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A survey on fuzzy semigroups

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Abstract

Fuzzy semigroup is an algebraic extension of semigroup. It has found application in fuzzy coding theory, fuzzy finite state machines and fuzzy languages. In this paper, a comprehensive literature review on fuzzy semigroup theory is realized. We will begin with a review of fuzzy groups which heavily inspired the notion of fuzzy semigroups put forth by the algebraists. Subsequently, a brief tour of semigroup theory is considered as a precursor to the emerging subject.

Keywords: Fuzzy group; semigroup; fuzzy semigroup.

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1 Introduction

Semigroup theory is a thriving field of modern abstract algebra. As the name suggests, a semigroup is a generalization of a group; because a semigroup need not in general have an element which has an inverse. The earliest major contributions to the theory of semigroups are strongly motivated by comparisons with groups and rings. Semigroup theory can be considered as one of the most successful off-springs of ring theory in the sense that the ring theory gives a clue how to develop the ideal theory of semigroups. The algebraic structure enjoyed by a semigroup is a non-empty set together with an associative binary operation. However, the fuzzy algebraic structures and their extensions are very important. Nowadays, a lot of extensions of fuzzy algebraic structures have been introduced by many authors and have been applied to real life problems in different fields of science.

Many crisp concepts of algebraic structures have been extended to the non-classical structures. Fuzzy groups were first considered by Rosenfeld (1971). In 1971, he defined fuzzy subgroup and established some of its properties. His definition of fuzzy group is a turning point for pure mathematicians. Since then, the study of fuzzy algebraic structure has been pursued in many directions such as groups, rings, modules, vector space and so on. Aktas and Cagman (2007) gave a definition of soft groups and derived their basic properties. Rough groups were defined by Biswas and Nanda (1994), and some other authors have studied the algebraic properties of rough set as well. Demirci (2001) introduced the concept of smooth groups by using fuzzy binary operation. Multigroups were first described by Marty and several scholars put forth different definitions in an attempt to generalize group concept (see Dresher and Ore (1938), Griffiths (1938), Schein (1987) Barlotti and Strambach (1991), Nazmul et al. (2013), Tella and Daniel (2013)).

Moreover, the algebraic extensions of a semigroup have been been studied by many authors. Among others are the notion of ternary semigroups known to Banach (cf. Los (1955)) who is credited with an example of a ternary semigroup which does not reduce to a semigroup. kazim and Naseeruddin (1972) introduced left almost semigroups (LA-semigroups). The structure is also known as AG-groupoid and modular groupoid and has a variety of applications in topology, matrices, flock theory, finite mathematics and geometry. Sen (1981) introduced the concept of Γ-semigroups as a generalization of semigroups.

The purpose of this paper is to promote research and disseminate fuzzy proficiency by presenting a comprehensive and up to date literature review of the fuzzy semigroup theory.

2 A brief review of fuzzy groups

The important concept of a fuzzy set put forth by Zadeh (1965) has opened up keen insights and applications in a wide range of scientific fields. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids and topology.

The study of fuzzy algebraic structures started in the pioneering paper of Rosenfeld (1971). Rosenfeld introduced the notion of fuzzy groups and successfully extended many results from groups to fuzzy groups. Though some other definitions of fuzzy groups are available in the literature (for example, Anthony and Sherwood (1979) redefined the fuzzy groups in terms of a t-norm which replaced the minimum operation), Rosenfeld's definition seems to be the most conventional and accepted one.

Most of the recent contributions in the field are the validations of Rosenfeld's definition where a fuzzy subset A of a group X is called a fuzzy subgroup of X if and only if $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\mu_A(x^{-1}) \geq \mu_A(x)$. Das (2014) defined a level subgroup of a fuzzy subgroup A of a group X as an ordinary subgroup A_t of X, where $t \in [0, 1]$.

Wu (1981) studied fuzzy normal subgroup. Also, fuzzy normal subgroups were studied by Liu (1982) and Kumar et al. (1992). In line with this, Ajmal and Jahan (2012) introduced the notion of a characteristic fuzzy subgroup of a group and related results.

Mukherjee and Bhattacharya (1984) introduced the concept of fuzzy cosets and their relation with fuzzy normal subgroups. Moreover, the authors proved fuzzy generalizations of some important theorems like Lagranges and Cayleys theorems. Also, the authors initiated the notions of a fuzzy normalizer of a fuzzy subgroup and fuzzy solvable in Mukherjee and Bhattacharya (1986) and Mukherjee and Bhattacharya (1987).

The effect of group homomorphism on fuzzy groups was studied by Rosenfeld Rosenfeld (1971) and proved that a homomorphic image of a fuzzy subgroup is a fuzzy subgroup provided the fuzzy subgroup has $\bigvee$-property, while a homomorphic pre image of a fuzzy subgroup is always a fuzzy subgroup. Anthony and Sherwood Anthony and Sherwood (1979) later proved that even without the $\bigvee$-property the homomorphic image of a fuzzy subgroup is a fuzzy subgroup. Sidky and Mishref (1990) proved that if $f : X \longrightarrow Y$ is a group homomorphism and A is a fuzzy subgroup of X "with respect to a continuous t-norm T, then $f(A)$

is a fuzzy subgroup of Y with respect to T". Since $\bigwedge$ is a continuous t-norm (Anthony and Sherwood), it follows that $f(A) \in FG(Y)$ whenever $A \in X$. It was proved by Akgul (1988) that $f^{-1}(B)$ is a fuzzy subgroup of X whenever B is a fuzzy subgroup of Y. Fang (1994) introduced the concepts of fuzzy homomorphism and fuzzy isomorphism by a natural way, and study some of their properties. Ajmal (1994) defined a notion of 'containment' of an ordinary kernel of a group homomorphism in a fuzzy subgroup and provided the long-awaited solution of the problem of showing a one-to-one correspondence between the family of fuzzy subgroups of a group, containing the kernel of a given homomorphism, and the family of fuzzy subgroups of the homomorphic image of the given group. Yong (2004) constructed a quotient group induced by a fuzzy normal subgroup and proved the corresponding isomorphism theorems.

Demirci and Racasens (2004) initiated fuzzy equivalence relation associated with a fuzzy subgroup and showed that a fuzzy subgroup is normal if only if the operation of the group is compatible with its associated fuzzy equivalence relation. kondo (2004) modified the idea of Demirci and Recasens and defined a fuzzy congruence on a group.

Ngcibi et al. (2010) obtained a formula for the group $Z_{p^m} \times Z_{p^n}$ when $n = 1, 2, 3$ and Sehgal et al. (2016) extended the concept for all values of n.

3 A tour of semigroup theory

The term semigroup was first coined in a French group theory textbook (de Seguier (1904)) with a more stringent definition than the modern one, before being introduced to the English-speaking mathematical world by Leonard Dickson the following year Dickson (1905). Three decades after, the only semigroup theory being done was that done in near-obscurity (at least from the Western perspective) by a Russian mathematician, Anton Kazimirovich Suschkewitsch. Suschkewitsch (1928) was essentially doing semigroup theory before the rest of the world knew that there was such a thing, thus many of his results were rediscovered by later researchers who were unaware of his achievements.

The study of semigroups exploded after the publication of a series of highly influential papers in the early 1940s. Ree (1940) obtained the structure of finite simple semigroups and proved that the minimal ideal (Green's relation) of a finite semigroup is simple. Clifford (1941) introduced semigroups admitting relative inverses. Dubreil (1941) studied semigroup theory from the concept of lattice of equivalence relations on sets. Preston (1954) defined and developed the concept of inverse semigroups. Furthermore, Preston described congruences on completely 0-simple semigroup and free inverse semigroups were also studied by the author (see Preston (1961) and Preston (1973)). Munn (1955), having carried out research in a different direction, introduced the notion of semigroup algebras.

kimura (1957) studied semigroups very widely and vividly and carried studies on idempotent semigroups. He further researched idempotent semigroups which satisfies some identities. Moreover, idempotent semigroups was earlier studied by McLean (1954).

Yamada (1997) analyzed idempotent semigroups. Green (1951) authored a classical paper on the structure of semigroups and with Rees to study those semigroups in which $x^r = x$. (Green and Rees, 1952)

For over three decades, Howie (1976)-Howie (1995) worked on embedding theorems for semigroups in his book Howie (1976). He collaborated with Munn and Weirert and edited a proceeding of the conference on semigroups and their applications. Howie et al. (1992). His contributions to semigroup theory is very significant. In this period of three decades, semigroup theorists like Petrich (1973)-Petrich (1984), McAlister and McFadden (1974)-McAlister (1974), Alan (1998), Lawson (1998) and Lajos (1971) have done lots of research on special class of semigroups in the vein of inverse semigroups, free semigroups, etc. and their properties. Okninski (1998) published a book on semigroups of matrices.

Several researchers have worked on the types of semigroup mentioned earlier and developed more properties with applications across a broad spectrum of areas (see Eilenberg (1974), Eilenberg (1976), Hopcropt and Ullman (1979), Howie (1991), Lallement (1979), Straubing (1994)).

In a conference, Meakin (2005) delivered a lecture on groups and semigroups exploring their connections and contrasts. He clearly acknowledged that in the past several decades, group theory and semigroup theory have developed in different directions. Cayley's theorem enables one to view groups as groups of permutations of some set while semigroups are represented as semigroups of functions from a set to itself. However, significant research has been carried out both in group theory and semigroup theory beyond the early viewpoints. In reality, several concepts in modern semigroup theory are closely related to group theory. For instance, automata theory and formal language theory turn out to be related (see Hopcropt and Ullman (1979), Howie (1991)).

Very recently, Gould and Yang (2014) presented a piece of research work titled "Every group is a maximal subgroup of a naturally occurring free idempotent generated semigroup". The structures of generalized inverse semigroups by kudryavtseva and Lausa (2014) is also a recent work on inverse semigroups. Haggins (1992) carried out a research on permutations of a semigroup that maps to inverses. The variety of unary semigroups with associate inverse subsemigroup by Billhardt et al. (2014) is however an additional view on inverse subsemigroups. Thus, semigroup theory has developed rapidly to become the extremely prolific area of research for scholars.

4 Development of fuzzy semigroup theory

In this section, we systematically provides research work done on fuzzy semigroup analogue of some basic notions from semigroup theory as well as record some elementary properties and applications of fuzzy semigroups.

4.1 Fuzzy Semigroup

The study of fuzzy algebraic structures started with the introduction of the concepts of fuzzy subgroup (subgroupoid) and fuzzy (left, right) ideal in the pioneering paper by Rosenfeld Rosenfeld (1971). In 1979, fuzzy semigroups were introduced by ((Kuroki (1981), Kuroki (1982)), which is a generalization of classical semigroups. He had published a series of papers Kuroki (1981)-Kuroki (1997), in which he laid the foundation of an algebraic theory of semigroup in the fuzzy framework. In literature, many related works vis-$\acute{a}$-vis fuzzy ideals of semigroups can be found in (Lajos (1979), Mclean and Kummer (1992), Xue-Ping et al. (1992), Ahsan et al. (1995), Zhi-Wen and Xue-Ping (1995), Dib and Galhum (1997), Xiang-Yun (1999b), Das (1999), Xiang-Jun (2001b)-Xiang-Jun (2002), Ahsan et al. (2001), Lee and Shun (2001), Ahsan et al. (2002), Jun and Seok-Zun (2016a)-Jun and Seok-Zun (2016b), kazanci and Yamak (2008), Zhan and Jun (2010), khan et al.).

Shen (1990) initiated the concepts of fuzzy regular subsemigroups, fuzzy weakly regular subsemigroups fuzzy completely regular subsemigroups, fuzzy weakly completely regular subsemigroup and investigated some of their algebraic properties. Based on the definition of fuzzy regular subsemigroup given by Shen (1990), Xue-Ping and Wang-Jin (1993) defined fuzzy (left, intra-) regular subsemigroup in semigroups and studied some related properties. Furthermore, point-wise depiction of fuzzy regularity of semigroups was introduced by Zhi-

Wen and Xue-Ping (1993). They also proposed the concept of a fuzzy weakly left (right, intra-) regular subsemigroup and exhibited some algebraic properties. Shabir et al. (2010) in the twentieth century characterized regular semigroups by (α, β) - fuzzy ideals.

Samhan (1993) discussed the fuzzy congruence relation generated by a given fuzzy relation on a semigroup. He also studied the lattice of fuzzy congruence relation on a semigroup and gave some lattice theoretic properties. Kuroki (1997) introduced the notion of a quotient semigroup induced by a fuzzy congruence relation on a semigroup and obtained homomorphism theorems with respect to the fuzzy congruence. Earlier before his published paper in Kuroki (1997), Kuroki (1995) studied fuzzy congruences on T^*- pure semigroups. Moreover, he had earlier proposed the concept of an idempotent-separating fuzzy congruence on inverse semigroups before Das (1997) developed fuzzy congruences in an inverse semigroup and established some important results. The notions of fuzzy kernel and fuzzy trace of a fuzzy congruence on an inverse semigroup were introduced by Al-Thurkair (1993). He established a one-to-one correspondence between fuzzy congruence pair and fuzzy congruences on an inverse semigroup. Xiang-Yun (1999a) introduced fuzzy Rees congruences on semigroups and obtained that a homomorphic image of a fuzzy Rees congruences semigroup is a fuzzy Rees congruences semigroup. Tan (2001) studied fuzzy congruences on a regular semigroup. Zhang (2000) introduced the concept of fuzzy group congruences on a semigroup and investigated some of its properties. Two years after, he examined fuzzy congruences on completely 0-simple semigroups. Recently, Ma and Tian (2011) introduced the notion of fuzzy congruence triple on a completely simple semigroup and used it to characterize fuzzy congruence on a completely simple semigroup.

The concept of fuzzy semiprimality in a semigroup as an extension of semiprimality in a semigroup was introduced by Kuroki (1982). He described a semigroup that is a semilattice of simple semigroups in terms of fuzzy semiprimality. Kuroki (1993) characterized a completely regular semigroup and a semigroup that is semilattice of groups in terms of fuzzy semiprime quasi-ideals. Xiang-Jun (2000) defined and studied prime fuzzy ideals of a semigroup. Subsequently, Xiang-Jun (2001a) introduced and studied the quasi-prime and weakly quasi-prime fuzzy left ideals of a semigroup. kehayopulu et al. (2001) worked on characterization of prime and semiprime ideals of semigroups in terms of fuzzy subsets. Shabir (2015) characterized semigroups in which each fuzzy ideal is prime. kazanci and Yamak (2009) defined φ-semiprime fuzzy ideals of a fuzzy semigroup and described all of φ-semigroups in which every φ-fuzzy ideal is φ-semiprime. Manikantan and Peter (2015) proposed some new kind of fuzzy subsets of a semigroup by using fuzzy magnified translation, fuzzy translation, fuzzy multiplication and extension of a fuzzy subset and obtained some results on fuzzy semiprime ideals of semigroups.

Among other authors who reported work done on fuzzy semigroup are Lizasoain and Gomez (2017) who showed that the direct of two fuzzy transformation is again a fuzzy transformation semigroup if and only if the lattice is distributive. Budimirovic et al. (2014) introduced fuzzy semigroups with respect to a fuzzy equality. Sen and Choudhury (2006) studied the intersection graphs of fuzzy semigroups and showed related results.

4.2 Elemantary Properties of Fuzzy Semigroup

Here, we refer readers to Mordeson et al. (2003) for more details.

4.2.1 Fuzzy set

Let S be a non empty set. A fuzzy set in S is a function $f : S \longrightarrow [0, 1]$.

4.2.2 Semigroup

A semigroup is an algebraic structure $(S, .)$ consisting of a non empty set together with an associative binary operation " $\cdot$ ".

4.2.3 Fuzzy ideals in semigroups

Let S be a semigroup and f, g be two fuzzy subsets of S. The product of $f \circ g$ is defined by

$$f \circ g\,(x) = \begin{cases} \bigvee_{x=yz}\{f(y) \wedge g(z)\}, & if\ \exists\, y, z \in S\ such\ that\ x = yz, \\ 0, & otherwise. \end{cases}$$

for all $x \in S$.

A fuzzy subset f of S is called a fuzzy subsemigroup of S if $f(ab) \geq f(a) \bigwedge f(b)$ for all $a, b \in S$, and is called a fuzzy left (right) ideal of S if $f(ab) \geq f(b) \quad (f(ab) \geq f(a))$ for all $a, b \in S$. A fuzzy subset f of S is called a fuzzy two-sided ideal (or a fuzzy ideal) of S if it is both a fuzzy left and a fuzzy right of S.

Lemma 4.1. *Let f be a fuzzy subset of a semigroup S. Then the following properties hold.*

(i) *f is a fuzzy subsemigroup of S if and only if $f \circ f \subseteq f$.*

(ii) *f is a left ideal of S if and only if $S \circ f \subseteq f$.*

(iii) *f is a right ideal of S if and only if $f \circ S \subseteq f$.*

(iv) *f is a two-sided ideal of S if and only if $S \circ f \subseteq f$ and $f \circ S \subseteq f$.*

Proof. See Mordeson et al. (2003) □

Lemma 4.2. *Let S be a semigroup. Then the following properties hold.*

(i) *Let f and g be two fuzzy subsemigroups of S. Then $f \cap g$ is a fuzzy subsemigroup of S.*

(ii) *Let f and g be (left, two-sided) ideal of S. Then $f \cap g$ is also a fuzzy left (right two-sided) ideal of S.*

Proof. See Mordeson et al. (2003) □

Lemma 4.3. *If f is a fuzzy left (right) ideal of S. Then $f \cup (S \circ f)$ $(f \cup (f \circ S))$ is a fuzzy two-sided ideal of S.*

Proof. See Mordeson et al. (2003) □

4.2.4 Fuzzy regular subsemigroup and homomorphism

If f is a fuzzy subsemigroup of S and $\forall\, x \in S$, there exists $x^{'} \in R_x$ such that $f(x^{'}) \geq f(x)$ provided $f(x) \neq 0$, then f is called a fuzzy regular subsemigroup of S.

Proposition 4.1. *f is a fuzzy regular subsemigroup of S if and only if $\forall\, t \in (0, 1]$, f_t is a regular subsemigroup of S provided $f_t \neq \emptyset$.*

Proof. See Mordeson et al. (2003) □

Proposition 4.2. *If f is a fuzzy regular subsemigroup of S, then $f \circ f = f$.*

Proof. See Mordeson et al. (2003) □

Proposition 4.3. *Let α be a semigroup surjection homomorphism from S onto T.*

(i) If f is a fuzzy regular subsemigroup of S, then $\alpha(f)$ is a fuzzy regular subsemigroup of T.

(ii) If g is a fuzzy regular subsemigroup of T, then $\alpha^{-1}(g)$ is a fuzzy regular subsemigroup of S.

Proof. See Mordeson et al. (2003) □

4.2.5 Fuzzy congruences on semigroups and fuzzy factor semigroups

A fuzzy equivalence relation on a semigroup S which is compatible is called a fuzzy congruence relation on S.

Theorem 4.1. *Let μ and ν be fuzzy congruences on semigroup S. Then the following conditions are equivalent.*

(i) $\mu \circ \nu$ is a fuzzy congruence.

(ii) $\mu \circ \nu$ is a fuzzy equivalence.

(iii) $\mu \circ \nu$ is fuzzy symmetric.

(iv) $\mu \circ \nu = \nu \circ \mu$.

Proof. See Mordeson et al. (2003) □

Let μ be a fuzzy congruence on S. Then $S/\mu = \{\mu_a \mid a \in S\}$, where $\mu_a = \mu(a, x)$ for all $x \in S$.

Theorem 4.2. *The binary relation $*$ on S/μ is well-defined.*

Proof. See Mordeson et al. (2003) □

Theorem 4.3. *Let μ be a fuzzy congruence on a semigroup S. Then $\mu^{-1}(1) = \{(a, b) \in S \times S \mid \mu(a, b) = 1\}$ is a congruence on S.*

Proof. See Mordeson et al. (2003) □

4.3 Applications of Fuzzy Semigroups

There are some important areas in which the fuzzy semigroup-theoretic approach is quite substantial and more completely utilized. The most significant such areas are the theories of fuzzy codes, fuzzy finite state machines and fuzzy languages. For greater details on the subject, the readers are also directed to the monograph by Mordeson et al. (2003).

4.3.1 Fuzzy codes

Let X be an alphabet with $1 \leq \mid X \mid < \infty$ and $X^*(X^+)$ is the free monoid (semigroup) generated by X with operation of concatenation. If A is a fuzzy submonoid of X^* and $B \in \mathcal{FP}(X^*)$ such that $B \subseteq A$, then B is its fuzzy base with $B(e) = 0$ and

(B1) $\forall\, x \in Supp(A) \backslash e,\ B^*(x) \geq A(x)$;

(B2) $\forall\, x \in Supp(A) \backslash e,\ x_i y_j \in X^*,\ i = 1, ..., n;\ j = 1, ..., m$ and $x = x_i ... x_n = y_1 ... y_m,\ \bigwedge\{B(x_1), ..., B(x_n), B(y_1), ..., B(y_n)\} \propto \bigwedge\{[m = n], [x_1 = y_1], ..., [x_n = y_n\} \geq A(x)$, where e and $\mathcal{FP}(X^*)$ denote the empty string and the class of all fuzzy subsets of X^*. This explains the origin of the concept. A fuzzy code A over X^+ is such that $A \neq \emptyset$ and A is a fuzzy base of A^*.

4.3.2 Fuzzy finite state machine

A fuzzy finite state machine is an ordered triple $M = (Q, X, \mu)$, where Q and X are non-empty finite sets and $\mu : Q \times X \times X \longrightarrow [0, 1]$. The elements of Q are called states and those of X are called inputs. However, a fuzzy finite state machine can be regarded as a finite state machine when $M \subseteq \{0, 1\}$.

Fuzzy finite state machines can be divided into four categories:

(i) M is called a deterministic fuzzy finite state if μ is a partial fuzzy function.

(ii) M is called a non-deterministic fuzzy finite state if μ is a fuzzy relation.

(iii) M is called a complete deterministic fuzzy finite state if μ is a complete partial fuzzy function.

(iv) M is called a complete non-deterministic fuzzy finite state if μ is a complete fuzzy relation.

4.3.3 Fuzzy languages

A fuzzy formal language or a fuzzy language $\mu : T^* \longrightarrow [0,1]$ can serve to indicate the degree of meaningfulness of each string in T^*, namely, for $x \in T^*$, $\mu(x)$ near 1 implies that x is meaningful and $\mu(x)$ near 0 implies that x is not meaningful. A language L is defined to be a sequential fuzzy language if there is a finite fuzzy automata A_f and a cut-point t such that L is the set of coded words that yield at least one path from the initial state to a final state of A_f whose fuzzy measure is greater than t.

5 Conclusion

We have presented a comprehensive literature survey on the concept of fuzzy semigroups with some basic properties outlined and significant notable applications highlighted. For future research, we can hybridize non-classical structures to study their algebraic structures in semigroups.

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Integrity of generalized transformation graphs

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Abstract

The values of vulnerability helps the network designers to construct such a communication network which remains stable after some of its nodes or communication links are damaged. The transformation graphs considered in this paper are taken as model of the network system and it reveals that, how network can be made more stable and strong. For this purpose the new nodes are inserted in the network. This construction of new network is done by using the definition of generalized transformation graphs of a graphs. Integrity is one of the best vulnerability parameter. In this paper, we investigate the integrity of generalized transformation graphs and their complements. Also, we find integrity of semitotal point graph of combinations of basic graphs. Finally, we characterize few graphs having equal integrity values as that of generalized transformation graphs of same structured graphs.

Keywords: Vulnerability; connectivity; integrity; generalized transformation graphs; semitotal point graph.
2020 AMS subject classifications: 05C40, 90C35.[1]

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1 Introduction

The stability of a communication network composed of processing nodes and communication links are of prime importance to network designers. As the network begins losing links or nodes, eventually there will be a decrease to certain extent in its effectiveness. Thus, communication networks must be constructed as stable as possible; not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. In the analysis of vulnerability of a communication network we often consider the following quantities: 1. the number of members of the largest remaining group within mutual communication can still occur, 2. the number of elements that are not functioning.

The communication network can be represented as an undirected graph. Consequently, a number of other parameters have recently been introduced in order to attempt to cope up with this difficulty. Tree, mesh, hypercube and star graphs are popular communication networks. If we think of graph as modeling a network, there are many graph theoretical parameters used in the past to describe the stability of communication networks. Most notably, the vertex-connectivity and edge-connectivity have been frequently used. The best known measure of reliability of a graph is its vertex-connectivity. The difficulty with these parameters is that they do not take into account what remains after the graph has been disconnected. To estimate these quantities, the concept of integrity was introduced by Barefoot et al. in [6] as a measure of the stability of a graph.
The integrity of a graph G is defined in [6] as

$$I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\},$$

where $m(G - S)$ denotes the order of the largest component of $G - S$. In [6], the authors have compared integrity, connectivity, toughness and binding number for several classes of graphs. In 1987, Barefoot et al. [7] have investigated the integrity of trees and powers of cycles. In 1988, Goddard et al. [14] have obtained integrity of the join, union, product and composition of two graphs. The integrity of a small class of regular graphs was studied by Atici et al. [1].The authors in [3, 20] have studied the integrity of cubic graphs. For more details on integrity of a graph refer to [2, 4, 5, 11–13, 15].

In this paper, we are concerned with nontrivial, simple, finite, undirected graphs. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. The degree of a vertex $d_G(v)$ is the number of edges incident to it in G. The symbol $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x and $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x. For undefined graph theoretic terminologies and notations refer to [16] or [17].

2 Preliminaries

2.1 Basic results on integrity

In this subsection, we review some of the known results about integrity of graphs.

Theorem 2.1. *[5] The integrity of*

(i) complete graph K_n, $I(K_n) = n$,

(ii) null graph $\overline{K_n}$, $I(\overline{K_n}) = 1$,

(iii) star $K_{1,b}$, $I(K_{1,b}) = 2$,

(iv) path P_n, $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$,

(v) cycle C_n, $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$,

(vi) complete bipartite graph $K_{a,b}$, $I(K_{a,b}) = 1 + min\{a, b\}$,

(vii) wheel W_n, $I(W_n) = \lceil 2\sqrt{n-1} \rceil$.

3 Generalized transformation graphs

Sampathkumar and Chikkodimath [19] defined the semitotal-point graph $T_2(G)$ as the graph whose vertex set is $V(G) \cup E(G)$, and where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) one is a vertex of G and other is an edge of G incident with it. Inspired by this definition, Basavanagoud et al. [9] introduced some new graphical transformations. These generalize the concept of semitotal-point graph.

Let $G = (V, E)$ be a graph, and let α, β be two elements of $V(G) \cup E(G)$. We say that the associativity of α and β is $+$ if they are adjacent or incident in G, otherwise is $-$. Let xy be a 2-permutation of the set $\{+, \}$. We say that α and β correspond to the first term x of xy if both α and β are in $V(G)$, whereas α and β correspond to the second term y of xy if one of α and β is in $V(G)$ and the other is in $E(G)$. The generalized transformation graph G^{xy} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{xy} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of xy.

We denote the complement of the generalized transformation graph G^{xy} by $\overline{G^{xy}}$.

In view of above, one can obtain four graphical transformations of graphs, since there are four distinct 2-permutations of $\{+, -\}$. Note that G^{++} is just the semitotalpoint graph $T_2(G)$ of G, whereas the other generalized transformation graphs are G^{+-}, G^{-+} and G^{--}.

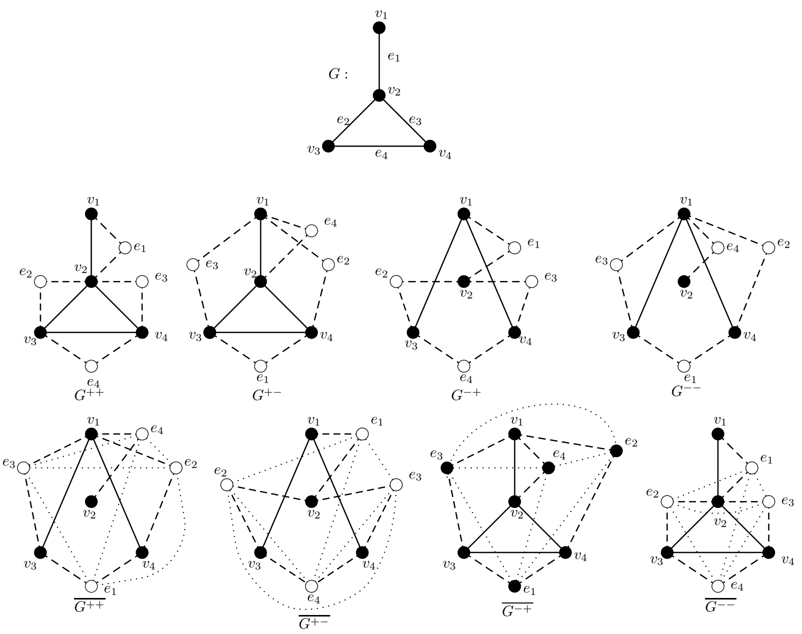

Figure 1: Graph G, its generalized transformation G^{xy} and their complements $\overline{G^{xy}}$

The *generalized transformation graph* G^{xy}, introduced by Basavanagoud et al. [9], is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{xy})$. The vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:

(a) $\alpha, \beta \in V(G)$, α, β are adjacent in G if $x = +$ and α, β are nonadjacent in G if $x = -$

(b) $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $y = +$ and α, β are nonincident in G if $y = -$

An example of generalized transformation graphs and their complements are shown in Figure 1. The vertex v of G^{xy} corresponding to a vertex v of G is referred to as a *point vertex*. The vertex e of G^{xy} corresponding to an edge e of G is referred to as a *line vertex*.
For more details on generalized transformation graphs, refer to [8–10, 17–19].

4 Main results

In this section, we determine the integrity of semitotal point graph(G^{++}) of some standard families of graphs. Also, the integrity of generalized transformation graphs G^{+-}, G^{-+}, G^{--}, $\overline{G^{++}}$, $\overline{G^{+-}}$, $\overline{G^{-+}}$ and $\overline{G^{--}}$ are obtained. Then, we calculate integrity of semitotal point graph of cartesian product and composition of some graphs

4.1 *Integrity of generalized transformation graphs*

Theorem 4.1. *For a graph P_n $(n \geq 4)$,*

$$I(P_n^{++}) = \begin{cases} \lceil 2\sqrt{2n} \rceil - 2, & if\ n\ is\ odd, \\ \lceil 2\sqrt{2n-1} \rceil - 1, & if\ n\ is\ even. \end{cases}$$

Proof. Let S be a subset of $V(P_n^{++})$. The number of remaining components after removing $|S| = r$ vertices is given in Table 1 and Table 2.
Case 1. Suppose n is even. The number of vertices in P_n^{++} is $2n-1$. If r vertices are removed from graph P_n^{++}, then one of the connected components has at least $\frac{2n-1-r}{r}$ vertices. So, the order of the largest component is $m(P_n^{++} - S) \geq \frac{2n-1-r}{r}$. So

$$I(P_n^{++}) \geq min\left\{r + \frac{2n-1-r}{r}\right\}.$$

The function $r + \frac{2n-1-r}{r}$ takes its minimum value at $r = \sqrt{2n-1}$. If we substitute the minimum value in the function, then we have $I(P_n^{++}) = 2\sqrt{2n-1} - 1$. Since the integrity is integer valued, we round this up to get a lower bound. So the integrity of P_n^{++} is, $I(P_n^{++}) = \lceil 2\sqrt{2n-1} \rceil - 1$.

Number of removing vertices	1	2	3	...	r
Number of remaining components	1	2	3	...	r

Table 1: n is even

Number of removing vertices	1	2	3	...	r
Number of remaining components	2	3	4	...	$r+1$

Table 2: n is odd

Case 2. Suppose n is odd. Since the number of vertices in P_n^{++} is $2n-1$ and $m(P_n^{++} - S) \geq \frac{2n-1-r}{r+1}$, we have $I(P_n^{++}) \geq min\{r + \frac{2n-1-r}{r+1}\}$. After the required elementary arithmetical operations, we get

$$I(P_n^{++}) = \lceil 2\sqrt{2n} \rceil - 2.$$

□

Example 4.1. *Consider a graph P_5 and its semitotal point graph P_5^{++}.*
Let $S = \{a, b\} \subset V(P_5^{++})$ (see Figure 2) such that $|S| = 2$ and $m(P_5^{++} - S) = 3$. So, $I(P_5^{++}) = 5$.

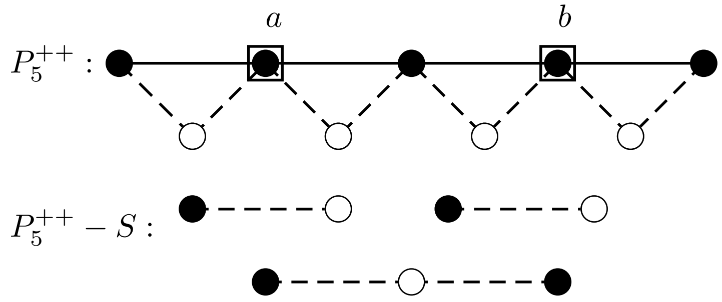

Figure 2: Graph $P_5^{++} - S$.

Theorem 4.2. *For a cycle C_n of length $n \geq 4$,*

$$I(C_n^{++}) = \begin{cases} \lceil \frac{n}{2} \rceil + 3, & if\ n(\leq 7)\ is\ odd\ and\ n(\leq 16)\ is\ even, \\ \lceil \frac{n}{3} \rceil + 5, & if\ n(\geq 9)\ is\ odd\ and\ n(\geq 18)\ is\ even. \end{cases}$$

Proof. C_n^{++} has $2n$ vertices and $3n$ edges. Let $S \subset V(C_n^{++})$.
Case 1. Suppose $n(\leq 7)$ is odd and $n(\geq 16)$ is even.
Choose a set S in such a way that it is an independent set of vertices of C_n. It is clear that $|S| = \lceil \frac{n}{2} \rceil = \beta_0(C_n)$ and $m(C_n^{++} - S) = 3$. So, $I(C_n^{++}) = \lceil \frac{n}{2} \rceil + 3$.
Case 2. Suppose $n(\geq 9)$ is odd and $n(\leq 18)$ is even.
Choose a set S in such a way that it is an independent set of vertices of C_n having distance 3 in between them. It is clear that $|S| = \lceil \frac{n}{3} \rceil$ and $m(C_n^{++} - S) = 5$. So, $I(C_n^{++}) = \lceil \frac{n}{3} \rceil + 5$. □

Example 4.2. *Consider a graph C_6 and its semitotal point graph C_6^{++}.*
Let $S = \{a, b, c\} \subset V(C_6^{++})$ (see Figure 3) such that $|S| = 3$ and $m(C_6^{++} - S) = 3$. So, $I(C_6^{++}) = 6$.

Theorem 4.3. *For a complete graph K_n of order $n \geq 2$,*

$$I(K_n^{++}) = n + 1.$$

Proof. K_n^{++} has $\frac{n(n+1)}{2}$ vertices and $\frac{3n(n-1)}{2}$ edges. Let $S \subset V(K_n^{++})$ be a set containing all the vertices of K_n. So, $|S| = n$. The removal of vertices from K_n^{++} leaves a totally disconnected graph with $\frac{n(n-1)}{2}$ vertices. Hence, $m(K_n^{++} - S) = 1$. Therefore, $|S| + m(K_n^{++} - S) = n + 1$ is minimum for above set S. Then it is clear that, $I(K_n^{++}) = n + 1$. □

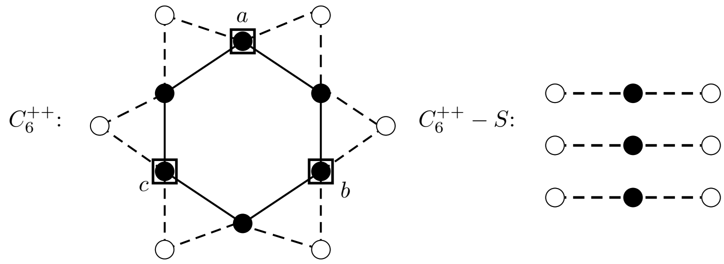

Figure 3: Graph $C_6^{++} - S$.

Example 4.3. *Consider a graph K_4 and its semitotal point graph K_4^{++}. Let $S = \{a, b, c, d\} \subset V(K_4^{++})$ (see Figure 4). It is clear that $m(K_4^{++} - S) = 1$. So, $I(K_4^{++}) = 5$.*

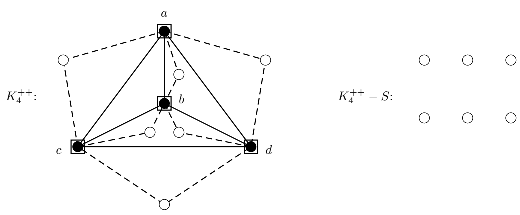

Figure 4: Graph $K_4^{++} - S$.

Corolary 4.1. *$I(K_p) = I(K_q^{++})$ if and only if $p = q + 1$.*

Theorem 4.4. *For a complete bipartite graph $K_{a,b}$ of order $a + b$,*

$$I(K_{a,b}^{++}) = 2min\{a, b\} + 1.$$

Proof. $K_{a,b}^{++}$ has $2ab$ vertices and $3ab$ edges. Let us select S in such a way that it should contain minimum number of vertices among two partite sets of $K_{a,b}$. So, $|S| = min\{a, b\}$. The deletion of vertices of S from $K_{a,b}^{++}$ results in union of stars $K_{1,min\{a,b\}}$. Hence, $m(K_{a,b}^{++} - S) = min\{a, b\} + 1$. The value of $|S| + m(K_{a,b}^{++} - S)$ whose sum is minimum for chosen S. Therefore, $I(K_{a,b}^{++}) = 2min\{a, b\} + 1$. $\square$

Example 4.4. *Consider a graph $K_{2,3}$. Let $S = \{a, b\} \subset V(K_{2,3}^{++})$ (see Figure 5). It is clear to write $|S| = 2$ and $m(K_{2,3}^{++} - S) = 3$. So, $I(K_{2,3}^{++}) = 5$.*

Corolary 4.2. *$I(K_{a_1,b_1}) = I(K_{a_2,b_2}^{++})$ if and only if $min\{a_1, b_1\} = 2min\{a_2, b_2\}$. $K_{2,2}$ and $K_{1,2}$ are the smallest graphs satisfying above condition such that $I(K_{2,2}) = I(K_{1,2}^{++})$.*

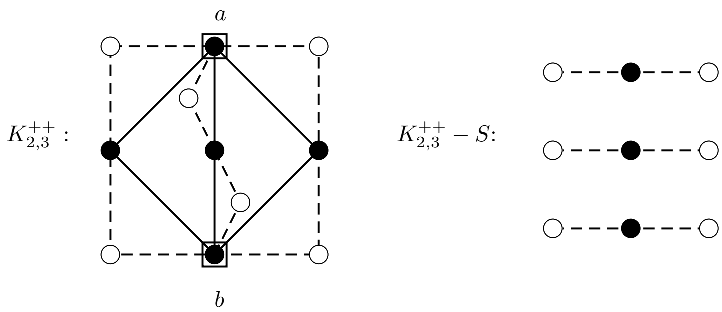

Figure 5: Graph $K_{2,3}^{++} - S$.

Theorem 4.5. *For a star $K_{1,b}$ of order $b + 1$,*

$$I(K_{1,b}^{++}) = 3.$$

Proof. $K_{1,b}^{++}$ has $2b + 1$ vertices and $3b$ edges. Let $S \subset V(K_{1,b}^{++})$ containing a central vertex of $K_{1,b}$. So, $|S| = 1$. The removal of a vertex of set S from $K_{1,b}^{++}$ results in graph bK_2. Hence, $m(K_{1,b}^{++} - S) = 2$. The value $|S| + m(K_{1,b}^{++} - S)$ is minimum for the chosen S. Therefore, $I(K_{1,b}^{++}) = 3$. □

Remark 4.1. *The values of integrity of star graph and integrity of semitotal point graph of star graph are never same, since $I(K_{1,b}) = 2$ and $I(K_{1,b}^{++}) = 3$.*

Example 4.5. *Consider a graph $K_{1,3}$. Let $S = \{a\} \subset V(K_{1,3}^{++})$ (see Figure 6). It is clear to write $|S| = 1$ and $m(K_{1,3}^{++} - S) = 2$ So, $I(K_{1,3}^{++}) = 3$*

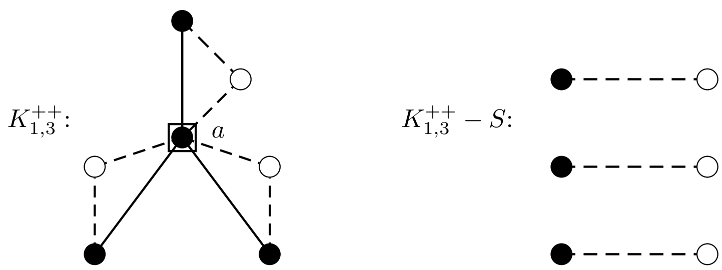

Figure 6: Graph $K_{1,3}^{++} - S$.

Theorem 4.6. *For a wheel W_n of order $n \geq 5$,*

$$I(W_n^{++}) = \left\lceil \frac{n-1}{2} \right\rceil + 5.$$

Proof. W_n^{++} has $3n-2$ vertices and $6(n-1)$ edges. Let $S \subset V(W_n^{++})$.
Case 1. Suppose n is odd
Clearly, the order of an outer cycle of wheel is $n-1$, which is even. Choose a set S_1 in such a way that it is an independent set of vertices of C_{n-1}. It is clear that $|S_1| = \frac{n-1}{2} = \beta_0(C_{n-1})$.
Case 2. Suppose n is even
Clearly, the order of an outer cycle of wheel is $n-1$, which is odd. Let S_2 be an independent set of vertices of C_{n-1} such that $|S_2| = \frac{n-2}{2}$. Let v_1 be a vertex of $V(C_{n-1}) \setminus S_2$ such that v_1 is adjacent to a vertex of S_2 as well as to a vertex $V(C_n) \setminus S_2$. Let us take $S_1 = S_2 \cup \{v\}$ and hence $|S_1| = \frac{n}{2}$.
Combining the above two cases we get, $|S_1| = \lceil \frac{n-1}{2} \rceil$, for all n, Let v_2 be a central vertex of W_n. Let us define a set S in such a manner that $S = S_1 \cup \{v_2\}$. It is to be noted that $|S| = \lceil \frac{n-1}{2} \rceil + 1$. The deletion of vertices of set S from W_n^{++} gives a graph whose components are P_4's and K_1's. Hence, $m(W_n^{++} - S) = 4$. The set S defined in this manner gives minimum value of $|S| + m(W_n^{++} - S)$. Therefore, $I(W_n^{++}) = \lceil \frac{n-1}{2} \rceil + 5$. □

Corolary 4.3. *$I(W_p) = I(W_q^{++})$ if and only if $\lceil 2\sqrt{p-1} \rceil = \lceil \frac{q-1}{2} \rceil + 5$. W_{11} and W_5 are the smallest graphs which satisfy the above condition such that $I(W_{11}) = I(W_5^{++})$.*

Example 4.6. *Consider a graph W_7. Let $S = \{a, b, c, d\} \subset V(W_7^{++})$ (see Figure 7). It is clear to write $|S| = 4$ and $m(W_7^{++} - S) = 4$. So, $I(W_7^{++}) = 8$*

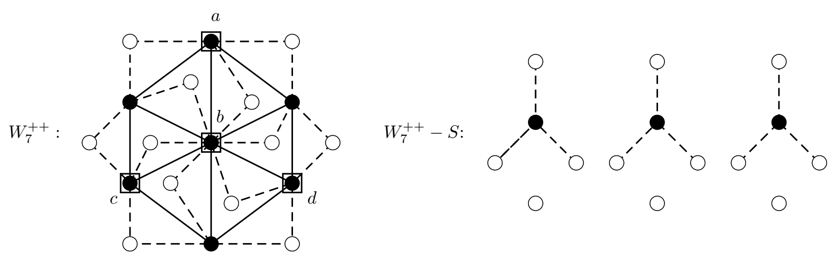

Figure 7: Graph $W_7^{++} - S$.

