \beta^*(y).$$

For every $c \in x \cdot y$ we have $\beta^*(c) = \beta^*(x) \cdot \beta^*(y)$. Therefore we get $a \cdot b \subseteq \beta^*(c)$ and $a \cdot b \subseteq X$.

Therefore $\beta^*(c) \cap X \neq \emptyset$ which yields $\beta^*(c) \in \overline{\beta^*(X)}$ or $\beta^*(x) \odot \beta^*(y) \in \overline{\beta^*(X)}$.

Finally, if $\beta^*(x) \in \overline{\beta^*(X)}$ then we show that $\beta^*(x)^{-1} \in \overline{\beta^*(X)}$. Since $\omega_H \cap X \neq \emptyset$ then there exists $h \in \omega_H \cap X$ and since $\beta^*(x) \cap X \neq \emptyset$ then there exists $y \in \beta^*(x) \cap X$. By reproduction axiom we get $h \in y \cdot X$ then there exists $a \in X$ such that $h \in y \cdot a$ which implies $\beta^*(h) = \beta^*(y) \odot \beta^*(a)$. Since $h \in \omega_H$ then $\beta^*(h) = \omega_H$. Therefore $\omega_H = \beta^*(y) \odot \beta^*(a)$ or $\omega_H = \beta^*(x) \odot \beta^*(a)$ which yields $\beta^*(a) = \beta^*(x)^{-1}$. Since $a \in X$ and $a \in \beta^*(a)$ then $\beta^*(a) \cap X \neq \emptyset$ and so $\beta^*(a) \in \overline{\beta^*(X)}$. Therefore $\overline{\beta^*(X)}$ is a subgroup of $H/\beta^*, \odot)$. $\square$

Proposition 8. If X and Y are non-empty subsets of H, then

$$\overline{\beta^*(X)} \odot \overline{\beta^*(Y)} \subseteq \overline{\beta^*(X \cdot Y)}.$$

Proof. We have

$$\begin{aligned} \overline{\beta^*(X)} \odot \overline{\beta^*(Y)} &= \{\beta^*(a) \odot \beta^*(b) |\ \beta^*(a) \in \overline{\beta^*(X)}\ ,\ \beta^*(b) \in \overline{\beta^*(Y)}\} \\ &= \{\beta^*(a) \odot \beta^*(b) |\ \beta^*(a) \cap X \neq \emptyset\ ,\ \beta^*(b) \cap Y \neq \emptyset\}. \end{aligned}$$

Therefore $(\beta^*(a) \cdot \beta^*(b)) \cap (X \cdot Y) \neq \emptyset$. Since $\beta^*(a) \cdot \beta^*(b) \subseteq \beta^*(a \cdot b)$. We obtain $\beta^*(a \cdot b) \cap (X \cdot Y) \neq \emptyset$. Thus $\beta^*(a \cdot b) = \beta^*(a) \odot \beta^*(b) \in \beta^*(X \cdot Y)$ and so $\overline{\beta^*(X)} \odot \overline{\beta^*(Y)} \subseteq \overline{\beta^*(X \cdot Y)}$. $\square$

Proposition 9. Let X and Y be two H_v-subgroups of H and let $f : X \longrightarrow Y$ be a strong homomorphism, then f induces a homomorphism $F : \overline{\beta^*(X)} \longrightarrow \overline{\beta^*(Y)}$ by setting

$$F(\beta^*(x)) = \beta^*(f(x)), \quad \forall x \in X.$$

Proof. First we prove that F is well-defined. Suppose that $\beta^*(a) = \beta^*(b)$ then there exist $x_1, \ldots, x_{m+1} \in H$ with $x_1 = a,\ x_{m+1} = b$ and $u_1, \ldots, u_m \in \mathcal{U}$ such that $\{x_i, x_{i+1}\} \subseteq u_i \quad (i = 1, \ldots, m)$ which implies $\{f(x_i), f(x_{i+1})\} \subseteq f(u_i) \quad (i = 1, \ldots, m)$. Since f is a strong homomorphism and $u_i \in \mathcal{U}$ we get $f(u_i) \in \mathcal{U}$. Therefore $f(a)\beta^* f(b)$ or $F(\beta^*(a)) = F(\beta^*(b))$. On the other hand if $\beta^*(a) \in \overline{\beta^*(X)}$ then $\beta^*(a) \cap X \neq \emptyset$ and so there exists $b \in \beta^*(a) \cap X$. Thus $b\beta^* a$ and $b \in X$ which yield $f(b)\beta^* f(a)$ and $f(b) \in Y$. So $f(b) \in \beta^*(f(a))$ and $f(b) \in Y$ then $\beta^*(f(a)) \cap Y \neq \emptyset$ and so $\beta^*(f(a)) \in \overline{\beta^*(Y)}$ or $F(\beta^*(a)) \in \overline{\beta^*(Y)}$. Thus F is well-defined. Now we have

$$\begin{aligned} F(\beta^*(a) \odot \beta^*(b)) &= F(\beta^*(a \cdot b)) \\ &= \beta^*(f(a \cdot b)) \\ &= \beta^*(f(a) \cdot f(b)) \\ &= \beta^*(f(a)) \odot \beta^*(f(b)) \\ &= F(\beta^*(a)) \odot F(\beta^*(b)). \end{aligned}$$

Therefore F is a homomorphism. □

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