Theorem 4.7. *For a connected graph $G \ncong K_{1,b}$ of order n and size m,*

$$I(G^{+-}) = n + 1.$$

Proof. For an (n, m) graph G, G^{+-} has $n + m$ vertices and $m(n-1)$ edges. The n vertices have degree m and m vertices have $n-2$ in G^{+-}. Let $S \subset V(G^{+-})$. Consider a set S consisting a vertices of G^{+-} which corresponds to the vertices of a graph G. Then it is clear that $|S| = n$. The removal of the vertices of set S from G^{+-} results in a null graph $\overline{K_m}$. Hence, $m(G^{+-} - S) = 1$. $|S| + m(G^{+-} - S)$ is minimum for above chosen S. Therefore, $I(G^{+-}) = n + 1$.. $\square$

Theorem 4.8. *For a star $K_{1,b}$ $(b \geq 3)$,*

$$I(K_{1,b}^{+-}) = b + 1.$$

Proof. For a star $K_{1,b}$ of order $b+1$ and size b, the graph $K_{1,b}^{+-}$ has $2b+1$ vertices and b^2 edges. Let $S \subset V(K_{1,b}^{+-})$. Choose as set S in such a way that it should contain the pendant vertices of $K_{1,b}$. So, $|S| = b$. The deletion of the vertices of set S from $K_{1,b}^{+-}$ results in null graph $\overline{K_{b+1}}$. So, $m(K_{1,b}^{+-} - S) = 1$. $|S| + m(K_{1,b}^{+-} - S)$ is minimum for above chosen S. Therefore, $I(K_{1,b}^{+-}) = b + 1$. $\square$

The column 2 and 4 of Table Table 3 shows integrity of basic graphs and integrity of transformation graph G^{+-} of graphs with same structure.

G	$I(G)$	G^{+-}	$I(G^{+-})$	G^{-+}	$I(G^{-+})$	G^{--}	$I(G^{--})$
P_6	4	P_3^{+-}	4	P_3^{-+}	4	P_3^{--}	4
P_{10}	5	P_{10}^{+-}	11	P_{10}^{-+}	8	P_{10}^{--}	11
C_5	4	C_3^{+-}	4	C_3^{-+}	4	C_3^{--}	4
C_7	5	C_7^{+-}	8	C_7^{-+}	8	C_7^{--}	8
K_n	n	K_n^{+-}	$n+1$	K_n^{-+}	$n+1$	K_n^{--}	$n+1$
$K_{5,5}$	6	$K_{2,3}^{+-}$	6	$K_{2,3}^{-+}$	6	$K_{2,3}^{--}$	6
$K_{5,6}$	6	$K_{5,6}^{+-}$	12	$K_{5,6}^{-+}$	12	$K_{5,6}^{--}$	12
$K_{1,b}$	2	$K_{1,b}^{+-}$	$b+1$	$K_{1,b}^{-+}$	$b+2$	$K_{1,b}^{--}$	$b+1$
W_8	6	W_5^{+-}	6	W_5^{-+}	6	W_5^{--}	6
W_9	6	W_9^{+-}	10	W_9^{-+}	10	W_9^{--}	10

Table 3:

Theorem 4.9. *For a connected graph $G \not\cong K_{1,b}$ of order n and size m,*

$$I(G^{-+}) = n + 1.$$

Proof. For an (n, m) graph G, G^{-+} has $n+m$ vertices and $\frac{n(n-1)}{2} + m$ edges. The n vertices have degree 2 in G^{-+}. Let $S \subset V(G^{-+})$. Consider a set S consisting a vertices of G^{-+} which corresponds to the vertices of a graph G. Then it is clear that $|S| = n$. The removal of the vertices of set S from G^{-+} results in a null graph $\overline{K_m}$. Hence, $m(G^{-+} - S) = 1$. The value $|S| + m(G^{-+} - S)$ is minimum for above chosen S. Therefore, $I(G^{-+}) = n + 1$. $\square$

The column 2 and 6 of Table 3 shows integrity of basic graphs and integrity of transformation graph G^{-+} of graphs with same structure.

Theorem 4.10. *For a star $K_{1,b}$ $(b \geq 3)$,*

$$I(K_{1,b}^{-+}) = b + 1.$$

Proof. The proof is similar to that of Theorem 4.8. $\square$

Theorem 4.11. *For a connected graph $G \ncong K_{1,b}$ of order n and size m,*

$$I(G^{--}) = n + 1.$$

Proof. For an (n, m) graph G, G^{--} has $n + m$ vertices and $\frac{n(n-1)}{2} + m(n-3)$ edges. Let $S \subset V(G^{--})$. Consider a set S consisting a vertices of G^{--} which corresponds to the vertices of a graph G. Then it is clear that $|S| = n$. Deleting the vertices of set S from G^{--} results in a null graph $\overline{K_m}$. Hence, $m(G^{--} - S) = 1$. $|S| + m(G^{--} - S)$ is minimum for above chosen S. Therefore, $I(G^{--}) = n + 1$. $\square$

Theorem 4.12. *For a star $K_{1,b}(b \geq 3)$,*

$$I(K_{1,b}^{--}) = b + 1.$$

Proof. For a star $K_{1,b}$ of order $b+1$ and size b, the graph $K_{1,b}^{--}$ has $2b+1$ vertices. Let $S \subset V(K_{1,b}^{--})$. Choose as set S in such a way that it should contain the pendant vertices of $K_{1,b}$. So, $|S| = b$. The deletion of the vertices of set S from $K_{1,b}^{--}$ results in null graph $\overline{K_{b+1}}$. So, $m(K_{1,b}^{--} - S) = 1$. $|S| + m(K_{1,b}^{--} - S)$ is minimum for above chosen S. Therefore, $I(K_{1,b}^{--}) = b + 1$. $\square$

The column 2 and 8 of Table 3 shows integrity of basic graphs and integrity of transformation graph G^{--} of graphs with same structure.

Theorem 4.13. *For any connected graph G of order n and size $m \geq 2$,*

$$I(\overline{G^{++}}) = n + m - 2.$$

Proof. For a connected graph G of order n and size $m \geq 2$, $\overline{G^{++}}$ has order $n + m$. Let $S \subset V(\overline{G^{++}})$. Consider a set S containing all the vertices and edges of G except one edge and its incident vertices. So, $|S| = n + m - 3$. The removal of the vertices of set S from $\overline{G^{++}}$ gives $3K_1$. Hence, $m(\overline{G^{++}} - S) = 1$. The value $|S| + m(\overline{G^{++}} - S)$ is minimum for the selected subset S. Therefore, $I(\overline{G^{++}}) = n + m - 2$.. $\square$

Theorem 4.14. *For any connected graph G of order n and size $m \geq 2$,*

$$I(\overline{G^{+-}}) = n + m - 2.$$

Proof. For a connected graph G of order n and size $m \geq 2$, $\overline{G^{+-}}$ has order $n + m$. Let $S \subset V(\overline{G^{+-}})$. Consider a set S containing all the vertices and edges of G except one edge and two nonincident vertices(adjacent vertices). So, $|S| = n + m - 3$. The removal of the vertices of set S from $\overline{G^{+-}}$ gives $3K_1$. Hence, $m(\overline{G^{+-}} - S) = 1$. The value of $|S| + m(\overline{G^{+-}} - S)$ is minimum for the selected subset S. Therefore, $I(\overline{G^{+-}}) = n + m - 2$.. □

Corolary 4.4. *The integrity of*

(i) *path P_n, $I(\overline{P_n^{++}}) = I(\overline{P_n^{+-}}) = 2n - 3$,*

(ii) *cycle C_n, $I(\overline{C_n^{++}}) = I(\overline{C_n^{+-}}) = 2n - 2$,*

(iii) *complete graph K_n, $I(\overline{K_n^{++}}) = I(\overline{K_n^{+-}}) = \frac{n(n+1)}{2} - 2$,*

(iv) *complete bipartite graph $K_{a,b}$, $I(\overline{K_{a,b}^{++}}) = I(\overline{K_{a,b}^{+-}}) = a + b + ab - 2$,*

(v) *star $K_{1,b}$, $I(\overline{K_{1,b}^{++}}) = I(\overline{K_{1,b}^{+-}}) = 2b - 1$,*

(vi) *wheel W_n, $I(\overline{W_n^{++}}) = I(\overline{W_n^{+-}}) = 3n - 4$.*

The Table 4 shows integrity of basic graphs and integrity of transformation graphs $\overline{G^{++}}$ and $\overline{G^{+-}}$ of graphs with same structure.

G	$I(G)$	$\overline{G^{++}}$ and $\overline{G^{+-}}$	$I(\overline{G^{++}}) = I(\overline{G^{+-}})$
P_4	3	$\overline{P_3^{++}}$ and $\overline{P_3^{+-}}$	3
P_5	3	$\overline{P_5^{++}}$ and $\overline{P_5^{+-}}$	7
C_5	4	$\overline{C_3^{++}}$ and $\overline{C_3^{+-}}$	4
C_6	4	$\overline{C_6^{++}}$ and $\overline{C_6^{+-}}$	10
K_n	n	$\overline{K_n^{++}}$ and $\overline{K_n^{+-}}$	$\frac{n(n+1)}{2} - 2$
$K_{8,8}$	9	$\overline{K_{2,3}^{++}}$ and $\overline{K_{2,3}^{+-}}$	9
$K_{8,9}$	9	$\overline{K_{8,9}^{++}}$ and $\overline{K_{8,9}^{+-}}$	87
$K_{1,b}$	2	$\overline{K_{1,b}^{++}}$ and $\overline{K_{1,b}^{+-}}$	$2b - 1$
W_{27}	11	$\overline{W_5^{++}}$ and $\overline{W_5^{+-}}$	11
W_{28}	11	$\overline{W_{28}^{++}}$ and $\overline{W_{28}^{+-}}$	80

Table 4:

Theorem 4.15. *For any connected graph G of order n and size m,*

$$I(\overline{G^{-+}}) = min\{n+m-1, m+I(G)\}.$$

Proof. For a connected graph G of order n and size m, $\overline{G^{-+}}$ has order $n+m$. Let $S_1 \subset V(\overline{G^{-+}})$. Choose a set S_1 containing the edges of G. So, $|S_1| = |E(G)| = m$. The removal of elements of set S_1 from a graph $\overline{G^{-+}}$ gives a graph G. Consider the value $|S_1| + I(G) = m + I(G)$.
Choose a set $S_2 \subset V(\overline{G^{-+}})$ consisting of all the elements of $\overline{G^{-+}}$ except an edge and two incident vertices. So, $|S_2| = n+m-3$. The removal of elements of set S_2 from $\overline{G^{-+}}$ gives $K_2 \cup K_1$. Hence, $m(\overline{G^{-+}} - S_2) = 2$. Consider, $|S_2| + m(\overline{G^{-+}} - S_2) = n+m-1$.
The minimum value among $m+I(G)$ and $n+m-1$ gives integrity of $\overline{G^{-+}}$. Therefore, $I(\overline{G^{-+}}) = min\{n+m-1, m+I(G)\}$. □

Corolary 4.5. *The integrity of*

(i) path $P_n(n \geq 3)$, $I(\overline{P_n^{-+}}) = n + \lceil 2\sqrt{n+1} \rceil - 3$,

(ii) cycle $C_n(n \geq 4)$, $I(\overline{C_n^{-+}}) = n + 2\lceil n \rceil - 1$,

(iii) complete graph K_n, $I(\overline{K_n^{-+}}) = \frac{n(n+1)}{2} - 1$,

(iv) complete bipartite graph $K_{a,b}(a, b \geq 2)$, $I(\overline{K_{a,b}^{-+}}) = ab + 1 + min\{a, b\}$,

(v) star $K_{1,b}(b \geq 2)$, $I(\overline{K_{1,b}^{-+}}) = b + 2$,

(vi) wheel $W_n(n \geq 5)$, $I(\overline{W_n^{-+}}) = 2n + \lceil 2\sqrt{n-1} \rceil - 2$.

The column 2 and 4 Table 5 shows integrity of basic graphs and integrity of transformation graphs $\overline{G^{-+}}$ of graphs with same structure.

Theorem 4.16. *For any connected graph G of order n and size m,*

$$I(\overline{G^{--}}) = m + I(G).$$

Proof. Let G be an (n, m) graph. Then $\overline{G^{--}}$ is a graph of order $n+m$. Let $S_1 \subset V(\overline{G^{--}})$. Consider a set S_1 containing all the edges of G. So, $|S| = |E| = m$. The removal of the vertices of set S_1 from $\overline{G^{--}}$ gives a graph G. Therefore, $I(\overline{G^{--}}) = m + I(G)$. □

Corolary 4.6. *The integrity of*

(i) path P_n, $I(\overline{P_n^{--}}) = n - 3 + \lceil 2\sqrt{n+1} \rceil$,

G	$I(G)$	$\overline{G^{-+}}$	$I(\overline{G^{-+}})$	$\overline{G^{--}}$	$I(\overline{G^{--}})$
P_6	4	$\overline{P_3^{-+}}$	4	$\overline{P_3^{--}}$	4
P_7	4	$\overline{P_7^{-+}}$	10	$\overline{P_7^{--}}$	10
C_{13}	7	$\overline{C_4^{-+}}$	6	$\overline{C_4^{--}}$	7
C_{14}	7	$\overline{C_{14}^{-+}}$	21	$\overline{C_{14}^{--}}$	21
K_n	n	$\overline{K_n^{-+}}$	$\frac{n(n+1)}{2}-1$	$\overline{K_n^{--}}$	$\frac{n(n+1)}{2}$
$K_{8,9}$	9	$\overline{K_{2,3}^{-+}}$	9	$\overline{K_{2,3}^{--}}$	9
$K_{9,9}$	9	$\overline{K_{9,9}^{-+}}$	91	$\overline{K_{9,9}^{--}}$	91
$K_{1,b}$	2	$\overline{K_{1,b}^{-+}}$	$b+2$	$\overline{K_{1,b}^{--}}$	$2b+1$
W_{32}	12	$\overline{W_5^{-+}}$	12	$\overline{W_5^{--}}$	12
W_{33}	12	$\overline{W_{33}^{-+}}$	76	$\overline{W_{33}^{--}}$	76

Table 5:

(ii) *cycle* C_n, $I(\overline{C_n^{--}}) = n - 1 + \lceil 2\sqrt{n} \rceil$,

(iii) *complete graph* K_n, $I(\overline{K_n^{--}}) = \frac{n(n+1)}{2}$,

(iv) *complete bipartite graph* $K_{a,b}$, $I(\overline{K_{a,b}^{--}}) = ab + 1 + min\{a, b\}$,

(v) *star* $K_{1,b}$, $I(\overline{K_{1,b}^{--}}) = 2b + 1$,

(vi) *wheel* W_n, $I(\overline{W_n^{--}}) = 2n - 2 + \lceil 2\sqrt{n-1} \rceil$.

The column 2 and 6 of Table 5 shows integrity of basic graphs and integrity of transformation graphs $\overline{G^{--}}$ of graphs with same structure.

4.2 *Integrity of semitotal point graph of combination of basic graphs*

Definition 4.1. *[16] The product* $G \times H$ *of two graphs* G *and* H *is defined as follows:*
Consider any two points $u = (u_1, u_2)$ *and* $v = (v_1, v_2)$ *in* $V = (V_1, V_2)$*. Then* u *and* v *are adjacent in* $G \times H$ *whenever [*$u_1 = v_1$ *and* u_2 *adj* v_2*] or [*$u_2 = v_2$ *and* u_1 *adj* v_1*].*
If G *and* H *are* (n_1, m_1) *and* (n_2, m_2) *graphs respectively. Then,* $G \times H$ *is* $(n_1n_2, n_1m_2 + n_2m_1)$ *graph.*

Theorem 4.17. *For a graph $K_2 \times P_n$ $(n \geq 3)$,*

$$I\big((K_2 \times P_n)^{++}\big) = \begin{cases} 7, & if\ n = 3, \\ 11, & if\ n = 5, \\ \frac{5n-7}{2}, & if\ n\ is\ odd\ and\ n \geq 7, \\ \frac{5n-2}{2}, & if\ n\ is\ even. \end{cases}$$

Proof. The graph $(K_2 \times P_n)^{++}$ has $5n-2$ vertices and $3(3n-2)$ edges.
Let $S \subset V\big((K_2 \times P_n)^{++}\big)$. The proof includes the following cases.
Case 1. Suppose n is odd and $n \geq 7$.
Choose a set S containing the two internal vertices adjacent to corresponding central vertices of each of two P_n's in $K_2 \times P_n$. So, $|S| = 4$. The deletion of vertices of set S from $(K_2 \times P_n)^{++}$ results in a graph with components of orders 1, 7, $\frac{5(n-3)}{2}$. Hence, $m\big((K_2 \times P_n)^{++} - S\big) = \frac{5(n-3)}{2}$, since $n \geq 7$. The value of $|S| + m\big((K_2 \times P_n)^{++} - S\big)$ for this set S is $\frac{5n-7}{2}$ and it is minimum.
Therefore, $I\big((K_2 \times P_n)^{++}\big) = \frac{5n-7}{2}$.
Case 2. Suppose n is even.
Let S be a set containing the two internal vertices which are central vertices of each of two P_n's in $K_2 \times P_n$. So, $|S| = 4$. The removal of vertices of set S from $(K_2 \times P_n)^{++}$ results in a graph with components of orders 1, $\frac{5(n-2)}{2}$. Hence, we can write $m\big((K_2 \times P_n)^{++} - S\big) = \frac{5(n-2)}{2}$. The value of $|S| + m\big((K_2 \times P_n)^{++} - S\big)$ for the above set S is minimum.
Therefore, $I\big((K_2 \times P_n)^{++}\big) = \frac{5n-2}{2}$.
Case 3. Suppose $n = 3, 5$.
By direct calculation using the definition of integrity, the result follows. $\square$

Theorem 4.18. *For a graph $K_2 \times C_n$ $(n \geq 4)$,*

$$I\big((K_2 \times C_n)^{++}\big) = \begin{cases} 2\left\lceil \frac{n}{2} \right\rceil + 7, & if\ n\ is\ even, \\ 2\left\lceil \frac{n}{2} \right\rceil + 7, & if\ n\ is\ odd\ and\ n \leq 7, \\ 2\left\lceil \frac{n}{3} \right\rceil + 12, & if\ n\ is\ odd\ and\ n \geq 9. \end{cases}$$

Proof. The graph $(K_2 \times C_n)^{++}$ has $5n$ vertices and $9n$ edges.
Let $S \subset V\big((K_2 \times C_n)^{++}\big)$.
Case 1. Suppose n is even.
Let S_1 be an independent set of vertices of C_n such that $|S_1| = \beta_0(C_n) = \frac{n}{2}$.
Case 2. Suppose n is odd.
Let S' be an independent set C_n such that $|S'| = \beta_0(C_n) = \frac{n-1}{2}$. Let v_1 be a vertex of $V(C_n) \setminus S'$ such that v_1 is adjacent to a vertex of S' as well as to a vertex of $V(C_n) \setminus S'$. Let $S_1 = S' \cup \{v_1\}$.
Combining the above two cases we get, $S_1 = \left\lceil \frac{n}{2} \right\rceil$. Choose a set S consisting of vertices of two C_n's of $K_2 \times C_n$ such that $|S| = 2|S_1| = 2\left\lceil \frac{n}{2} \right\rceil$. The removal of

vertices of set S from $(K_2 \times C_n)^{++}$ results in a graph with components of orders $1, 7$. Hence, $m\big((K_2 \times C_n)^{++} - S\big) = 7$. The value of $|S| + m\big((K_2 \times C_n)^{++} - S\big)$ for the above set S is minimum.
Therefore, $I((K_2 \times C_n)^{++}) = 2\left\lceil \frac{n}{2} \right\rceil + 7$.
Case 3. Suppose $n(\geq 9)$ is odd.
Let S be an independent set of vertices of two C_n's in a manner that the distance between the selected vertices is 3. Then, $|S| = 2\left\lceil \frac{n}{3} \right\rceil$. The removal of vertices of set S from $(K_2 \times C_n)^{++}$ results in a disconnected graph with components of orders $1, 12$. Hence, $m\big((K_2 \times C_n)^{++} - S\big) = 12$. Therefore, $I\big((K_2 \times C_n)^{++}\big) = 2\left\lceil \frac{n}{3} \right\rceil + 12$. □

Theorem 4.19. *For a graph $K_2 \times K_{1,b}$ $(b \geq 2)$,*

$$I\big((K_2 \times K_{1,b})^{++}\big) = 7.$$

Proof. The semitotal point graph $(K_2 \times K_{1,b})^{++}$ has $5b + 3$ vertices and $3(3b + 1)$ edges.
Let S be a subset of $V\big((K_2 \times K_{1,b})^{++}\big)$. Choose S such that it contains the two vertices corresponding to central vertices of each of two stars of $K_2 \times K_{1,b}$. Clearly, $|S| = 2$. The removal of vertices of S from $(K_2 \times K_{1,b})^{++}$ results in a graph with components of orders $1, 5$. Hence, $m\big((K_2 \times K_{1,b})^{++} - S\big) = 5$. This set S gives least value of $|S| + m\big((K_2 \times K_{1,b})^{++} - S\big)$. Therefore, $I\big((K_2 \times K_{1,b})^{++}\big) = 7$. □

Theorem 4.20. *For a graph $K_p \times K_q$ $(p = q \geq 2)$,*

$$I\big((K_p \times K_q)^{++}\big) = pq + 1.$$

Proof. The semitotal point graph $(K_p \times K_q)^{++}$ has $\frac{pq(p+q)}{2}$ vertices and $\frac{3pq(p+q-2)}{2}$ edges.
Select a set S such that the elements of it correspond to all the vertices of $K_p \times K_q$. So, $|S| = pq$. The removal of vertices of S from $(K_p \times K_q)^{++}$ results in a totally disconnected graph with $\frac{3pq(p+q-2)}{2}$ vertices. Clearly, $m\big((K_p \times K_q)^{++} - S\big) = 1$. Therefore, $I\big((K_p \times K_q)^{++}\big) = pq + 1$. □

Definition 4.2. *[16] The corona $G \circ H$ of graphs G and H is a graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and then joining by an edge each vertex of the i'th copy of H is named (H, i) with the i'th vertex of G.*

If G and H are (n_1, m_1) and (n_2, m_2) graphs respectively. Then, $G \circ H$ is $(n_1(1 + n_2), m_1 + n_1m_2 + n_1n_2)$ graph.

Theorem 4.21. *For a graph $K_2 \circ P_n$ $(n \geq 3)$,*

$$I\big((K_2 \circ P_n)^{++}\big) = \begin{cases} 7, & if\ n = 3, \\ 10, & if\ n = 5, \\ \frac{3(n+1)}{2}, & if\ n\ is\ odd\ and\ n \geq 7, \\ 3(\frac{n}{2}+1), & if\ n\ is\ even. \end{cases}$$

Proof. The graph $(K_2 \circ P_n)^{++}$ has $6n+1$ vertices and $3(4n-1)$ edges.
Let $S \subset V\big((K_2 \circ P_n)^{++}\big)$. The proof includes the following cases.
Case 1. Suppose n is odd and $n \geq 7$.
Choose a set S containing the two internal vertices adjacent to corresponding central vertices of each of two P_n's and the two vertices of K_2 in $K_2 \circ P_n$. So, $|S| = 6$. The removal of vertices of S results in a graph with components of orders 1, 4, $\frac{3(n-1)}{2}$. Hence, $m\big((K_2 \circ P_n)^{++} - S\big) = \frac{3(n-1)}{2}$, since $n \geq 7$. The value of $|S| + m\big((K_2 \circ P_n)^{++} - S\big)$ for this set S is $\frac{5n-7}{2}$ is least. Therefore, $I\big((K_2 \circ P_n)^{++}\big) = \frac{5n-7}{2}$.
Case 2. Suppose n is even.
Choose a set S containing the two internal vertices which are central vertices of each of two P_n's and the two vertices of K_2 in $K_2 \circ P_n$. So, $|S| = 6$. The removal of vertices of S from $(K_2 \circ P_n)^{++}$ results in a graph with components of orders 1, $\frac{3(n-2)}{2}$. Clearly, we can write $m\big((K_2 \circ P_n)^{++} - S\big) = \frac{3(n-2)}{2}$. The value of $|S| + m\big((K_2 \circ P_n)^{++} - S\big)$ for the above set S is minimum. Therefore, $I\big((K_2 \circ P_n)^{++}\big) = 3(\frac{n}{2}+1)$.
Case 3. Suppose $n = 3, 5$.
By direct calculation using the definition of integrity, we can obtain the result. □

Theorem 4.22. *For a graph $K_2 \circ C_n$ $(n \geq 4)$,*

$$I\big((K_2 \circ C_n)^{++}\big) = 2\left\lceil \frac{n}{2} \right\rceil + 6.$$

Proof. The graph $(K_2 \circ C_n)^{++}$ has $7n+2$ vertices and $15n$ edges.
Let $S \subset V\big((K_2 \circ C_n)^{++}\big)$.
Case 1. Suppose n is even.
Let S_1 be an independent set of vertices of C_n such that $|S_1| = \beta_0(C_n) = \frac{n}{2}$.
Case 2. Suppose n is odd.
Let S' be an independent set of vertices of C_n such that $|S'| = \beta_0(C_n) = \frac{n-1}{2}$. Let v_1 be a vertex of $V(C_n) \setminus S'$ such that v_1 is adjacent to a vertex of S' as well as to a vertex of $V(C_n) \setminus S'$. Let $S_1 = S' \cup \{v_1\}$ and $|S_1| = \frac{n+1}{2}$.
Combining the above two cases we get, $S_1 = \lceil \frac{n}{2} \rceil$. Choose a set S_2 consisting of vertices of two C_n's of $K_2 \circ C_n$ such that $|S_2| = 2|S_1| = 2\lceil \frac{n}{2} \rceil$. Select a set S_3 consisting of the vertices of K_2 of $K_2 \circ C_n$. So, $S = S_2 \cup S_3$. Hence,

$|S| = 2\left\lceil \frac{n}{2} \right\rceil + 2$. The removal of vertices of set S from $(K_2 \circ C_n)^{++}$ results in a graph with components of orders 1, 4. Hence, $m\big((K_2 \circ C_n)^{++} - S\big) = 4$. The value of $|S| + m\big((K_2 \circ C_n)^{++} - S\big)$ for the above set S is minimum. Therefore, $I\big((K_2 \circ C_n)^{++}\big) = 2\left\lceil \frac{n}{2} \right\rceil + 6$. □

Theorem 4.23. *For a graph $K_p \circ K_q$,*

$$I\big((K_p \circ K_q)^{++}\big) = pq + p + 1.$$

Proof. The semitotal point graph of $K_p \circ K_q$ has $p(q+1)$ vertices and $\frac{p}{2}[p+q(q+1)-1]$ edges. Let S be a subset of $V\big((K_p \circ K_q)^{++}\big)$. Choose S such that it contains vertices of K_p and vertices of p copies of K_q of $K_p \circ K_q$. So $|S| = p(q+1)$. The removal of vertices of set S from $(K_p \circ K_q)^{++}$ results in a totally disconnected graph with $\frac{p}{2}[p + q(q+1) - 1]$ vertices. Clearly, $m\big((K_p \circ K_q)^{++} - S\big) = 1$. Therefore, $I\big((K_p \circ K_q)^{++}\big) = pq + p + 1$. □

5 Conclusion

In this paper, we have computed the integrity of generalized transformation graphs in terms of elements of a graph G. Also, integrity of semitotal point graph of combinations of basic graphs are obtained. Finally, we have established the relation between integrity of basic graphs and integrity of their generalized transformation graphs. We conclude that integrity of generalized transformation graphs are greater than or equal to integrity of graphs that have same structure.

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Analysis of classical retrial queue with differentiated vacation and state dependent arrival rate

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Abstract

In the present paper, we have introduced the concept of differentiated vacations in a retrial queueing model with state-dependent arrival rates of customers. The arrival rate of customers is different in various states of the server. The vacation types are differentiated by means of their durations as well as the previous state of the server. In type I vacation, the server goes just after providing service to at least one customer whereas in type II, it comes after remaining free for some time. In a steady state, we have obtained the system size probabilities and other system performance measures. Finally, sensitivity and cost analysis of the proposed model is also performed. The probability generating function technique, parabolic method and MATLAB is used for this purpose.

Keywords: Retrial queue; Markov process; differentiated vacations; exponential distribution etc.

2010 AMS subject classification: **60K25, 60K30**

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1.Introduction

Retrial queues have wide applications in communication system, production system, computer networking system, telecommunication etc. Retrial queues are characterized by the fact that arriving customers on finding the server busy, leave the system and join the retrial group to complete their request for service after a random time period. A good survey on retrial queues have been done by Falin,Templeton [5] and Artalejo,Gomez-coral [1].

In queueing theory, many situations occur where arrival rate of customers depends upon the different states of server such as busy state, idle state or on vacation state etc. Singh et al.[14] studied M/G/1 queueing model with state dependent arrival of customers. Batch arrival queueing system under retrial policy with state dependent admission is analysed by Bagyam and chandrika [2]. Niranjan et al. [12] did the pioneer work on state dependent arrival in bulk retrial queues with Bernoulli feedback and multiple vacations.

Nowadays, Retrial queueing system with server vacation has become increasingly important due to wide applications in research area. In queueing system with vacation, server becomes unavailable from service station for random period of time due to some reasons like server breakdown, maintenance of server, service provided by server in secondary service station when primary station is empty or simply going for break etc. The time period during which the server is not available for primary customers is known as vacation. In single vacation queueing model, server goes for vacation of random time duration whenever there is no customer in the system and returns to the system after vacation completion. The idea of queueing system with server vacation was first discussed by Levy and Yechiali [9]. Doshi [3] had performed good survey on queueing model with vacation. Later on Takagi [16], Tian and Zhang [18] did the pioneer work on vacation queueing system.

In multiple vacation system, if server finds no customer in system on returning from vacation, then server immediately goes for another vacation otherwise server will serve the customers. Servi and Finn [13] introduced the concept of working vacation queueing system in which server works at slow rate during vacation period rather than completely stopping the service during vacation. In queueing literature, lot of work have been done on queueing model with working vacation by many researchers [8,23]. Li and Tian [10] analysed M/M/1 queueing model with working vacation and interruption. Retrial queueing model with working vacation was first studied by Do [4]. Later on Li et al. and Tao et al. [11,17] did pioneer work on retrial queueing model with working vacation and interruption.

In differentiated vacation queueing model, server takes vacation I i.e. vacation of longer duration after serving all the customers in system and vacation II i.e. vacation of shorter duration will be taken by server if there is no customer in system after completing the type I vacation. The concept of differentiated vacations in queuing literature was first introduced by Ibe and Isijola [6]. In this paper they considered two types of vacations with different durations. Further they extended their model by introducing the concept of vacation interruption [7]. M/M/1 single server queue with m kinds of differentiated working vacations was analyzed by Zhang and Zhou [22]. Vijayashree and Janani [21] performed transient solution of M/M/1 queueing system with differentiated vacation. Suranga Sampath and Liu [15] studied the customer's impatience behaviour on M/M/1 queueing system subject to differentiated vacation. Unni and Mary [19] studied queueing system with multiple servers under differentiated vacations. Further they extended their work by introducing differentiated working vacation [20].

In this paper, we have extended the concept of differentiated vacations to queueing system under classical retrial policy considering the state dependent arrival of customers. The organization of rest of the paper into different sections is as follows. The model description is given in section 2. Section 3 is devoted to steady state equations and solutions. The closed form expressions for some of the performance measures are derived in section 4. Section 5 represents the effect of various parameters on some important system performance measures graphically. Conclusion and future scope is discussed in section 6.

2. Model description

The model is outlined as follows.

1. Customers arrive according to Poisson process but with different rates depending on the present state of the server. The different arrival rates of customers are λ, α, γ, δ in busy, free, vacation I, vacation II states of the server, respectively.
2. The arriving customers are served on FCFS basis. If server is free in active period, the arriving customer is immediately served otherwise due to unavailability of waiting space in service area, he has to join a free pool of

infinite capacity known as orbit to wait for the service. From the orbit, customers retry for their turn with classical rate β. For convenience, the service time is supposed to follow exponential distribution with parameter μ.

3. As soon as the last customer is served i.e. system gets empty, the server leaves for type I vacation. At the end of type I vacation, if system is still empty, the server goes on type II vacation otherwise returns to active state to serve the waiting customers. On completion of vacation II, if there is a customer waiting in the system, server returns to free state in normal active period otherwise again goes on vacation II repeatedly. The vacation I is assumed to be of longer duration than vacation II. The time period of both vacations is assumed to follow exponential distribution with parameters $v_{1,}v_2$ respectively.

3. Steady state equations and solution

Denoting the probability of n customers in state k of the server by $p_{n\,k}$ and server states at time t by S(t) were

$$S(t)=\begin{cases} 1, server\ is\ busy\ in\ active\ period \\ 2, server\ is\ free\ in\ active\ period \\ 3, server\ is\ on\ type\ I\ vacation \\ 4, server\ is\ on\ type\ II\ vacation \end{cases}$$

Let N(t) be the number of customers in the orbit at time t. Then the quasi birth-death process is a Markovian process represented by {N(t),S(t)} with state space {(n, k), n ≥ 0, k=1,3,4} U {(n, 2), n ≥1}.

Using Markov Process, the differential difference equations for the proposed model are

$$\frac{d}{dt}p_{0\,1}(t) = \beta p_{1\,2}(t) - (\lambda + \mu)p_{0\,1}(t) \tag{1}$$

$$\frac{d}{dt}p_{n\,1}(t) = \lambda p_{n-1\,1}(t) + (n+1)\beta p_{n+1\,2}(t) + \alpha p_{n\,2}(t) - (\lambda + \mu)p_{n\,1}(t), \quad n \geq 1 \tag{2}$$

$$\frac{d}{dt}p_{n\,2}(t) = v_1 p_{n\,3}(t) + v_2 p_{n\,4}(t) + \mu p_{n\,1}(t) - (\alpha + n\beta)p_{n\,2}(t), \ \ n \geq 1 \tag{3}$$

$$\frac{d}{dt}p_{0\,3}(t) = \mu p_{0\,1}(t) - (\gamma + v_1)p_{0\,3}(t) \qquad (4)$$

$$\frac{d}{dt}p_{n\,3}(t) = \gamma p_{n-1\,3}(t) - (\gamma + v_1)p_{n\,3}(t), \qquad n \geq 1 \qquad (5)$$

$$\frac{d}{dt}p_{0\,4}(t) = v_1 p_{0\,3}(t) - \delta p_{0\,4}(t) \qquad (6)$$

$$\frac{d}{dt}p_{n\,4}(t) = \delta p_{n-1\,4}(t) - (\delta + v_2)p_{n\,4}(t), \qquad n \geq 1 \qquad (7)$$

To obtain steady state equations, taking limit t → ∞ and using

$$\left.\begin{aligned} \lim_{t\to\infty} p_{n\,i}(\mathrm{t}) &= p_{n\,i} \\ \lim_{t\to\infty} \frac{d}{dt}p_{n\,i}(t) &= 0 \end{aligned}\right\} \quad i = 1, 2, 3, 4$$

The steady state equations are

$$(\lambda + \mu)p_{0\,1} = \beta p_{1\,2} \qquad (8)$$

$$(\lambda + \mu)p_{n\,1} = \lambda p_{n-1\,1} + (n+1)\beta p_{n+1\,2} + \alpha p_{n\,2}, \qquad n \geq 1 \qquad (9)$$

$$(\alpha + n\beta)p_{n\,2} = v_1 p_{n\,3} + v_2 p_{n\,4} + \mu p_{n\,1}, \qquad n \geq 1 \qquad (10)$$

$$(\gamma + v_1)p_{0\,3} = \mu p_{0\,1} \qquad (11)$$

$$(\gamma + v_1)p_{n\,3} = \gamma p_{n-1\,3}, \qquad n \geq 1 \qquad (12)$$

$$\delta p_{0\,4} = v_1 p_{0\,3} \qquad (13)$$

$$(\delta + v_2)p_{n\,4} = \delta p_{n-1\,4}\ , \qquad n \geq 1 \qquad (14)$$

Defining the probability generating functions as

$$P_i(z) = \sum_{n=0}^{\infty} p_{n\,i} z^n\,, \qquad i = 1,3,4 \qquad (15)$$

$$P_2(z) = \sum_{n=1}^{\infty} p_{n\,2} z^n \qquad (16)$$

Using equations (10), (11), (13)and P.G.Fs defined in(15) and (16), we get

$$z\beta P_2'(z) + \alpha P_2(z) = v_1 P_3(z) + v_2 P_4(z) + \mu P_1(z) - \left(\gamma + 2v_1 + \frac{v_1 v_2}{\delta}\right) p_{0\,3} \qquad (17)$$

From equations (8), (9), (15) and (16) we obtain

$$(\lambda+\mu-\lambda z)P_1(z) = \beta P_2'(z)+\alpha P_2(z) \tag{18}$$

Similarly using equations (11) and (12) along with (15), we get

$$(\gamma+v_1-\gamma z)P_3(z)=(\gamma+v_1)p_{0\,3}$$

$$P_3(z)=\frac{(\gamma+v_1)p_{0\,3}}{(\gamma+v_1-\gamma z)} \tag{19}$$

On similar steps from equations (13), (14) and (15) we obtain

$$P_4(z)=\frac{v_1(\delta+v_2)}{\delta(\delta+v_2-\delta z)}p_{0\,3} \tag{20}$$

Taking z=1 in equation (20),we obtain

$$P_4(1)=\frac{v_1(\delta+v_2)}{\delta v_2}p_{0\,3} \tag{21}$$

From equation (17)

$$z\beta P_2'(z)+\alpha P_2(z)=v_1P_3(z)+v_2P_4(z)+\mu P_1(z)-Ap_{0\,3} \tag{22}$$

where $A=\left(\gamma+2v_1+\frac{v_1v_2}{\delta}\right)$

Using equations (18), (22) together, after some rearrangement of terms we obtain

$$P_2'(z)+\frac{\alpha\lambda}{\beta(\lambda z-\mu)}P_2(z) = \frac{(\lambda+\mu-\lambda z)}{\beta(1-z)(\lambda z-\mu)}\left(v_1P_3(z)+v_2P_4(z)-Ap_{0\,3}\right) \tag{23}$$

To solve the differential equation (23)

Taking $\mathrm{I.F}=(\lambda z-\mu)^{\frac{\alpha}{\beta}}$

$$P_2(z)=(\lambda z-\mu)^{\frac{-\alpha}{\beta}}\int_0^z(\lambda x-\mu)^{\frac{\alpha}{\beta}}\frac{(\lambda+\mu-\lambda x)}{\beta(1-x)(\lambda x-\mu)}\left(v_1P_3(x)+v_2P_4(x)-Ap_{0\,3}\right)dx \tag{24}$$

Substituting value of $P_2'(z)$ in equation (18) and solving, we get

$$P_1(z) = \frac{(\alpha - \alpha z)P_2(z) - v_1 P_3(z) - v_2 P_4(z) + Ap_{0\,3}}{(\mu - z(\lambda + \mu - \lambda z))} \tag{25}$$

On differentiating equations (19),(20) we get

$$P_3'(z) = \frac{\gamma(\gamma+v_1)p_{0\,3}}{(\gamma + v_1 - \gamma z)^2} \tag{26}$$

$$P_4'(z) = \frac{v_1(\delta + v_2)}{\delta(\delta + v_2 - \delta z)^2} p_{0\,3} \tag{27}$$

Again, differentiating equations (26) and (27), we get

$$P_3''(z) = \frac{2\gamma^2(\gamma+v_1)p_{0\,3}}{(\gamma + v_1 - \gamma z)^3} \tag{28}$$

$$P_4''(z) = \frac{2\delta v_1(\delta + v_2)}{(\delta + v_2 - \delta z)^3} p_{0\,3} \tag{29}$$

Also, from equation (18), we get

$$P_2'(z) = \frac{(\lambda + \mu - \lambda z)P_1(z) - \alpha P_2(z)}{\beta} \tag{30}$$

Taking limit $z \to 1$in equations (19), (20), (24), (26), (27), (28) and (29) we get

$$P_3(1) = \frac{(\gamma+v_1)p_{0\,3}}{v_1} \tag{31}$$

$$P_4(1) = \frac{v_1(\delta + v_2)}{\delta v_2} p_{0\,3} \tag{32}$$

$$P_2(1) = (\lambda - \mu)^{\frac{-\alpha}{\beta}} \int_0^1 (\lambda x - \mu)^{\frac{\alpha}{\beta}} \frac{(\lambda + \mu - \lambda x)}{\beta(1 - x)(\lambda x - \mu)} (v_1 P_3(x) + v_2 P_4(x) - Ap_{0\,3})\, dx \tag{33}$$

$$P_3'(1) = \frac{\gamma(\gamma+v_1)}{(v_1)^2} p_{0\,3} \tag{34}$$

$$P_4'(1) = \frac{v_1(\delta + v_2)}{{v_2}^2} p_{0\,3} \tag{35}$$

$$P_3''(1) = \frac{2\gamma^2(\gamma+v_1)}{{v_1}^3} p_{0\,3} \tag{36}$$

$$P_4''(1) = \frac{2\delta v_1(\delta+v_2)}{{v_2}^3} p_{0\,3} \tag{37}$$

Taking limit $z \to 1$ in equation (25) and using L-Hospital rule, we get

$$P_1(1) = \frac{\alpha P_2(1) + v_1 P_3'(1) + v_2 P_4'(1)}{\mu - \lambda} \tag{38}$$

Taking limit $z \to 1$ in equation (30)

$$P_2'(1) = \frac{\mu P_1(1) - \alpha P_2(1)}{\beta} \tag{39}$$

On differentiating equation (25) and taking limit $z \to 1$ we get

$$P_1'(1) = \frac{\left(2\alpha P_2'(1) + v_1 P_3''(1) + v_2 P_4''(1)\right)(\mu-\lambda) + 2\lambda\left(\alpha P_2(1) + v_1 P_3'(1) + v_2 P_4'(1)\right)}{2(\mu-\lambda)^2} \tag{40}$$

All the P.G. F's are expressed in terms of $p_{0\,3}$ which is obtained by using normalization condition

$$\sum_{i=1}^{4} P_i(1) = 1 \tag{41}$$

It follows that,

$$p_{0\,3}\left[\left(\frac{\alpha+\mu-\lambda}{\mu-\lambda}\right)(\lambda-\mu)^{\frac{-\alpha}{\beta}} \int_0^1 (\lambda-\mu)^{\frac{\alpha}{\beta}} \frac{(\lambda+\mu-\lambda z)}{\beta(1-z)(\lambda z-\mu)} \left\{v_1\left(\frac{\gamma+v_1}{\gamma+v_1-\gamma z}\right) + \frac{v_1 v_2}{\delta}\left(\frac{\delta+v_2}{\delta+v_2-\delta z}\right) - A\right\} dz + \frac{\gamma+v_1}{v_1} + \frac{\gamma(\gamma+v_1)}{v_1(\mu-\lambda)} + \frac{v_1(\delta+v_2)}{v_2(\mu-\lambda)} + \frac{v_1(\delta+v_2)}{v_2\delta}\right] = 1 \tag{42}$$

$$p_{0\,3} = \left[\left(\frac{\alpha+\mu-\lambda}{\mu-\lambda}\right)(\lambda - \mu)^{\frac{-\alpha}{\beta}} \int_0^1 (\lambda-\mu)^{\frac{\alpha}{\beta}} \frac{(\lambda+\mu-\lambda z)}{\beta(1-z)(\lambda z-\mu)} \left\{ v_1 \left(\frac{\gamma+v_1}{\gamma+v_1-\gamma z}\right) + \frac{v_1 v_2}{\delta}\left(\frac{\delta+v_2}{\delta+v_2-\delta z}\right) - A \right\} dz + \frac{\gamma+v_1}{v_1} + \frac{\gamma(\gamma+v_1)}{v_1(\mu-\lambda)} + \frac{v_1(\delta+v_2)}{v_2(\mu-\lambda)} + \frac{v_1(\delta+v_2)}{v_2\delta}\right]^{-1} \tag{43}$$

4. Important performance measures

In this section, we present some of the important performance measures of the system as follows.

The expected number of customers in the orbit is

$$\mathrm{E}[L_0] = \sum_{i=1}^{4} P_i'(1) \tag{44}$$

The expected number of customers in the system is

$$\mathrm{E}[L_s] = \mathrm{E}[L_0] + P_1(1) \tag{45}$$

Probability of server in type I vacation

$$Pr_{V1} = P_3(1)$$

$$= \sum_{n=0}^{\infty} p_{n\,3}$$

$$= \frac{(\gamma+v_1)p_{0\,3}}{v_1} \tag{46}$$

Probability of server in type II vacation

$$Pr_{V2} = P_4(1)$$

$$= \sum_{n=0}^{\infty} p_{n\,4}$$

$$= \frac{v_1(\delta + v_2)}{\delta v_2} p_{0\,3} \tag{47}$$

Probability of server on vacations

$$Pr_V = Pr_{V1} + Pr_{V2}$$

$$= \frac{(\gamma + v_1)p_{0\,3}}{v_1} + \frac{v_1(\delta + v_2)}{\delta v_2} p_{0\,3} \tag{48}$$

Probability of server in working (active) state

$$Pr_N = P_1(1) + P_2(1)$$

$$= \sum_{n=0}^{\infty} p_{n\,1} + \sum_{n=1}^{\infty} p_{n\,2} \tag{49}$$

5. Graphical results

In this section, we illustrate the effect of various parameters on some of the performance measures of system. We have also optimized the cost with respect to service rate.

In the below graphs, we have set λ=1.2, μ=3, β=2, γ=0.6, α=1, $v_1 = 0.6$, $v_2 = 1$, δ=0.8 unless they are varied in the graphs.

5.1 Sensitivity analysis

For qualitative analysis of the proposed model, we represent some of the numerical results graphically.

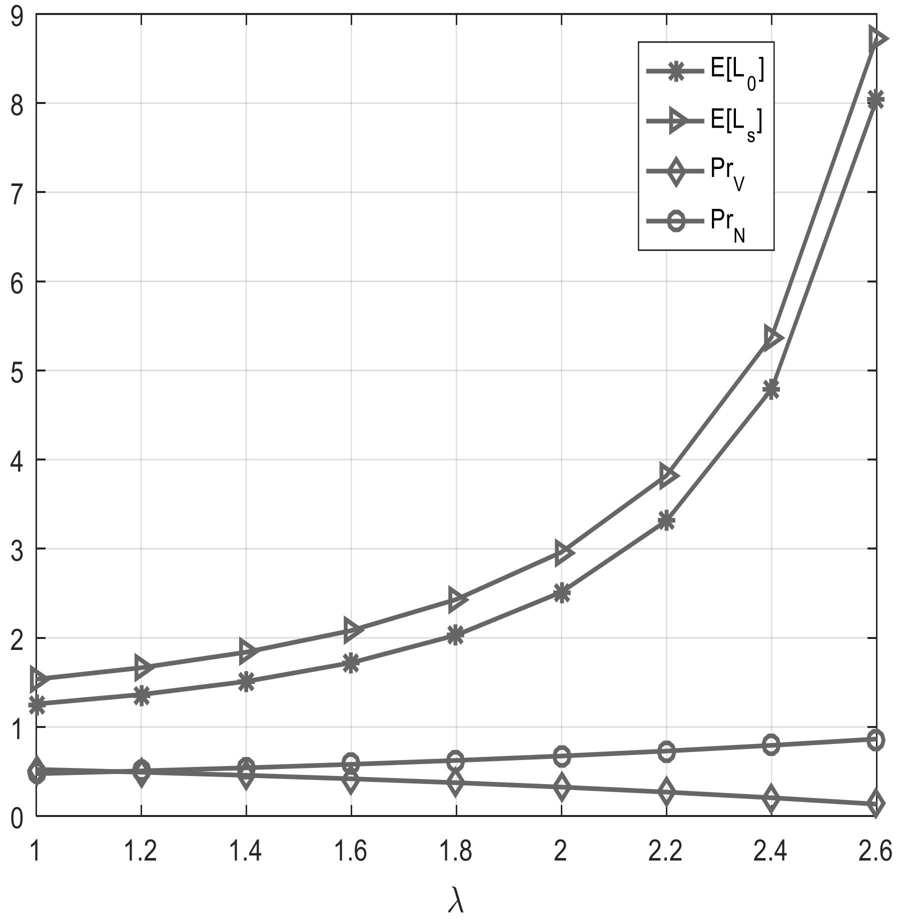

Figure1. Effect of active state arrival rate (λ) on system performance measures.

From figure1, we observe that with the increase in arrival rate λ, expected orbit length, system length and probability of normal state increase, whereas the probability of vacation decreases. This is explained by the fact that with the increase in arrival rate, the number of customers increases in orbit and in system. Hence, the probability of normal state increases and thereby, the probability of vacation decreases.

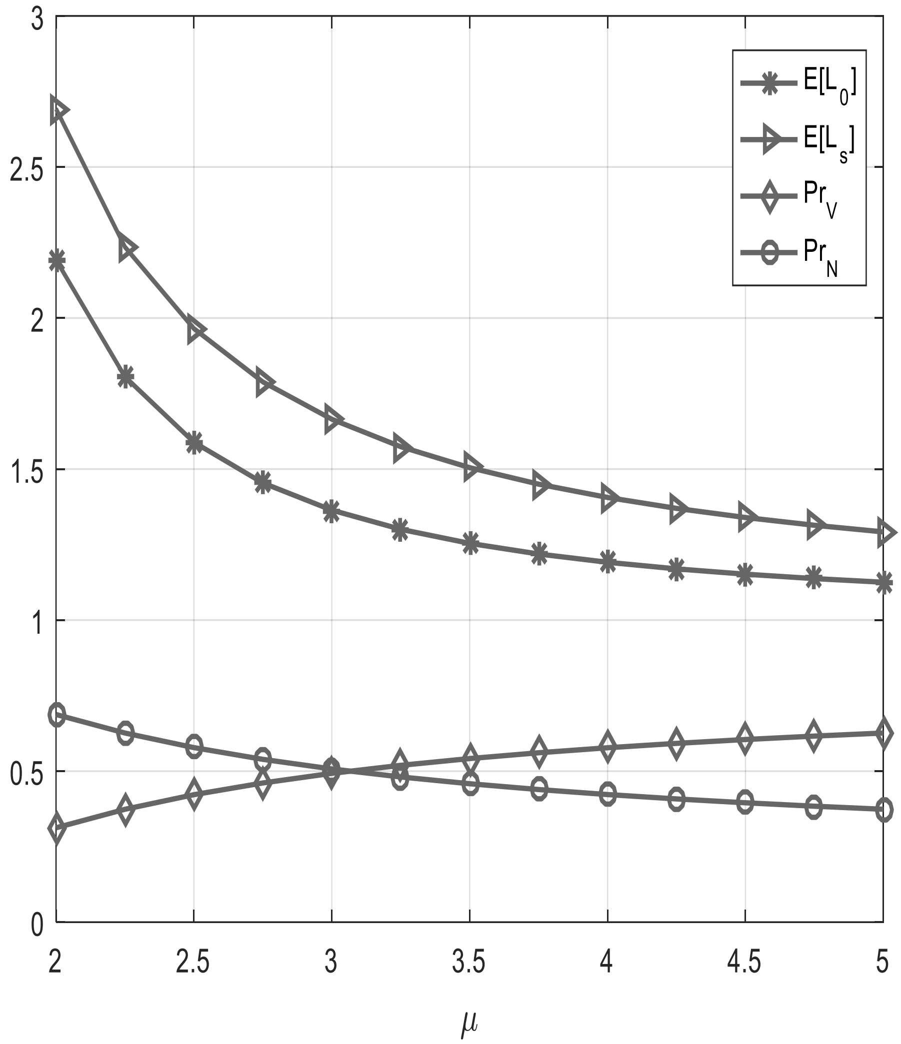

Figure 2. Effect of service rate (μ) on system performance measures.

Figure 2 reveals that the expected length of orbit, system and probability of normal (active) state decrease, but the probability of vacation state increases with an increase in service rate μ. The reason being that with the increase in μ, the customers will be served fasterand this reduces the number of customers in orbit and hence in the system. Also, due to faster service, the probability of normal period decreases and this increases the probability of a vacation period.

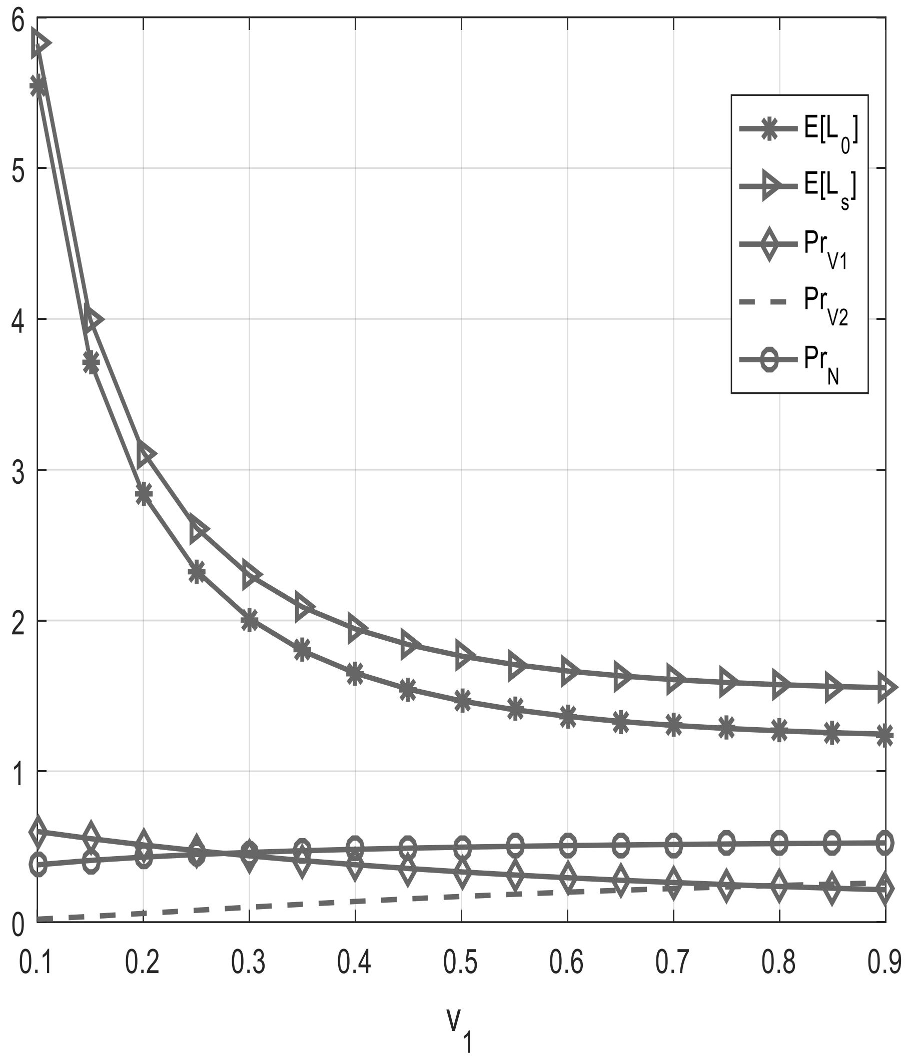

Figure3.Effect of rate of type I vacation (v_1) on system performance measures.

From figure 3, we see that as the type I vacation rate increases, the expected length of orbit, expected length of system and probability of type I vacation decrease but the probability of type II vacation and probability of normal (active) state increase. The fact behind the observation is that with the increase in type I vacation rate, the duration of type I vacation decreases and this causes increase in probability of normal state and the probability of type II vacation. Due to which the expected number of customers in orbit and that in the system decrease.

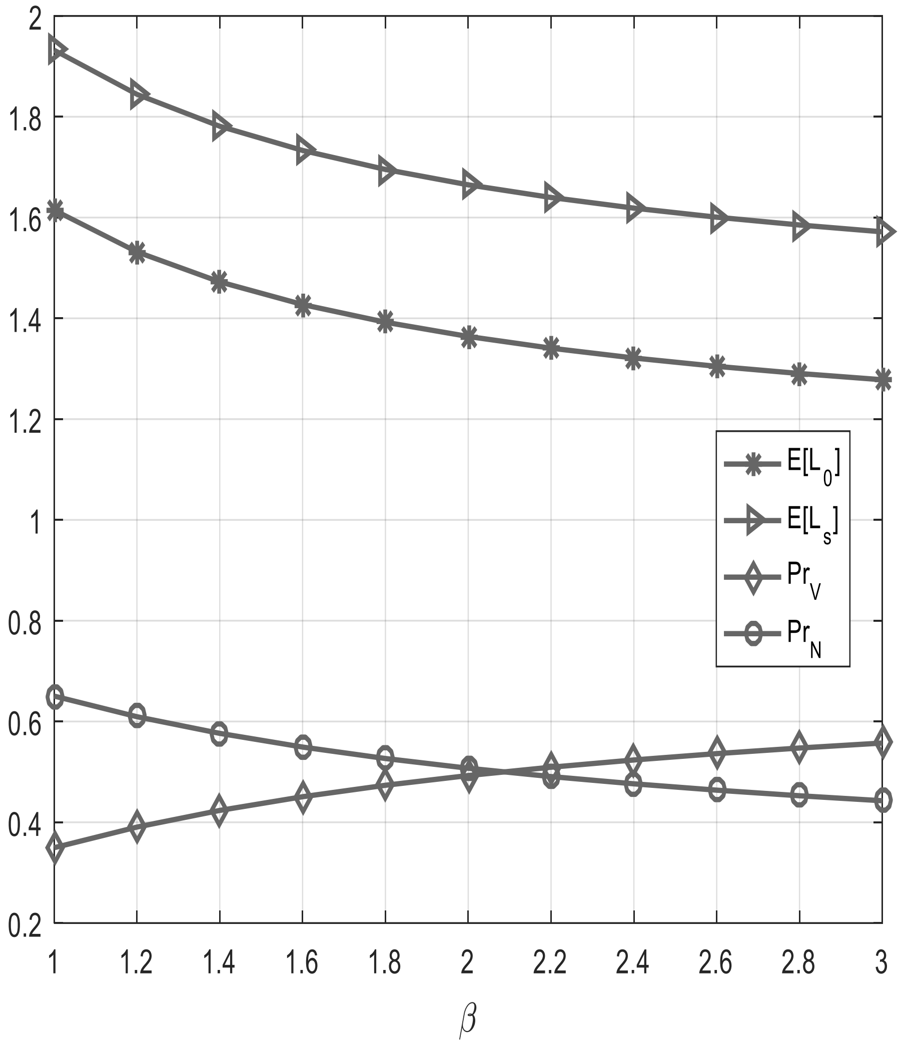

Figure 4. Effect of variation in retrial rate (β) on system performance measures.

Figure 4 shows the effect of change in retrial rate on expected orbit length, system length, probability of vacation and active server states. The graphical results obtained here matches the intuitive expectations.

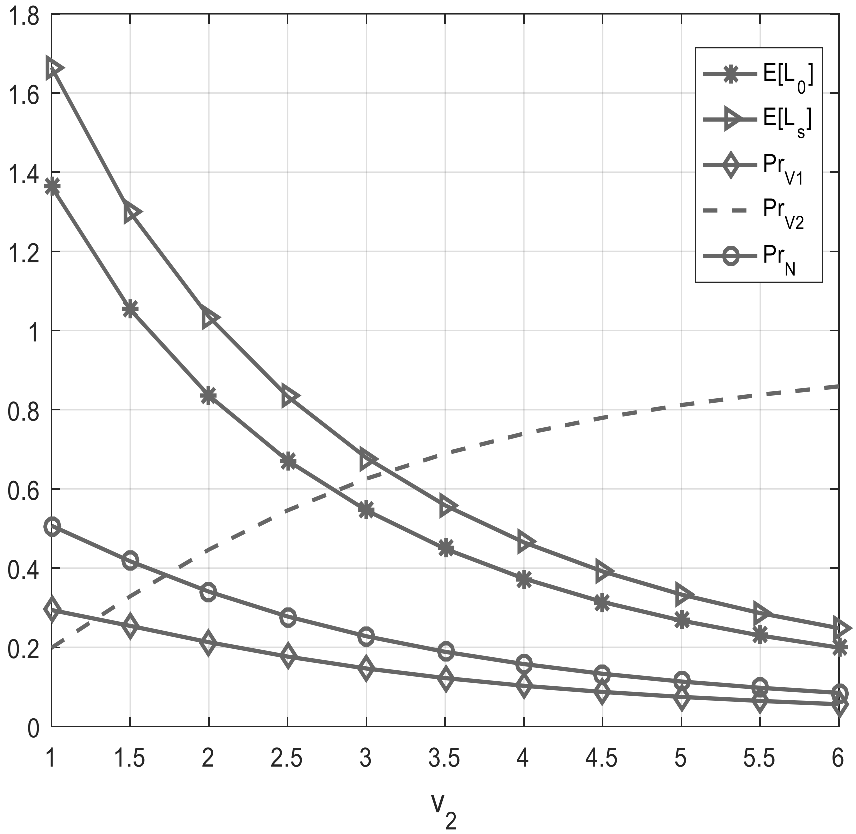

Figure 5. Effect of variation in rate of type II vacation (v_2) on system performance measures.

Figure5 represents that expected orbit length,expected system length, probability of server in normal state and probability of type I vacation decrease as the rate of type II vacation increase. As the type II vacation rate increases, the duration of type II vacation decreases: hence, the expected queue length and system length decrease.

5.2 Cost analysis

In this subsection, we optimize the operating cost function with respect to service rate in working state. To obtain the optimal value ofμ, some cost elements are taken as

C_L = Cost per unit time for each customer present in the orbit.

C_μ = Cost per unit time for service in working state.

C_{v1}= Cost per unit time in type I vacation.

C_{v2}= Cost per unit time in type II vacation.

The corresponding cost function per unit time is defined as

F(μ) = $C_L E[L_0]$ + μC_μ + $v_1 C_{v1}$+ $v_2 C_{v2}$

We takeC_L= 20, C_μ=28, C_θ=10, C_ϕ=8 in the parabolic method for obtaining optimal cost F(x) and the corresponding value of x. Parabolic-method works by generating quadratic function through calculated points in every iteration to which the function F(x) can be approximated. The point at which F(x) is optimum in three- point pattern $\{x_1, x_2, x_3\}$ is given by

$$x_L = \frac{0.5(F(x_1)(x_2{}^2 - x_3{}^2) + F(x_2)(x_3{}^2 - x_1{}^2) + F(x_3)(x_1{}^2 - x_2{}^2))}{F(x_1)(x_2 - x_3) + F(x_2)(x_3 - x_1) + F(x_3)(x_1 - x_2)}$$

The new value obtained replaces one of the three points to improve the current three-point pattern. The process is repeatedly applied until optimum value is obtained up to the desired degree of accuracy.

Table 1 shows that optimum value $F(\mu)$ =112.83101corresponding to μ= 2.15566 with the permissible error of 10^{-4}, which is verified by Figure 6.

Table 1. Optimization of cost by parabolic method

x_1	x_2	x_3	$F(x_1)$	$F(x_2)$	$F(x_3)$	x_L
1.70	**2.00**	**2.50**	**127.37253**	**113.85024**	**115.77571**	**2.21852**
2.00	**2.21852**	**2.50**	**113.85024**	**112.95905**	**115.77571**	**2.18165**
2.00	**2.18165**	**2.21852**	**113.85024**	**112.85383**	**112.95905**	**2.16269**
2.00	**2.16269**	**2.18165**	**113.85024**	**112.83273**	**112.85383**	**2.15844**
2.00	**2.15844**	**2.16269**	**113.85024**	**112.83128**	**112.83273**	**2.15648**
2.00	**2.15648**	**2.15844**	**113.85024**	**112.83103**	**112.83128**	**2.15594**
2.00	**2.15594**	**2.15648**	**113.85024**	**112.83101**	**112.83103**	**2.15572**
2.00	**2.15572**	**2.15594**	**113.85024**	**112.83100**	**112.83101**	**2.15566**

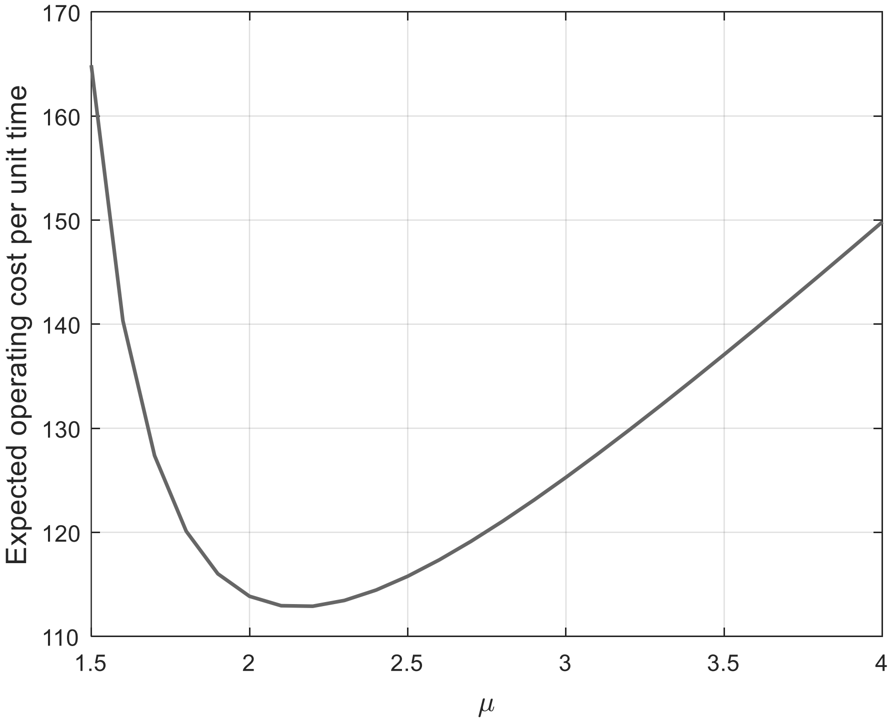

Figure 6. Variation in expected operating cost per unit time with service rate (μ)

6. Conclusion and future scope

In this paper, we have analyzed single server Markovian queueing model with state dependent arrival rates of customers under differentiated vacations and classical retrial policy. The closed form expressions for various performance measures are derived with the help of probability generating functions. The performance of the proposed model is represented graphically using MATLAB software. The operating cost of the queueing system is optimized with respect to service rate of the server. The model can be extended to multiple servers.

Conflicts of interests

The authors declare that there is no conflict of interests.

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Certain results on metric and norm in fuzzy multiset setting

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Abstract

Fuzzy multiset is an extension of fuzzy set in multiset framework. In this paper, we review the concept of fuzzy multisets and study the notions of metric and norm on fuzzy multiset. Some results on metric and norm are established in fuzzy multiset context.

Keywords: fuzzy set; fuzzy multiset; metric; norm.

2010 AMS subject classification: 03E72, 08A72, 20N25.

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1. Introduction

The theory of set proposed by Cantor in 1915, is a collection of well-defined objects of thought and intuition. The limitation of set theory is its inability to deal with the vague properties of its member or element, and likewise its distinctness property which does not allows repetition in the collection. In other to handle the vague property of a set, Zadeh [19] proposed a mathematical model that deal with vagueness of a set known as fuzzy sets. The distinct property of crisp set has been violated by allowing repetition of an element in a collection. This gave birth to a set called multiset. The term multiset was first suggested by De Bruijn to Knuth in a private correspondence as noted in [13]. The theory of multisets has been studied [3, 4, 10, 11, 12, 16]. Lake [14] presented an abridge account on sets, fuzzy sets, multisets and functions.

By synthesizing the concepts of fuzzy sets and multisets, Yager [18] introduced the concept of fuzzy multiset (FMS) that deal with vagueness property of a set and allowed the repetition of its membership function. In fact, fuzzy bag or fuzzy multiset generalizes fuzzy sets in such a way that the membership degree of a fuzzy set is allowed to repeat. Some fundamentals properties of fuzzy bags have been studied [6, 15]. The concept of fuzzy bags has been applied in multi-criteria decision-making [1, 2], sequences [5] and computational science [17].

Metric is a function that defines a concept of distance between any members of the set, which are usually called points. The notions of metric and norm have been extended to the environment of fuzzy sets [7, 8, 9]. In this work, we present the notions of norm and metric in fuzzy multiset context.

2. Preliminaries

In this section, we review some definitions and result that are important for the main work.

Definition 2.1 [18]. Assume X is a set of elements. Then, a fuzzy bag/multiset, A drawn from X can be characterized by a count membership function CM_A such that $CM_A : X \to Q$, where Q is the set of all crisp bags or multisets from the unit interval, $I = [0,1]$.

According to Syropoulos [17], a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0,1] \to N,$$

where $I = [0,1]$ and $N = \mathbb{N} \cup \{0\}$.

The count membership degrees, $CM_A(x)$ for $x \in X$ is given as

$CM_A(x) = \{ \mu_A^1(x), \mu_A^2(x), \dots, \mu_A^n(x), \dots \}$,

where $\mu_A^1(x), \mu_A^2(x), \dots, \mu_A^n(x), \dots \in [0,1]$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq \mu_A^3(x)$, $\geq \dots \geq \mu_A^n(x) \geq \dots$, whereas in a finite case, we write

$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), \dots, \mu_A^n(x)\}$ for $\mu_A^1(x) \geq \mu_A^2(x) \geq \dots \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented in the form

$$A = \{< \frac{CM_A(x)}{x} > \mid x \in X\} \text{ or } A = \{< x, CM_A(x) > \mid x \in X\}.$$

In a simple term, a fuzzy multiset, A of X is characterized by the count membership function, $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0,1]$. We denote the set of all fuzzy multisets by $FMS(X)$.

Example 2.2. Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{0.7, 0.6, 0.1\}, CM_A(b) = \{0.9, 0.7, 0.5\}, CM_A(c) = \{0.5, 0.4, 0.2\}$, A is a fuzzy multiset of X written as

$$A = \{< \frac{0.7, 0.6, 0.1}{a} >, < \frac{0.9, 0.7, 0.5}{b} >, < \frac{0.5, 0.4, 0.2}{c} >\}.$$

Definition 2.3 [15]. Let $A, B \in FMS(X)$. Then, A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x)\ \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.4 [15]. Let $A, B \in FMS(X)$. Then, A and B are comparable to each other if and only if $A \subseteq B$ *or* $B \subseteq A$, *and* $A = B \Leftrightarrow CM_A(x) = CM_B(x)\ \forall x \in X.$

Definition 2.5 [17]. Let $A, B \in FMS(X)$.Then, the intersection and union of A *and* B, denoted by $A \cap B$ and $A \cup B$, are defined by

(i) $CM_{A\cap B}(x) = CM_A(x) \wedge CM_B(x)\ \forall x \in X.$
(ii) $CM_{A\cup B}(x) = CM_A(x) \vee CM_B(x)\ \forall x \in X.$

Definition 2.6 [17]. Let $A, B \in FMS(X)$. Then, the sum of A and B, denoted by $A \bigoplus B$, is defined by the addition operation in $X \times [0,1]$ for crisp multiset. That is,
$CM_{A\bigoplus B}(x) = CM_A(x) + CM_B(x)\ \forall x \in X$

The addition operation is carry out by merging the membership degree in a decreasing order.
Definition 2.7 [6]. Let $A, B \in FMS(X)$. Then, the difference of B from A is a fuzzy multiset $A \ominus B$ such that $\forall x \in X,\ CM_{A\ominus B}(x) = CM_A(x) - CM_B(x) \vee 0.$

Definition 2.8 [6]. Let $A, B \in FMS(X)$. Then, the complement of A is a fuzzy multiset A' such that $\forall x \in X,\ CM_{A'}(x) = 1 - CM_A(x)$. **Metric and Norm defined over Fuzzy Multisets**

3. Metric and norm defined over fuzzy multisets

In this section, we present metrics and norm defined over fuzzy multiset.
Definition 3.1. Let X be an arbitrary non-empty set and let $A, B \in FMS(X)$. A metric or distance function between A and B on X is a function $d: X \times X \rightarrow [0,1]$ with the following properties:

(i) $d\ (CM_A(x), CM_B(x)\ ) \geq 0\ \forall\ x\ \in X.$

(ii) $d\ \left(CM_A(X), CM_B(x)\right) = 0$ iff $CM_A(x) = CM_B(x)\ \forall\ x\ \in X.$

(iii) $d\ \left(CM_A\ (x), CM_B(x)\right) = d\ \left(CM_B(x), CM_A(x)\right)\ \forall\ x\ \in X.$

(iv)
$$d\ \left(CM_A(x), CM_C(x)\right) \leq d\ (CM_A(x), CM_B(x)) + d\ \left(\ CM_B(x), CM_C(x)\right)$$

$\forall\ x \in X$ if $C \in FMS(X)$.

Note:
(i) The distance is a non-negative function and only zero at a single point.
(ii) The distance is a symmetric function.
(iii) The distance satisfy triangle.
Proposition 3.2. Let $A, B, C\ \in FMS(X)$. Then $d(A, B) =\ |A - B|$ is a metric defined on $\mathrm{FMS(X)}$.

Proof. We use Definition 3.1:
Axiom (i)

$$d(A, B) = d\big(CM_A(x), CM_B(x)\big) =\ |A - B| =\ |CM_A(x) - CM_B(x)|$$

$$= V\{CM_A(x) - CM_B(x), 0\ \} \geq 0.$$

Axiom (ii)
If
$d(A, B) = 0\ \Longrightarrow |CM_A(x) - CM_B(x)| = 0 \Longrightarrow CM_A(x) - CM_B(x) = 0 \Longrightarrow$ $CM_A(x) = CM_B(x)$

.
Conversely, if $A = B \Longrightarrow d(A,A)\ =\ |A - A|\ =\ |0|$.

Axiom (iii)
$d(A, B)\ =\ |A - B| =\ |-1||B - A|\ = |B - A| = d\ (B, A)$.

Axiom (iv)

$d(A, B) =\ |A - B| =\ |A - C + C - B| \geq |A - C| +\ |C - B| \geq d(A, C) + d(C, B)$.

The following are distances between fuzzy multisets:
Hamming distance;

$$d(A, B) =\ \sum_{i=1}^{n} |CM_A(x) - CM_B\ (x)|.$$

Euclidean distance;

$$d\ (A, B) =\ \sqrt{\sum_{i=1}^{n} (CM_A(x) - CM_B(x))^2}$$

Normalized Hamming distance;

$$d(A, B) = \frac{1}{n} \sum_{i=1}^{1} |CM_A(x) - CM_B(x)|.$$

Normalized Euclidean distance;

$$d(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(CM_A(x) - CM_B(x))^2}.$$

Theorem 3.3. Let X be non-empty set and $A, B \in FMS(X)$, then

$d(A,B) = d(A',B')$.

Proof. We show that $d(A,B) = d(A',B')$ or

$d(CM_A(x), CM_B(x)) = d(CM_{A'}(x), CM_{,}(x))$. But $CM_{A'}(x) = 1 - CM_A(x)$ and $CM_{B'}(x) = 1 - CM_B(x)$.

Thus, $d(CM_A(x), CM_B(x)) = |CM_A(x) - CM_B(x)|$

$$\begin{aligned} &= |[1 - CM_{A'}(x)] - [1 - CM_{B'}(x)]| \\ &= |-CM_{A'}(x) + CM_{B'}(x)| \\ &= |-1||CM_{A'}(x) - CM_{B'}(x)| \\ &= |CM_{A'}(x) - CM_{B'}(x)| \\ &= d(CM_{A'}(x), CM_{B'}(x)) \end{aligned}$$

Hence $d(CM_A(x), CM_B(x)) = d(CM_{A'}(x), CM_{B'}(x))$.

Corollary 3.4. If $d(CM_A(x), CM_B(x))$ is a distance of fuzzy multiset of A and B, then

$d^*(CM_A(x), CM_B(x)) = \frac{1}{2}[d(CM_A(x), CM_B(x)) + d(CM_A(x), CM_B(x))]$**Proof.** Clearly,

$$\begin{aligned} d^*(CM_A(x), CM_B(x)) &= \frac{1}{2}[d(CM_A(x), CM_B(x)) + d(CM_A(x), CM_B(x))] \\ &= d(CM_A(x), CM_B(x)). \end{aligned}$$

Proposition 3.5. If $d(CM_A(x), CM_B(x))$ is a metric of fuzzy multiset A and B, then $d(CM_A(x), CM_B(x)) - d(CM_B(x), CM_A(x)) = 0$.

Proof. By Definition 3.1, if $d(CM_A(x), CM_B(x)) = d(CM_B(x), CM_A(x))$, so it follows that $d(CM_A(x), CM_B(x)) - d(CM_B(x), CM_A(x)) = 0$.

Proposition 3.6. Let $\lambda \in \mathcal{R}$ and $d(CM_A(x), CM_B(x))$ is a metric defined on $FMS(x)$. Then $\lambda d(CM_A(x), CM_B(x))$ is also a metric.

Proof. The proof is obvious, since $\lambda d\left(CM_A(x), CM_B(x)\right) = \ d(\lambda(CM_A(x), CM_B(x))) =$

$d\left(\lambda CM_A(x), \lambda CM_B(x)\right)$.

Hence $\lambda d\left(CM_A(x), CM_B(x)\right)$ is a metric.

Corollary 3.7. If $\lambda > 1$, then $\lambda d\left(CM_A(x), CM_B(x)\right) \geq d\left(CM_A(x), CM_B(x)\right)$.

Proof. The proof is straightforward.

Corollary 3.8. If $\lambda < 1$ then $\lambda d\left(CM_A(x), CM_B(x)\right) < d\left(CM_A(x), CM_B(x)\right)$.

Proof. The proof is straightforward.

Corollary 3.9. If $\lambda < 0$ then $\lambda d\left(CM_A(x), CM_B(x)\right) = d\left(CM_B(x), CM_A(x)\right)$

Proof. The proof is straightforward.

Definition 3.10. Let X be a non-empty set and A be a fuzzy multiset of X. A non-negative real-valued function $\|.\|$ defined on A is called a norm if the following properties are satisfied:

(i) $\|A\| = 0$ iff $A = 0$ that is, $\|CM_A(x)\| = 0$ iff $CM_A(x) = 0$.

(ii) $\|\alpha A\| = \ |\alpha|\|A\|$ which implies that

$\|CM_{\alpha A}(x)\| = \ |\alpha|\,\|CM_A(x)\|\ \forall\, \alpha \in \mathcal{R}$.

(iii) $\|A + B\| \ \leq \ \|A\| + \ \|B\|$ which implies

that $\|CM_{A+B}(x)\| \leq \ \|CM_A(x)\| + \ \|CM_B(x)\|$.

The Fuzzy multiset equipped with a norm is called Normed Fuzzy multiset.

Proposition 3.11. Let $A, B \ \in FMS(x)$, then $\|A + B\| = \ \|A - B\|$.

Proof. We show that $\|A \ + B\ \| \ = \ \|A - \ B\|$. Now,

$$\|A - \ B\| \ = \ \|CM_{A-\ B}(x)\| = \left\|CM_{A\ +(-B)}(x)\right\|$$

$$= \ \|CM_A(x)\| + \ |-1|\|CM_B(x)\| = \|CM_A(x)\| + \|CM_B(x)\|$$

$$= \ \|CM_{A\ +B}(x)\| = \|A \ + B\|.$$

Proposition 3.12. Let $A \in FMS(x)$ and a norm $\|.\|$ define over $A\ as\ \|A\| = |A|$.

Proof.

(i) $\|A\| \ = \ |CM_A(x)| \ = CM_A(x) > 0$.

(ii) $\|\alpha A\| \ = \ |\alpha CM_A(x)| = |\alpha|\|CM_A(x)\| \ = \ |\alpha||CM_A(x)|$.

(iii) $\|A+B\| = \|CM_{A+B}(x)\| = |CM_{A+B}(x)| \leq |CM_A(x)| + |CM_B(x)|$

$= \|A\| + \|B\|$.

Hence $\|A\| = |A|$ is a norm defined over fuzzy multiset A.

Corollary 3.13. If $\alpha > 0$, then $\|\alpha A\| > \|A\|$ and if $\alpha \in [0,1]$, then $\|\alpha A\| < \|A\|$.

Proof. The proof is obvious.

4 Conclusions

We have presented a brief review on the concept of fuzzy multisets and explored metric and norm in fuzzy multiset context. A number of results on metric and norm were established, respectively.

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Recognizable Hexagonal Picture Languages and xyz Domino Tiles

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Rosini Babu Rajendran Pillai†

Abstract

In this paper we introduced xyz local hexagonal picture languages, where the usual notion of hexagonal tiles of size (2, 2, 2) are replaced by xyz dominoes, motivated by the studies of xyz domino systems. This new formalism is used for checking recognizability of hexagonal pictures. It is noticed that non- regular hexagonal pictures can also be studied in the place of regular pictures. Recognizability of xyz local hexagonal picture were studied and the fact that every recognizable hexagonal p[icture languages can be obtained as a projection of xyz local hexagonal picture languages.
Keywords: xyz domino system, Local hexagonal picture languages, recognizability of xyz domino system, mapping, hexagonal pictures, x domino, y domino, z domino.[1]

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1 Introduction

A picture is used to understand things in a better way. Hexagonal pictures and tiles has got many significances. A lot of technologies are there to compute pictures with the help of computers. This resulted in the introduction of picture generating models. In [D.Giammarresi, 1992] Giammerresi et.al proposed the recognizability of picture languages. The languages in recognizable pictures were defined using tiling systems. Hexagonal pictures have got various uses particularly in picture processing and image analysis. Hexagonal arrays on triangular grid are viewed as two dimensional representation of three dimensional blocks and perceptual twins of pictures of a given set of blocks[KS, 2005]. Since late seventies, formal models to generate or recognize hexagonal pictures has been found in literature in the frame work of pattern recognition and image analysis.

Recently searching for a new method for defining hexagonal pictures has moved towards the new definition for recognizable languages generated by hexagonal pictures which inherits many properties from existing cases, in [Giammarresi, 1966],. Local and recognizable hexagonal picture languages in terms of hexagonal tiling system were introduced and studied in [KS, 2005]. In [Dersanambika, 2004] K S Dersanambika et.al. define xyz- domino systems and charecterised hexagonal pictures using this. Subsequently hexagonal hv-local picture languages via hexagonal domino systems were introduced in the light of two dimensional domino system introduced by Latturex et.al [Latteurx, 1997]. Hexagonal arrays and hexagonal patterns are found in picture processing and image analysis [Dersanambika, 2004].

It is very natural to consider hexagonal tiles on triangular grid, we require certain hexagonal tiles only to present in each hexagonal pictures of a hexagonal picture languages. This leads to recognizable hexagonal pictures and the hexagonal tiling systems. The xyz domino tiling characterize the hexagonal picture languages. So we define xyz local hexagonal picture languages over the usual notion of a hexagonal picture of size (2, 2, 2) and proved that every recognizable hexagonal picture language can be obtained as some projection of these languages.

2 Recognizability of hexagonal pictures

In this part we review the notions of formal language theory and some of the basic concepts on hexagonal pictures and hexagonal picture languages [Anitha, 2011].
Let Σ be a finite alphabet of symbols. A hexagonal picture p over Σ is a hexagonal array of symbols of Σ. The set of all hexagonal arrays of the alphabet Σ is denoted by Σ^{**H}. A hexagonal picture over the alphabet a, b, c d is shown in Figure 1.

$$\begin{array}{ccccccccc} & & a & & a & & b & & \\ & a & & b & & f & & c & \\ b & & c & & c & & a & & c \\ & d & & a & & a & & d & \\ & & c & & c & & a & & \end{array}$$

Figure 1

With respect to a triad of triangular axes (x, y, z) the co-ordinates of each element of the hexagonal picture in Figure 2(a) and Figure 2(b) respectively are given below [KS, 2005].

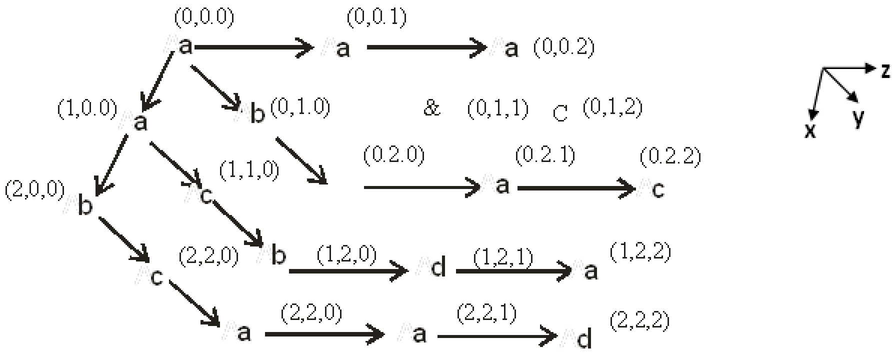

Figure2(a)

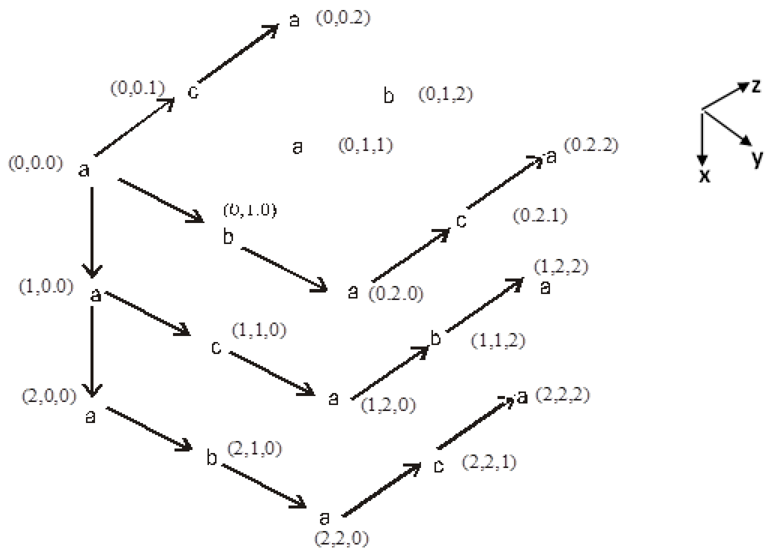

Figure2(b)

If $p \in \Sigma^{**H}$, then $\hat{p}$ is the hexagonal picture obtained by surrounding p with a special boundary $\#$ is called a bordered hexagonal picture which is shown in Figure 3.

$$
\begin{array}{ccccccccccccc}
 & & & \# & & \# & & \# & & \# & & & \\
 & & \# & & a & & b & & c & & \# & & \\
 & \# & & b & & b & & a & & b & & \# & \\
\# & & c & & a & & b & & c & & f & & \# \\
 & \# & & d & & b & & e & & d & & \# & \\
 & & \# & & c & & f & & a & & \# & & \\
 & & & \# & & \# & & \# & & \# & & &
\end{array}
$$

Figure 3

Let $l_1(p) = l, l_2(p) = m, l_3(p) = n$ be the size of the hexagonal arrays. We write$p = (l, m, n)$, the size of a picture. For a picture p of size (l, m, n) we have the bordered picture $\hat{p}$ is of size$(l + 1, m + 1, n + 1)$.

Now we see the projections of hexagonal picture and projections of a language. Γ and Σ be two finite alphabets and $\pi : \Gamma \rightarrow \Sigma$ be a mapping, this mapping π is called a projection.

A hexagonal tile is of the form as shown in Figure 4.

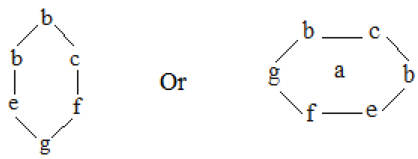

Figure 4

Given a hexagonal picture p of size (l, m, n) we denote the set of hexagonal subpicture of p of size (2, 2, 2) is called a hexagonal tile of size (2, 2, 2). Figure 4 denote a hexagonal tile of size (2, 2, 2).

A hexagonal tiling system [Dersanambika, 2004] T is a 4-tuple $(\Sigma, \Gamma, \pi, \theta)$ where Σ and Γ are two finite set of symbols. $\pi : \Gamma \longrightarrow \Sigma$ is a projection and θ is the set of hexagonal tiles over the alphabet $\Gamma \cup \{\#\}$.

Definition 2.1. *A hexagonal sub picture $\hat{p'}$ is a picture which is a hexagonal sub array of the picture $\hat{p}$. Given a hexagonal picture $\hat{p}$ then $B_{l,m,n}(\hat{p})$ denotes the set of hexagonal sub pictures of size l, m, n.*

For hexagonal pictures there are three types of concatenations namely type 1, type 2, and type 3 refer [Anitha, 2011]. A hexagonal picture language is recognizable if there exist a local language L' over an alphabet Σ and a mapping $\pi : \Gamma \longrightarrow \Sigma$ such that L $\subseteq \pi(L')$.

3 xyz local hexagonal picture languages

In this section we introduce the notion of xyz local hexagonal picture languages where the hexagonal tiles of size (2, 2, 2) are replaced by xyz dominos.

Definition 3.1. *L be a hexagonal picture language included in* Σ^{**H}. *L is said to be xyz local if there exist a set* Δ *of x,y,z dominoes over* $\Sigma \cup \{\#\}$ *such that L =* $\{q \in \Sigma^{**H} | T_{1,1,2}(q) \cup T_{1,2,1}(q) \cup T_{2,1,1}(q) \subseteq \Delta\}$

Example 3.1. *If we consider the hexagonal picture langauages L over the alphabet* $\Sigma = \{0, 1\}$ *then all the hexagonal picture of can be obtained by the concatenation of xy,yz,xz dominoes.*

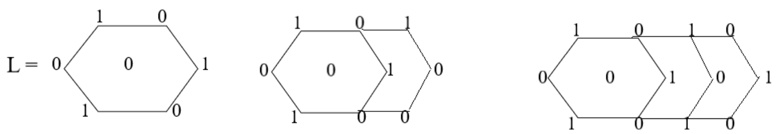

Figure 5

The hexgonal picture so obtained is local as we can associate a set of xyz dominoes Δ as follows which generates the whole picture. Here figure 6 show the picture generated by x dominoes, figure 7 shows the y dominoes, while figure 8 shows the z dominoes.

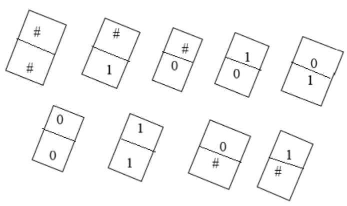

Figure 6

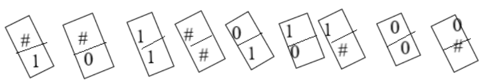

Figure 7

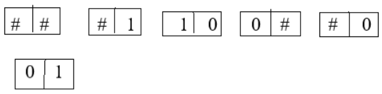

Figure 8

Theorem 3.1. *Let L $\subseteq \Sigma^{**H}$ be a hexagonal picture language. If L is xyz local, then L is local.*

Proof. **proof** Let L $\subseteq \Sigma^{**H}$ be a xyz local hexagonal picture language. For provingt L is local, construct a local hexagonal picture language L' and show that L = L'. We know that there exist a set Δ of xyz dominoes over
$\Sigma \cup \{\#\}(\Delta \subseteq (\Sigma \subseteq \{\#\})^{(1,1,2)} \cup (\Sigma \subseteq \{\#\})^{(1,2,1)} \cup (\Sigma \subseteq \{\#\})^{(2,1,1)})$
such that
$L = \{p \in \Sigma^{**H} | T_{1,1,2}(\hat{p}) \cup T_{1,2,1}(\hat{p}) \cup T_{2,1,1}(\hat{p}) \subseteq \Delta\}$.
We define the set of hexagonal pictures $\Delta\,'$,
$\Delta\,'$ = $q \in (\Sigma \cup \{\#\})^{(2,2,2)} | T_{1,2,1}(\hat{q}) \cup T_{1,1,2}(\hat{q}) \cup T_{2,1,1}(\hat{q}) \subseteq \Delta\}$.
Let $L' = \{p \in \Sigma^{**H} | T_{2,2,2}(\hat{p}) \subseteq \Delta\,'\}$.
Clearly L' is local. Now let $p \in L'$.
Then $T_{2,2,2}(\hat{p}) \subseteq \Delta\,'$ and $T_{1,1,2}(\hat{p}) \subseteq T_{1,1,2}(T_{2,2,2}(\hat{p})) \subseteq T_{1,1,2}(\Delta\,') \subseteq \Delta$.
Similarly $T_{2,1,1}(\hat{p}) \subseteq \Delta$ and $T_{1,2,1}(\hat{p}) \subseteq \Delta$. Hence $p \in$ L.
Therefore $L' \in$ L.
To show that L $\in L'$. Let $p \in$ L, let $q \in$ L and $a \in T_{2,2,2}(\hat{q})$. Then $T_{1,2,1}(a) \subseteq T_{1,2,1}(\hat{q}) \subseteq \Delta$,
$T_{2,1,1}(a) \subseteq T_{2,1,1}(\hat{q}) \subseteq \Delta$,
and $T_{1,1,2}(a) \subseteq T_{1,1,2}(\hat{q}) \subseteq \Delta$.
So $a \in \Delta'$, and $q \in L'$. Therefore L $\in L'$.
Hence L = L'. That is if L is an xyz local hexagonal picture language then L is a local hexagonal picture language. ☐

Example 3.2. *For instance, the hexagonal picture language defined above is local with,*

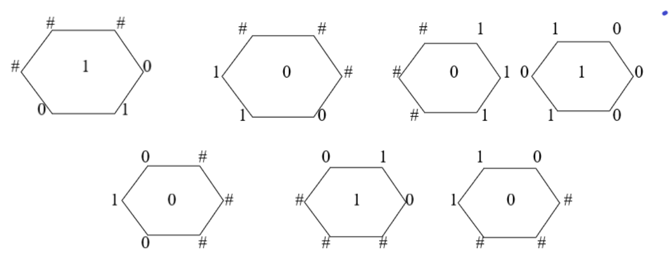

Figure 9

Theorem 3.2. *Let L $\subseteq \Sigma^{**H}$ be a hexagonal picture language. If L is local there exists a xyz local hexagonal picture language L′ over Σ' and a mapping $\pi : \Sigma' \to \Sigma$ such that L = $\pi(L')$.*

Proof. We define an extended alphabet from Σ. We denote this alphabet $E(\Sigma)$ = $(\Sigma \cup \{\#\})^{3,3,3}$. Now we define a mapping π, $\pi : \Sigma^{**H} \to E(\Sigma^{**H})$

$p \to p^E \in E(\Sigma^{**H})$.
We define p and p^E with same size and for all $1 \leqslant i \leqslant l+1, 1 \leqslant j \leqslant m+1, 1 \leqslant k \leqslant n+1$ where (l, m, n) be the size of$p$.

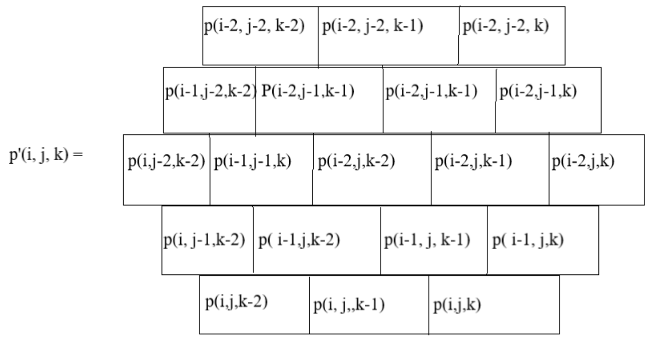

Figure 10

It can be verified that every hexagonal tile of size (2, 2, 2) in p^E where $p \in \Sigma^{**H}$ appear in $\pi(p)$ and vice versa.
$T_{2,2,2}(p^E) = \cup T_{2,2,2}(a)$.
If $p \in \Sigma^{**H}$ where $\Sigma = \{0, 1\}$ then

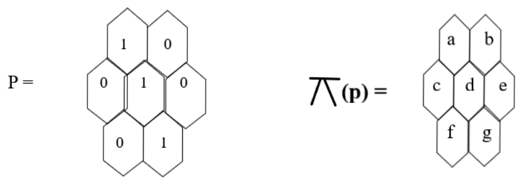

Figure 11

where

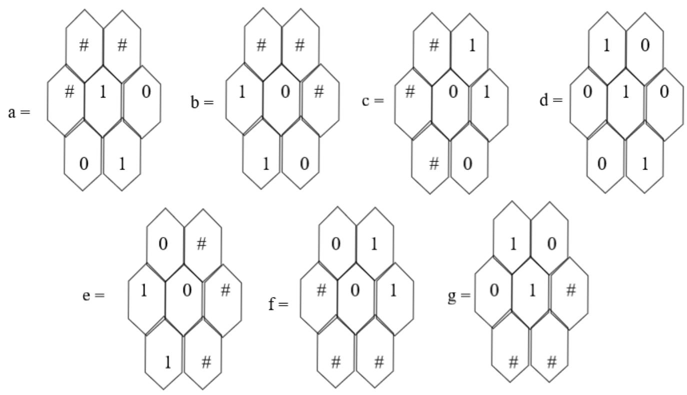

Figure 12

We also define a mapping ϕ from $E(\Sigma^{**H})$ onto Σ^{**H} by $\pi(a) = a(2,2,2)$ for a $\in E(\Sigma)$. By using figure 10 it is clear that for all $p \in \Sigma^{**H}$ we have $p = \phi(\pi(p))$. Hence we conclude that L = $\pi(L')$ $\square$

Theorem 3.3. *Let $L \cup \Sigma^{**H}$ be a hexagonal picture language L is recognizable if and only if there exist a xyz local hexagonal picture language L' over $\Sigma^{'}$ and a mapping $\pi : \Sigma^{'} \rightarrow \Sigma$ such that $L = \pi(L')$.*

Proof. Let L be a recognizable hexagonal picture language over Σ. By definition of recognizable picture languages we know that there exist a local hexagonal picture language L' over an alphabet $\Sigma^{'}$ and a mapping $\pi : \Sigma^{'} \rightarrow \Sigma$ such that $L = \pi(L')$.
According to theorem 1 there exist a xyz local hexagonal picture language L'' over an alphabet $\Sigma^{''}$ and a mapping $\phi : \Sigma^{''} \rightarrow \Sigma^{'}$ such that $L' = \phi(L'')$.
From the above two results we get $L = \pi(L'') = \pi(\phi(L''))$ where L'' is xyz local.

Now let L' be a xyz local hexagonal picture language over $\Sigma^{'}$, then $\pi : \Sigma^{'} \rightarrow \Sigma$ be a mapping. Applying theorem 2 it follows that L' is local and hence the hexagonal picture $\pi(L')$ is recognizable. $\square$

4 Conclusion

xyz local recognizable hexagonal picture languages provides a new formalism of using xyz dominos instead of usual notation of hexagonal tile.We tried to prove that a hexagonal picture language L is recognizable and is the projection of a xyz local hexagonal picture language. In a similar way we can extend the various other properties of recognizable rectangular picture to recognizable hexagonal picture.

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On odd integers and their couples of divisors

Giuseppe Buffoni*

Abstract

A composite odd integer can be expressed as the product of two odd integers. Possibly, this decomposition is not unique. From $2n+1 = (2i+1)(2j+1)$ it follows that $n = i + j + 2ij$. This form of n characterizes the composite odd integers. It allows the formulation of simple algorithms to compute all the couples of divisors of odd integers and to identify the odd integers with the same number of couples of divisors (including the primes, with the number of non trivial divisors equal to zero). The distributions of odd integers $\leq 2n+1$ vs. the number of their couples of divisors have been computed up to $n \simeq 5\ 10^7$, and the main features are illustrated.

Keywords: divisor computation; odd integer distribution vs. divisor number. **2020 AMS subject classifications**: 11Axx, 11Yxx. [1]

1 Introduction: characterization of composite odd and prime numbers

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{P}$ the set of prime numbers with the exception of 2. Composite odd integers $2n+1$, $n \in \mathbb{N}$, may be expressed as product of two odd integers,

$$2n + 1 = (2i + 1)(2j + 1),\ i, j \in \mathbb{N}, \tag{1}$$

or of more than two odd integers, i.e. as product of two odd integers in different ways. The decomposition (1) implies that

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$$n = k_{ij} = i + j + 2ij, \tag{2}$$

which may also be rewritten in the form

$$n = k_{ij} = i(j+1) + (i+1)j. \tag{3}$$

Either equation (2) or (3) specifies the structure of a composite odd integer $2n+1$.

Let $\mathbb{K} \subset \mathbb{N}$ be the set of the integers k_{ij} $\forall i, j \in \mathbb{N}$. Since any odd integer $2n+1$ greater than one is either a composite or a prime number, it follows that

$$n \in \mathbb{K} \quad \Longleftrightarrow \quad 2n+1 \in \mathbb{N} \setminus \mathbb{P},$$

or, equivalently,

$$n \in \mathbb{N} \setminus \mathbb{K} \quad \Longleftrightarrow \quad 2n+1 \in \mathbb{P}.$$

Remark. More involved characterizations of prime numbers can be formulated. They are obtained starting from the observation that all prime numbers greater than $c \in \mathbb{N}$, are of the form $c\#h + \iota$, where $c\#$ represents c primorial, $h, \iota \in \mathbb{N}$, and $\iota < c\#$ is coprime to $c\#$, i.e. $gcd(\iota, c\#) = 1$. As an example, let $c = 4$, $c\# = 6$; thus, all prime numbers > 4 may be expressed as $6h + \iota$ with $\iota = 1, 5$. Since $6h + 5 = 6(h+1) - 1$, then all prime numbers may be expressed in the form $6h \pm 1$, with the exception of 2 and 3. Let the odd integer $2n+1$ be written as $2n+1 = 6h \pm 1$, so that either $n = 3h$ or $n = 3h - 1$. For composite integers $n = k_{ij}$, and consequently 3 should be a dvisor of either k_{ij} or $k_{ij} + 1$.

The paper is organized as follows. In section 2 varios formulations of the relationship between n and the pair (i, j) are viewed. An algorithm to compute the divisors of an odd integer is described; it can also be used as a primality test. In sections 3 and 4 it is shown how odd integers with the same number of couples of divisors can be identified. Moreover, the distributions of odd integers $\leq 2n+1$ vs. the number of their couples of divisors are computed up to $n = 5\ 10^7$ and illustrated. Some concluding remarks can be found in section 5. Details of calculations are reported in appendix.

2 The relationship between n and the pair (i, j)

The functional relationship between a composite integer $2n+1$ and the factors $2i+1$, $2j+1$ of its decompositions, or between n, i, j, can be written in different forms. The decomposition (1) is an inverse proportional relationship (hyperbolic relation) between $2i+1$ and $2j+1$. Here and in the following it is assumed that $i \leq j$, so that $2i+1 \leq \sqrt{2n+1} \leq 2j+1$ (equality holds iff $i = j$), or equivalently

$$i \leq I_n = \frac{1}{2}(-1 + \sqrt{2n+1}) \leq j. \tag{4}$$

The relation (1) has been written in the forms (2) and (3). These equations define the entries of the matrix $K = \{k_{ij}\}$, used for the computation of the distribution of odd integers vs. the number of their couples of divisors. Properties of K can be found in appendix 1.

By making explicit the variable j, (2) can be written in the form of an homographic function

$$j = \phi_n(i) = \frac{n-i}{2i+1}, \quad 1 \leq i \leq I_n. \tag{5}$$

Thus, $2i+1$ is a divisor of both $2n+1$ and $n-i$. From (12) in appendix 1, it follows that $2i+1$ is also a divisor of $n - k_{ii}$.

Equation (5) can be used to compute the couples of divisors of an integer $2n+1$ by means of the following algorithm:

given n, compute $\phi_n(i)$ for $i = 1, 2, ..., [I_n]$,
where $[\cdot]$ is the integer part of the real argument;
if for some $i = i_q$ we obtain that $j_q = \phi_n(i_q) \in \mathbb{N}$,
then $2i_q + 1 \leq \sqrt{2n+1} \leq 2j_q + 1$ is a couple of divisors of $2n+1$.

The order of the number of operations is $\sqrt{n/2}$. The algorithm can also be used as a primality test: if the computed $\phi_n(i) \notin \mathbb{N}\ \forall i$, then $2n+1$ is a prime.

The functions $y = \phi_n(x)$, $x+y$, xy, $y-x$, of the real variable x, are monotone for $0 \leq x \leq I_n$ (figure 1). I_n, defined in (4), is the unique positive solution to the equation $\phi_n(x) = x$, i.e. $2x^2 + 2x - n = 0$. The point $x = I_n$ corresponds to the minimum of $x+y$, to the maximum of xy, and, obviously, to $y - x = 0$.

By means of a change of variables, the relationship (1) can be put in the form

$$2n+1 = (s+t)(s-t) = s^2 - t^2, \quad with\ s = i+j+1,\ t = j-i, \tag{6}$$

while (2) and (3), representing partitions of the integer n in two sections, can be put in linear forms

$$n = s + 2t, \quad with\ s = i+j,\ t = ij, \tag{7}$$

$$n = s + t, \quad with\ s = i(j+1),\ t = (i+1)j. \tag{8}$$

Equation (6) shows the well known fact that composite odd integers can be written as a difference of two squares in different ways, while for a prime only holds the decomposition $2n+1 = (n+1)^2 - n^2$.

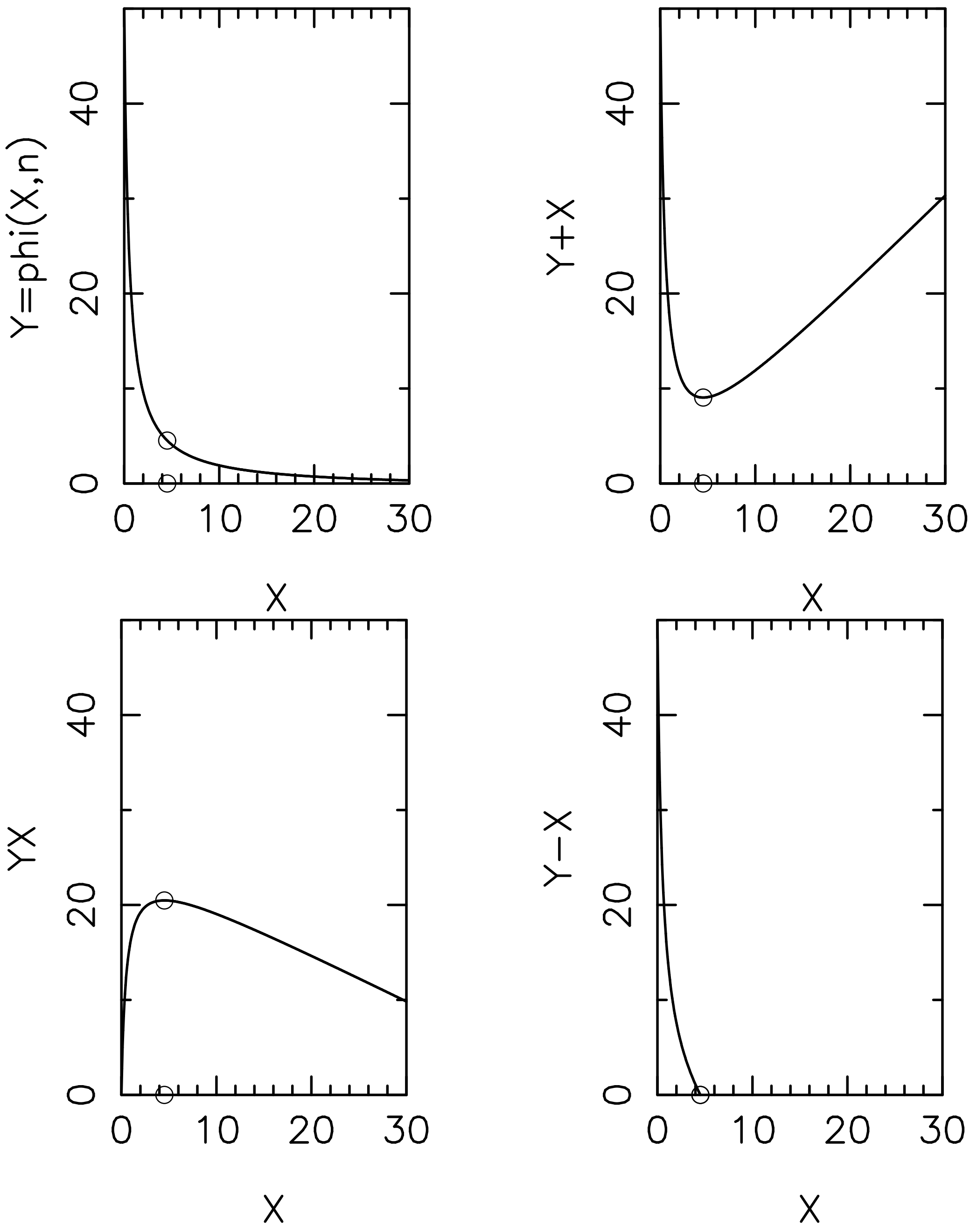


Figure 1: Top left $y = \phi_n(x)$, top right $x + y$, bottom left xy, bottom right $y - x$. Circle: point $x = I_n$ on the x axis, and corresponding points on the curves. $n = 50$, $I_n = 4.52$.

Given $n, s \in \mathbb{N}$, it is possible to prove when s and $t = n - s$ can be expressed as either in (7) or in (8). The details are reported in appendix 2: it is shown that i, j are solutions to second order equations, and they are integer satisfying either (7) or (8), iff the square root of a quadratic form in n and s is an integer,

3 Identification of odd integers $\leq 2n + 1$ with the same number of couples of divisors

Let $2m + 1$ be a composite integer and let

$\psi(m) =$ number of couples of divisors of $2m + 1$.

Obviously, $\psi(m)$ is also equal to the number of divisors of $2m + 1 \leq \sqrt{2m + 1}$. If $\psi(m) = \nu$, then the entry $m = k_{ij}$, with $i \leq j$, appears ν times in the matrix $K = \{k_{ij}\}$.

Composite integers $2m + 1$ with $m \leq n$ are identified by the pairs (i, j) such that

$$4 \leq m = k_{ij} \leq n. \tag{9}$$

By assuming $i \leq j$, it follows that (9) holds for the pairs

$$(i, j) \in \Omega(4, n) = \{i, j \in \mathbb{N} : \ i = 1, 2, ..., [I_n]; \ j = i, i + 1, ..., [\phi_n(i)]\}.$$

An estimation of the number of these pairs as $n \longrightarrow +\infty$ is given by

$$\kappa_n^* \simeq n(\frac{1}{4}\ln(n) + c). \tag{10}$$

with $c = -0.4415$. The details can be found at the end of appendix 1. In doing so we do not consider the couple $(0, n)$, corresponding to the couple of trivial divisors $(1, 2n + 1)$.

The odd integers $2m+1, \ m \leq n$, with the same number of couples of divisors can be identified by means of the following algorithm:

let $\psi(m) = 0, \ m = 1, ..., n$;
compute $k_{ij}, \ \forall (i, j) \in \Omega(4, n)$;
for $k_{ij} = m$ let $\psi(m) = \psi(m) + 1$.

When $\psi(m) = 0$, then the integer $2m + 1$ is a prime. All the integers $2m + 1$, with ν couples of divisors, are identified by the values of m for which $\psi(m) = \nu$.

Furthermore, let

$\Pi_n(\nu)$ = number of odd integers $\leq 2n+1$ with ν couples of divisors.

$\Pi_n(0)$ is the number of primes $\leq 2n+1$, except 2. $\Pi_n(\nu)$ is estimated as follows:

for $\nu = 0$: $\Pi_n(0)$ = number of $\psi(m) = 0$,

for $\nu > 0$: $\Pi_n(\nu) = \frac{1}{\nu}\sum_{\psi(m)=\nu}\psi(m)$.

This approach, used to identify the prime numbers, is an equivalent formulation of the common implementation of the Eratostene's sieve (see for example the C program source in (2), section 6.3). In this case $\psi(m)$ could be a logical variable.

The algorithm may be easily applied to the integers in a generic set $[2a+1, 2n+1]$, with $4 < a < n$, to identify either the odd integers in this interval with the same number of couples of divisors or the primes. The inequalities identifying these integers,

$$a \leq k_{ij} \leq n, \quad with\ i \leq j,$$

hold for the pairs

$$(i,j) \in \Omega(a,n) = \{i,j \in \mathbb{N} :\ i = 1,2,...,[I_n];\ j = J_a(i), J_a(i)+1, ..., [\phi_n(i,)]\},$$

where:

$$when\ i \leq [I_a] :\ either\ J_a(i) = [\phi_a(i)] + 1,\ \phi_a(i) \notin \mathbb{N},\ or\ J_a(i) = \phi_a(i) \in \mathbb{N};$$

$$when\ i > [I_a] : J_a(i) = i.$$

The set of the points $(i,j) \in \Omega(a,n)$, with integer coordinates, is contained in a closed and convex set $\Omega^*(a,n)$ of a plane. See figure 2, where the boundaries of this set are plain defined.

Some remarks on the case with large n and $n - a << n$ can be found in appendix 3.

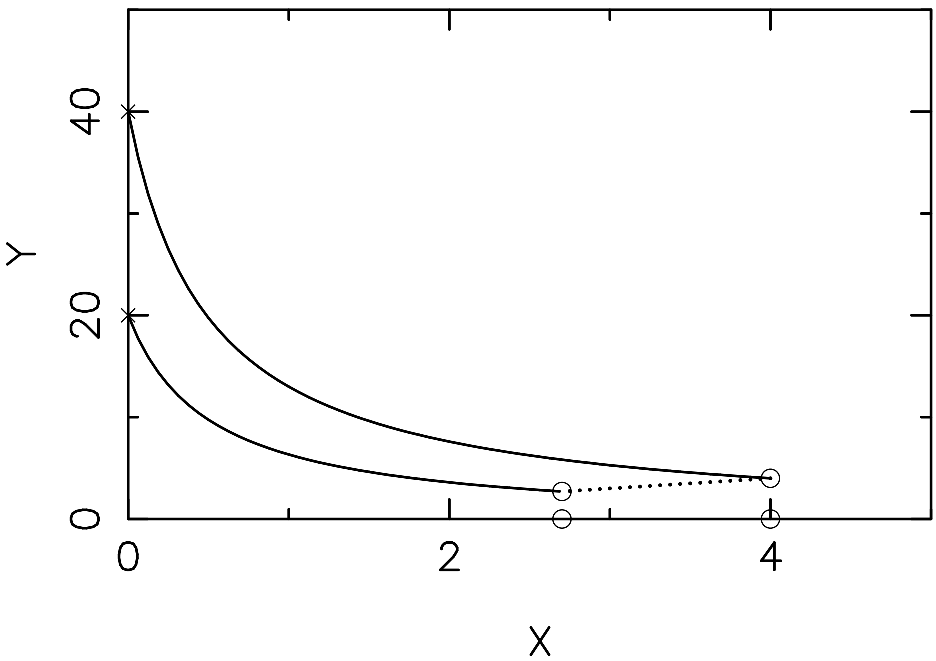


Figure 2: Set $\Omega^*(a,n)$ in the plane (x,y). Continuous lines: $y = \phi_a(x) < y = \phi_n(x)$; dotted line: $y = x$; asterisk: points $(0,a),(0,n)$; circle: points $(I_a,0),(I_n,0)$, and corresponding points on the curves. Different scales for x and y.

4 Distributions of odd numbers vs. the number of their couples of divisors

The computation of the distributions $\Pi_n(\nu)$ has been performed, by means of the algorithm described in the previous section, for $n \leq 5\ 10^7$, i.e. for odd integers $2n+1 \leq 10^8+1$ (see the tables 1, 2 for some values of n).

	$n=5$		$n=50$		$n=5\ 10^2$		$n=5\ 10^3$		$n=5\ 10^4$	
ν										
0	4		25		167		1228		9591	
1		1		20		207		1964		18259
2			5		52		382		2824	
3						56		925		11380
4					5		50		308	
5						12		264		3200
6					0		4		32	
7						1		128		2785
8							22		265	
9								9		188
10							0		3	
11								24		826
12									1	
13										18
14									27	
15										195
16									0	
17										66
18									0	
19										13
20									0	
21										0
22									1	
23										18
tot.	4	1	30	20	224	276	1686	3314	13052	36948

Table 1: Distribution $\Pi_n(\nu)$ of odd integers $\leq 2n+1$, with ν couples of divisors, for $n = 5, 50, 5\ 10^2, 5\ 10^3, 5\ 10^4$.

Let

ν_n^* = maximum number of couples of divisors of odd integers $\leq 2n+1$.

	$n = 5\ 10^5$	$n = 5\ 10^6$		$n = 5\ 10^5$	$n = 5\ 10^6$
ν			ν		
0	78497	664578	1	168522	1555858
2	21711	174188	3	126518	1336044
4	2030	14919	5	32314	309137
6	236	1758	7	42022	542740
8	2228	17481	9	2171	21649
10	16	74	11	13521	172181
12	2	12	13	238	2343
14	295	2376	15	5733	105676
16	0	4	17	1403	17487
18	0	0	19	545	8847
20	24	268	21	0	15
22	11	60	23	1537	34648
24	6	57	25	0	0
26	50	705	27	17	566
28	0	0	29	67	1503
30	0	0	31	179	8098
32	0	0	33	0	0
34	0	7	35	88	3589
36	0	0	37	0	4
38	0	0	39	13	922
40	0	11	41	0	69
42	0	0	43	0	0
44	0	65	45	0	0
46	0	0	47	6	1693
48		0	49		7
50		0	51		0
52		0	53		118
54		0	55		16
56		0	57		0
58		0	59		86
60		0	61		0
62		1	63		91
64		0	65		0
66		0	67		0
68		0	69		0
70		0	71		46
72		0	73		0
..		..	..		..
78		0	79		3
tot.	105106	876564	tot.	394894	4123436

Table 2: Distribution $\Pi_n(\nu)$ of odd integers $\leq 2n+1$, with ν couples of divisors, for $n = 5\ 10^5, 5\ 10^6$. (Since $\nu_n^* = 143$ for $n = 5\ 10^7$, this case is not reported here).

Thus, $\Pi_n(\nu) = 0$ for $\nu > \nu_n^*$. It has been estimated (table 3, figure 3) that ν_n^* increases as a power of n. The following approximation has been found by a fitting procedure

$$\nu_n^* = \mu n^\lambda, \quad \mu = e^{0.3992 \pm 0.1050}, \ \lambda = 0.2586 \pm 0.0083, \quad 5\ 10^2 \le n \le 5\ 10^7.$$

n	$5\ 10^2$	$5\ 10^3$	$5\ 10^4$	$5\ 10^5$	$5\ 10^6$	$5\ 10^7$
ν_n^*	8	12	24	48	80	144
$\mu\ n^\lambda$	7.43	13.48	24.46	44.37	80.48	145.99

Table 3: Computed values of ν_n^* and those produced by $\nu_n^* = \mu n^\lambda$ for some values of n.

A visual inspection of the patterns of $\Pi_n(\nu)$ (the scattered plots of $\ln(\Pi_n(\nu))$ vs. ν are shown in the figures 4, 5) suggests that the odd integers with even and odd numbers of couples of divisors should belong to different populations. This view has to be considered only as a guess of the author, trying to interpret special features of $\Pi_n(\nu)$. Anyhow, to avoid repetitions, we nickname these integers as

$ravens$ the odd integers with 2ν couples of divisors,
$cods$ the odd integers with $2\nu + 1$ couples of divisors.

The primes, identified by $\nu = 0$, are included in the ravens. We have that

$\Pi_n(2\nu) =$ number of $ravens \le 2n + 1$,
$\Pi_n(2\nu + 1) =$ number of $cods \le 2n + 1$.

For $n > 150$, in general

$$\Pi_n(2\nu) < \Pi_n(2\nu + 1). \tag{11}$$

Only for few values of ν this inequality is not satisfied in the computed distributions (table 4). The number of $ravens$ is less large than that of $cods$ (see the last row in the tables 1, 2).

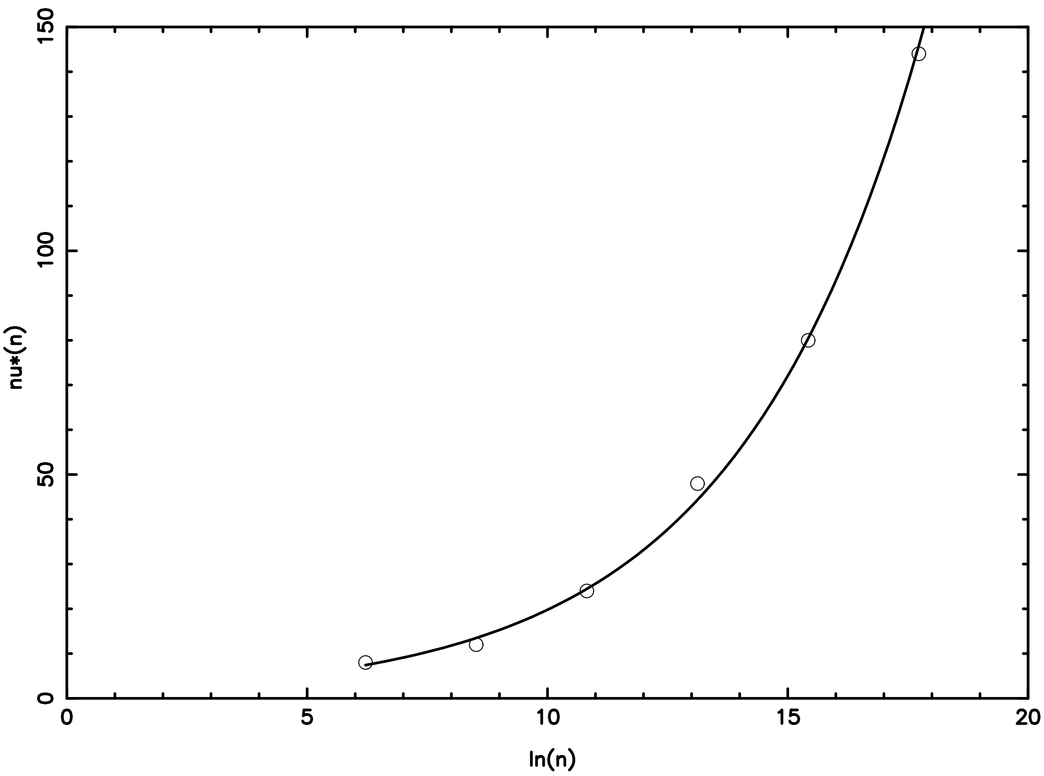


Figure 3: ν_n^* vs. $ln(n)$. Circles: computed values, continuous line: approximation $\nu_n^* = \mu\ exp(\lambda\ ln(n))$ for $2\ 10^2 \le n \le 5\ 10^7$.

n	$5\ 10^2$	$5\ 10^3$	$5\ 10^4$	$5\ 10^5$	$5\ 10^6$	$5\ 10^7$
		8-9	8-9	8-9 20-21 24-25 26-27	20-21 24-25 44-45	20-21 24-25 32-33 44-45 74-75 80-81
N. couples	0	1	1	4	3	6

Table 4: Couples of ν for which inequality (11) is not satisfied.

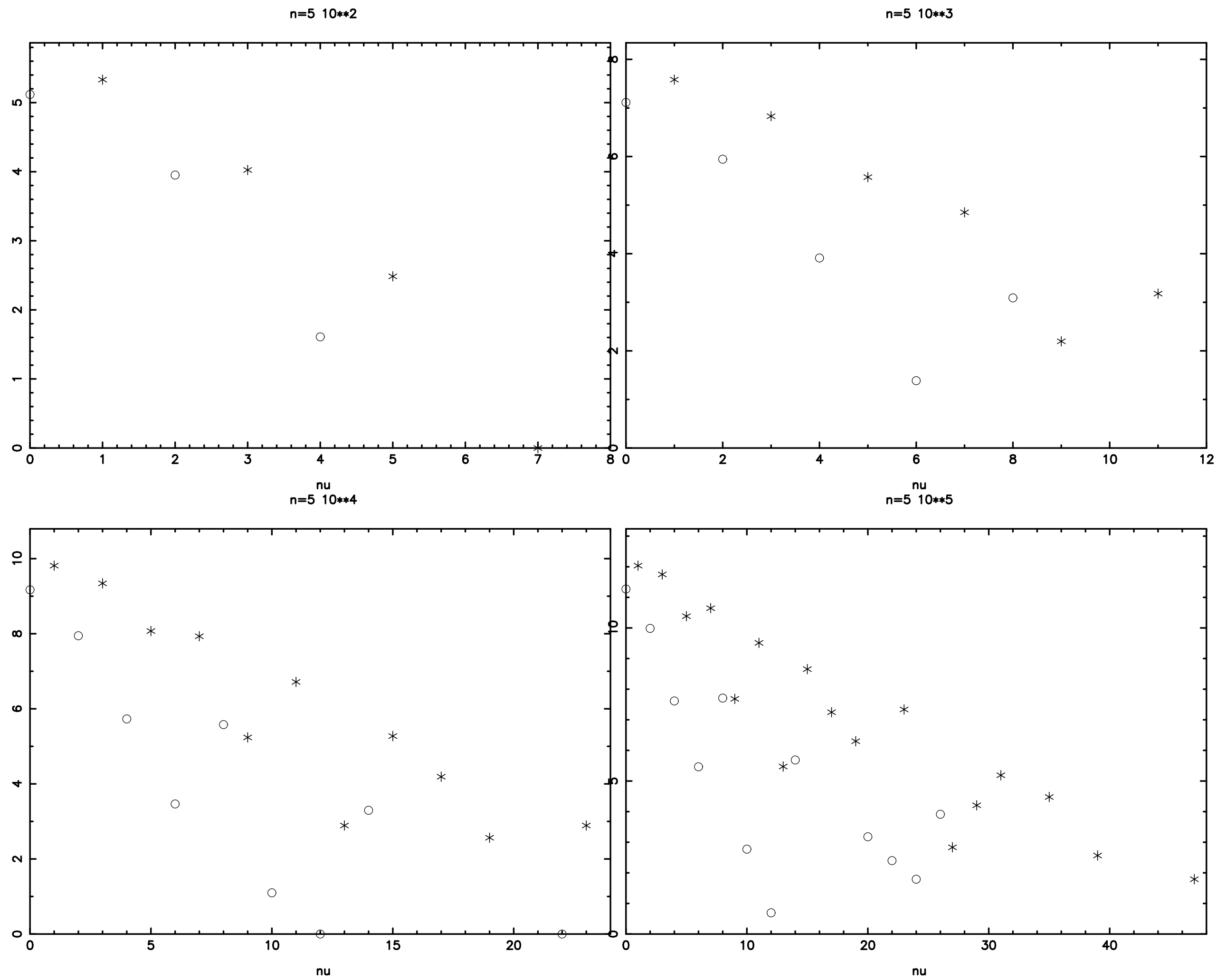


Figure 4: Distributions $ln(\Pi_n(\nu)$ vs. ν for $n = 5\ 10^2,\ 5\ 10^3\ 5\ 10^4,\ 5\ 10^5$. Circle: $ravens$, asterisk: $cods$.

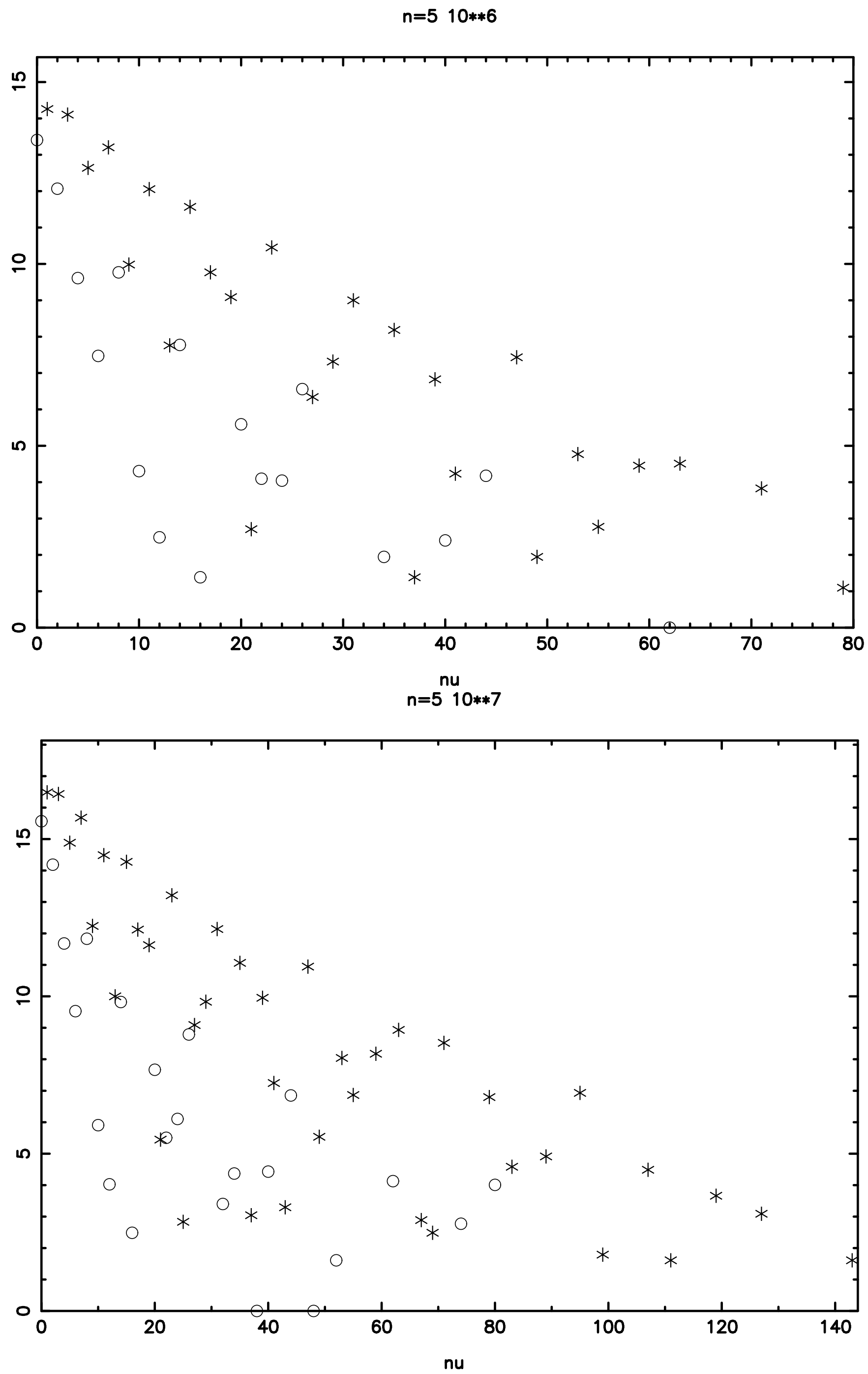


Figure 5: Distribution $ln(\Pi_n(\nu))$ vs. ν for $n = 5\ 10^6$, $n = 5\ 10^7$. Circle: $ravens$, asterisk: $cods$.

For $\simeq 5\ 10^2 \leq n \leq\simeq 5\ 10^4$ both the points $\Pi_n(2\nu)$ and $\Pi_n(2\nu+1)$ show well distinct decreasing trends with ν (figure 4). However, points not belonging to the initial trends begin to appear for $n \simeq 5\ 10^3$. Indeed, new branches (generally decreasing with ν) grow for increasing n, beginning at ν values not detected in the previous branches (figure 5).

A branch may be roughly defined as a sequence of points in the plane $(\nu, \ln \Pi)$ which approximately lay on a straight line. For example, in the plot for $n = 5\ 10^5$ in figure 4, we can recognize two $raven$ branches: the initial at the points $\nu = 0, 2, 4, 6, 8$ and a second branch at $\nu = 8, 14, 20, 22, 24$ (the point at $\nu = 8$ is already present in the distribution for $n = 5\ 10^3$), and a single point at $\nu = 26$. Moreover, four *cod* branches: the initial at the points $\nu = 1, 3, 7, 11, 15, 23, 31, 35$, and then at $\nu = 5, 9, 13$, at $\nu = 17, 19, 27$, and at $\nu = 29, 39$. The attribution of a point to a branch is sometimes uncertain. Indeed, the interpretation of the evolution of the distributions $\Pi_n(\nu)$ with n in terms of growing branches is arbitrary. The straight lines in the plane $(\nu, \ln \Pi)$ approximating the initial trends of both $ravens$ and $cods$ are estimated by a fitting procedure (table 5, figure 6).

	n	$\alpha \pm \sigma$	$\beta \pm \sigma$	σ
$ravens$				
	$5\ 10^5$	11.3131 ± 0.2663	-0.8846 ± 0.0377	0.3901
	$5\ 10^6$	13.4700 ± 0.2292	-0.9257 ± 0.0324	0.3358
	$5\ 10^7$	15.7156 ± 0.1925	-0.9823 ± 0.0272	0.2820
$cods$				
	$5\ 10^5$	12.2196 ± 0.1155	-0.2234 ± 0.0059	0.1971
	$5\ 10^6$	14.3920 ± 0.1244	-0.1770 ± 0.0063	0.2122
	$5\ 10^7$	16.4599 ± 0.2084	-0.1484 ± 0.0106	0.3557

Table 5: Initial branches of $\Pi_n(\nu)$: coefficients of the linear relationship $\ln(\Pi_n(\nu)) = \alpha + \beta\nu$ and their standard deviations σ. Last column: σ of $\ln(\Pi_n(\nu))$.

All the points $(\nu, \ln(\Pi_n(\nu))$ are contained in a bounded region of the plane $(\nu, \ln \Pi)$ (figures 4, 5). This region is bounded from the bottom by the initial branch of $ravens$, starting from the number of primes $\ln(\Pi_n(0))$ and ending in $\nu \simeq 20$, and then by the axis $\ln \Pi = 0$. From the top by the initial branch of $cods$, starting from $\ln(\Pi_n(1))$ and ending in $\nu \simeq 40$, and then by sparse decreasing *cod* points, belonging to different branches. The upper boundary can be approximated by a straight line with $\simeq$ the same slope of the initial *cod* trend.

A guess about the description of the evolution of $\Pi_n(\nu)$ with n has been sug-

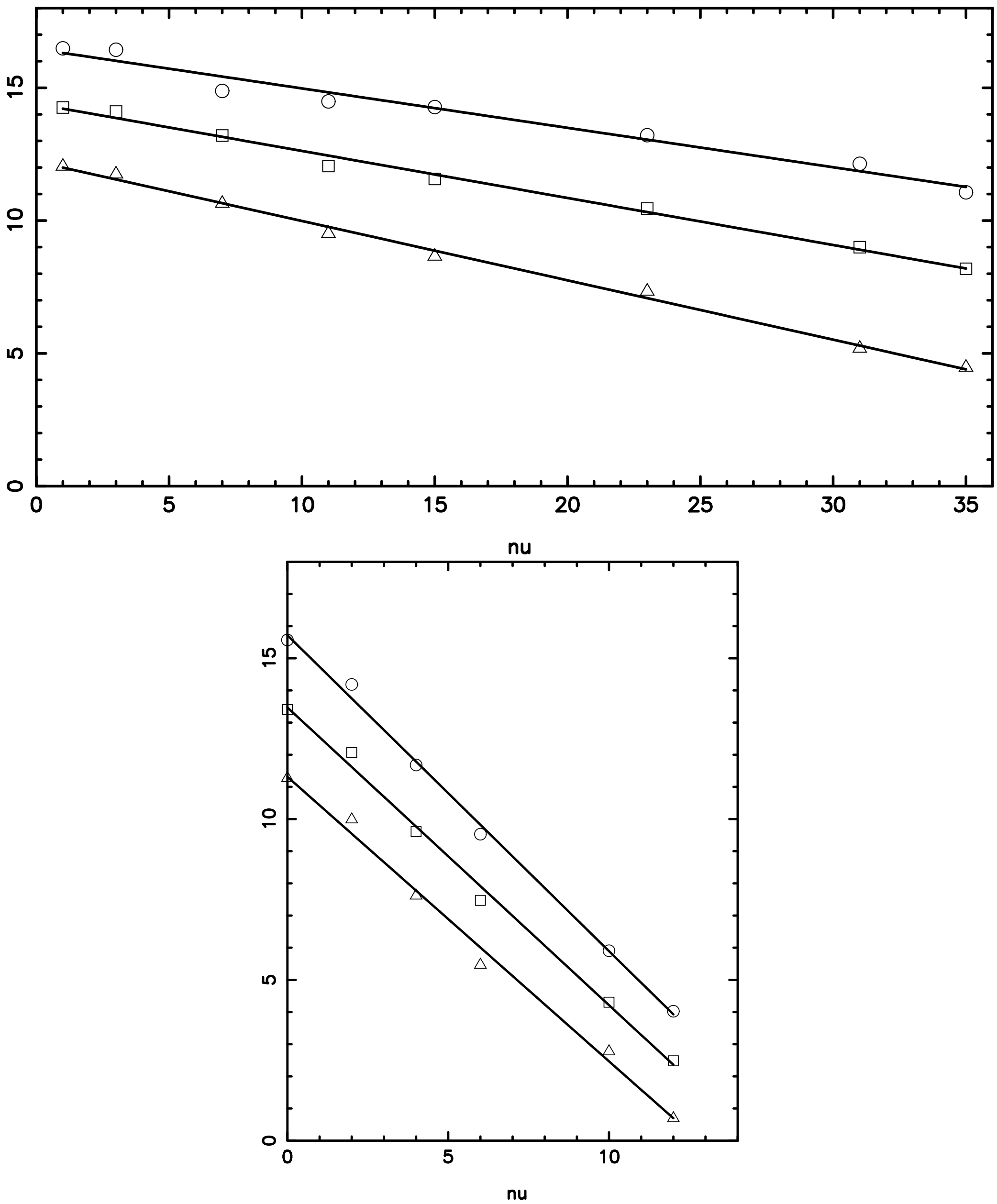


Figure 6: Top: initial branch of cods at $\nu = 1, 3, 7, 11, 15, 23, 31, 35$. Bottom: initial branch of ravens at $\nu = 0, 2, 4, 6, 10, 12$. Triangle: $n = 5\ 10^5$; square: $n = 5\ 10^6$; circle: $n = 5\ 10^7$. Continuous line: linear approximation.

gested by the most simple formula ((1), p. 8) approximating the number of primes $\leq 2n+1$:

$$\Pi_n(0) = \frac{2n+1}{\ln(2n+1)}.$$

Taking into account that $\ln(\ln(n)) \simeq 1.2334 + 0.0992\ln(n),\ 10^2 \leq n \leq 10^7$, this relationship can be approximated by $\ln(\Pi_n(0)) \simeq -0.5403 + 0.9008\ln(n)$. Fitting of $\ln(\Pi_n(\nu))$ to the linear expression $\alpha + \beta\ln(n)$ has been carried out for $\nu = 0, 1, 2, 3$ (table 6, figure 7). The straight lines are $\simeq$ parallel for the $ravens$ $\nu = 0, 2$, while the lines for the $codes$ $\nu = 1, 3$ show different slopes (the line for $\nu = 3$ is not shown in figure 7 for clearness of the figure). It is worth here to remind that a logarithmic approximation of a quantity may lead to a rough estimation of the quantity.

ν	$\alpha \pm \sigma$	$\beta \pm \sigma$	σ
0	-0.4902 ± 0.0755	0.8993 ± 0.0068	0.0847
1	-0.7130 ± 0.0282	0.9714 ± 0.0025	0.0317
2	-1.7076 ± 0.0550	0.8944 ± 0.0049	0.0617
3	-2.3972 ± 0.1632	1.0721 ± 0.0140	0.1528

Table 6: $\ln(\Pi_n(\nu))$ vs. $\ln(n)$: coefficients of the linear relationship $\ln(\Pi_n(\nu)) = \alpha + \beta\ln(n)$ and their standard deviations σ. Last column: σ of $\ln(\Pi_n(\nu))$. Ten $n-points$, from $n = 100$ to $n = 5\ 10^7$ are used in fitting $\ln(\Pi_n(\nu))$ for $\nu = 0, 1, 2$. Since $\Pi_n(3)$ is very small for $n = 100$, this point is not included for $\nu = 3$.

The "regularity" of some relationships between $\Pi_n(\nu)$ (figure 8) may arouse some surprise. We have performed a survey on the ratios between $\Pi_n(\nu)$ with $\nu = 0, 1, 2, 3$. The trends of the ratios $\Pi_n(1)/\Pi_n(0)$, $\Pi_n(3)/\Pi_n(0)$, $\Pi_n(3)/\Pi_n(1)$ (figure 8 top), increasing with n, seem to be reasonable. On the other hand, the trends of the ratios $\Pi_n(2)/\Pi_n(\nu),\ \nu = 0, 1, 3$, (figure 8 bottom), are disturbing. This might be due to the shortage of $ravens$ with $\nu = 2$ detected. Obviously, computations with n greater than $n = 5\ 10^7$, the maximum value here considered, should be carried out to confirm the results, and to try to explain the trends. A careful analysis to produce a thorough knowledge has to be hoped for.

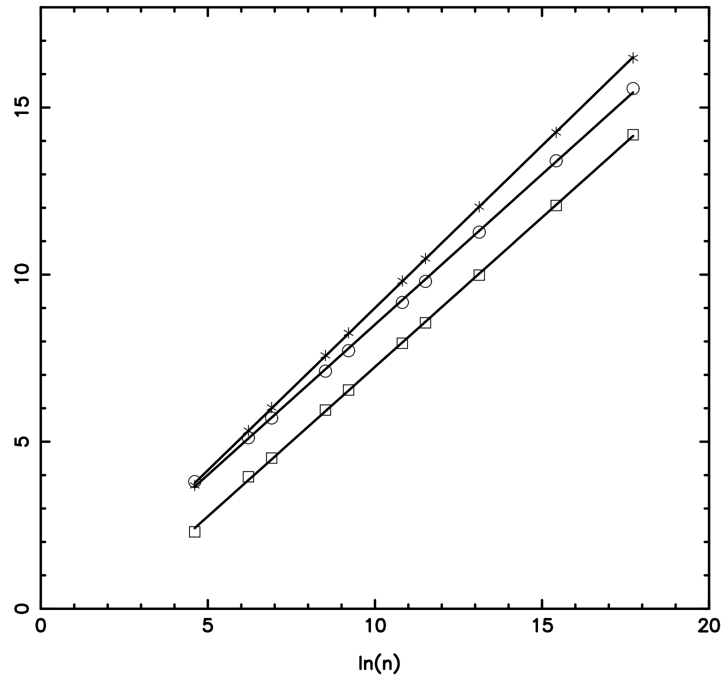


Figure 7: Distributions $\ln(\Pi_n(\nu))$ vs. $\ln(n)$. Circle: $\Pi_n(0)$, asterisk: $\Pi_n(1)$, square: $\Pi_n(2)$. Continuous line: linear approximation.

5 Concluding remarks

We have focused our attention on the computation of the couples of divisors of odd integers. Indeed, any even integer can be written in the form

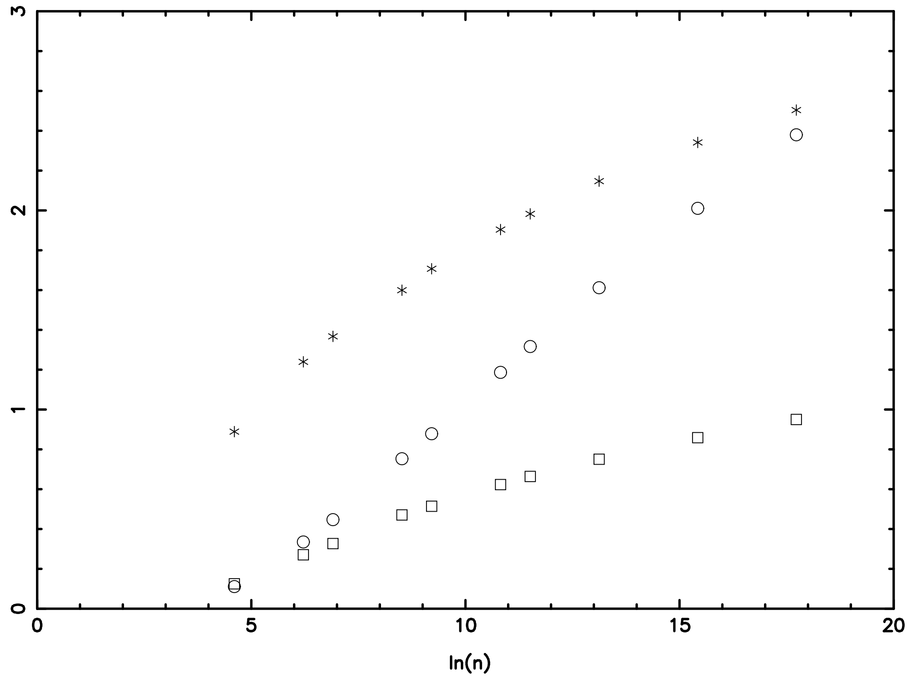
$$2^m \, (2n+1), \quad with \; m \geq 1, \; n \geq 0.$$

Thus, it is characterized by the power of two, and possibly by an odd integer with its divisors. The following considerations hold for the computed distributions for n up to $5 \; 10^7$.

For small n the following inequalities hold:

$$\Pi_n(0) > \Pi_n(1) > \Pi_n(2) > \Pi_n(\nu), \quad \nu > 2, \quad n < 149.$$

$\Pi_n(0)$ is the number of primes, $\Pi_n(1)$ is the number of $cods$ either products of two primes or primes cubed, while $\Pi_n(2)$ is the number of $ravens$ either products of primes by primes squared or primes to the fourth. The previous inequalities can be explained by the following reasonings: for small n, (1) the density of primes is high, and (2) the prime factors in the divisors of both $cods$ and $ravens$ should be small. As an example, the possible decompositions of $cods \leq 101$ with $\nu = 1$ are reported here:



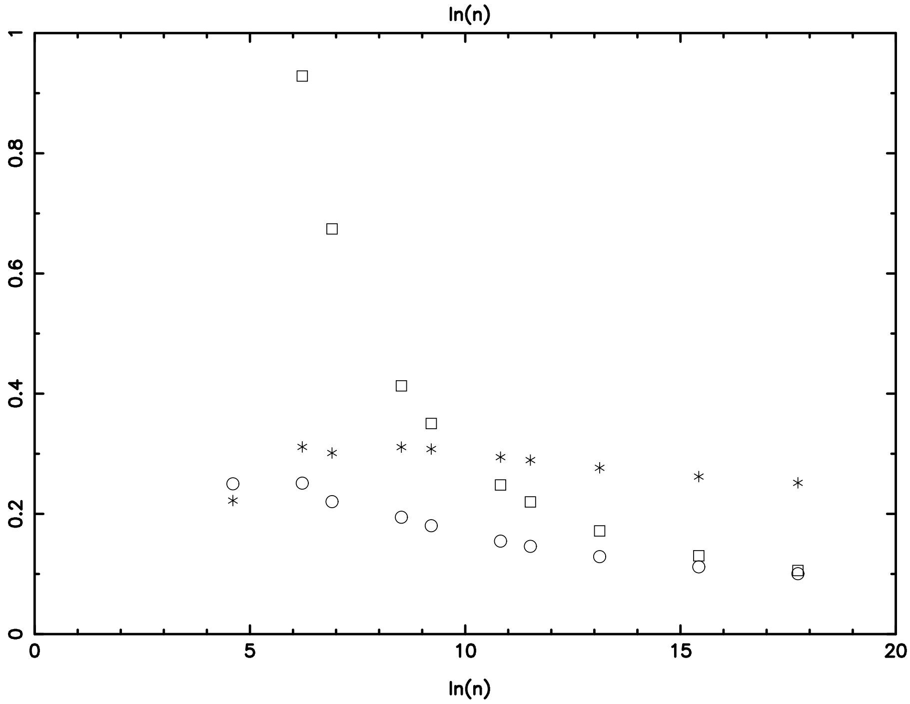


Figure 8: Ratios of distributions $\Pi_n(\nu)$ vs. $\ln(n)$. Top: $\Pi_n(1)/\Pi_n(0)$ asterisk, $\Pi_n(3)/\Pi_n(0)$ circle, $\Pi_n(3)/\Pi_n(1)$ square. Bottom: $\Pi_n(2)/\Pi_n(0)$ asterisk, $\Pi_n(2)/\Pi_n(1)$ circle, $\Pi_n(2)/\Pi_n(3)$ square.

$$p_1p_i,\ i=1,...,10;\ p_2p_i,\ i=2,3,...,7;\ p_3p_i,\ i=3,4,5;\ p_1p_1^2;$$

where $p_1=3,\ p_2=5,...$ are the primes. Thus, for small n, few factors produce *cod* integers $\leq 2n+1$. For increasing n, the inequality $\Pi_n(1)>\Pi_n(\nu),\ \nu\neq 1,$ hold. For $n=149$ we have $\Pi_n(0)=\Pi_n(1)$ (table 7.

n	$\Pi_n(0)$	$\Pi_n(1)$	$\Pi_n(2)$
50	25	20	5
100	45	40	10
148	61	60	16
149	61	61	16
150	61	62	16
200	78	83	21
250	94	104	25

Table 7: The transition from $\Pi_n(0)>\Pi_n(1)$ to $\Pi_n(0)<\Pi_n(1)$.

The distributions $\Pi_n(\nu)$ have been obtained by identifying all the couples $(2i+1,2j+1)$ of divisors of the integers $2m+1$ with $m=k_{ij}\leq n$. For large n the number κ_n^* of $k_{ij}\leq n$ is $n(\ln(n)/4+c)$ (10). The number k_{a+1n}^* of k_{ij} such that $a+1\leq k_{ij}\leq n$ can be estimated by

$$k_{a+1n}^*=k_n^*-k_a^*=\frac{1}{4}(n\ln(n)-a\ln(a))+c(n-a).$$

Under the assumption $n-a<<a<n$ it follows that

$$1<\frac{n}{a}=1+(\frac{n}{a}-1)<<2,\quad \textit{so that } 0<\frac{n}{a}-1<<1.$$

Thus,

$$k_{a+1n}^*=k_n^*-k_a^*=(n-a)(\frac{1}{4}\ln(n)+c)+\frac{1}{4}a\ln(\frac{n}{a})=(n-a)(\frac{1}{4}\ln(n)+c+\frac{1}{4}).$$

Some explanations on the numerical computations are given. The algorithms described in sections 2 and 3 can be easily implemented in Fortran language.

The algorithm (in section 2) for the computation of the couples of divisors of a given integer $2n+1$ does not require the storage of large dimension vectors. It has been successfully used to determine the couples of divisors of odd composite integers (and whether a number is prime or composite), up to input numbers of order $2^{60} \simeq 10^{18}$. Note that quadruple precision for floating point operations is necessary for numbers of order 2^{60}. We do not have recourse to computer algebra systems, with numbers of variable length (3; 4).

The algorithm (in section 3) for the computaion of the distributions $\Pi_n(\nu)$ requires the storage of an INTEGER*8 vector; the computation has been carried out up to the limit of the storage carrying capacity of the available computer (about a vector of $6\ 10^7$ of INTEGERS*8 entries). The computation of the primes in a given interval $[2a+1, 2n+1]$ has been performed either with small $n-a \in [5, 50]$ and a up to 10^{18}, or with large $n - a \in [10^2, 4\ 10^7]$ and a up to 10^9.

Acknowledgments

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[4] The GNU MP Bignum Library. Available from: http:77gmplib.org/

Appendix 1. The matrix $K = \{k_{ij}\}$

Obviously, the symmetry property holds for the elements k_{ij} of K: $k_{ij} = k_{ji}, \forall\ i, j \in \mathbb{N}$. Thus, they can be represented by means of a symmetric matrix. (see the table 8).

Since some composite odd numbers $2n + 1$ may be expressed as product of two odd numbers in different ways, it follows that

$$2n + 1 = (2i_1 + 1)(2j_1 + 1) = (2i_2 + 1)(2j_2 + 1) \Longrightarrow k_{i_1 j_1} = k_{i_2 j_2},$$

as it can be observed in the matrix K (table 8). The number of couples (i, j) such that $n = k_{ij}$, if they exist, is the number of decompositions of $2n + 1$ in two factors.

$-\downarrow i\ j \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	4	7	10	13	16	19	22	25	28	31	34	37	40	43	46
2		12	17	22	27	32	37	42	47	52	57	62	67	72	77
3			24	31	38	45	52	59	66	73	80	87	94	101	108
4				40	49	58	67	76	85	94	103	112	121	130	139
5					60	71	82	93	104	115	126	137	148	159	170
6						84	97	110	123	136	149	162	175	188	201
7							112	127	142	157	172	187	202	217	232
8								144	161	178	195	212	229	246	263
9									180	199	218	237	256	275	294
10										220	241	262	283	304	325
11											264	287	310	333	356
12												312	337	362	387
13													364	391	418
14														420	449
15															480

Table 8: Matrix $K = \{k_{ij}\}$ for $1 \leq i \leq j \leq 15$. i=row and j=column index.

Besides the symmetry identity, the elements k_{ij} satisfy other combinatorial properties, obtained from the equation

$$(2n + 1)(2k_{ij} + 1) = 2k_{nk_{ij}} + 1.$$

The identities

$$k_{pk_{qr}} = k_{qk_{rp}} = k_{rk_{pq}}, \quad p, q, r \in N$$

follow from the product of three odd numbers $(2p+1)(2q+1)(2r+1)$, while the identities

$$k_{k_{pq}k_{rs}} = k_{k_{pr}k_{qs}} = ... = k_{pk_{qk_{rs}}} = k_{qk_{pk_{rs}}} = ..., \quad p, q, r, s \in N$$

follow from the product of four odd numbers $(2p + 1)(2q + 1)(2r + 1)(2s + 1)$. Obviously, more involved identities are obtained from products of more than four odd numbers.

The quantities $k_{ij} - i = (2i + 1)j$ and $k_{ij} - j = i(2j + 1)$ are divisible by $2i + 1,\ j$ and by $i,\ 2j + 1$, respectively. Moreover, k_{ij} can be written in the form

$$k_{ij} = k_{ii}+(2i+1)(j-i), \quad i = 1, 2, ..., \ j = i, i+1, ... \ with \ k_{ii} = 2i^2+2i, \quad (12)$$

which leads to the following recurrence formula for the computation (by means of additions) of the entries of the $i-th$ row, with $i \leq j$, of the matrix K:

$$k_{ij} = k_{i\,(j-1)} + 2i + 1, \quad j = i+1, i+2, ...$$

Now we estimate
$\kappa_n^* =$ number of pairs (i, j) with $1 \leq i \leq j$ such that $4 \leq k_{ij} \leq n$.
It is given by

$$\kappa_n^* = \sum_{i=1}^{I_n} [q_n(i)].$$

where

$$q_n(i) = \phi_n(i) - i + 1 = \frac{1}{2}(\frac{2n+1}{2i+1} - (2i-1)),$$

and here I_n denotes the integer part of the quantity defined in (4). $q_n(i)$ are decreasing with i, and

$$q_n(I_n) = 1 \leq q_n(i) \leq q_n(1) = \frac{n-1}{3}.$$

By direct calculation we have that

$$Q_n = \sum_{i=1}^{I_n} q_n(i) =$$

$$= (n+\frac{1}{2})\sum_{i=1}^{I_n} \frac{1}{2i+1} - \frac{1}{2}I_n^2 = (n+\frac{1}{2})\sum_{i=1}^{I_n} \frac{1}{2i+1} - \frac{1}{4}(1+n-\sqrt{2n+1}).$$

Taking into account the logarthmic growth of the harmonic series, we have for n large enough

$$\sum_{i=1}^{I_n} \frac{1}{2i+1} \simeq \ln(\frac{2I_n+1}{\sqrt{I_n}}) + \frac{\gamma}{2} - 1,$$

where $\gamma \simeq 0.5772$ is the Euler-Mascheroni constant. It follows that

$$\frac{Q_n}{n} \longrightarrow \ln(2\sqrt{I_n} + \frac{1}{\sqrt{I_n}}) + \frac{\gamma}{2} - \frac{5}{4} \simeq \frac{1}{4}\ln(n) + c \quad as\ n \longrightarrow +\infty,$$

with $c = 0.5(1.5\ln(2) + \gamma - 2.5) = -0.4415$.

Since the following inequalities

$$Q_n - I_n \leq k_n^* \leq Q_n.$$

hold, and $I_n/n \longrightarrow 0$ as $n \longrightarrow +\infty$, for the increasing function κ_n^*/n we have that

$$\frac{\kappa_n^*}{n} \longrightarrow \frac{1}{4}\ln(n) + c \quad as\ n \longrightarrow +\infty.$$

A linear fit of κ_n^*/n vs. $\ln(n)$, for $10 \leq n \leq 10^{17.5}$, produces the line $\kappa_n^*/n \simeq (-0.4017 \pm 0.0102) + (0.2486 \pm 0.0004)\ln(n)$.

Appendix 2. The partitions $n = s + 2t$ and $n = s + t$

Equation (2) represents a partition of the integer n in two sections

$$s = i + j \quad and \quad n - s = 2ij, \quad with \quad 2s \leq n\ (2s = n\ iff\ i = j = 1). \tag{13}$$

Since $i \in [1, I_n]$ and $j = \phi_n(i)$, the bounds for s in the partition (13) are given by $I_n + \phi_n(I_n) = 2I_n$ and $1 + \phi_n(1)$ (see figure 1, plot of $x + y$). Therefore, the set of admissible values for s is

$$\Omega_1 = [2I_n, \frac{n+2}{3}]. \tag{14}$$

From (13) it follows that i and j are the positive integer solutions,if they exist, to the equation

$$x^2 - sx + \frac{n-s}{2} = 0. \tag{15}$$

The solutions to (15) are

$$x^{\pm} = \frac{1}{2}(s \pm \sqrt{\Delta}, \quad with \quad \Delta = s^2 + 2s - 2n. \tag{16}$$

Since Δ and s have the same parity, positive integer solutions exist iff

$$\exists s \in \Omega_1 : \sqrt{\Delta} \in \mathbb{N}.$$

Equation (3) represents another partition of the integer n in two sections

$$s = i(j+1) \quad and \quad n-s = (i+1)j, \quad with \quad 2s \leq n \ (2s = n \ iff \ i = j). \quad (17)$$

The set of admissible values for s is

$$\Omega_2 = [\frac{n+2}{3}, \frac{n}{2}]. \quad (18)$$

From (17) it follows that i is solution to the following equation

$$i^2 + (n - 2s + 1)i - s = 0,$$

and

$$j = i + n - 2s.$$

The results are given by

$$i = \frac{1}{2}[-1 - (n - 2s) + \sqrt{\Delta}],$$

$$j = \frac{1}{2}[-1 + (n - 2s) + \sqrt{\Delta}],$$

where

$$\Delta = 2n + 1 + (n - 2s)^2.$$

Since Δ and s have the same parity, positive integer solutions to the system (17) exist iff

$$\exists s \in \Omega_2 : \sqrt{\Delta} \in \mathbb{N}.$$

We can summarize the reasonings on the partitions of n in the following implications:

$$n \in \mathbb{K}, \ \Delta = s^2 + 2s - 2n \Longleftrightarrow \exists s \in \Omega_1 : \sqrt{\Delta} \in \mathbb{N},$$

$$n \in \mathbb{K}, \Delta = 4s^2 - 4ns + 2n + 1 \Longleftrightarrow \exists s \in \Omega_2 : \sqrt{\Delta} \in \mathbb{N}.$$

Appendix 3. Remarks on the sets $\Omega(a, n)$ and $\Omega^*(a, n)$

Here we consider the case

$$n - a << I_a = min(a, n, I_a, I_n). \qquad (19)$$

For example, this situation happens when we are looking for very few primes in $[2a + 1, 2n + 1]$ with large a. In virtue of the prime distribution ((1), p. 8) we should choose

$$n - a \simeq \frac{\iota}{2} \ln(2a + 1),$$

with $1 \leq \iota \leq 10$.

The difference between the top and bottom boundary lines of $\Omega^*(a, n)$ (figure 2) is

$$\phi_n(i) - \phi_a(i) = \frac{n - a}{2i + 1}.$$

It is decreasing with i, and

$$\phi_n(i) - \phi_a(i) < 1 \quad for\ i \geq [I_0],\ I_0 = \frac{n - a - 1}{2} + 1 < I_a. \qquad (20)$$

When (20) holds, at most only one integer is in $[\phi_a(i), \phi_n(i)]$. It follows that for $i \geq [I_0]$ the points of $\Omega^*(a, n)$ with integer coordinates are the points (i, j) with $j = [\phi_a(i) + 1] = [\phi_n(i)]$.

Furthermore, since $I_a < 2\sqrt{n + a}$, from (19) we have that also $n - a < 2\sqrt{n + a}$, which implies $I_n - I_a < 1$. Thus, the boundary of $\Omega^*(a, n)$ between the points (I_a, I_a) and (I_n, I_n) (figure 2) does not contain points with integer coordinates.

In the limit case $a = n$, the set $\Omega^*(a, n)$ (figure 2) reduces to the curve $y = \phi_n(x)$ for $0 \leq x \leq I_n$, and $\Omega(n, n)$ is then defined by

$$(i, j) \in \Omega(n, n) = \{i = 1, 2, ..., [I_n] : \phi_n(i) \in \mathbb{N},\ j = \phi_n(i)\}.$$

The algorithm described in section 2 identifies the points of the set $\Omega(n, n)$.

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On Characterization of δ-Topological Vector Space

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Abstract

The main objective of this paper is to present the study of δ-topological vector space. δ-topological vector space is defined by using δ-open sets and δ-continuous mapping which was introduced by J.H.H. Bayati[3] in 2019. In this paper, along with basic inherent properties of the space, δ-closure and δ-interior operators are discussed in detail. We characterize some important properties like translation, dilation of the δ-topological vector space and an example of δ-topological vector space is also established.

Keywords: Regular open set, δ-open set, δ-closed set, δ-continuous mapping and δ-topological vector space.
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1 Introduction

In functional analysis, topological vector space is one of the fundamental space being investigated by mathematicians due to the significant role played by it in other branches of mathematics such as operator theory, fixed point theory, variational inequality etc. The formalism of topological vector space belongs to Kolmogroff [4] who was the first to introduce a well structured notion of topological vector space in his pioneering work done in 1934. Since then, it has evolved further and many mathematicians have developed different generalizations of topological vector space. In 2015, the notion of s-topological vector space is being developed by M. Khan et.al.[5], which is one of the generalization of topological vector space. Later, many other significant generalizations of topological vector space are being introduced such as irresolute topological vector space [2], β-topological vector space [11], strongly preirresolute topological vector space [9], almost s-topological vector spaces [10], etc.

2 Preliminaries

In this paper, (X, τ)(or simply X) always means topological space on which no separation axioms are assumed unless stated explicitly. For a subset D of a space X , we denote closure and interior by $Cl(D)$ and $Int(D)$ respectively and neighborhood and δ-neighborhood of an element x in any topological space X is denoted by $N(x)$ and $N_\delta(x)$.

Definition 2.1. *Let B be a subset of a topological space (X, τ). Then B is said to be*
(a) Regular open [12] if $B = Int(Cl(B))$
(b) Pre-open [6] if $B \subseteq Int(Cl(B))$
(c) β-open [1] if $B \subseteq Cl(Int(Cl(B)))$.

Definition 2.2. *A subset C of a topological space X is called*
(a) Regular closed if $X \setminus C$ is open i.e. $C = Cl(Int(C))$
(b) Pre-closed if $Cl(Int(C)) \subseteq C$
(e) β-closed if $Int(Cl(Int(C))) \subseteq C$.

Definition 2.3. *A subset D of a topological space X is said to be δ-open [13] if for each $x \in D$, there exist a regular open set R such that $x \in R \subseteq D$.*

Remark 2.1. *Every regular open set is open and every open set is pre-open, while the converse need not be true.*

Example 2.1. *Let $\mathbb{R}$ be a set of real numbers with usual topology. Then $Int(Cl(\mathbb{Z})) = \emptyset$, which implies $\mathbb{Z}$ is not regular in topological space $(\mathbb{R}, \tau_u)$. Also, set of rational number denoted by $\mathbb{Q}$ is pre-open but neither regular open nor open set in topological space $(\mathbb{R}, \tau_u)$.*

The complement of δ-open set is δ-closed. The concept of δ-closure and δ-interior are introduced by Velicko [13] in 1968. The intersection of all δ-closed sets in X containing a subset $D \subseteq X$ is called δ-closure of D and is denoted by $Cl_\delta(D)$. A point $x \in Cl_\delta(D)$ if and only if $D \cap R \neq \emptyset$, for a regular open set R in X containing x. A subset C of X is δ-closed if and only if $C = Cl_\delta(C)$. The union of all δ-open sets in X that are contained in $D \subseteq X$ is called δ-interior of D and is denoted by $Int_\delta(D)$. A point $x \in X$ is called δ-interior of $D \subseteq X$ if there exist a δ-open set U in X such that $x \in U \subseteq D$.

Definition 2.4. *[4] Let X be a vector space over the field $\mathbb{F}(\mathbb{R}$ or $\mathbb{C})$. Let τ be a topology on X such that*
1) for each $x, y \in X$, and for each open neighborhood W of $x+y$ in X, there exist open neighborhoods U and V of x and y respectively in X such that $U + V \subseteq W$
2) for each $\lambda \in \mathbb{F}$, $x \in X$ and for each open neighborhood W of $\lambda \cdot x$ in X, there exist open neighborhoods U of λ in $\mathbb{F}$ and V of x in X such that $U \cdot V \subseteq W$.
Then, the $(X_{(\mathbb{F})}, \tau)$ is called topological vector space.

3 δ-Topological Vector Space

In this section, we give an examples of δ-topological vector space and further illustrate the properties of this space.

Definition 3.1. *[3] Let X be a vector space over field $\mathbb{F}(\mathbb{R}$ or $\mathbb{C})$ with standard topology. Let τ be a topology over X such that the following conditions hold:*
(a) For each $x, y \in X$ and each open set R containing $x + y$, there exist δ-open set P and Q containing x and y respectively, such that $P + Q \subseteq R$;
(b) For each $\lambda \in \mathbb{F}$, $x \in X$ and each open set R containing $\lambda x \in X$, there exist δ-open set P and Q containing λ and x respectively such that $P.Q \subseteq R$.
Then the pair $(X_{(\mathbb{F})}, \tau)$ is said to be δ-topological vector space.

Following are some examples of δ-topological vector.

Example 3.1. *Let $\mathbb{K} = \mathbb{R}$ with usual topology. Let $X = \mathbb{R}$ with base $B = \{(a, b) : a, b \in \mathbb{R}\}$. We shall show that $(X_{(\mathbb{K})}, \tau)$ is δ-topological vector space.*
For, we will check the following:
(i) Let $x, y \in X$. Consider open set $R = (x + y - \epsilon, x + y + \epsilon)$ in X containing

$x+y$. Then we can choose δ-open sets $P=(x-\eta,x+\eta)$ and $Q=(y-\eta,y+\eta)$ in X containing x and y respectively such that $P+Q\subseteq R$, for each $\eta<\frac{\epsilon}{2}$. This establish the first condition of the definition of δ-topological vector space.
(ii) Let $\lambda\in\mathbb{R}$ and $x\in X$. Consider an open set $R=(\lambda x-\epsilon,\lambda x+\epsilon)$ in X containing λx. Then, we have the following cases:
Case I: If $\lambda>0$ and $x>0$, then we can choose δ-open set $P=(\lambda-\eta,\lambda+\eta)$ in $\mathbb{R}$ containing λ and $Q=(x-\eta,x+\eta)$ in X containing x such that $P.Q\subseteq R$, for each $\eta<\frac{\epsilon}{\lambda+x+1}$.
Case II: If $\lambda<0$ and $x<0$, then $\lambda x>0$. We choose δ-open set $P=(\lambda-\eta,\lambda+\eta)$ of λ in $\mathbb{R}$ and $Q=(x-\eta,x+\eta)$ of x in $\mathbb{R}$ such that $P.Q\subseteq R$, for $\eta\leq\frac{-\epsilon}{\lambda+x-1}$.
Case III: If $\lambda>0$ and $x=0$, ($\lambda=0$ and $x>O$). We can choose δ-open sets as $P=(\lambda-\eta,\lambda+\eta)$ ($resp.(-\eta,\eta)$) containing λ in $\mathbb{K}$ and $Q=(\eta,\eta)$ (resp.$(x-\eta,x+\eta)$) containing x in $\mathbb{R}$ such that $P.Q\subseteq R$, for each $\eta<\frac{\epsilon}{\lambda+1}(resp.(\eta<\frac{\epsilon}{x+1}$)).
Case IV: If $\lambda=0$ and $x<0,(\lambda=0$ and $x>0)$. We can choose δ-open sets as $P=(\eta,\eta)(resp.(\lambda-\eta,\lambda+\eta))$ containing λ in $\mathbb{K}$ and $Q=(x-\eta,x+\eta)(resp.(-\eta,\eta))$ containing x in $\mathbb{R}$, we have $P.Q\subseteq R$, for every $\eta<\frac{\epsilon}{1-x}(resp.(\eta<\frac{\epsilon}{1-\lambda}))$.
Case V: If $\lambda=0$ and $x=0$. Then, for δ-open set $P=(\eta,\eta)$ and $Q=(\eta,\eta)$ of λ and x respectively such that $P.Q\subseteq R$, for each $\eta<\sqrt{\epsilon}$.
This proves that the pair $(X_{(\mathbb{K})},\tau)$ is δ-TVS.

Example 3.2. *Consider a vector space $X=\mathbb{R}$ of real number over the field $\mathbb{K}$ with the topology $\tau=\{\phi,Q^c,\mathbb{R}\}$, where Q^c denotes the set of irrational numbers and the field $\mathbb{K}$ is endowed with standard topology. Then $(X_{(\mathbb{K})},\tau)$ is not δ-topological vector space. For $x,y\in Q^c$ and open neighborhood Q^c of $x+y$ in X, there doesn't exist any δ-open sets P and Q containing x and y respectively such that $P+Q\subseteq Q^c$.*

Theorem 3.1. *[3] Let D be any open subset of δ-topological vector space X. Then*
(a) $x+D\in\delta O(X)$, for each $x\in X$;
(b) $\lambda D\in\delta O(X)$, for each non-zero scalar λ.

Theorem 3.2. *Let C be any closed subset of a δ-topological vector space X, then*
(a) $x+C\in\delta C(X)$, for each $x\in X$;
(b) $\lambda C\in\delta C(X)$, for each non-zero scalar λ.

Proof: (a) Let $y\in Cl_\delta(x+C)$, $z=-x+y$ and R be an open set in X containing z. Then there exist δ-open set P and Q containing $-x$ and y respectively, such that $P+Q\subseteq R$. Also, $y\in Cl_\delta(x+C)$, $y\in Q$ and Q is δ-open implies there exist regular open set Q' such that $y\in Q'\subseteq Q$. So $(x+C)\cap Q'\neq\emptyset$. Let $a\in(x+C)\cap Q'\Rightarrow -x+a\in C\cap(P'+Q')\subseteq C\cap(P+Q)\subseteq C\cap R\neq\emptyset$.

Hence $z \in Cl(C) = C \Rightarrow -x + y \in C \Rightarrow y \in x + C$. Thus, $x + C \in \delta C(X)$, for each $x \in X$.

(b) Assume that $x \in Cl_\delta(\lambda C)$ and R be open neighborhood of $y = \frac{1}{\lambda}x \in X$. Since X is $\delta - TVS$, there exist δ-open neighborhood P of $\frac{1}{\lambda}$ in $\mathbb{F}$ and Q of x in X such that $P.Q \subseteq R$. By hypothesis, $(\lambda C) \cap Q' \neq \emptyset$, for regular open set Q' subset of Q containing x. Let $a \in (\lambda C) \cap Q'$. Now $\frac{1}{\lambda}a \in C \cap P'.Q' \subseteq C \cap P.Q \subseteq C \cap R$ $\Rightarrow C \cap R \neq \emptyset$ i.e. y is a limit point of C and so $y = \frac{1}{\lambda}x \in Cl(C) = C$, since C is closed subset of X. Hence $y \in \lambda C$. Since the inclusion $\lambda C \subseteq Cl_\delta(\lambda C)$ holds generally, so $Cl_\delta(\lambda C) = \lambda C$. Therefore, λC is δ-closed set in X. This completes the proof.

Theorem 3.3. *For any subset D of δ-topological vector space X,*
(a) $Cl_\delta(x + D) \subseteq x + Cl(D)$, for each $x \in X$.
(b) $x + Cl_\delta(D) \subseteq Cl(x + D)$, for each $x \in X$.

Proof: (a) Let $y \in Cl_\delta(x + D)$ and consider $z = -x + y$ in X. Let R be open neighborhood of z. By hypothesis, there exist δ-open set P and Q containing -x and y respectively such that $P + Q \subseteq R$. Existence of δ-open set confirms the existence of regular open set P' and Q' such that $-x \in P' \subseteq P$ and $y \in Q' \subseteq Q$. Since $y \in Cl_\delta(x + D)$, $(x + D) \cap Q' \neq \emptyset$. Let $a \in (x + D) \cap Q'$. Now, $-x+a \in D\cap(P'+Q') \subseteq D\cap(P+Q) \subseteq D\cap R \neq \emptyset$, which implies $z \in Cl(D)$. Hence $y \in x + Cl(D)$. This completes the proof.

(b) Let $z \in x + Cl_\delta(D)$. Then $z = x + y$, for some $y \in Cl_\delta(D)$. Let R be any open neighborhood of z in X, then there exist δ-open neighborhood P and Q of x and y respectively such that $P + Q \subseteq R$. Also, $D \cap Q' \neq \emptyset$, for regular open set $Q' \subseteq Q$ containing y which implies $D \cap Q \neq \emptyset$. Let $a \in D \cap Q$. Then $x + a \in (x + D) \cap (P + Q) \subseteq (x + D) \cap R \neq \emptyset$, which implies z is a limit point of $x+D$ i.e $z \in Cl(x+D)$. Hence the inclusion holds for each $x \in X$.

Theorem 3.4. *For a subset D of δ-topological vector space X, the following are valid:*
(a) $x + Int(D) \subseteq Int_\delta(x + D)$, for each $x \in X$.
(b) $Int(x + D) \subseteq x + Int_\delta(D)$, for each $x \in X$.

Proof: (a) Assume $y \in x + Int(D)$. Then,$-x + y \in Int(D)$. Since X is δ-topological vector space, there exist δ-open sets P containing -x and Q containing y in X such that $Q \subseteq Int(D)$. Also, δ-openness of P and Q implies the existence of regular open set P' and Q' such that $-x \in P' \subseteq P$ and $y \in Q' \subseteq Q$ satisfying $P' + Q' \subseteq P + Q \subseteq Int(D)$. In particular,

$-x + Q' \subseteq Int(D) \subseteq D \Rightarrow Q' \subseteq x + D$. Thus there exist regular open set Q' containing y such that $y \in Q' \subseteq x + D$, which implies y is δ-interior point of $x + D$ i.e. $y \in Int_\delta(x + D)$. Hence the proof.

(b) Let $y \in Int(x + D)$, then $y = x + a$, for some $a \in D$. By definition of δ-topological vector space, there exist δ-open set P and Q such that $x \in P, a \in Q$ satisfying $P + Q \subseteq Int(x + D)$. Hence, $x + a \in P' + Q' \subseteq P + Q \subseteq Int(x + D)$, for each regular open set P' and Q' such that $x \in P' \subseteq P$ and $a \in Q' \subseteq Q$. Now $x + Q' \subseteq x + Q \subseteq Int(x + D) \subseteq x + D$, which implies $y \in x + Int_\delta(D)$. Hence the inclusion $Int(x + D) \subseteq x + Int_\delta(D)$ holds.

Theorem 3.5. *Let D be any subset of δ-topological vector space X. Then the following holds:*
(a) $\lambda Cl_\delta(D) \subseteq Cl(\lambda D)$, for every non-zero scalar λ.
(b) $Cl_\delta(\lambda D) \subseteq \lambda Cl(D)$, for every non-zero scalar λ.

Proof The proof is trivial, omitted.

Theorem 3.6. *Let X be a δ-topological vector space and D be any subset of X. Then the following holds:*
(a) $Int(\lambda D) \subseteq \lambda Int_\delta(D)$, for every non-zero scalar λ.
(b) $\lambda Int(D) \subseteq Int_\delta(\lambda D)$, for every non-zero scalar λ.

Proof The proof is trivial, omitted.

Theorem 3.7. *Let C and D be any subset of a δ-topological vector space X. Then $Cl_\delta(C) + Cl_\delta(D) \subseteq Cl(C + D)$.*

Proof: Let $z \in Cl_\delta(C) + Cl_\delta(D)$. Then $z = x + y$, where $x \in Cl_\delta(C)$ and $y \in Cl_\delta(D)$. Let R be an open neighborhood of z in X. By definition of δ-topological vector space, there exist δ-open neighborhood P and Q of x and y respectively such that $P + Q \subseteq R$. Since $x \in Cl_\delta(C)$, $C \cap P' \neq \emptyset$ for regular open set P' such that $x \in P' \subseteq P$ and also $y \in Cl_\delta(D)$, $D \cap Q' \neq \emptyset$ for regular open set Q' satisfying $y \in Q' \subseteq Q$.
Let $a \in C \cap P'$ and $b \in D \cap Q' \Rightarrow (a + b) \in (C + D) \cap (P' + Q') \subseteq (C + D) \cap (P + Q) \subseteq (C + D) \cap R \Rightarrow (C + D) \cap R \neq \emptyset$. Thus z is a closure point of $(C + D)$ i.e. $z \in Cl(C + D)$. Hence the inclusion holds.

Theorem 3.8. *For any subsets C and D of δ-topological vector space X. Then $C + Int(D) \subseteq Int_\delta(C + D)$.*

Proof: Let $z \in C + Int(D)$ be arbitrary. Then $z = x + y$, for some $x \in C, y \in Int(D)$, which results in $-x + z \in Int(D)$. By definition of δ-TVS, there exist δ-open neighborhood P and Q containing -x and z respectively such that $P + Q \subseteq Int(D)$. Hence, there exist regular open sets P' and Q' containing -x and z respectively satisfying $P' \subseteq P$, $Q' \subseteq Q$ and $P' + Q' \subseteq P + Q \subseteq Int(D)$. In particular, $-x + Q' \subseteq Int(D) \Rightarrow Q' \subseteq x + Int(D) \subseteq C + D$. Hence, there exist regular open set Q' containing z such that $z \in Q' \subseteq C + D$. Therefore, z is δ-interior point of $A + B$. Hence the proof.

Definition 3.2. *[8] A function $f : X \to Y$ is called δ-continuous if for each $x \in X$ and each open neighborhood Q of f(x), there exist open neighborhood P of x such that $f(Int(Cl(P)) \subseteq Int(Cl(Q))$.*

Lemma 3.1. *[8] For a function $f : X \to X$, the following are equivalent:*
(a) f is δ-continuous.
(b) For each $x \in X$ and each regular open set V containing f(x), there exist a regular open set U containing x such that $f(U) \subseteq V$.
(c) $f([A]_\delta) \subset [f(A)]_\delta$, for every $A \subset X$.
(d) $[f^{-1}(B)]_\delta \subset f^{-1}([B]_\delta)$, for every $B \subset X$.
(e) For every regular closed set F of Y, $f^{-1}(F)$ is δ-closed in X.
(f) For every δ-closed set V of Y, $f^{-1}(V)$ is δ-closed in X.
(g) For every δ-open set V of Y, $f^{-1}(V)$ is δ-open in X.
(h) For every regular-open set V of Y, $f^{-1}(V)$ is δ-open in X.

Theorem 3.9. *[3] Let X be δ-topological vector space, then the following are true:*
(a) the translation mapping $g_a : X \to X$ defined by $g_a(b) = a + b$, $\forall b \in X$ is δ-continuous.
(b) the mapping $g_\lambda : X \to X$ defined by $g_\lambda(a) = \lambda a$, $\forall a \in X$ is δ-continuous, where λ is a fixed scalar.

Theorem 3.10. *For a δ-topological vector space X, the mapping $\Phi : X \times X \to X$ defined by $\Phi(x, y) = x + y$, $\forall x \in X \times X$ is δ-continuous.*

Proof: Take arbitrary elements x, y in X and let R be regular open neighborhood of $x + y$ which implies R is open neighborhood of x+y. Then by hypothesis, there exist δ-open neighborhood P and Q of x and y respectively such that $P + Q \subseteq R$.
Also, by definition of δ-open set, there exist regular open neighborhood P' and Q' such that $x \in P' \subseteq P$ and $y \in Q' \subseteq Q$. This implies that $\Phi(P' \times Q') = P' + Q' \subseteq P + Q \subseteq R$. Since $P \times Q$ is regular open in $X \times X$(with respect to product topology), it follows that Φ is δ-continuous.

Theorem 3.11. *For δ-topological vector space X, the mapping $\Psi : \mathbb{K} \times X \to X$ defined by $\Psi(\lambda, x) = \lambda x$, $\forall(\lambda, x) \in \mathbb{K} \times X$ is δ-continuous.*

Proof: Let $\lambda \in \mathbb{K}$ and $x \in X$ and R be a regular open neighborhood of λx in X. Then there exist δ-open neighborhood P of λ in $\mathbb{K}$ and δ-open neighborhood Q of x in X such that $P.Q \subseteq R$. Also, $P'.Q' \subseteq P.Q \subseteq R$, for regular open set P' and Q' contained in P and Q containing λ and x respectively. Since $P \times Q$ is regular in $\mathbb{K} \times X$, $\Psi(P'.Q') \subseteq R$. Hence, it follows that Ψ is δ-continuous for arbitrary element $\lambda \in \mathbb{K}$ and $x \in X$.

Theorem 3.12. *Let X be δ-topological vector space and Y be topological vector space over the same field $\mathbb{K}$. Let $f : D_1 \to D_2$ be a linear map such that f is continuous at 0. Then f is δ-continuous.*

Proof: Let $0 \neq x \in X$ and V be regular open set and hence open in Y containing f(x). Since translation of an open set is open in topological vector space, which implies $V - f(x)$ is open in Y containing 0. Since f is continuous at 0, there exist open set U in X containing 0 such that $f(U) \subseteq V - f(x)$. Also, by linearity of f implies that $f(x + U) \subseteq V$. By theorem 3.1, $x + U$ is δ-open and hence there exist regular open set Q such that $Q \subseteq x + U$. Hence, $f(Q) \subseteq V$.

4 Conclusions

δ-Topological vector space is an extension of topological vector space and this paper give an insight into this space. We presented the space with new examples and inherent properties. Moreover, important characterization of the space is studied in this paper.

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Reliability estimation of Weibull-exponential distribution via Bayesian approach

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Abstract

Weibull-exponential distribution is considered. Bayesian method of estimation is employed in order to estimate the reliability function of Weibull-exponential distribution by using non-informative and beta priors. In this paper, the Bayes estimators of the reliability function have been obtained under squared error, precautionary and entropy loss functions.

Keywords: Weibull-exponential distribution. Reliability. Bayesian method. Non-informative and beta priors. Squared error, precautionary and entropy loss functions.
2010 AMS subject classification: 60E05, 62E15, 62H10, 62H12.§

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1. Introduction

The Weibull-exponential distribution was proposed by Oguntunde et al. [1]. They obtained some of its basic Mathematical properties. This distribution is useful as a life testing model and is more flexible than the exponential distribution. The probability function $f(x;\theta)$ and distribution function $F(x;\theta)$ of Weibull-exponential distribution are respectively given by

$$f(x;\theta) = a\lambda\theta\left(1-e^{-\lambda x}\right)^{a-1} e^{a\lambda x} exp\left[-\theta\left(e^{\lambda x}-1\right)^{a}\right] \quad ; x \geq 0. \tag{1}$$

$$F(x;\theta) = 1-e^{-\theta\left(e^{-\lambda x}-1\right)^{a}} \quad ; x \geq 0,\ \theta > 0. \tag{2}$$

Let $R(t)$ denote the reliability function, that is, the probability that a system will survive a specified time t comes out to be

$$R(t) = e^{-\theta\left(e^{\lambda t}-1\right)^{a}} \quad ; t > 0,\ \theta > 0. \tag{3}$$

And the instantaneous failure rate or hazard rate, h(t) is given by

$$h(t) = a\lambda\theta e^{a\lambda t}\left(1-e^{-\lambda t}\right). \tag{4}$$

From equation (1) and (3), we get

$$f(x;R(t)) = \frac{a\lambda e^{a\lambda x}}{\left(e^{\lambda t}-1\right)^{a}}\left(1-e^{-\lambda x}\right)^{a-1}\left[-log\,R(t)\right]\left[R(t)\right]^{\left(\frac{e^{\lambda x}-1}{e^{\lambda t}-1}\right)^{a}} \quad ;\ 0<R(t)\leq 1. \tag{5}$$

The joint density function or likelihood function of (5) is given by

$$f\left(\underline{x}\,|\,R(t)\right) = \frac{(a\lambda)^{n}\, e^{a\lambda\sum_{i=1}^{n}x_i}}{\left(e^{\lambda t}-1\right)^{na}}\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)\left[-log\,R(t)\right]^{n}\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}} \tag{6}$$

The log likelihood function is given by

$$\log f\left(\underline{x} \mid R(t)\right) = \log\left(\frac{(a\lambda)^n}{\left(e^{\lambda t}-1\right)^{na}}\right) + a\lambda\sum_{i=1}^{n} x_i + \log\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)$$
$$+ n\log\left[-\log R(t)\right] + \frac{1}{\left(e^{\lambda t}-1\right)^a}\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a \log\left[R(t)\right] \tag{7}$$

Differentiating (7) with respect to R (t) and equating to zero, we get the maximum likelihood estimator of R (t) as

$$\hat{R}(t) = \exp\left[-n\left\{\left(e^{\lambda t}-1\right)^a \Big/ \sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a\right\}\right]. \tag{8}$$

2. Bayesian method of estimation

The Bayesian estimation procedure have been developed generally under squared error loss function

$$L\left(\hat{R}(t), R(t)\right) = \left(\hat{R}(t) - R(t)\right)^2. \tag{9}$$

where $\hat{R}(t)$ is an estimate of $R(t)$. The Bayes estimator under the above loss function, say $\hat{R}(t)_S$, is the posterior mean, i.e.,

$$\hat{R}(t)_S = E\left[R(t)\right]. \tag{10}$$

The squared error loss function is often used also because it does not lead extensive numerical computation but several authors (Zellner [2], Basu & Ebrahimi [3]) have recognized the inappropriateness of using symmetric loss function. Canfield [4] points out that the use of symmetric loss function may be inappropriate in the estimation of reliability function. Norstrom [5] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is

$$L\left(\hat{R}(t),R(t)\right)=\frac{\left(\hat{R}(t)-R(t)\right)^2}{\hat{R}(t)} \tag{11}$$

The Bayes estimator of $R(t)$ under precautionary loss function is denoted by $\hat{R}(t)_P$, and is obtained by solving the following equation

$$\hat{R}(t)_P=\left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}. \tag{12}$$

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\dfrac{\hat{R}(t)}{R(t)}$. In this case, Calabria and Pulcini [6] points out that a useful asymmetric loss function is the entropy loss

$$L(\delta)\propto\left[\delta^p-p\ log_e(\delta)-1\right],$$

where $\delta=\dfrac{\hat{R}(t)}{R(t)}$, and whose minimum occurs at $\hat{R}(t)=R(t)$ when $p>0$, a positive error $\left(\hat{R}(t)>R(t)\right)$ causes more serious consequences than negative error, and vice-versa. For small $|p|$ value, the function is almost symmetric when both $\hat{R}(t)\,and\ R(t)$ are measured in a logarithmic scale, and approximately

$$L(\delta)\propto\frac{p^2}{2}\left[log_e\hat{R}(t)-log_e R(t)\right]^2.$$

Also, the loss function $L(\delta)$ has been used in Dey et al. [7] and Dey and Liu [8], in the original form having $p=1$. Thus $L(\delta)$ can be written as

$$L(\delta)=b\left[\delta-log_e(\delta)-1\right];\quad b>0. \tag{13}$$

The Bayes estimator of $R(t)$ under entropy loss function is denoted by $\hat{\theta}_E$ and is obtained as

$$\hat{R}(t)_E = \left[E\left(\frac{1}{R(t)} \right) \right]^{-1}. \tag{14}$$

For the situation where we have no prior information about $R(t)$, we may use non-informative prior distribution

$$h_1(R(t)) = \frac{1}{R(t)\ logR(t)};\quad 0 < R(t) \le 1. \tag{15}$$

The most widely used prior distribution for $R(t)$ is a beta distribution with parameters $\alpha, \beta > 0,$ given by

$$h_2(R(t)) = \frac{1}{B(\alpha,\beta)} \left[R(t)\right]^{\alpha-1} \left[1 - R(t)\right]^{\beta-1};\quad 0 < R(t) \le 1. \tag{16}$$

3. Bayes estimators of $R(t)$ under $h_1(R(t))$

Under $h_1(R(t))$, the posterior distribution is defined by

$$f(R(t) \mid \underline{x}) = \frac{f(\underline{x} \mid R(t))\, h_1(R(t))}{\int_0^1 f(\underline{x} \mid R(t))\, h_1(R(t))\, dR(t)} \tag{17}$$

Substituting the values of $h_1(R(t))$ and $f(\underline{x} \mid R(t))$ from equations (15) and (6) in (17), we get

$$f\left(R(t)\,|\,\underline{x}\right)=\frac{\left[\dfrac{(a\lambda)^n}{\left(e^{\lambda t}-1\right)^{na}}e^{a\lambda\sum_{i=1}^{n}x_i}\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)\left[-log\,R(t)\right]^n \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a}\dfrac{1}{R(t)\;log\,R(t)}\right]}{\int_0^1\left[\dfrac{(a\lambda)^n}{\left(e^{\lambda t}-1\right)^{na}}e^{a\lambda\sum_{i=1}^{n}x_i}\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)\left[-log\,R(t)\right]^n \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a}\dfrac{1}{R(t)\;log\,R(t)}\right]dR(t)}$$

$$=\frac{\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1}\left[\text{-}log\,R(t)\right]^{n-1}}{\int_0^1\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1}\left[\text{-}log\,R(t)\right]^{n-1}dR(t)}$$

or, $$f\left(R(t)\,|\,\underline{x}\right)=\frac{\left(\sum_{i=1}^{n}\left(\dfrac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1}\left[\text{-}log\,R(t)\right]^{n-1} \tag{18}$$

Theorem 1. Assuming the squared error loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_S=\left(1+\frac{\left(e^{\lambda t}-1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a}\right)^{-n} \tag{19}$$

Proof. From equation (10), on using (18),

$$\hat{R}(t)_S=E\left[R(t)\right]$$

$$=\int_0^1 R(t)\, f\left(R(t)\,|\,\underline{x}\right) dR(t)$$

$$=\int_0^1 R(t)\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1}\left[-log\, R(t)\right]^{n-1} dR(t)$$

$$=\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\int_0^1\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a}\left[-log\, R(t)\right]^{n-1} dR(t)$$

$$=\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\frac{\Gamma(n)}{\left(\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+1\right)^n}$$

or, $$\hat{R}(t)_S=\left(1+\frac{\left(e^{\lambda t}-1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a}\right)^{-n}.$$

Theorem 2. Assuming the precautionary loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_P=\left[1+\frac{2\left(e^{\lambda t}-1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a}\right]^{-\frac{n}{2}}. \tag{20}$$

Proof. From equation (12), on using (18),

$$\hat{R}(t)_P=\left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}$$

$$=\left[\int_0^1 \left(R(t)\right)^2 f\left(R(t\,|\,\underline{x})\right) dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\int_0^1 \left(R(t)\right)^2 \frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)} \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1} \left[-log\, R(t)\right]^{n-1} dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)} \int_0^1 \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a+1} \left[-log\, R(t)\right]^{n-1} dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)} \frac{\Gamma(n)}{\left(\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+2\right)^n}\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\left(\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+2\right)^n}\right]^{\frac{1}{2}}$$

or, $$\hat{R}(t)_P=\left[1+\frac{2\left(e^{\lambda t}-1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a}\right]^{-\frac{n}{2}}.$$

Theorem 3. Assuming the entropy loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_E = \left[1 - \frac{\left(e^{\lambda t}-1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a}\right]^n \tag{21}$$

Proof. From equation (14), on using (18),

$$\hat{R}(t)_E = \left[E\left(\frac{1}{R(t)}\right)\right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} f\left(R(t\,|\,\underline{x})\right) dR(t)\right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} \frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)} \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1} \left[-log\, R(t)\right]^{n-1} dR(t)\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)} \int_0^1 \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-2} \left[-log\, R(t)\right]^{n-1} dR(t)\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)} \frac{\Gamma(n)}{\left(\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)-1\right)^n}\right]^{-1}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)^{n}}{\left(\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)-1\right)^{n}}\right]^{-1}$$

or, $$\hat{R}(t)_E=\left[1-\frac{\left(e^{\lambda t}-1\right)^{a}}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{a}}\right]^{n}.$$

4. Bayes estimators of $R(t)$ under $h_2\left(R(t)\right)$

Under $h_2\left(R(t)\right)$, the posterior distribution is defined by

$$f\left(R(t)\,|\,\underline{x}\right)=\frac{f\left(\underline{x}\,|\,R(t)\right)h_2\left(R(t)\right)}{\int_0^1 f\left(\underline{x}\,|\,R(t)\right)h_2\left(R(t)\right)dR(t)} \tag{22}$$

Substituting the values of $h_2\left(R(t)\right)$ and $f\left(\underline{x}\,|\,R(t)\right)$ from equations (16) and (6) in (22), we get

$$f\left(R(t)\,|\,\underline{x}\right)=\frac{\left[\begin{array}{l}\frac{(a\lambda)^n}{\left(e^{\lambda t}-1\right)^{na}}e^{a\lambda\sum_{i=1}^{n}x_i}\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)\left[-\log R(t)\right]^{n}\\ \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}}\frac{1}{B(\alpha,\beta)}\left[R(t)\right]^{\alpha-1}\left[1-R(t)\right]^{\beta-1}\end{array}\right]}{\int_0^1\left[\begin{array}{l}\frac{(a\lambda)^n}{\left(e^{\lambda t}-1\right)^{na}}e^{a\lambda\sum_{i=1}^{n}x_i}\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)\left[-\log R(t)\right]^{n}\\ \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}}\frac{1}{B(\alpha,\beta)}\left[R(t)\right]^{\alpha-1}\left[1-R(t)\right]^{\beta-1}\end{array}\right]dR(t)}$$

$$= \frac{\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}+\alpha-1}\left[-log\,R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}}{\int_0^1\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}+\alpha-1}\left[-log\,R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}dR(t)}$$

or, $$f\left(R(t)\,|\,\underline{x}\right) = \frac{\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}+\alpha-1}\left[-log\,R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha+k\right)^{n+1}\right]} \tag{23}$$

Theorem 4. Assuming the squared error loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_S = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha+1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha+k\right)^{n+1}}\right] \tag{24}$$

Proof. From equation (10), on using (23),

$$\hat{R}(t)_S = E\left[R(t)\right]$$

$$= \int_0^1 R(t)\, f\left(R(t)\,|\,\underline{x}\right) dR(t)$$

$$= \int_0^1 R(t)\frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha-1}\left[-log\,R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha+k\right)^{n+1}\right]}dR(t)$$

$$= \frac{\int_0^1 \left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha} \left[\text{-}log\, R(t)\right]^n \left[1-R(t)\right]^{\beta-1} dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}\right]}$$

or, $$\hat{R}(t)_S = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}}\right].$$

Theorem 5. Assuming the precautionary loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_P = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+2+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}}\right]^{\frac{1}{2}} \tag{25}$$

Proof. From equation (12), on using (23),

$$\hat{R}(t)_P = \left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\int_0^1 \left(R(t)\right)^2 f\left(R(t\,|\,\underline{x})\right) dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\int_0^1 \left(R(t)\right)^2 \frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha-1}\left[\text{-}log\, R(t)\right]^n\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}\right]}dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\frac{\int_0^1\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+1}\left[\text{-}log\, R(t)\right]^n\left[1-R(t)\right]^{\beta-1}dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}\right]}\right]^{\frac{1}{2}}$$

or, $$\hat{R}(t)_P=\left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+2+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}}\right]^{\frac{1}{2}}.$$

Theorem 6. Assuming the entropy loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_E=\left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha-1+k\right)^{n+1}}\right] \tag{26}$$

Proof. From equation (14), on using (23),

$$\hat{R}(t)_E = \left[E\left(\frac{1}{R(t)} \right) \right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} f\left(R(t \mid \underline{x}) \right) dR(t) \right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} \frac{\left[R(t) \right]^{\left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right) + \alpha - 1} \left[\text{-}log\, R(t) \right]^n \left[1 - R(t) \right]^{\beta - 1}}{\Gamma(n+1) \left[\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta - 1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right) + \alpha + k \right)^{n+1} \right]} dR(t) \right]^{-1}$$

$$= \left[\frac{\int_0^1 \left[R(t) \right]^{\left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right) + \alpha - 2} \left[\text{-}log\, R(t) \right]^n \left[1 - R(t) \right]^{\beta - 1} dR(t)}{\Gamma(n+1) \left[\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta - 1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right) + \alpha + k \right)^{n+1} \right]} \right]^{-1}$$

$$= \left[\frac{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta - 1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right) + \alpha - 1 + k \right)^{n+1}}{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta - 1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right) + \alpha + k \right)^{n+1}} \right]^{-1}$$

or, $$\hat{R}(t)_E = \left[\frac{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta - 1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right) + \alpha + k \right)^{n+1}}{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta - 1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right) + \alpha - 1 + k \right)^{n+1}} \right] .$$

5. Conclusion

We have obtained a number of Bayes estimators of reliability function $R(t)$ of Weibull-exponential distribution. In equations (19), (20), and (21), we have obtained the Bayes estimators by using non-informative prior and in equations (24), (25), and (26), under beta prior. From the above said equation, it is clear that the Bayes estimators of $R(t)$ depend upon the parameters of the prior distribution. In this case the risk function and corresponding Bayes risks do not exist.

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$gp\alpha$- Kuratowski closure operators in topological spaces

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Abstract

In this paper, we introduce and study topological properties of gpα-limit points, $gp\alpha$-derived sets, $gp\alpha$-interior and $gp\alpha$-closure using the concept of $gp\alpha$-open set. Further, the relationships between these concepts are investigated. Also, Kuratowski axioms are discussed.
Keywords: $gp\alpha$-open set, $gp\alpha$-closed set, $gp\alpha$-limit point, $gp\alpha$-derived set, $gp\alpha$-closure, $gp\alpha$-interior points.
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1 Introduction

Topology is an indispensable object of study in Mathematics with open sets as well as closed sets being the most fundamental concepts in topological spaces. General topology plays an important role in many branches of mathematics as well as many fields of applied sciences.

Introduction to the concept of pre-open sets and pre-closed sets were made by Mashhour et al. (7) and the idea of α-open sets was introduced by Njastad (6).The concepts and characterizations of semi open and semi pre open sets are studied in (4) and (1) repectively. The concept, generalized closed sets of Levine (5) opened the flood gates of research in generalizations of closed sets in general topology. Many researchers (3), (8), (10), (11) worked on weaker forms of closed sets. Recently, Benchalli et al.(2) and Patil et al. (9) introduced and studied the concept of $\omega\alpha$-open sets and $gp\alpha$-open sets in topological spaces.

The present authors continued the study of $gp\alpha$-closed sets and their properties. We study the $gp\alpha$-closure, $gp\alpha$-interior, $gp\alpha$-neighbourhood, $gp\alpha$-limit points and $gp\alpha$-derived sets by using the concept of $gp\alpha$-open sets and their topological properties. We provide the relationship between $gp\alpha$-derived set (resp. $gp\alpha$-limit points, $gp\alpha$-interior) and pre-derived set (resp. pre-limit points and pre-interior). Also, we studied Kuratowski closure axioms with respect to $gp\alpha$-open sets.

2 Preliminaries

Throught the paper, let X and Y (resp. (X,τ) and (Y,σ)) always denotes non-empty topological spaces on which no separation axioms are assumed unless explicitly mentioned.

The following definitions are useful in the sequel:

Definition 2.1. *A subset A of X is said to be a*
(a) $\omega\alpha$-closed (2) if α-cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is ω-open in X.
(b) $gp\alpha$-closed (9) if p-cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is α-open in X.

3 $gp\alpha$-Interior and $gp\alpha$-closure in topological spaces

This section deals with $gp\alpha$-interior and $gp\alpha$-closure and some of their properties.

Definition 3.1. *(9) Let $A \subset X$, then $gp\alpha$-interior of A is denoted by $gp\alpha$-int(A), and is defined as $gp\alpha$-int(A) = $\cup\{G : A \subseteq G :$ G is $gp\alpha$-open in X $\}$.*

Definition 3.2. *(9) Let $A \subset X$, then $gp\alpha$-closure of A is denoted by $gp\alpha$-cl(A), and is defined as $gp\alpha$-cl(A) = $\cap\{G : A \subseteq G :$ G is $gp\alpha$-closed in X $\}$.*

Theorem 3.1. *If A is $gp\alpha$-closed then $gp\alpha$-cl(A) = A.*
Proof: *Let A be $gp\alpha$-closed in X. Since $A \subseteq A$, and A is $gp\alpha$-closed in X. Then $A \in \{G : A \subseteq G$ and G is $gp\alpha$-closed in X $\}$, that is $A = \cap\{G : A \subseteq G$ and G is $gp\alpha$-closed $\}$. Hence $gp\alpha$-cl(A) $\subseteq A$. But $A \subseteq gp\alpha$-cl(A) is always true. Therefore $gp\alpha$-cl(A) = A.*

In general the converse of Theorem 3.1 is not true.

Example 3.1. *Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$. Let $A = \{a, b\}$. Then we can observe that $gp\alpha$-cl(A) = $\{a, b\}$. therefore $gp\alpha$-cl(A) = A. But A is not $gp\alpha$-closed in X.*

Remark 3.1. *(9) Let $A \subseteq X$ and A is $gp\alpha$-closed in X. Then $gp\alpha$-cl(A) is the smallest $gp\alpha$-closed set containing A.*

However, the converse of the Remark 3.1 is not true in general.

Example 3.2. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, c\}\}$. Consider $A = \{a, c\}$, then $gp\alpha$-cl(A) = X , which is the smallest $gp\alpha$-closed set containing A. But A is not $gp\alpha$-closed in X.*

Remark 3.2. *(9) For subsets A,B of X, then $gp\alpha$-cl$(A \cap B)$$\subseteq$$gp\alpha$-cl(A) $\cap$ $gp\alpha$-cl(B).*

Remark 3.3. *(9) For subsets A,B of X, then $gp\alpha$-cl$(A \cup B)$ = $gp\alpha$-cl(A) $\cup$$gp\alpha$-cl(B).*

Remark 3.4. *For any $A, B \subseteq X$ then we have the following properties:*
(i) $gp\alpha$-int(X) = X and $gp\alpha$-int$(\phi) = \phi$. (9)
(ii) $gp\alpha$-int(A) $\subseteq A$.(9)
(iii) If B is any $gp\alpha$-open set contained in A then $B \subseteq gp\alpha$-int(A).(9)
(iv) if $A \subseteq B$, $gp\alpha$-int(A) $\subseteq$ $gp\alpha$-int(B).
(v) $gp\alpha$-int($gp\alpha$-int(A)) = $gp\alpha$-int(A).

Remark 3.5. *For any $A, B \subseteq X$, $gp\alpha$-int$(A \cup B) =$ $gp\alpha$-int(A) $\cup$ $gp\alpha$-int(B).*

4 $gp\alpha$-Neighbourhood points in topological spaces

This section deals with the properties of gpα-neighbourhood points in topological spaces.

Definition 4.1. *(9) A subset N of X is said to be gpα-neighbourhood of a point $x \in X$, if there exists an gpα-open set G containing x such that $x \in G \subseteq N$.*

Theorem 4.1. *Every neighbourhood N of X is a gpα-neighbourhood of X.*
Proof: *Follows from the definition 4.1 and every open set is gpα-open (9).*

From the following example converse of the Theorem 4.1is not true.

Example 4.1. *Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a, b\}, \{a, b, c\}\}$.*
Let $a \in X$. Consider $A = \{a, d\}$. Since A is gpα-neighbourhood of the point a, but A is not a neighbourhood of the point a.

Remark 4.1. *If $N \subseteq X$ is gpα-open then, N is gpα-neighbourhood of each of its points.*

Converse of the Theorem 4.1 need not true in general, as seen from the following example.

Example 4.2. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$. Here the gp$\alpha$-open sets are: X, ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{c, d\}$, $\{a, c, d\}$. Then the set $A = \{a, b\}$ is gpα-neighbourhod of the points a and b, but the set $A = \{a, b\}$ is not gpα-open in X.*

Theorem 4.2. *Let A be gpα-closed set in X and $x \in A^c$. Then there exists gpα-neighbourhood N of x such that $N \cap A = \phi$.*
Proof: *Let A be gpα-closed in X and $x \in A^c$. Then $A^c = X \setminus A$ which is gpα-open in X. Then from Remark 4.1, A^c is gpα-neighbourhood of each of its points. Hence, for every point $x \in A^c$, there exists gpα-neighbourhood N of X such that $N \subseteq A^c$. Hence $N \cap A = \phi$.*

Theorem 4.3. *Let $x \in X$ and $gp\alpha N(x)$ be the collection of all gpα-neighbourhoods of X. Then the following results holds:*
(i) $gp\alpha N(x) \neq \phi$, $\forall\, x \in X$.
(ii) $N \in gp\alpha N(x)$ implies $x \in N$.
(iii) Let $N \in gp\alpha N(x)$ and $N \subseteq M$, then $M \in gp\alpha N(x)$.
(iv) $N \in gp\alpha N(x)$ and $M \in gp\alpha N(x)$ then $N \cap M \in gp\alpha N(x)$.
Proof: *(i) We have X is always gpα-open in X. Hence X is in gpαN(x) of every point $x \in X$. Therefore gpα-N(X) $\neq \phi$ for every point $x \in X$.*
(ii) From definition of $N \in gp\alpha N(x)$, it follows that $x \in N$.

(iii) Let $N \in gp\alpha N(x)$ and $N \subseteq M$. Then there exists $gp\alpha$-open set G such that $x \in G \subseteq N$. Since $N \subseteq M$, then $x \in G \subseteq M$. So by definition of 4.1, M is a $gp\alpha$-neighbourhood point of x. Hence $M \in gp\alpha N(x)$.
(iv) Let $N \in gp\alpha N(x)$ and $M \in gp\alpha N(x)$. Then, there exist $gp\alpha$-open sets U and V such that $x \in U \subseteq N$ and $x \in V \subseteq M$. That is $x \in U \cap V \subseteq N \cap M$. Therefore, for every point $x \in X$, there exists $gp\alpha$-open set $U \cap V$ such that $x \in U \cap V \subseteq N \cap M$. So, $N \cap M$ is a $gp\alpha$-neighbourhood of a point x. Hence, intersection of two $gp\alpha$-neighbourhood of a point is again a $gp\alpha$-neighbourhood of point.

Corollary 4.1. *For any subset A of X, every α-interior point of A is $gp\alpha$-interior point of A.*
Proof: *It follows from the fact that every α-open set is $gp\alpha$-open in X (9).*

Theorem 4.4. *For any subset A of X, every pre-interior point of A is $gp\alpha$-interior point of A.*
Proof: *For any pre-interior point x of A. Then there exists pre-open set G containing x such that $G \subseteq A$. Since every pre-open set is $gp\alpha$-open (9), then G is $gp\alpha$-open in X. Hence x is a $gp\alpha$-interior point of A.*

5 $gp\alpha$-Kuratowski closure operators in topological spaces

Theorem 5.1. *If P-C(X,τ) is closed under finite union, then $gp\alpha$-C(X,τ) is closed under finite union, where P-C(X,τ) and $gp\alpha$-C(X,τ) are the families of pre-closed sets and $gp\alpha$-closed sets in (X,τ) respectively.*
Proof: *Let A and B are $gp\alpha$-closed sets in X and $A \cup B \subseteq G$, where G is $\omega\alpha$-open in X. Then $A \subseteq G$ and $B \subseteq G$. Since A and B are $gp\alpha$-closed, then $pcl(A) \subseteq G$ and $pcl(B) \subseteq G$. Then $pcl(A) \cup pcl(B) = pcl(A \cup B) \subseteq G$ from (7). Thus, from hypothesis, $pcl(A \cup B) \subseteq G$. Hence $A \cup B$ is $gp\alpha$-closed in X.*

Definition 5.1. *Let $\tau^*_{gp\alpha}$ be the topology on X generated by $gp\alpha$-closure in the usual manner, $\tau^*_{gp\alpha} = \{G \subset X : gp\alpha\text{-}cl(X \setminus G) = X \setminus G\}$.*

Definition 5.2. *Let $\tau^*_{g^*p}$ be the topology on X generated by g^*p-closure in the usual manner, that is*
$\tau^*_{g^*p} = \{G \subset X : g^*p\text{-}cl(X \setminus G) = X \setminus G\}$

Theorem 5.2. *Let $A \subseteq X$. Then the following statements holds:*
(i) $\tau \subseteq \tau^*_{gp\alpha}$
(ii) $\tau \subseteq \tau^p \subseteq \tau^*_{gp\alpha}$ (τ^p is family of pre-open sets.(12))

Proof: *(i) Let $A \in \tau$, then A^c is closed in X. We have $A^c \subseteq gp\alpha$-cl$(A^c) \subseteq cl(A^c)$. Since, A^c is closed, then $cl(A^c)$ is also closed in X. Hence $A^c \subseteq gp\alpha$-cl$(A^c) \subseteq A^c$. Therefore $gp\alpha$-cl$(A^c) \subseteq A^c$. But $A^c \subseteq gp\alpha$-cl(A^c) is always true. Thus $gp\alpha$-cl$(A^c) = A^c$. Hence $A \in \tau^*_{gp\alpha}$.*
(ii) Since, every pre-closed set is $gp\alpha$-closed in X, proof follows.

Theorem 5.3. *For any subset A of a topological space X, $\tau \subseteq \tau^*_{gp\alpha} \subseteq \tau^*_{g^*p}$.*
Proof: *Let us consider $A \in \tau$, then A^c is closed in X. Then $A^c \subseteq gp\alpha$-cl$(A^c) \subseteq cl(A^c)$. Since, A^c is closed, then $cl(A^c) = A^c$. Therefore $A^c \subseteq gp\alpha$-cl$(A^c) \subseteq A^c$. Hence, $gp\alpha$-cl$(A^c) \subseteq A^c$. But $(A^c) \subseteq gp\alpha$-cl(A^c) is always true. Hence $(A^c) = gp\alpha$-cl(A^c). Thus $A \in \tau^*_{gp\alpha}$ and hence $\tau \subseteq \tau^*_{gp\alpha}$.*
Let $A \in \tau^*_{gp\alpha}$. Then $gp\alpha$-cl$(A^c) = A^c$. But $A^c \subseteq g^*p$-cl$(A^c) \subseteq gp\alpha$-cl$(A^c) = A^c$ from (9). Hence g^*p-cl$(A^c) = A^c$. Hence $A \in \tau^*_{g^*p}$. Thus $A \in \tau^*_{gp\alpha}$ implies that $A \in \tau^*_{g^*p}$. Hence $\tau \subseteq \tau^*_{gp\alpha} \subseteq \tau^*_{g^*p}$.

Theorem 5.4. *The following statements are equal for the space X:*
(i) Every $gp\alpha$-closed set is pre-closed.
(ii) $\tau^p = \tau^*_{gp\alpha}$.
(iii) For each $x \in X$, $\{x\}$ is $\omega\alpha$-open or pre-open.
Proof: *$(i) \to (ii)$ Let $G \in \tau^*_{gp\alpha}$. Then from (9) and by the Theorem 3.1, we have $gp\alpha$-cl(A) = A. Hence $X \setminus G = gp\alpha$-cl$(X \setminus G)$ = p-cl$(X \setminus G)$. Therefore $X \setminus G$ is pre-closed and so G is pre-open. Therefore $\tau^*_{gp\alpha} \subseteq \tau^p$ and from (9), $\tau^p \subseteq \tau^*_{gp\alpha}$. Hence $\tau^p = \tau^*_{gp\alpha}$.*
$(ii) \to (iii)$ Let $\{x\} \in X$. By (9), we have $X \setminus \{x\} = gp\alpha$-cl$(X \setminus \{x\})$ is true only when $\{x\}$ is not $\omega\alpha$-closed. Hence $\{x\} \in \omega\alpha$-C(X, τ) or $x \in \tau^p$.
$(iii) \to (i)$ Let A be $gp\alpha$-closed in X and $x \in p - cl(A)$. Then, we have $x \in A$.
case I: If $\{x\}$ is $\omega\alpha$-closed. Suppose $x \notin A$, then $p - cl(A) \setminus A$ contains $\omega\alpha$-closed set $\{x\}$, which is contradiction. Hence $x \in A$.
case II: If $\{x\}$ is pre-open. Since $x \in pcl(A)$, then $\{x\} \cap A = \phi$. Hence, we have pcl(A) = A and thus A is pre-closed. Therefore $gp\alpha$-C$(X, \tau) \subset$ p-C(X, τ).

Theorem 5.5. *Every $gp\alpha$-closed set closed if and only if $\tau = \tau^*_{gp\alpha}$.*
Proof: *Suppose every $gp\alpha$-closed set is closed. Let A be $gp\alpha$-closed then, $gp\alpha$-cl(A) = cl(A). Thus $\tau = \tau^*_{gp\alpha}$.*
Conversely, let A be $gp\alpha$-closed then from (9), $A = gp\alpha$-cl(A). Hence $X \setminus A \in \tau^*_{gp\alpha}$. Hence, A is closed in X.

Theorem 5.6. *Every $gp\alpha$-closed set is pre-closed if and only if $\tau^p = \tau^*_{gp\alpha}$.*
Proof: *Suppose that every $gp\alpha$-closed set is pre-closed. Let A be $gp\alpha$-closed in X. Then from hypothesis, $gp\alpha$-cl(A) = pcl(A). Thus $\tau^p = \tau^*_{gp\alpha}$.*
Conversely, let A be $gp\alpha$-closed in X. Then $A = gp\alpha$-cl(A). Thus $X \setminus A \in \tau^*_{gp\alpha}$. Hence, A is pre-closed in X.

Remark 5.1. *Let A be any subset of X. Then $gp\alpha$-int(A) is the largest $gp\alpha$-open set contained in A if A is $gp\alpha$-open.*

Theorem 5.7. *$gp\alpha$-closure is a Kuratowski closure operator on X.*
Proof:*Follows from the Definition 3.2 and (9)*

6 Characterizations of gpα-closed sets in topological spaces

Definition 6.1. *(9) A point $x \in X$ is a $gp\alpha$-limit point of a subset A of X, if and only if every $gp\alpha$-neighbourhood of x contains a point of A distinct from x. That is, $[N \setminus \{x\}] \cap A \neq \phi$ for each $gp\alpha$-neighbourhood N of x.*

Definition 6.2. *(9) The set of all $gp\alpha$-limit points of A is a $gp\alpha$-derived set of A and is denoted by $gp\alpha$-d(A).*

Example 6.1. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$.*
Let $A = \{c\}$. Then the only limit point with respect to the set $A = c$ is point b.
Therefore $d(A) = \{b\}$.
But $gp\alpha$-limit point with respect to the set A is ϕ. Therefore $gp\alpha$-d(A) = ϕ.

Theorem 6.1. *Let A be any subset of X, Then, A is $gp\alpha$-closed if and only if $gp\alpha$-d(A)$\subseteq A$.*
Proof: *Let A be $gp\alpha$-closed in X, then A^c is $gp\alpha$-open in X such that $x \in A^c$. Then for each point $x \in X$ and from Definition 4.1 , there exist $gp\alpha$-open set G such that $x \in G \subseteq X \setminus A$. Then $A \cap (X \setminus A) = \phi$. Therefore, $gp\alpha$-neighbourhood of G contains no points of A. Hence, x is not a $gp\alpha$-limit point of A. Thus, no point of $X \setminus A$ is a $gp\alpha$-limit point of A, that is A contains all the $gp\alpha$-limit points. Therefore A contains the $gp\alpha$-derived points. Hence $gp\alpha$-d(A) $\subseteq A$.*
Conversely, suppose $gp\alpha$-d(A)$\subseteq A$ and let $x \in A^c$. So $x \notin A$. Hence $x \notin gp\alpha$-d(A). Therefore x is not a limit point of A. Then, there exists $gp\alpha$-open set G such that $G \cap (A \setminus \{x\}) = \phi$, that is $G \subseteq X \setminus A$. Therefore for each $x \in X \setminus A$, there exists $gp\alpha$-open set G such that $x \in G \subseteq X \setminus A$. Therefore $X \setminus A$ is $gp\alpha$-open in X and hence A is $gp\alpha$-closed.

Theorem 6.2. *Let τ_1 and τ_2 be any two topologies on a set X such that $gp\alpha$-O$(X, \tau_1) \subseteq gp\alpha$-O$(X, \tau_2)$. Then for every subset A of X, every $gp\alpha$-limit point of A with respect to τ_2 is $gp\alpha$-limit point of A with respect to τ_1.*
Proof: *Let x be a $gp\alpha$-limit point of A with respect to τ_2. Then by definition of $gp\alpha$-limit point $(G \cap A) \setminus \{x\} \neq \phi$, this is true for every $G \in gp\alpha$-O(X, τ_2) and $x \in G$.*

But by hypothesis, gpα-O$(X, \tau_1) \subseteq$ gpα-O(X, τ_2). Hence $(G \cap A) \setminus \{x\} \neq \phi$ for every $G \in$ gpα-O(X, τ_1) such that $x \in G$. Hence x is a gpα-limit point of A with respect to the topology τ_1.

Theorem 6.3. *Let A and B be any two subsets of (X, τ). Then the following assertions are valid:*
(i) gpα-d(A) $\subseteq d_p(A)$, where d_p is a pre-derived set (12).
(ii) gpα-d($A \cup$ gpα-d(A)) $\subseteq A \cup$ gpα-d(A).
Proof: *(i) It clearly observed from the fact that every pre-open set is gpα-open in X.*
Then $y \in G$ and $y \in$ gpα-d(A) $\setminus\{x\}$. That is $y \in G$ and $y \in$ gpα-d(A). Hence $G \cap (A \setminus \{y\}) \neq \phi$. Let $z \in G \cap (A \setminus \{y\})$, then $x \neq z$ as $x \notin A$. Thus $G \cap (A \setminus \{x\}) \neq \phi$.
(ii) Let $x \in$ gpα-d($A \cup$ gpα-d(A)). If $x \in A$, then $x \in$ gpα-d(A). Therefore $x \in A \cup$ gpα-d(A). On the contrary assume that $x \notin A$. Then $G \cap (A \cup$ gpα-d(A)) $\setminus\{x\}) \neq \phi$, is true for all $G \in$ gpα-d(A) and $x \in G$. Therefore $(G \cap A) \setminus \{x\} \neq \phi$ or $G \cap ($gpα-d(A)) $\setminus\{x\}) \neq \phi$. Thus $x \in$ gpα-d(A).
If $G \cap ($gpα-d(A)) $\setminus\{x\} \neq \phi$, then will get $x \in$ gpα-d(gpα-d(A)). Since $x \notin A$, then $x \in$ gpα-d(gpα-d(A)) $\setminus A$. Therefore gpα-d($A \cup$ gpα-d(A)) $\subseteq A \cup$ gpα-d(A).

Remark 6.1. *We can see the following implification with respect to gpα-open sets.*

Example 6.2. *Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$. In (X, τ) we have,*
pre-open sets are: X, ϕ, $\{a\}, \{c\}, \{d\}, \{a, c\}, \{b, c\}, \{c, d\}, \{d, e\}, \{a, c, d\}$, $\{a, c, e\}, \{a, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}$.
gpα-open sets are: X, ϕ, $\{a\}, \{c\}, \{d\}, \{a, b\}, \{b, d\}, \{c, d\}, \{a, c\}$, $\{b, c\}, \{d, e\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c\}, \{a, c, e\}, \{a, d, e\}$, $\{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}$.
(i)Let $A = \{b, c, d\}$.
Then pre-limit point of the set A is $\{b, e\}$ and
gpα-limit point of A is $\{e\}$.
Hence $d_p(A) = \{b, e\}$ and gpα-d(A) = $\{e\}$.
Hence, gpα-d(A) $\subseteq d_p(A)$.

(ii)Let $A = \{a, c, d\}$, then gpα-d(A) = $\{b, e\}$.
Consider $A \cup$ gpα-d(A) = $\{a, c, d\} \cup \{b, e\} = X$.
But gpα-d(X) = $\{b, e\}$.
Therefore gpα-d($A \cup$ gpα-d(A)) = gpα-d(X) = $\{b, e\}$.
Now consider $A \cup$ gpα-d(A) = $\{a, c, d\} \cup \{b, e\} = X$.
Hence gpα-d($A \cup$ gpα-d(A)) $\neq A \cup$ gpα-d(A), that is $\{b, e\} \neq X$, but gpα-d($A \cup$ gpα-d(A) $\subset A \cup$ gpα-d(A).

Remark 6.2. *if $A \subseteq B$, then gpα-d(A) $\subseteq$ gpα-d(B).*

Example 6.3. *Consider the Example 6.2,*
Let $A = \{b, c, d\}$. Then we have gpα-d(A) = $\{e\}$.
Let $B = \{a, e\}$. Then gpα-limit point of the set B is ϕ, that is gpα-d(B) = ϕ.
Thus we can observe that gpα-d(B) $\subset$ gpα-d(A) but $B \nsubseteq A$.

Remark 6.3. *gpα-d$(A \cap B) \subseteq$ gpα-d(A) $\cap$ gpα-d(B).*

Example 6.4. *From the Example 6.2,*
Let $A = \{b, c, d\}$ and $B = \{b, c\}$ are any two subsets of X.
Then gpα-d(A) = $\{e\}$ and gpα-d(B) = ϕ.
Also gpα-d$(A \cap B)$ = ϕ.
Therefore gpα-d$(A \cap B) \subseteq$ gpα-d(A) $\cap$ gpα-d(B)

Theorem 6.4. *Let A be any subset of X and $x \in X$. Then the following statements are equal:*
(i) For each $x \in X$, $A \cap G \neq \phi$ where G is gpα-open in X.
(ii) $x \in$ gpα-cl(A).
Proof: *Let A be any subset of X.*
$(i) \rightarrow (ii)$: On the contrary assume that $x \notin$ gpα-cl(A). Then there exists gpα-closed set F such that $A \subseteq F$ and $x \notin F$. Then $X \setminus F$ is gpα-open in X containing a point x. Hence $A \cap (X \setminus F) \subseteq A \cap (X \setminus A) = \phi$, which is contradiction to the assumption. Hence $x \in$ gpα-cl(A).
$(ii) \rightarrow (i)$: Follows from the Definition 3.2.

Corollary 6.1. *For any subset A of a space X, gpα-d(A) $\subseteq$ gpα-cl(A).*

Theorem 6.5. *Let A be any subset of X, then gpα-cl(A) = $A \cup$ gpα-d(A).*
Proof: *Let $x \in$ gpα-cl(A). On the contrary assume that $x \notin A$. Let G be any gpα-open set containing a point x. Then $(G \setminus \{x\}) \cap A \neq \phi$. Therefore x is gp$\alpha$-limit point of A and hence x is gpα-derived set of A, that is $x \in$ gpα-d(A).*
Hence gpα-cl(A) $\subseteq A \cup$ gpα-d(A).
From the Corollary 6.1, we have gpα-d(A) $\subseteq$ gpα-cl(A) and $A \subseteq$ gpα-cl(A) is always true. Hence $A \cup$ gpα-d(A) $\subseteq$ gpα-cl(A).
Therefore gpα-cl(A) = $A \cup$ gpα-d(A).

Theorem 6.6. *Let A be gpα-open set in X and B be any subset of X. Then $A \cap$gpα-cl(B) $\subseteq$ gpα-cl$(A \cap B)$.*
Proof: *Let $x \in A \cap$gpα-cl(B). Then $x \in A$ and $x \in$ gpα-cl(B). From the Theorem 6.5, we have gpα-cl(B) = $B \cup$ gpα-d(B).*
If $x \in B$ then $x \in A \cap B$. Then $A \cap B \subseteq$ gpα-cl$(A \cap B)$.
If $x \notin B$ then $x \in$ gpα-d(B). From the definition of gpα-limit point, we have

$G \cap B \neq \phi$ for every $gp\alpha$-open set G containing x. Therefore $G \cap (A \cap B) = (G \cap A) \cap B \neq \phi$. Hence $x \in gp\alpha$-$d(A \cap B) \subseteq gp\alpha$-$cl(A \cap B)$. Therefore $A \cap gp\alpha$-cl(B) $\subseteq gp\alpha$-$cl(A \cap B)$.

However the equality does not holds in general

Example 6.5. *Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}\}$.*
Let $A = \{a, b\}$ and $B = \{a, c\}$ are two subsets of X.
Then $A \cap gp\alpha$-cl(B) = $\{a, b\}$
and $gp\alpha$-$cl(A \cap B) = X$
This implies $A \cap gp\alpha$-cl(B) $\neq gp\alpha$-$cl(A \cap B)$. But, $A \cap gp\alpha$-cl(B) $\subset gp\alpha$-$cl(A \cap B)$.

7 Conclusions

In this present work, we have analyzed the notion of generalized pre α-closed sets in topological spaces. We have established the results of $gp\alpha$-closure, $gp\alpha$-interior, $gp\alpha$-neighbourhood and $gp\alpha$-limit points. Moreover, we have characterized these concepts with suitable examples. Finally, we apply $gp\alpha$-open sets for Kuratowski closure axioms. There is a scope to study and extend these newly defined concepts.

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Regular generalized fuzzy b-separation axioms in fuzzy topology

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Abstract

Regular generalized fuzzy b-closure and regular generalized fuzzy b-interior are stated and their characteristics are examined, also Regular generalized fuzzy b -τ_i separation axioms have been introduced and their interrelations are examined. The characterization of regular generalized fuzzy b –separation axioms are analyzed.

Keywords: rgfbCS; rgfbOS; rgfbCl; rgfbInt; rgfbT$_0$; rgfbT$_{1;}$ rgfbT$_2$; rgfbT$_{2\frac{1}{2}}$ and fuzzy topological spaces X (in short fts).
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1. Introduction

The fundamental theory of fuzzy sets were introduced by Zadeh [16] and Chang [9] studied the theory of fuzzy topology. After this Ghanim.et.al [10] introduced separation axioms, regular spaces and fuzzy normal spaces in fuzzy topology. The theory of regular generalized fuzzy b-closed set (open set) presented by Jenifer et. al [11]. In this study we define rgfb-closure, rgfb-interior and rgfb-separation axioms and their implications are proved. Effectiveness nature of the various concepts of fuzzy separation ideas are carried out. Characterizations are obtained.

2. Preliminary

(X_1, τ),(X_2, σ) (or simply X_1, X_2) states fuzzy topological spaces(in short, fts) in this article.

Definition 2.1[1, 3]: In fts X_1, α be fuzzy set.

(i) If α =IntCl(α)then α is fuzzy regular open(precisely, frOS).
(ii) If α =ClInt(α)then α is fuzzy regular closed (precisely, frCS).
(iii) If $\alpha \leq (\mathrm{IntCl}\alpha) \vee (\mathrm{ClInt}\alpha)$ then α is f b-open set (precisely, fbOS).
(iv) If $\alpha \geq (\mathrm{IntCl}\alpha) \wedge (\mathrm{ClInt}\alpha)$ then α is f b-closed set (precisely, fbCS).

Remark 2.2 [1]: In a fuzzy topological space X, The following implication holds good

$$\boxed{\text{frOS}} \longrightarrow \boxed{\text{f-open}} \longrightarrow \boxed{\text{fbOS}}$$

Figure1. Interrelations between some fuzzy open sets

Definition 2.3[3]: Let α be a fuzzy set in a fts X_1. Then ,
(i) $\mathrm{bCl}(\alpha) = \wedge\{\beta : \beta \text{ is a fbCS}(X_1), \geq \alpha\}$.
(ii) $\mathrm{bInt}(\alpha) = \vee\{\lambda : \lambda \text{ is a fbOS}(X_1), \leq \alpha\}$.

Definition 2.4[11]: In a fts X_1, if bCl(α) $\leq \beta$, at any time when $\alpha \leq \beta$, then fuzzy set α is named as regular generalized fuzzy b-closed (rgfbCS).Where β is fr- open.
.
Remark 2.5[11]: In a fts X_1, if 1-α is rgfbCS(X_1) then fuzzy set α is rgfbOS.

Definition 2.6[11]: In a fts X_1, if bInt(α)$\geq \beta$, at any time when $\alpha \geq \beta$, then fuzzy set α is named as regular generalized fuzzy b-open (rgfbOS).Where β is fr- closed.

Definition 2.7[13]: Let (X_1, τ), (X_2, σ) be two fuzzy topological spaces. Let f : $X_1 \rightarrow X_2$ be mapping,

(i) if $f^{1}(\alpha)$ is rgfbCS(X_1), for each closed fuzzy set α in X_2, then f is said to be regular generalized fuzzy b-continuous (briefly, rgfb-continuous).

(ii) if $f^{1}(\alpha)$ is open fuzzy in X_1, for each rgfbOS α in X_2, then f is called strongly rgfb-continuous.

(iii) if $f^{1}(\alpha)$ is rgfbCS in X_1, for each rgfbCS α in X_2, then f is called rgfb-irresolute.

Definition 2.8[10]: X_1 is a fts which is named as

(i) fuzzy T_0(in short, fT_0) if and only if for each pair of fuzzy singletons p_1 and p_2 with various supports there occurs open fuzzy set U such that either $p_1 \leq U \leq 1- p_2$ *or* $p_2 \leq U \leq 1-p_1$.

(ii) fuzzy T_1(in short fT_1)if and only if for each pair of fuzzy singletons p_1 and p_2 with
various supports, there occurs open fuzzy sets U and V such that $p_1 \leq U \leq 1- p_2$ and $p_2 \leq V \leq 1- p_1$.

(iii) fuzzy T_2(in short, fT_2) or f-Hausdorff if and only if for each pair of fuzzy singletons p_1 and p_2 with various supports ,there occurs open fuzzy sets U and V such that $p_1 \leq U \leq 1- p_2$, $p_2 \leq V \leq 1- p_1$ and $U \leq 1- V$.

(iv) fuzzy $T_{2\frac{1}{2}}$ (in short, $fT_{2\frac{1}{2}}$) or f-Urysohn if and only if for each pair of fuzzy singletons p_1 and p_2 with various supports, there occurs open fuzzy sets U and V such that $p_1 \leq U \leq 1- p_2$, $p_2 \leq V \leq 1- p_1$ and $clU \leq 1-cl\ V$.

3. Regular generalized fuzzy b-closure (rgfbCl) and Regular generalized fuzzy b-Interior (rgfbInt)

Definition 3.1: The regular generalized fuzzy b-closure is denoted and defined by, rgfbCl (α) = Λ { λ : λ is a rgfbCS(X_1), $\geq \alpha$ }. Where α be a fuzzy set in fts X_1.

Theorem 3.2: Let X_1 be fts, then the properties that follows are occurs for rgfbCl of a set

i. rgfbCl(0) = 0
ii. rgfbCl(1) = 1
iii. rgfbCl(α) is rgfbCS in X_1
iv. rgfbCl[rgfbCl(α)] = rgfbCl(α)

Definition 3.3: Let α and β be fuzzy sets in fuzzy topological space X_1. Then regular generalized fuzzy b-closure of (α V β) and regular generalized fuzzy b-closure of ($\alpha \wedge \beta$) are denoted and defined as follows

i. rgfbCl (α V β) = $\wedge$ { λ : λ is a rgfbCS(X_1) , where $\lambda \geq$ (α V β) }
ii. rgfbCl ($\alpha \wedge \beta$) = $\wedge$ { λ : λ is a rgfbCS(X_1), where $\lambda \geq$ ($\alpha \wedge \beta$) }

Theorem 3.4: Let α and β be fuzzy sets in fts X_1, then the following relations occurs

i. rgfbCl(α) V rgfbCl(β) $\leq$ rgfbCl(α V β)
ii. rgfbCl(α) $\wedge$ rgfbCl(β) $\geq$ rgfbCl($\alpha \wedge \beta$)

Proof: (i) We know that $\alpha \leq$ (α V β) or $\beta \leq$ (α V β)
$\Rightarrow$ rgfbCl (α) $\leq$rgfbCl (α V β) orrgfbCl(β) $\leq$rgfbCl (α V β)
Hence, rgfbCl (α) V rgfbCl (β) $\leq$rgfbCl (α V β).
(ii) We know that $\alpha \geq$ ($\alpha \wedge \beta$) or $\beta \geq$ ($\alpha \wedge \beta$)
$\Rightarrow$ rgfbCl(α) $\geq$rgfbCl ($\alpha \wedge \beta$) orrgfbCl(β) $\geq$rgfbCl ($\alpha \wedge \beta$)
Hence, rgfbCl(α) $\wedge$ rgfbCl(β) $\geq$rgfbCl ($\alpha \wedge \beta$).

Theorem 3.5: α is rgfbCS in a fts X_1, if and only if α =rgfbCl(α).
Proof: Suppose α is rgfbCS. Since $\alpha \leq \alpha$ and $\alpha \in$ { β: β is rgfbCS(X_1) and $\alpha \leq \beta$ }, α is the smallest and contained in β,therefore α =Λ\{ β: β is rgfbCS(X_1)and $\alpha \leq \beta$}=rgfbCl(α). Hence, α =rgfbCl (α).
On the other hand, Suppose α =rgfbCl (α), then
α = Λ\{ β: β is rgfbCS, $\alpha \leq \beta$ } $\Rightarrow$ $\alpha \in \Lambda$ { β: β is rgfbOS, $\alpha \leq \beta$ }.
Hence, α is rgfbCS.

Definition 3.6: The regular generalized fuzzy b-interior is denoted and defined by, rgfbInt(α) = V { δ: δ is a rgfbOS(X_1), $\leq \alpha$}. Where α be a fuzzy set in fts X_1.

Theorem 3.7: Let X_1 be fts, then the properties that follows are occurs for rgfbInt of a set

i. rgfbInt(0) = 0
ii. rgfbInt(1) = 1
iii. rgfbInt(α) is rgfbOS in X_1
iv. rgfbInt[rgfbInt(α)] = rgfbInt(α) .

Definition 3.8: Let α and β are fuzzy sets in fts X_1. Then regular generalized fuzzy b-interior of ($\alpha \vee \beta$) and regular generalized fuzzy b-interior of ($\alpha \wedge \beta$) are denoted and defined as follows

i. rgfbInt ($\alpha \vee \beta$) = V {δ: δ is a rgfbOS(X_1), where $\delta \leq (\alpha \vee \beta)$ }.
ii. rgfbInt($\alpha \wedge \beta$) = V { δ: δ is a rgfbOS(X_1), where $\delta \leq (\alpha \wedge \beta)$}.

Theorem 3.9: Let α and β are fuzzy sets in fts X_1, then the following relations occurs

i. rgfbInt(α) V rgfbInt(β) $\leq$ rgfbInt($\alpha \vee \beta$)
ii. rgfbInt(α) $\wedge$ rgfbInt(β) $\geq$ rgfbInt($\alpha \wedge \beta$)

Proof: (i) We know that, $\alpha \leq (\alpha \vee \beta)$ or $\beta \leq (\alpha \vee \beta)$
$\Rightarrow$ rgfbInt(α) $\leq$ rgfbInt ($\alpha \vee \beta$) or rgfbInt(β) $\leq$ rgfbInt ($\alpha \vee \beta$)
Hence, rgfbInt(α) V rgfbInt(β) $\leq$ rgfbInt($\alpha \vee \beta$).
(ii)We know that $\alpha \geq (\alpha \wedge \beta)$ or $\beta \geq (\alpha \wedge \beta)$
$\Rightarrow$ rgfbInt(α) $\geq$ rgfbInt ($\alpha \wedge \beta$) or rgfbInt(β) $\geq$ rgfbInt ($\alpha \wedge \beta$)
Hence, rgfbInt(α) $\wedge$ rgfbInt(β) $\geq$ rgfbInt ($\alpha \wedge \beta$).

Theorem 3.10: Let X_1 be fts, α is rgfbOS if and only if α =rgfbInt(α).
Proof: Suppose α is rgfbOS. Since $\alpha \leq \alpha$, $\alpha \in$ { δ: δ is rgfbOS and $\delta \leq \alpha$} Since biggest α contains δ. Therefore, α = V{ δ: δ is rgfbOS $\delta \leq \alpha$} = rgfbInt (α). Hence, α =rgfbInt(α).
On the other hand, Suppose α =rgfbInt (α).Then, α =V{ δ: δ is rgfbOS, $\delta \leq \alpha$} $\Rightarrow \alpha \in$ V { δ: δ is rgfbOS $\delta \leq \alpha$}. Hence, α is rgfbOS.

Theorem 3.11: Let α be a fuzzy set in a fts X_1, in that case following relations holds good

i. rgfbInt(1-α) = 1-rgfbCl(α)
ii. rgfbCl(1-α) = 1- rgfbInt(α)

Proof: (i) Let α be a fuzzy set in fts X_1. Then we have
rgfbCl (α) = Λ { λ : λ is a rgfbCS(X_1), $\geq \alpha$}. Where α be a fuzzy set in fts X_1.
1-rgfbCl (α) = 1- Λ { λ : λ is a rgfbCS(X_1), $\geq \alpha$}.
= V { $1-\lambda$: λ is a rgfbCS(X_1), $\geq \alpha$}.
= V{ $1-\lambda$: $1-\lambda$ is a rgfbOS(X_1), $\leq$ 1-α}.
= rgfbInt (1-α)
Hence, 1-rgfbCl(α) = rgfbInt (1-α).

(ii) Let α be a fuzzy set in fts X_1. Then we have
rgfbInt(α) = V { δ: δ is a rgfbOS(X_1), $\leq \alpha$}. Where α be a fuzzy set in fts X_1.

1-rgfbInt (α) = 1-V { δ : $\delta \leq \alpha$ and δ is rgfbOS (X_1)}
= Λ{1- δ: $\delta \leq \alpha$ and δ is rgfbOS(X_1)}
= Λ {1- δ : 1-$\alpha \leq$ 1- δ and 1- δ is rgfbCS(X_1)}
= rgfbCl (1-α)
Hence 1-rgfbInt (α) = rgfbCl (1-α).

4. rgfb-separation axioms

Definition 4.1: A fts is known as rgfbT_0, that is regular generalized fuzzy bT_0, iff for each pair of fuzzy singletons q_1 and q_2 with various supports, there occurs rgfbOS δ such that either $q_1 \leq \delta \leq 1- q_2$ or $q_2 \leq \delta \leq 1- q_1$.

Theorem 4.2: A fts is rgfbT_0, that is regular generalized fuzzy bT_0, if and only if rgfbCl of crisp fuzzy singletons q_1 and q_2 with various supports are different.
Proof: To prove the necessary condition: Let a fuzzy topological space be rgfbT_0 and two crisp fuzzy singletons be q_1 & q_2 with various supports x_1 & x_2 respectively i.e. $x_1 \neq x_2$. Since fts is rgfbT_0 ,there exist a rgfbOS δ such that, $q_1 \leq \delta \leq 1- q_2 \Rightarrow q_2 \leq 1- \delta$, but $q_2 \leq$ rgfbCl(q_2) $\leq 1- \delta$, where $q_1 \leq$ rgfbCl(q_2) $\Rightarrow q_1 \leq 1- \delta$ where $1- \delta$ is rgfbCS. But, $q_1 \leq$ rgfbCl(q_1). This shows that, rgfbCl(q_1) $\neq$ rgfbCl(q_2).
To prove the sufficiency: Let p_1 & p_2 be fuzzy singletons with various supports x_1 & x_2 respectively, q_1 & q_2 be crisp fuzzy singletons such that $q_1(x_1)=1$, $q_2(x_2)=1$. But, $q_1 \leq$ rgfbCl(q_1) $\Rightarrow$ 1-rgfbCl(q_1) $\leq 1-q_1 \leq 1-p_1$. As each crisp fuzzy singleton is rgfbCS, 1- rgfbCl(q_1) is rgfbOS and $p_2 \leq$ 1- rgfbCl(q_1) $\leq 1-p_1$.This proves, fts is rgfbT_0 space.

Definition 4.3: A fts is known as rgfbT_1, that is regular generalized fuzzy bT_1, iff for each pair of fuzzy singletons q_1 & q_2 with various supports x_1 & x_2 respectively, there occurs rgfbOSs δ_1 & δ_2 such that, $q_1 \leq \delta_1 \leq 1- q_2$ and $q_2 \leq \delta_2 \leq 1- q_1$.

Theorem 4.4: A fts is rgfbT_1, that is regular generalized fuzzy bT_1, if and only if each crisp fuzzy singleton is rgfbCS.
Proof: To prove the necessary condition: Let rgfbT_1 be fts and crisp fuzzy singleton with supports x_0 be q_0 .There occurs, rgfbOSs δ_1 and δ_2 for any fuzzy singleton q with supports x ($\neq x_0$), such that, $q_0 \leq \delta_1 \leq 1- q$ and $q \leq \delta_2 \leq 1- q_0$. Since, it includes each fuzzy set as the collection of fuzzy singletons. So that, $1-q_0 = \bigvee_{q \leq 1-q_0} q = 0$. Thus, $1-q_0$ is rgfbOS. This shows that, q_0 (crisp fuzzy singleton) is rgfbCS.

To prove the sufficiency: Assume p_1 and p_2 be pair of fuzzy singletons with various supports x_1& x_2 .Further on fuzzy singletons with various supports x_1 & x_2 be q_1 & q_2, such that $q_1(x_1) = 1$ and $q_2(x_2)=1$. As each crisp fuzzy singleton is rgfbCS, the fuzzy sets $1\text{-}q_1$ & $1\text{-}q_2$ are rgfbOSs such that, $p_1 \leq 1\text{-} q_1 \leq 1\text{-} p_2$ and $p_2 \leq 1\text{-}q_2 \leq 1\text{-} p_1$.This proves, fts is rgfbT_1 space.

Remark 4.5: In a fts X_1, each rgfbT_1 space is rgfbT_0 space.
Proof: It follows the above definition.
The opposite of this theorem is in correct. This is shown as follows –

Example 4.6: Let $X_1=\{a, b\}$,$p_1=\{(a,0),(b,1)\}$ and $p_2=\{(a,0.4),(b,0)\}$ are fuzzy singletons. U= $\{(a, 0.5),(b, 1)\}$ be rgfbOS . Let $\tau= \{ 0,p_1 , p_2 , U,1 \}$. The space is rgfb$T_0$ and it is not rgfbT_1.

Definition 4.7: A fts is known as rgfbT_2 , that is regular generalized fuzzy bT_2 or rgfb-Hausdorff iff, for each pair of fuzzy singletons q_1 & q_2 with various supports x_1 & x_2 respectively, there occurs, rgfbOS δ_1 & δ_2 such that, $q_1 \leq \delta_1 \leq 1\text{-} q_2$, $q_2 \leq \delta_2 \leq 1\text{-} q_1$ and $\delta_1 \leq 1\text{-} \delta_2$.

Theorem 4.8: A fts is known as rgfbT_2, that is regular generalized fuzzy bT_2 or rgfb-Hausdorff if and only if for each pair of fuzzy singletons q_1 & q_2 with various supports x_1 & x_2 respectively, there occurs an rgfbOS δ_1 such that, $q_1 \leq \delta_1 \leq$ rgfbCl $\delta_1 \leq 1\text{-} q_2$.
Proof: To prove the necessary condition: Let rgfbT_1 be fts and fuzzy singletons q_1 & q_2 with various supports .Let δ_1 & δ_2 be rgfbOS such that, $q_1 \leq \delta_1 \leq 1\text{-} q_2$, $q_2 \leq \delta_2 \leq 1\text{-} q_1$ and $\delta_1 \leq 1\text{-} \delta_2$ where $1\text{-} \delta_2$ is rgfbCS. We have by definition, rgfbCl$(\delta_1)=\wedge \{(1\text{-} \delta_2) : (1\text{-} \delta_2)$ rgfbCS$\}$ where $\delta_1 \leq 1\text{-} \delta_2$ .Also rgfbCl$(\delta_1) \geq \delta_1$.This shows that, $q_1 \leq \delta_1 \leq$ rgfbCl $(\delta)_1 \leq 1\text{-} \delta_2 \leq 1\text{-} q_2 \Rightarrow q_1 \leq \delta_1 \leq$ rgfbCl $(\delta_1) \leq 1\text{-} q_2$.
To prove the sufficiency: Assume q_1 and q_2 are pair of fuzzy singletons with various supports and δ_1 be rgfbOS. Since, $q_1 \leq \delta_1 \leq$ rgfbCl $(\delta_1) \leq 1\text{-} q_2 \Rightarrow q_1 \leq \delta_1 \leq 1\text{-} q_2$. Also $q_1 \leq$rgfbCl$(\delta_1) \leq 1\text{-} q_2 \Rightarrow q_2 \leq 1\text{-}$ rgfbCl $(\delta_1) \leq 1\text{-}q_1$. This shows that, 1- rgfbCl (δ_1) is rgfbOS. Also rgfbCl $(\delta_1) \leq 1\text{-}$ rgfbCl (δ_2) . This proves that, fts is rgfbT_2 space.

Remark 4.9: In a fts X_1,each rgfbT_2 space is rgfbT_1 space.
Proof: It follows the above definition.
The opposite of this theorem is in correct. This is shown as follows –

Example 4.10: Let $X_1=\{a,b\}$. $q_1=\{(a, 0.2),(b, 0)\}$ and $q_2=\{(a,0), (b,0.4)\}$ are fuzzy singletons, $O_1= \{(a,0.3),(b,0.4)\}$ and $O_2= \{(a,0.8),(b,0.7)\}$ are rgfbOS .Let $\tau = \{ 0, p_1 , p_2 , O_1 , O_2, 1\}$. The space is rgfb$T_1$ and it's not rgfbT_2.

Definition 4.11: A fts is known as rgfb$T_{2\frac{1}{2}}$, that is regular generalized fuzzy b$T_{2\frac{1}{2}}$ or rgfb-Urysohn iff for each pair of fuzzy singletons q_1 & q_2 with various supports x_1 & x_2 respectively, there occurs, rgfbOSs δ_1 & δ_2 such that, $q_1 \leq \delta_1 \leq 1- q_2$, $q_2 \leq \delta_2 \leq 1- q_1$ and rgfbCl (δ_1)$\leq$ 1-rgfbCl (δ_2).

Remark 4.12: In a fts X_1,each rgfb$T_{2\frac{1}{2}}$ space is rgfbT_2 space.
Proof: It follows from the above definition.
The opposite of this theorem is in correct. This is shown as follows –

Example 4.13: Let $X_1=\{a, b\}$. $q_1=\{(a, 0.1),(b,0)\}$ and $q_2=\{(a,0),(b,0.3)\}$ are fuzzy singletons, $O_1= \{(a,0.2),(b,0.3)\}$ and $O_2= \{(a,0.7),(b,0.5)\}$ are rgfbOSs. Let $\tau =\{ 0, p_1 , p_2 , O_1 , O_2, 1 \}$. The space is rgfb$T_2$ and it's not rgfb$T_{2\frac{1}{2}}$.

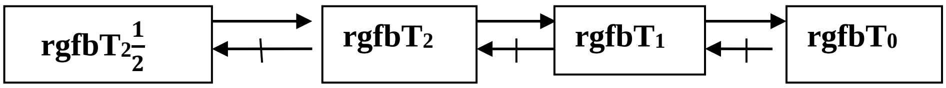

Figure2. From the above definition and examples one can notice that the above chains of implication.

Theorem 4.14: An injective function f: $X_1 \to X_2$ is rgfb-continuous, and X_2 is fT_0, then X_1 is rgfbT_0.
Proof: Assume α & β be fuzzy singletons in X_1 with various support then f(α) & f(β) belongs to X_2, As f is injective and $f(\alpha) \neq f(\beta)$. As X_2 is fT_0, there occurs, a open set O in X_2 such that, $f(\alpha) \leq O \leq 1 - f(\beta)$or $f(\beta) \leq O \leq 1 - f(\alpha), \Rightarrow \alpha \leq f^{-1}(O) \leq 1 - \beta$ or $\beta \leq f^{-1}(O) \leq 1 - \alpha$. Since, f: $X_1 \to X_2$ is rgfb-continuous, $f^{-1}(O)$ is rgfbOS in X_1. This shows that, X_1 is rgfbT_0-space[4.1].

Theorem 4.15: An injective function f: $X_1 \to X_2$ is rgfb-irresolute, and X_2 is rgfbT_0, then X_1 is rgfbT_0.
Proof: Assume α & β be fuzzy singletons in X_1 with various support. As f is injective $f(\alpha)$ & $f(\beta)$ belongs to X_2 and $f(\alpha) \neq f(\beta)$. As, X_2 is rgfbT_0, there occurs rgfbOS O in X_2 so that $f(\alpha) \leq O \leq 1 - f(\beta)$ or $f(\beta) \leq O \leq 1 - f(\alpha) \Rightarrow \alpha \leq f^{-1}(O) \leq 1 - \beta$ or $\beta \leq f^{-1}(O) \leq 1 - \alpha$. As, f is rgfb-irresolute $f^{-1}(O)$ is rgfbOS(X_1). This shows that, X_1 is rgfbT_0 space[4.1].

Theorem 4.16: An injective function f: $X_1 \to X_2$ is strongly rgfb-continuous, and X_2 is rgfbT_0, then X_1 is fT_0.

Proof: Assume α & β be fuzzy singletons in X_1 with various support. Since f is injective $f(\alpha)$ & $f(\beta)$ belongs to X_2 and $f(\alpha) \neq f(\beta)$. As, X_2 is rgfbT_0, there occurs rgfbOS O in X_2 so that, $f(\alpha) \leq O \leq 1 - f(\beta)$ or $f(\beta) \leq O \leq 1 - f(\alpha), \Rightarrow \alpha \leq f^{-1}(O) \leq 1 - \beta$ or $\beta \leq f^{-1}(O) \leq 1 - \alpha$. Since, f is strongly rgfb-continuous, $f^{-1}(O)$ is fuzzy-open in X_1. This shows that, X_1 is fT_0-space[2.8].

Theorem 4.17: An injective function f: $X_1 \rightarrow X_2$ is rgfb-continuous, and X_2 is fT_1, then X_1 is rgfbT_1.
Proof: Assume α and β be fuzzy singletons in X_1 with various supports. $f(\alpha)$ and $f(\beta)$ belongs to X_2, Since, f is injective. As, X_2 is fT_1 space hence, by the statement there occurs fuzzy-open sets O_1 & O_2 in X_2 such that, $f(\alpha) \leq O_1 \leq 1 - f(\beta)$ and $f(\beta) \leq O_2 \leq 1 - f(\alpha) \Rightarrow \alpha \leq f^{-1}(O_1) \leq 1 - \beta$ and $\beta \leq f^{-1}(O_2) \leq 1 - \alpha$.
Since, f is rgfb-continuous $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are rgfb-open in X_1. This shows that, X_1 is rgfbT_1 space[4.3].

Theorem 4.18: An injective function f: $X_1 \rightarrow X_2$ is rgfb-irresolute, and X_2 is rgfbT_1, then X_1 is rgfbT_1.
Proof: Assume α & β be fuzzy singletons in with various supports. Since f is injective, $f(\alpha)$ & $f(\beta)$ belongs to X_2. As X_2 is rgfbT_1, there occurs two rgfbOS O_1& O_2 in X_2 so that $f(\alpha) \leq O_1 \leq 1 - f(\beta)$ and $f(\beta) \leq O_2 \leq 1 - f(\alpha) \Rightarrow \alpha \leq f^{-1}(O_1) \leq 1 - \beta \; and \; \beta \leq f^{-1}(O_2) \leq 1 - \alpha$. Since, f is rgfb-irresolute, then $f^{-1}(O_1) \; and \; f^{-1}(O_2)$ are rgfbOS(X_1). This shows that, X_1 is rgfbT_1 space[4.3].

Theorem 4.19: If f : $X_1 \rightarrow X_2$ is strongly rgfb-continuous and X_2 is rgfbT_1, then X_1 is fT_1.
Proof: Assume α & β be fuzzy singletons in X_1 with various supports. Since, f is injective, $f(\alpha)$ &$f(\beta)$ belong to X_2. As, X_2 is rgfbT_1, there occurs two rgfbOSs O_1 and O_2 in X_2 so that, $f(\alpha) \leq O_1 \leq 1 - f(\beta)$ and $f(\beta) \leq O_2 \leq 1 - f(\alpha) \Rightarrow \alpha \leq f^{-1}(O_1) \leq 1 - \beta$ and $\beta \leq f^{-1}(O_2) \leq 1 - \alpha$. Since, f is strongly rgfb-continuous, therefore $f^{-1}(O_1)$ & $f^{-1}(O_2)$ are fuzzy-open in X_1. This shows that, X_1 is fT_1 space[2.8].

Theorem 4.20: An injective function f: $X_1 \rightarrow X_2$ is rgfb-continuous, and X_2 is fT_2, then X_1 is rgfbT_2.
Proof: Assume α & β be fuzzy singletons in X_1 with various supports. Since, f is injective, so $f(\alpha)$ & $f(\beta)$ belongs to X_2 and $f(\alpha) \neq (\beta)$. Since, X_2 is fT_2, therefore there occurs open fuzzy set O in X_2 so that, $f(\alpha) \leq O \leq$

$\mathrm{Cl}(O) \leq 1 - f(\beta) \Rightarrow \alpha \leq f^{-1}(O) \leq f^{-1}[Cl(O)] \leq 1 - \beta$. Since, f is rgfb-continuous $f^{-1}(O)$ is rgfbCS(X_1). Hence, $\alpha \leq f^{-1}(O) \leq f^{-1}[\mathrm{Cl}(O)] \leq f^{-1}[\mathrm{rgfbCl}(O)] \leq \mathrm{rgfbCl}[f^{-1}[(O)] \leq 1 - \beta$. That is, $\alpha \leq f^{-1}(O) \leq \mathrm{rgfbCl}[f^{-1}[(O)] \leq 1 - \beta$. This shows that, X_1 is rgfbT_2[4.7].

Theorem 4.21: An injective function $f: X_1 \rightarrow X_2$ is rgfb-irresolute, and X_2 is rgfbT_2. Then, X_1 is rgfbT_2.
Proof: Obvious.

Theorem 4.22: An injective function $f: X_1 \rightarrow X_2$ is strongly rgfb-continuous, and X_2 is rgfbT_2. Then, X_1 is fT_2.
Proof: Obvious.

Theorem 4.23: An injective function $f: X_1 \rightarrow X_2$ is rgfb-continuous, and X_2 is f$T_{2\frac{1}{2}}$. Then, X_1 is rgfb$T_{2\frac{1}{2}}$.
Proof: Assume α & β be fuzzy singletons in X_1 with various supports. Since, f is injective, then $f(\alpha)$ and $f(\beta)$ belongs to X_2 and $f(\alpha) \neq f(\beta)$. Since, X_2 is f$T_{2\frac{1}{2}}$, then there occurs open fuzzy sets O_1 and O_2 in X_2 such that, $f(\alpha) \leq O_1 \leq 1 - f(\beta)$, $f(\beta) \leq O_2 \leq 1 - f(\alpha)$and $\mathrm{Cl}O_1 \leq 1 - \mathrm{Cl}O_2 \Rightarrow \alpha \leq f^{-1}(O_1) \leq 1 - \beta$,$\beta \leq f^{-1}(O_2) \leq 1 - \alpha$ and $\mathrm{Cl}f^{-1}(O_1) \leq 1 - \mathrm{Cl}f^{-1}(O_2)$. Since, f is rgfb-continuous $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are rgfbOS(X_1). $\mathrm{Cl}(f^{-1}(O_1)) \leq \mathrm{rgfbCl}(f^{-1}(O_1))$ and$1 - Cl(f^{-1}(O_2)) \leq 1 - \mathrm{rgfbCl}(f^{-1}(O_2))$. Hence, $\mathrm{rgfbCl}(f^{-1}(O_1)) \leq 1 - \mathrm{rgfbCl}(f^{-1}(O_2))$. This shows that, X_1 is rgfb$T_{2\frac{1}{2}}$[4.11].

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Generalized double Fibonomial numbers

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Abstract

From the beginning of 20th century, generalization of binomial coefficient has been deliberated broadly. One of the most famous generalized binomial coefficients are Fibonomial coefficients, obtained by substituting Fibonacci numbers in place of natural numbers in the binomial coefficients. In this paper, we further generalize the concept of Fibonomial coefficient and called it Generalized double Fibonomial number and obtain interesting properties of it. We also discuss its special case, double Fibonomial number along with the situation in which they give integer values. Other properties of it have also been discussed along with its upper and lower bounds.
Keywords: Fibonacci numbers, Lucas numbers, Fibonomial numbers, Binomial coefficient, Double factorial.
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1 Introduction

In combinatorics, the factorial of a positive integer n, denoted by $n!$, is defined by $n! = n \times n-1 \times \cdots \times 2 \times 1; n \geq 1$ and $0! = 1$. Whereas the double factorial of a positive integer n, usually denoted by $n!!$ is defined as

$$n!! = \begin{cases} n \times n-2 \times \cdots \times 3 \times 1; & n\,is\,odd \\ n \times n-2 \times \cdots \times 4 \times 2; & n\,is\,even \\ 1; & n = 0 \end{cases}$$

Note that $n!!$ is not the same as the iterated factorial $(n!)!$, which grows much faster. We do not know precisely when, where, or by whom, the double factorial notation was devised. It was used by Meserve [6] in 1948, and it is not mentioned by Cajori in his very detailed work in history of mathematical notations during 1928 – 1929 [2]. Thus, we summaries that the notation was introduced at some times during the period 1928 – 1948.

In this definition of $n!!$, if we replace the natural numbers by the terms of the generalized Fibonacci numbers w_n defined by the recurrence relation $w_n = pw_{n-1} + qw_{n-2}$, for $n \geq 2$; $w_0 = a$ and $w_1 = b$, where a, b, p and q are any integers, then what we get will be called Generalized double Fibonorial $n!!_w$ and is defined as

$$n!!_w \equiv \begin{cases} w_n \times w_{n-2} \times \cdots \times w_3 \times w_1 & n > 0\,is\,odd \\ w_n \times w_{n-2} \times \cdots \times w_4 \times w_2 & n > 0\,is\,even \\ 1 & n = 0 \end{cases} \tag{1}$$

Here note that when we substitute $p = q = b = 1$ and $a = 0$ in the definition of w_n, we get regular Fibonacci numbers F_n. The definition of $n!!_w$ helps to express the generalized double Fibonorial in terms of regular generalized Fibonorial as shown in the following lemma.

Lemma 1.1. $n!!_w = \frac{n!_w}{(n-1)!!_w} = \frac{(n+1)!_w}{(n+1)!!_w}; n \geq 1.$

In 1964 Fontene [3] generalized the notion of binomial coefficients and introduce the new concept of Fibonomial coefficients. In the definition of binomial coefficients $\binom{m}{k}$, he replaced the natural numbers by the terms of an arbitrary sequence $\{A_n\}$ of real or complex numbers. Since then there has been an accelerated interest in the study of Fibonomial coefficients. When the sequence $\{A_n\}$ is considered as the sequence $\{F_n\}$ of Fibonacci numbers, the Fibonomial coefficients $\begin{bmatrix} m \\ k \end{bmatrix}_F$, for $1 \geq k \geq m$, is defined as $\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{m!_F}{k!_F(m-k)!_F}$. The elaborated study on the generalized Fibonomial coefficients can be found in literature. (See [5])

This quantity will always be an integer, which can be shown by an induction argument by replacing F_m in the numerator with $F_k F_{m-k+1} + F_{k-1} F_{m-k}$, resulting in

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = F_{m-k+1} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F + F_{k-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_F \tag{2}$$

We use the concept of generalized double Fibonorial to further generalize the concept of generalized Fibonomial coefficients. We define the generalized double Fibonomial numbers $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w$ as

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = \frac{m!!_w}{k!!_w \, (m-k)!!_w} \tag{3}$$

It is easy to observe that

$$\left[\begin{bmatrix} m \\ 0 \end{bmatrix}\right]_w = 1, \left[\begin{bmatrix} m \\ 2 \end{bmatrix}\right]_w = w_m and \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = \left[\begin{bmatrix} m \\ m-k \end{bmatrix}\right]_w \tag{4}$$

2 Generalized double Fibonomial numbers:

2.1 Some properties of generalized double Fibonomial numbers:

The following results are now easy consequences from this definition:

Lemma 2.1. *For any positive integers k, m and n,*

1. $(Iterative\ rule) \left[\begin{bmatrix} n \\ k \end{bmatrix}\right]_w \left[\begin{bmatrix} k \\ m \end{bmatrix}\right]_w = \left[\begin{bmatrix} n \\ m \end{bmatrix}\right]_w \left[\begin{bmatrix} n-m \\ k-m \end{bmatrix}\right]_w.$
2. $w_{m-k} \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = w_m \left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_w.$
3. $w_k \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = w_{m-k+2} \left[\begin{bmatrix} m \\ k-2 \end{bmatrix}\right]_w.$
4. $w_k \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = w_m \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_w.$

Lemma 2.2. $(m-1)!!_w \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w$ *will always give an integer value.*

This result is an easy derived from the definition of generalized Fibonorial and generalized double Fibonomial numbers. The basic recurrence relations for the generalized double Fibonomial numbers is as follows:

Lemma 2.3. $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w - \left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_w = \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_w \left\{\frac{w_m - w_{m-k}}{w_k}\right\}.$

By changing k to $m-k$ and using (4), we get

Lemma 2.4. $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w - \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_w = \left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_w \left\{\frac{w_m - w_k}{w_{m-k}}\right\}.$

The following result can be easily obtained when we apply the sum on both sides with respect to the upper index such that m and k have the same parity.

Lemma 2.5. $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = \sum_{j=k}^{m} \left\{\frac{w_j - w_{j-k}}{w_k}\right\} \left[\begin{bmatrix} j-2 \\ k-2 \end{bmatrix}\right]_w$ *; where the sum is taken over integers starting from k with spacing of 2 up to m.*

2.2 Star of David theorem:

In 1972, Gould gave a result related to one interesting arithmetic property of binomial coefficients which was named as the Star of David theorem, which was stated as "The greatest common divisors of the binomial coefficients forming each of the two triangles in the Star of David shape in Pascal's triangle are equal:

$$\gcd\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\}.$$

The two sets of three numbers, which the Star of David theorem says, have equal greatest common divisors and equal products. Interestingly, Gould's result can be imitated for generalized double Fibonomial numbers too as shown in the following result.

Theorem 2.1. $\left[\begin{bmatrix} m-a \\ k-b \end{bmatrix}\right]_w \left[\begin{bmatrix} m \\ k+b \end{bmatrix}\right]_w \left[\begin{bmatrix} m+b \\ k \end{bmatrix}\right]_w$
$= \left[\begin{bmatrix} m-a \\ k \end{bmatrix}\right]_w \left[\begin{bmatrix} m+b \\ k+a \end{bmatrix}\right]_w \left[\begin{bmatrix} m \\ k-b \end{bmatrix}\right]_w$ *; where a, b are positive integers.*

Proof. Using the definition of generalized double Fibonomial numbers, the left side of the result becomes

$$\left[\begin{bmatrix} m-a \\ k-b \end{bmatrix}\right]_w \left[\begin{bmatrix} m \\ k+b \end{bmatrix}\right]_w \left[\begin{bmatrix} m+b \\ k \end{bmatrix}\right]_w$$
$$= \frac{(m-a)!!_w}{(k-b)!!_w (m-k-a+b)!!_w} \times \frac{(m)!!_w}{(k+a)!!_w (m-k-a)!!_w} \times \frac{(m+b)!!_w}{(k)!!_w (m-k+b)!!_w}$$

$= \frac{(m-a)!!_w}{(k)!!_w(m-k-a)!!_w} \times \frac{(m+b)!!_w}{(k+a)!!_w(m-k-a+b)!!_w} \times \frac{(m)!!_w}{(k-b)!!_w(m-k+b)!!_w}$

$= \left[\begin{bmatrix} m-a \\ k \end{bmatrix}\right]_w \left[\begin{bmatrix} m+b \\ k+a \end{bmatrix}\right]_w \left[\begin{bmatrix} m \\ k-b \end{bmatrix}\right]_w$, as required.□

Corolary 2.1. *If $a = b = 1$, we get the product of six generalized double Fibonomial numbers, which are equally spaced around $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w$.*

2.3 Generalized Double Multinomial Numbers:

Let $m = k_1 + k_2 + \cdots + k_r$ then we can define generalized double multinomial number as

$$\left[\begin{bmatrix} m \\ k_1, k_2, \cdots, k_r \end{bmatrix}\right]_w = \frac{m!!_w}{k_1!!_w k_2!!_w \cdots k_r!!_w}$$

Following result expresses generalized double multinomial numbers as the multiplication of generalized double Fibonomial numbers.

Lemma 2.6. *Generalized double multinomial numbers can be expressed as the multiplication of generalized double Fibonomial numbers.*

Proof. In the definition of generalized double multinomial numbers, consider $r = 2$, then we have $\left[\begin{bmatrix} m \\ k_1, k_2 \end{bmatrix}\right]_w = \left[\begin{bmatrix} m \\ k_1 \end{bmatrix}\right]_w$; where $k_1 + k_2 = m$.
For $r = 3$ and $m = k_1 + k_2 + k_3$, $\left[\begin{bmatrix} m \\ k_1, k_2, k_3 \end{bmatrix}\right]_w = \left[\begin{bmatrix} m \\ k_1 \end{bmatrix}\right]_w \left[\begin{bmatrix} m-k_1 \\ k_2 \end{bmatrix}\right]_w$.
Let us now consider $r = n$ and $m = k_1 + k_2 + \cdots + k_n$. Thus
$\left[\begin{bmatrix} m \\ k_1, k_2, \cdots, k_r \end{bmatrix}\right]_w = \frac{m!!_w}{k_1!!_w k_2!!_w \cdots k_n!!_w} = \frac{m!!_w}{k_1!!_w k_2!!_w \cdots k_{n-2}!!_w} \times \frac{1}{k_{n-1}!!_w k_n!!_w}$
$= \frac{m!!_w}{k_1!!_w k_2!!_w \cdots k_{n-2}!!_w \times (m-k_1-k_2-\cdots-k_{n-2})!!_w} \times \frac{(m-k_1-k_2-\cdots-k_{n-2})!!_w}{k_{n-1}!!_w (m-k_1-k_2-\cdots-k_{n-2}-k_{n-1})!!_w}$
$= \left[\begin{bmatrix} m \\ k_1 \end{bmatrix}\right]_w \left[\begin{bmatrix} m-k_1 \\ k_2 \end{bmatrix}\right]_w \cdots \left[\begin{bmatrix} m-k_1-k_2-\cdots-k_{n-2} \\ k_{n-1} \end{bmatrix}\right]_w$.
Hence, by the principle of Mathematical induction, we get the required result.□
It is obvious that all the above results related to generalized double Fibonorials and generalized double Fibonomial numbers are also true for double Fibonomials $n!!_F$ and double Fibonomial coefficients $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$. But there are some additional results related to them, which are discussed in the following article.

3 Double Fibonomial numbers:

3.1 Definition and some properties of double Fibonomial numbers:

Using the definitions (1) and (3), Double Fibonorials and double Fibonomial numbers can be respectively expressed as

$$n!!_F \equiv \begin{cases} F_n \times F_{n-2} \times \cdots \times F_3 \times F_1 & n > 0 \, is \, odd \\ F_n \times F_{n-2} \times \cdots \times F_4 \times F_2 & n > 0 \, is \, even \\ 1 & n = 0 \end{cases}$$

and

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \tfrac{m!!_F}{k!!_F (m-k)!!_F}.$$

The following table shows first few terms of double Fibonorials for some initial values of n.

n	0	1	2	3	4	5	6	7	8	9	10
$n!!_F$	1	1	1	2	3	10	24	130	504	4420	27720

Table 1: Double Fibonorial numbers

Also by (4), double FIbonomial numbers have the symmetry property. Thus Table 2 shows the first few terms of double Fibonomial numbers of one side only.

								1
							1	
						1		1
					1		2	
				1		3/2		3
			1		10/3		5	
		1		24/10		8		6
	1		65/12		13		65/3	
1		252/65		21		126/5		56

Table 2: Double Fibonomial numbers

We further show how Double Fibonorial and Double Fibonomial numbers are connected with the sequence $\{L_n\}$ of Lucas numbers. This sequence is famously known as the twin sequence of Fibonacci sequence, which can be obtained by substituting $p = q = b = 1$ and $a = 2$ in the definition of w_n. That is $L_n = L_{n-1} + L_{n-2}$; $L_0 = 2$ and $L_1 = 1$. It is easy to observe that $F_{2n} = F_n L_n$. If we define $n!_L = L_n \times L_{n-1} \times \cdots \times L_2 \times L_1$, then the following lemma follows easily.

Lemma 3.1. *$n!!_F = k!_F \times k!_L$, for even positive integer $n = 2k$.*

If we consider $\begin{bmatrix} m \\ k \end{bmatrix}_L = \frac{m!_L}{k!_L(m-k)!_L}$,then the following is an easy consequence of lemma 3.1.

Lemma 3.2. $\left[\begin{bmatrix} 2m \\ 2k \end{bmatrix}\right]_F = \begin{bmatrix} m \\ k \end{bmatrix}_F \times \begin{bmatrix} m \\ k \end{bmatrix}_L$

From the Table 2 it is clear that double Fibonomial numbers are not always an integer. Obviously, for any integer m, $\left[\begin{bmatrix} m \\ 0 \end{bmatrix}\right]_F = \left[\begin{bmatrix} m \\ m \end{bmatrix}\right]_F = 1$, will always have an integer value. Also $\left[\begin{bmatrix} m \\ 2 \end{bmatrix}\right]_F = \left[\begin{bmatrix} m \\ m-2 \end{bmatrix}\right]_F = F_m$ will be integers. These two will serve as the trivial cases. Following theorem speaks about when double Fibonomial numbers attain integer values.

Theorem 3.1. *The nontrivial double Fibonomial numbers $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ are integers only when either m and k both are even integers together or $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right]_F$.*

Proof. We prove the result in four cases depending on the parity of m and k.
Case 1: When m and k both are even integers, we have

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 2n \\ 2l \end{bmatrix}\right]_F = \frac{(2n)!!_F}{(2l)!!_F(2n-2l)!!_F} = \frac{F_{2n} \times F_{2n-2} \times \cdots \times F_{2n-2l+2}}{F_{2l} \times \cdots \times F_4 \times F_2}$$

Note that number of elements in numerator and denominator are same. Also, they are Fibonacci numbers with even subscripts, such that in the denominator we have first l even subscripted Fibonacci numbers. Since these numbers always divide multiplication of any l consecutive even subscripted Fibonacci numbers, it follows that $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ will always be an integer.
Case 2: When m and k both are odd integers, we have In this case, we have

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 2n+1 \\ 2l+1 \end{bmatrix}\right]_F = \frac{(2n+1)!!_F}{(2l+1)!!_F(2n-2l)!!_F} = \frac{F_{2n+1} \times F_{2n-1} \times \cdots \times F_{2l+3}}{F_{2n-2l} \times \cdots \times F_4 \times F_2}$$

In the numerator, every Fibonacci number is with odd subscript. Consequently, none of them will be divisible by $F_4 = 3$. Thus $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ will not be an integer in this case.
Case 3: When m is odd integer and k is even integer, we have In this case, we have

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 2n+1 \\ 2l \end{bmatrix}\right]_F = \frac{(2n+1)!!_F}{(2l)!!_F(2n-2l+1)!!_F} = \frac{F_{2n+1}\times F_{2n-1}\times\cdots\times F_{2n-2l+3}}{F_{2l}\times\cdots\times F_4\times F_2}$$

Here again in the numerator, every Fibonacci number is with odd subscript, so none of them will be divisible by $F_4 = 3$. And therefore, in this case $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ will not be an integer.
Case 4: When m is even integer and k is odd integer, we have

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 2n \\ 2l+1 \end{bmatrix}\right]_F = \frac{(2n)!!_F}{(2l+1)!!_F(2n-2l-1)!!_F} = \frac{F_{2n}\times F_{2n-2}\times\cdots\times F_4\times F_2}{(F_{2l+1}\times\cdots\times F_3\times F_1)(F_{2n-2l-1}\times\cdots\times F_3\times F_1)}$$

Here number of terms in the numerator and denominator are same. Also, the Fibonacci numbers in the numerator are with only even subscripts and in the denominator with only odd subscripts. But, for any Fibonacci number F_n, there exists a prime p such that if $p \mid F_n$, then p will only divide F_{mn}; for every $m \geq 1$.

Since $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} m \\ m-k \end{bmatrix}\right]_F$, for convenience we take $k > m-k$, that is, $2k > m$. Then there will not be the same Fibonacci numbers in the numerator and denominator. Also, there will not be any multiple subscripts of k in the numerator. Thus, there will exist a prime p in the denominator such that $p \mid F_k$ which will not divide any of the Fibonacci number in the numerator.

Likewise, when $k = m-k$, then except for $k = 3$, there will be a prime p such that $p \mid F_k$ as well as $p \mid F_m$, but it will appear in the denominator only once where as in the numerator twice. Thus in this case, except for $\left[\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right]_F = 6$, $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ will not be an integer.□

In the following theorem we obtain the recurrence relation for the double Fibonomial numbers.

Theorem 3.2. $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{k-1}\left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_F + F_{m-k+1}\left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_F.$

Proof. From [4], we observe that the Fibonomial coefficients $\begin{bmatrix} m \\ k \end{bmatrix}_F$ has the recurrence relation

$$\begin{bmatrix}m\\k\end{bmatrix}_F = F_{k-1}\begin{bmatrix}m-1\\k\end{bmatrix}_F + F_{m-k+1}\begin{bmatrix}m-1\\k-1\end{bmatrix}_F$$

Now, using this relation and lemma 3.2, we get

$$\left[\begin{bmatrix}m\\k\end{bmatrix}\right]_F = \tfrac{m!_F}{(m-1)!!_F} \times \tfrac{(k-1)!!_F}{k!_F} \times \tfrac{(m-k-1)!!_F}{(m-k)!_F} = \begin{bmatrix}m\\k\end{bmatrix}_F \times \tfrac{(k-1)!!_F \times (m-k-1)!!_F}{(m-1)!!_F}.$$

$$= \left\{F_{k-1}\begin{bmatrix}m-1\\k\end{bmatrix}_F + F_{m-k+1}\begin{bmatrix}m-1\\k-1\end{bmatrix}_F\right\} \times \tfrac{(k-1)!!_F \times (m-k-1)!!_F}{(m-1)!!_F}.$$

$$= \left\{\tfrac{F_{k-1}(m-1)!_F}{(m-1)!!_F} \times \tfrac{(k-1)!!_F}{k!_F} \times \tfrac{(n-k-1)!!_F}{(n-k-1)!_F}\right\} + \left\{\tfrac{F_{n-k+1}(m-1)!_F}{(m-1)!!_F} \times \tfrac{(k-1)!!_F}{(k-1)!_F} \times \tfrac{(n-k-1)!!_F}{(n-k-1)!_F}\right\}$$

$$= \tfrac{F_{k-1}(n-2)!!_F}{k!!_F \times (n-k-2)!!_F} + \tfrac{F_{n-k+1}(n-2)!!_F}{(k-2)!!_F \times (n-k)!!_F}$$

$$\left[\begin{bmatrix}m\\k\end{bmatrix}\right]_F = F_{k-1}\left[\begin{bmatrix}m-2\\k\end{bmatrix}\right]_F + F_{m-k+1}\left[\begin{bmatrix}m-2\\k-2\end{bmatrix}\right]_F$$, as required. □

Lemma 3.3. $\left[\begin{bmatrix}m\\k\end{bmatrix}\right]_F = \sum_{j=1}^{\left|\frac{m-k}{2}\right|} F_{k-1}^{j-1} F_{m-k+1-2(j-1)}\left[\begin{bmatrix}m-2j\\k-2\end{bmatrix}\right]_F + F_{k-1}^{\left|\frac{m-k}{2}\right|}A;$

where $A = \begin{cases} 1; \textit{when m and k both are even or both are odd integers} \\ \left[\begin{bmatrix}k+1\\k\end{bmatrix}\right]_F; \textit{otherwise}\end{cases}$

Proof. From above theorem, we have

$$\left[\begin{bmatrix}m\\k\end{bmatrix}\right]_F = F_{m-k+1}\left[\begin{bmatrix}m-2\\k-2\end{bmatrix}\right]_F + F_{k-1}\left[\begin{bmatrix}m-2\\k\end{bmatrix}\right]_F$$

$$= F_{m-k+1}\left[\begin{bmatrix}m-2\\k-2\end{bmatrix}\right]_F + F_{k-1}\left\{F_{m-k-1}\left[\begin{bmatrix}m-4\\k-2\end{bmatrix}\right]_F + F_{k-1}\left[\begin{bmatrix}m-4\\k\end{bmatrix}\right]_F\right\}$$

$$= F_{m-k+1}\left[\begin{bmatrix}m-2\\k-2\end{bmatrix}\right]_F + F_{k-1}F_{m-k-1}\left[\begin{bmatrix}m-4\\k-2\end{bmatrix}\right]_F$$

$$. \quad + F_{k-1}^2\left\{F_{m-k-3}\left[\begin{bmatrix}m-6\\k-2\end{bmatrix}\right]_F + F_{k-1}\left[\begin{bmatrix}m-6\\k\end{bmatrix}\right]_F\right\}$$

Continuing this process, we get

$$\left[\begin{bmatrix}m\\k\end{bmatrix}\right]_F = \begin{cases} \sum_{j=1}^{\left|\frac{m-k}{2}\right|} F_{k-1}^{j-1}F_{m-k+1-2(j-1)}\left[\begin{bmatrix}m-2j\\k-2\end{bmatrix}\right]_F + F_{k-1}^{\left|\frac{m-k}{2}\right|}\left[\begin{bmatrix}k\\k\end{bmatrix}\right]_F; \\ \quad \textit{when n and k both are even or odd} \\ \sum_{j=1}^{\left|\frac{m-k}{2}\right|} F_{k-1}^{j-1}F_{m-k+1-2(j-1)}\left[\begin{bmatrix}m-2j\\k-2\end{bmatrix}\right]_F + F_{k-1}^{\left|\frac{m-k}{2}\right|}\left[\begin{bmatrix}k+1\\k\end{bmatrix}\right]_F; \\ \quad \textit{otherwise}\end{cases}$$

,as required. □

To illustrate the result, we consider $m = 9$ and $k = 5$. Then

$$\left[\begin{bmatrix}m\\k\end{bmatrix}\right]_F = \sum_{j=1}^{\left|\frac{m-k}{2}\right|} F_{k-1}^{j-1}F_{m-k+1-2(j-1)}\left[\begin{bmatrix}m-2j\\k-2\end{bmatrix}\right]_F + F_{k-1}^{\left|\frac{m-k}{2}\right|}A$$

$$= \sum_{j=1}^{2} F_4^{j-1}F_{5-2(j-1)}\left[\begin{bmatrix}9-2j\\3\end{bmatrix}\right]_F + F_4^2\left[\begin{bmatrix}5\\5\end{bmatrix}\right]_F$$

$= F_5 \left[\left[\begin{matrix}7\\3\end{matrix}\right]\right]_F + F_4F_3 \left[\left[\begin{matrix}5\\3\end{matrix}\right]\right]_F + F_4^2 = \left(5 \times \frac{65}{3}\right) + (3 \times 2 \times 5) + (3^2)$
$= \frac{442}{3} = \left[\left[\begin{matrix}9\\5\end{matrix}\right]\right]_F$, as expected.

The following result is an easy consequence from the definition of double Fibonomial numbers and the basic identity $F_mL_n + F_nL_m = 2F_{m+n}$ relating both Fibonacci numbers and Lucas numbers.

Lemma 3.4. $\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F = \frac{1}{2}\left(L_k \left[\left[\begin{matrix}m-2\\k\end{matrix}\right]\right]_F + L_{m-k} \left[\left[\begin{matrix}m-2\\k-2\end{matrix}\right]\right]_F\right).$

Proof. Since $2F_m = F_kL_{m-k} + F_{m-k}L_k$, we have $2\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F F_m =$
$\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F F_kL_{m-k} + \left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F F_{m-k}L_k$
$= \left[\left[\begin{matrix}m-2\\k-2\end{matrix}\right]\right]_F F_mL_{m-k} + \left[\left[\begin{matrix}m-2\\k\end{matrix}\right]\right]_F F_mL_k$. Thus
$2\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F = L_k \left[\left[\begin{matrix}m-2\\k\end{matrix}\right]\right]_F + L_{m-k} \left[\left[\begin{matrix}m-2\\k-2\end{matrix}\right]\right]_F$, as required. □

Using lemma 3.4 and applying the same logic of lemma 3.3, the following result can be proved easily.

Lemma 3.5. $\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F = \sum_{j=1}^{\lfloor\frac{m-k}{2}\rfloor} \frac{L_k^{j-1}L_{m-k-2(j-1)}}{2^j} \left[\left[\begin{matrix}m-2j\\k-2\end{matrix}\right]\right]_F + \left(\frac{L_k}{2}\right)^{\lfloor\frac{m-k}{2}\rfloor} A;$

where $A = \begin{cases} 1; \textit{when } m \textit{ and } k \textit{ both are even or odd integers} \\ \left[\left[\begin{matrix}k+1\\k\end{matrix}\right]\right]_F; \textit{otherwise} \end{cases}$

To illustrate the result, we consider $m = 10$ and $k = 3$. Then $\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F =$
$\sum_{j=1}^{\lfloor\frac{m-k}{2}\rfloor} \frac{L_k^{j-1}L_{m-k-2(j-1)}}{2^j} \left[\left[\begin{matrix}m-2j\\k-2\end{matrix}\right]\right]_F + \left(\frac{L_k}{2}\right)^{\lfloor\frac{m-k}{2}\rfloor} A$
$= \sum_{j=1}^{3} \frac{L_3^{j-1}L_{7-2(j-1)}}{2^j} \left[\left[\begin{matrix}10-2j\\3\end{matrix}\right]\right]_F + \left(\frac{L_3}{2}\right)^3 \left[\left[\begin{matrix}4\\3\end{matrix}\right]\right]_F$
$= \frac{L_7}{2} \left[\left[\begin{matrix}8\\1\end{matrix}\right]\right]_F + \frac{L_3L_5}{2^2} \left[\left[\begin{matrix}6\\1\end{matrix}\right]\right]_F + \frac{L_3^2L_3}{2^3} \left[\left[\begin{matrix}4\\1\end{matrix}\right]\right]_F + \frac{L_3^3}{2^3} \left[\left[\begin{matrix}4\\3\end{matrix}\right]\right]_F$
$= \left(\frac{29}{2} \times \frac{252}{65}\right) + \left(\frac{4\times 11}{2^2} \times \frac{24}{10}\right) + \left(\frac{4^3}{2^3}\right)\left(\frac{3}{2} + \frac{3}{2}\right)$
$= \frac{1386}{13} = \left[\left[\begin{matrix}10\\3\end{matrix}\right]\right]_F$, as expected.

In the following section we find the bounds of these numbers.

3.2 Bounds of double Fibonomial numbers:

The Binet formula for the Fibonacci number is given by $F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$; where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The following theorem gives us the bounds of double Fibonomial numbers in terms of α.

Theorem 3.3. *For* $\chi(n) = \begin{cases} 0; when\ n\ is\ even \\ 1; when\ n\ is\ odd \end{cases}$,
$\alpha^{\frac{(k-\chi(k))(m-k-\chi(m(m-k-1)-1))}{2}} \leq \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \leq \alpha^{\frac{(k+\chi(k))(m-k+\chi(m(m-k-1)-1))}{2}}$.

Proof. It is well-known that $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$; for all $n \geq 1$.
Then it is easy to observe that

$$\frac{F_{m-2t}}{F_{2t+2}} \leq \alpha^{m-4t-1} \tag{5}$$

and

$$\frac{F_{m-2t}}{F_{2t+2}} \geq \alpha^{m-4t-3} \tag{6}$$

Here we consider the four cases depending on the parity of m and k. When both m and k are even, using the definition of double Fibonomial numbers and (5), we have
$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \frac{m!!_F}{k!!_F \times (m-k)!!_F} = \frac{F_m \times F_{m-2} \times \cdots \times F_{m-k+2}}{F_2 \times F_4 \times \cdots \times F_k}$
$\leq \alpha^{m-1} \times \alpha^{m-5} \times \cdots \times \alpha^{(m-2k+3)} = \alpha^{\frac{k(m-k+1)}{2}}$
Thus $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \leq \alpha^{\frac{k(m-k+1)}{2}}$.
Again using (6) in the definition of double Fibonomial number, we get

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \geq \alpha^{m-3} \times \alpha^{m-7} \times \cdots \times \alpha^{m-2k+1} = \alpha^{\frac{k(m-k-1)}{2}}.$$

This shows that $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \geq \alpha^{\frac{k(m-k-1)}{2}}$. Thus when m and n both are even, we have

$$\alpha^{\frac{k(m-k-1)}{2}} \leq \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \leq \alpha^{\frac{k(m-k+1)}{2}}.$$

Considering $\chi(n) = \begin{cases} 0; when\ n\ is\ even \\ 1; when\ n\ is\ odd \end{cases}$, this result can be written as

$$\alpha^{\frac{(k-\chi(k))(m-k-\chi(m(m-k-1)-1))}{2}} \leq \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \leq \alpha^{\frac{(k+\chi(k))(m-k+\chi(m(m-k-1)-1))}{2}}.$$

The required result can be proved using the similar technique for all the remaining cases.

To illustrate it, we consider $m = 9$ and $k = 4$. Then $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = \frac{442}{3}$.
Also, $\alpha^{\frac{(k-\chi(k))(m-k-\chi(m(m-k-1)-1))}{2}} = \alpha^{\frac{k(m-k-1)}{2}} = \alpha^8 = 46.97$
and $\alpha^{\frac{(k+\chi(k))(m-k+\chi(m(m-k-1)-1))}{2}} = \alpha^{\frac{k(m-k+1)}{2}} = \alpha^12 = 321$, which shows that

$$\alpha^{\frac{(k-\chi(k))(m-k-\chi(m(m-k-1)-1))}{2}} \leq \left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F \leq \alpha^{\frac{(k+\chi(k))(m-k+\chi(m(m-k-1)-1))}{2}}.$$

4 Double Fibonomial numbers and Fibonacci numbers:

By [1], it is known that a primitive divisor of a Fibonacci number F_n is any prime integer p such that $p \mid F_n$ but $p \nmid F_m$; where $m < n$. Also, primitive divisor theorem says that for $n \geq 13$, every F_n has a primitive divisor. We use this result to prove many interesting relations between generalized double Fibonomial numbers and Fibonacci numbers.

4.1 Double Fibonomial number as a power of Fibonacci number:

In literature, there are many results involving Fibonomial numbers and Fibonacci numbers. From (4), it is clear that $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_m$ for $k = 2$. Thus, the Diophantine equation $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_n$ will always have a trivial solution $(m, k, n) = (m, 2, m)$. Following result claims that there is no other solution for the considered Diophantine equation.

Lemma 4.1. *The Diophantine equation* $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_n$ *has no solution for* $k > 2$.

Proof. We know that except for $\left[\left[\begin{matrix} 6 \\ 3 \end{matrix}\right]\right]_F = 6$, which is not a Fibonacci number, and trivial cases, $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F$ is an integer only when both m and k are even integers. Thus, $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_n$ implies

$$\frac{F_m \times F_{m-2} \times \cdots \times F_{m-k+2}}{F_k \times F_{k-2} \times \cdots \times F_2} = F_n \tag{7}$$

If we consider $n \geq 13$ and $n > m$, then by the primitive divisor theorem, there exists a prime p such that $p \mid F_n$ but $p \nmid F_m$. That is, (7) has no solution possible. Similarly, for $m \geq 13$ and $m > n$, primitive divisor theorem implies that (7) has no solution.
Thus, we can narrow down the range of m and n as $max\,(m, n)$. A quick look at the Table 2 reveals that for $k > 2$, the Diophantine equation $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_n$ has no solution. □

The following result can be proved through the similar arguments.

Theorem 4.1. *For any positive integer* t*, the Diophantine equation* $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_n^t$ *has no solution for* $k > 2$*.*

Though the double Fibonomial numbers do not possess the value of a Fibonacci number except for the trivial cases, they do stand in the neighborhood of Fibonacci number. We present this fact in the following final result.

Theorem 4.2. *The only solutions of the Diophantine equation* $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \pm 1 = F_n$ *are* $(m, k, n) = (3, 1, 2), (3, 2, 2), (4, 2, 3), (6, 3, 5), (8, 4, 10)$ *for* $'+'$ *case and* $(3, 1, 4), (3, 2, 4)$ *for* $'-'$ *case.*

Poof. From the Table 2, it is easy to observe that the Diophantine equation $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \pm 1 = F_n$ has solution $(m, k, n) = (3, 1, 2), (3, 2, 2), (4, 2, 3)$ for $'+'$ case and $(m, k, n) = (3, 1, 4), (3, 2, 4)$ for the $'-'$ case for $m \leq 5$. Now for $m > 5$, when m is an odd integer, double Fibonomial number will not be an integer. And when m is an even integer such that k is an odd integer, $\left[\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right]_F = 6$ is the only possibility integer value of double Fibonomial. Thus $(m, k, n) = (6, 3, 5)$ will be a solution of the given Diophantine equation for $'+'$ case.

Now, we can narrow down our possible solution to the even integers for both m and k. Since $F_a L_b = F_{a+b} + (-1)^b F_{a-b}$, the different factorizations for $F_n \pm 1$ depending on the class of $nmodulo4$ can be written as:

$$F_{4l} + 1 = F_{2l-1}L_{2l+1} \quad F_{4l} - 1 = F_{2l+1}L_{2l-1}$$

$$F_{4l+1} + 1 = F_{2l+1}L_{2l} \quad F_{4l+1} - 1 = F_{2l}L_{2l+1}$$

$$F_{4l+2} + 1 = F_{2l+2}L_{2l} \quad F_{4l+2} - 1 = F_{2l}L_{2l+2}$$

$$F_{4l+3} + 1 = F_{2l+1}L_{2l+2} \quad F_{4l+3} - 1 = F_{2l+2}L_{2l+1}$$

Therefore, the considered Diophantine equation, which can also be written as $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_n \mp 1$, can be factorized for the $'+'$ case as
$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{2l+1}L_{2l-1}$ $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{2l}L_{2l+1}$ $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{2l}L_{2l+2}$
$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{2l+2}L_{2l+1}$;
and for the $'-'$ case as
$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{2l-1}L_{2l+1}$ $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{2l+1}L_{2l}$ $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{2l+2}L_{2l}$
$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{2l+1}L_{2l+2}$;

It is obvious that all these eight cases can be handled in the similar manner. Thus, we shall only focus on the proof of the first case. Now, $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F =$ $F_{2l+1}L_{2l-1}$ implies $\frac{F_m \times F_{m-2} \times \cdots \times F_{m-k+2}}{F_k \times F_{k-2} \times \cdots \times F_2} = F_{2l+1}L_{2l-1}$. Thus, we have

$$F_m \times F_{m-2} \times \cdots \times F_{m-k+2} = F_{2l+1} \times L_{2l-1} \times F_k \times F_{k-2} \times \cdots \times F_2$$

Since $F_2n = F_nL_n$, we write $L_{2l-1} = \frac{F_{4l-2}}{F_{2l-1}}$. Thus

$$F_m \times F_{m-2} \times \cdots \times F_{m-k+2} \times F_{2l-1} = F_{2l+1} \times F_{4l-2} \times F_k \times F_{k-2} \times \cdots \times F_2.$$

Since $l = \left\lfloor \frac{n}{4} \right\rfloor > 2$, we have$4l - 2 > 2l + 1$. Therefore, from primitive divisor theorem, we can write $m = 4l - 2$. Thus,

$$F_{m-2} \times \cdots \times F_{m-k+2} \times F_{2l-1} = F_{2l+1} \times F_k \times F_{k-2} \times \cdots \times F_2 \tag{8}$$

If we assume that $m \geq max\{14, k+1\}$, we have $m - 2 \geq 12$. So, again by primitive divisor theorem, we get $m - 2 = max\{2l+1, k\}$. But $m - 2 = 4l - 4 > 2l + 1$, which implies $m - 2 = k$ and from (8), we get $F_{2l-1} = F_{2l+1}$, which is not possible. Thus, we only need to consider the range $4 \leq k \leq 10$ and $k + 2 \leq m \leq 12$.

Again, from Table 2, we can easily claim that the only solution of the Diophantine equation $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \pm 1 = F_n$ are $(m, k, n) = (3, 1, 2), (3, 2, 2), (4, 2, 3),$ $(6, 3, 5), (8, 4, 10)$ for $'+'$ case and $(3, 1, 4), (3, 2, 4)$ for $'-'$ case. □

5 Conclusion:

In this paper, we have defined double Fibonorial numbers and double Fibonomial numbers. We have proved many properties for these numbers including recursive equations in terms of Fibonacci numbers and Lucas numbers. We have extended the star of David theorem for double Fibonomial numbers and also discussed various Diophantine equations related to double Fibonomial numbers and Fibonacci numbers.

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Intuitionistic FWI-ideals of residuated lattice Wajsberg algebras

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Abstract

The notions of intuitionistic Fuzzy Wajsberg Implicative ideal (FWI –ideal) and intuitionistic fuzzy lattice ideal of residuated Wajsberg algebras are introduced. Also, we show that every intuitionistic FWI- ideal of residuated lattice Wajsberg algebra is an intuitionistic fuzzy lattice ideal of residuated lattice Wajsberg algebra. Further, we discussed its converse part.

Keywords: Wajsberg algebra; Lattice Wajsberg algebra; Residuated lattice Wajsberg algebra; WI–ideal; FWI –ideal; Intuitionistic FWI –ideal; Intuitionistic fuzzy lattice ideal.
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1. Introduction

The concept of fuzzy set was introduced by Zadeh [13] in 1935. The concept of intuitionistic fuzzy set was introduced by Atanassov [1, 2]. The idea of Wajsberg algebra was introduced by Mordchaj Wajsberg [10]. The author [8] introduced the notions of FWI-ideals and investigated their properties with illustrations.

In the present paper, we introduce the notions of intuitionistic FWI –ideal and intuitionistic fuzzy lattice ideal of residuated lattice Wajsberg algebras. Also, we show that every intuitionistic FWI–ideal of residuated lattice Wajsberg algebra is an intuitionistic fuzzy lattice ideal of residuated lattice Wajsberg algebra. Further, we verify its converse part.

2. Preliminaries

In this section, we recall some basic definitions and properties which are helpful to develop our main results.

Definition 2.1[3]. Let $(A, \to, *, 1)$ bean algebra with a binary operation " $\to$ " and a quasi-complement " $*$ ". Then it is called a Wajsberg algebra, if the following axioms are satisfied for all $x, y, z \in A$,

(i) $1 \to x = x$

(ii) $(x \to y) \to y = ((y \to z) \to (x \to z)) = 1$

(iii) $(x \to y) \to y = (y \to x) \to x$

(iv) $(x^* \to y^*) \to (y \to x) = 1$.

Definition 2.2[3]. Let$(A, \to, *, 1)$ be a Wajsberg algebra. Then the following axioms are satisfied for all $x, y, z \in A$,

(i) $x \to x = 1$

(ii) If $(x \to y) = (y \to x) = 1$ then $x = y$

(iii) $x \to 1 = 1$

(iv) $(x \to (y \to x)) = 1$

(v) If $(x \to y) = (y \to z) = 1$ then $x \to z = 1$

(vi) $(x \to y) \to \big((z \to x) \to (z \to y)\big) = 1$

(vii) $x \to (y \to z) = y \to (x \to z)$

(viii) $x \to 0 = x \to 1^* = x^*$

(ix) $(x^*)^* = x$

(x) $(x^* \to y^*) = y \to x$.

Definition 2.3[3]. Let $(A, \to, *, 1)$ be a Wajsberg algebra. Then it is called a lattice Wajsberg algebra, if the following axioms are satisfied for all $x, y \in A$,

(i) The partial ordering " $\leq$ " on a Wajsberg algebra such that $x \leq y$ if and only if $x \to y = 1$
(ii) $x \vee y = (x \to y) \to y$
(iii) $x \wedge y = ((x^* \to y^*) \to y^*)^*$.

Thus, $(A, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Proposition 2.4[3]. Let$(A, \to, *, 1)$ be a lattice Wajsberg algebra. Then the following axioms are satisfied for all $x, y, z \in A$,

(i) If $x \leq y$ then $x \to z \geq y \to z$ and $z \to x \leq z \to y$
(ii) $x \leq y \to z$ if and only $if\ y \leq x \to z$
(iii) $(x \vee y)^* = (x^* \wedge y^*)$
(iv) $(x \wedge y)^* = (x^* \vee y^*)$
(v) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$
(vi) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
(vii) $(x \to y) \vee (y \to x) = 1$
(viii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$
(ix) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$
(x) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$
(xi) $(x \wedge y) \to z = (x \to y) \to (x \to z)$.

Definition 2.5[11]. Let$(A, \vee, \wedge, \otimes, \to, 0, 1)$ be an algebra of type (2, 2, 2, 2, 0, 0). Then it is called a residuated lattice, if the following axioms are satisfied for all $x, y, z \in A$,
(i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,
(ii) $(A, \otimes, 1)$ is commutative monoid,
(iii) $x \otimes y \leq z$ if and only if $x \leq y \to z$.

Definition 2.6[3]. Let $(A, \vee, \wedge, *, \to, 1)$ be a lattice Wajsberg algebra. If a binary operation " $\otimes$ " on A satisfies $x \otimes y = (x \to y^*)^*$ for all $x, y \in A$. Then $(A, \vee, \wedge, \otimes, \to, *, 0, 1)$ is called a residuated lattice Wajsberg algebra.

Definition 2.7[4]. Let A be a lattice Wajsberg algebra. Let I be a non-empty subset of A, then I is called a*WI*-ideal of lattice Wajsberg algebra A, if the following axioms are satisfied for all$x, y \in A$,
(i) $0 \in I$
(ii) $(x \to y)^* \in I$and $y \in I$ imply $x \in I$.

Definition 2.8[4]. Let L be a lattice. An ideal I of L is a non-empty subset of L is called a lattice ideal, if the following axioms are satisfied for all $x, y \in A$,
(i) $x \in I, y \in L$ and $y \leq x$ imply $y \in I$

(ii) $x, y \in I$ implies $x \vee y \in I$.

Definition 2.9[7]. Let A be a residuated lattice Wajsberg algebra and I be a non-empty subset of A.Then I is called a WI-ideal of residuated lattice Wajsberg algebra A, if the following axioms are satisfiedfor all $x, y \in A$,

(i) $0 \in I$

(ii) $x \otimes y \in I$ and $y \in I$ imply $x \in I$

(iii) $(x \to y)^* \in I$ and $y \in I$ imply $x \in I$.

Definition 2.10[13]. Let A be a set. A function $\mu: A \to [0,1]$ is called a fuzzy subset on A for each $x \in A$, the value of $\mu(x)$ describes a degree of membership of x in μ.

Definition 2.11[5]. Let A be a lattice Wajsberg algebra. Then the fuzzy subset μ of A is called a fuzzy WI-ideal of A, if the following axioms are satisfied for all $x, y \in A$,

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu((x \to y)^*), \mu(y)\}$.

Definition 2.12[5].A fuzzy subsetμ of a lattice Wajsberg algebra A is called a fuzzy lattice ideal if for all $x, y \in A$,

(i) If $y \leq x$ then $\mu(y) \geq \mu(x)$

(ii) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$.

Definition 2.13[8]. Let A be a residuated lattice Wajsberg algebra. Then the fuzzy subset μ of A is called a FWI-ideal of residuated lattice Wajsberg algebra A, if the following axioms are satisfied for all $x, y \in A$,

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu(x \otimes y), \mu(y)\}$

(iii) $\mu(x) \geq \min\{\mu((x \to y)^*), \mu(y)\}$.

Definition 2.14[2]. An intuitionistic fuzzy subset S is a non-empty set X is an object having the form $S = \{(x, \mu_s(x), \gamma_s(x)) | x \in X\} = (\mu_s, \gamma_s)$where the functions $\mu_s(x): X \to [0,1]$denote the degree of membership and the degree of non-membership respectively and $0 \leq \mu_s(x) + \gamma_s(x) \leq 1$ for any $x \in X$.

Definition 2.15[13]. If μ and v are fuzzy sets in A, define $\mu \leq \mathrm{v}$ if and only if $\mu(x) \leq \mathrm{v}(x)$ for all $x \in A$.

Definition 2.16[13]. The level set μ_t defined by $\mu_t = \{x \in A / \mu(x) \geq t\}$, where$t \in [0,1]$, then μ_t is also denoted by $U(\mu; t)$.

3. Properties of Intuitionistic *FWI*-ideal of a residuated lattice Wajsberg algebra

In this section, we introduce the concept of an intuitionistic *FWI*-ideal and intuitionistic fuzzy lattice ideals. Also, we obtain some properties of an intuitionistic FWI-ideal.

Definition 3.1. Let A be a residuated lattice Wajsberg algebra. An intuitionistic fuzzy set $S = (\mu_S, \gamma_S)$ of A is called an intuitionistic *FWI*-ideal of residuated lattice Wajsberg algebra A if it satisfies the following inequalities for all $x, y \in A$,

(i) $\mu_S(0) \geq \mu_S(x)$ and $\gamma_S(0) \leq \gamma_S(x)$
(ii) $\mu_S(x) \geq \min\{\mu_S(x \otimes y),\ \mu_S(y)\}$
(iii) $\gamma_S(x) \leq \max\{\gamma_S(x \otimes y),\ \gamma_S(y)\}$
(iv) $\mu_S(x) \geq \min\{\mu_S((x \to y)^*, \mu_S(y)$
(v) $\gamma_S(x) \leq \max\{\gamma_S((x \to y)^*, \gamma_S(y)\}$.

Example 3.2. Consider a set A={$0, a, b, c, d, r, s, t, 1$}. Define a partial ordering "$\leq$" on A, such that $0 \leq a \leq b \leq c \leq d \leq r \leq s \leq t \leq 1$ with a binary operations"$\otimes$"and"$\to$"and a quasi-complement "$*$"on A as in following tables 3.1 and 3.2.

x	x^*
0	1
a	t
b	b
c	r
d	d
r	c
s	b
t	a
1	0

Table 3.1: Complement

$\to$	0	a	b	c	d	r	s	t	1
0	1	1	1	1	1	1	1	1	1
a	t	1	1	t	1	1	t	1	1
b	b	t	1	s	t	1	s	t	1
c	r	r	r	1	1	1	1	1	1
d	d	r	r	t	1	1	t	1	1
r	c	d	r	s	t	1	s	t	1
s	b	b	b	r	r	r	1	1	1
t	a	b	b	d	r	r	t	1	1
1	0	a	b	c	d	r	s	t	1

Table 3.2: Implication

Define $\vee$ and $\wedge$ operations on A as follows:
$(x \vee y) = (x \to y) \to y$,
$(x \wedge y) = (x^* \to y^*) \to y^*)^*$,

$x \otimes y = (x \to y^*)^*$ for all $x, y \in A$.
Then, A is a residuated lattice Wajsberg algebra.
Consider an intuitionistic fuzzy set $S = (\mu_s, \gamma_s)$ on A as,

$$\mu_s(x) = \begin{cases} 1 & \text{if } x \in (0, q) \quad \text{for all } x \in A \\ 0.54 & \text{otherwise} \quad \text{for all } x \in A \end{cases};$$

$$\gamma_s(x) = \begin{cases} 0 & \text{if } x \in (0, q) \quad \text{for all } x \in A \\ 0.36 & \text{otherwise} \quad \text{for all } x \in A \end{cases}$$

Then, S is an intuitionistic FWI-ideal of A.
In the same Example 3.2, we consider an intuitionistic fuzzy set $S = (\mu_s, \gamma_s)$ on A as,

$$\mu_s(x) = \begin{cases} 1 & \text{if } x \in \{0, a, b\} \quad \text{for all } x \in A \\ 0.55 & \text{otherwise} \quad \text{for all } x \in A \end{cases};$$

$$\gamma_s(x) = \begin{cases} 0 & \text{if } x \in \{0, a, b\} \quad \text{for all } x \in A \\ 0.42 & \text{otherwise} \quad \text{for all } x \in A \end{cases}$$

Then, S is not an intuitionistic FWI-ideal of A.
Since $\mu_s(x) \ngeq \min\{\mu_s(s \otimes b), \mu_s(b)\}$ and $\gamma_s(x) \nleq \max\{\gamma_s(s \otimes b),\ \gamma_s(b)\}$.

Proposition 3.3. Every intuitionistic FWI-ideal $S = (\mu_s, \gamma_s)$ of residuated lattice Wajsberg algebra A is an intuitionistic monotonic. That is, if $x \le y$, then $\mu_s(x) \ge \mu_s(y)$ and $\gamma_s(x) \le \gamma_s(y)$.

Proof. Let $S = (\mu_s, \gamma_s)$ be an intuitionistic FWI-ideal of A.
Let $x, y \in A, x \le y$.
Then $x \otimes y = (x \to y^*)^*$ [From the definition 2.6]
$= (x \to x)^* = 1^* = 0$ [From (i) of definition 2.2]
$\mu_s(x) \ge \min\{\mu_s(x \otimes y), \mu_s(y)\}$ [From (ii) of definition 3.1]
We have $\mu_s(x) \ge \mu_s(y)$
Now, $\gamma_s(x) \le \max\{\gamma_s(x \otimes y), \gamma_s(y)\}$ [From (iii) of definition 3.1]
$= \max\{\gamma_s(0), \gamma_s(y)\} = \gamma_s(y)$ [From the definition 2.6]
Hence $\gamma_s(x) \le \gamma_s(y)$
And $\mu_s(x) \ge \min\{\mu_s(x \to y)^*, \mu_s(y)\}$ [From (iv) of definition 3.1]
$= \min\{\mu_s(0), \mu_s(y)\} = \mu_s(y)$ [From (ii) of definition 2.7]
We have $\mu_s(x) \ge \mu_s(y)$
Now, $\gamma_s(x) \le \max\{\gamma_s(x \to y)^*, \gamma_s(y)\}$ [From (v) of definition 3.1]
$= \max\{\gamma_s(0), \gamma_s(y)\} = \gamma_s(y)$ [From (ii) of definition 2.7]
Therefore, $\gamma_s(x) \le \gamma_s(y)$. ■

Example 3.4. Let A be a residuated lattice Wajsberg algebra defined in example 3.2, define an intuitionistic fuzzy set $S = (\mu_s, \gamma_s)$ of A as follows,

(i) $\mu_s(0) = \mu_s(c) = 1$
(ii) $\mu_s(x) = m$ for any $x \in \{a,\ b,\ c,\ d,\ r,\ s,\ t,\ 1\}$
(iii) $\gamma_s(0) = \gamma_s(c) = 0$
(iv) $\gamma_s(x) = n$ for any $x \in \{a,\ b,\ c,\ d,\ r,\ s,\ t,\ 1\}$.

Where $m, n \in [0, 1]$ and $m + n \le 1$. Then $S = (\mu_s, \gamma_s)$ is an intuitionistic FWI-ideal of A.

Example 3.5. Consider a set$A = \{a,\ b,\ p,\ q,\ c,\ d, 1\}$. Define a partial ordering "$\le$" on A, such that $0 \le a \le b \le p \le q \le c \le d \le 1$ with a binary operations"$\otimes$"and "$\to$"and a quasi-complement "$*$"on A as in following tables 3.3 and 3.4.

x	x^*
0	1
a	b
b	a
p	0
q	0
c	0
d	0
1	0

Table 3.3: Complement

$\to$	0	a	b	p	q	c	d	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	a	a	1	1	1	1	1	1
p	0	a	b	1	1	1	1	1
q	0	a	b	p	1	1	1	1
c	0	a	b	p	d	1	d	1
d	0	a	b	p	c	c	1	1
1	0	a	b	p	q	c	d	1

Table 3.4: Implication

Define $\vee$ and $\wedge$ operations on A as follows:
$(x \vee y) = (x \to y) \to y$,
$(x \wedge y) = (x^* \to y^*) \to y^*)^*$,
$x \otimes y = (x \to y^*)^*$ for all $x, y \in A$.
Then, A is a residuated lattice Wajsberg algebra.
Consider an intuitionistic fuzzy set $S = (\mu_s, \gamma_s)$ on A as,

$$\mu_s(x) = \begin{cases} 1 & \text{if } x \in (0, q) \quad \text{for all } x \in A \\ 0.54 & \text{otherwise} \quad \text{for all } x \in A \end{cases};$$

$$\gamma_s(x) = \begin{cases} 0 \ \text{if } x \in (0, q) & \text{for all } x \in A \\ 0.36 \ \text{otherwise} & \text{for all } x \in A \end{cases}$$

Then, S is an intuitionistic FWI-ideal of A.
In the same Example 3.5, we consider an intuitionistic fuzzy set $S = (\mu_s, \gamma_s)$ on A as,

$$\mu_s(x) = \begin{cases} 1 & \text{if } x \in \{0, a, b\} \quad \text{for all } x \in A \\ 0.55 & \text{otherwise} \quad \text{for all } x \in A \end{cases};$$

$$\gamma_s(x) = \begin{cases} 0 & \text{if } x \in \{0, a, b\} \quad \text{for all } x \in A \\ 0.42 & \text{otherwise} \quad \text{for all } x \in A \end{cases}$$

Then, S is not an intuitionistic FWI-ideal of A.
Since $\mu_s(x) \ngeq \min\{\mu_s(c \otimes a), \mu_s(a)\}$ and $\gamma_s(x) \nleq \max\{\gamma_s(c \otimes a),\ \gamma_s(a)\}$.

Proposition 3.6. Let $S = (\mu_s, \gamma_s)$ be an intuitionistic FWI-ideal of residuated lattice Wajsberg algebra A. For any $x, y, z \in A$ which satisfies $x \leq y^* \to z$ then $\mu_s(x) \geq \min\{\mu_s(y), \mu_s(z)\}$ and $\gamma_s(x) \leq \max\{\gamma_s(y), \gamma_s(z)\}$.

Proof. Let$S = (\mu_s, \gamma_s)$ be an intuitionistic FWI-ideal of A. If $x \leq y^* \to z$

Then, we have $1 = x \to (y^* \to z) = z^* \to (x \to y)$
$= (x \to y)^* \to z$ for all $x, y, z \in A$ [From (x) of definition 2.2]
And $((x \to y)^* \to z)^*) = 0$.
It follows that,
$\mu_s(x) \geq \min\{\mu_s(x \otimes y), \mu_s(y)\}$ [From (ii) of definition 3.1]
$\geq \min\{\min\{\mu_s\big((x \otimes y) \to z), \mu_s(z)\big\}, \mu_s(y)\}$
$= \min\big\{\min\{\mu_s\big((0) \to z\big), \mu_s(z)\}, \mu_s(y)\big\}$ [From the definition 2.6]
$= \min\{\min\{\mu_s(0), \mu_s(z)\}, \mu_s(y)\} = \min\{\mu_s(y), \mu_s(z)\}$
[From (ii) of definition 3.1]
We have $\mu_s(x) \geq \min\{\mu_s(y), \mu_s(z)\}$ for all $x, y, z \in A$
Now,$\gamma_s(x) \leq \max\{\max\big\{\gamma_s((x \otimes y), \gamma_s(y)\big)\big\}$
$\leq \max\{\max\{\gamma_s\Big(\big((x \otimes y) \to z\big), \gamma_s(z)\Big\}, \gamma_s(y)\}$
$= \max\big\{\max\{\gamma_s\big((0) \to z\big), \gamma_s(z)\}, \gamma_s(y)\big\}$ [From the definition 2.6]
$= \max\{\max\{\gamma_s(0), \gamma_s(z)\}, \gamma_s(y)\}$
$= \max\{\gamma_s(y), \gamma_s(z)\}$ [From (iii) of definition 3.1]
Hence $\gamma_s(x) \leq \max\{\gamma_s(y), \gamma_s(z)\}$ for all $x, y, z \in A$
Now, $\mu_s(x) \geq \min\{\mu_s((x \to y)^*), \mu_s(y)\}$ [From (iv) of definition 3.1]
$\geq \min\{\min\{\mu_s(x \to y)^* \to z)^*), \mu_s(z)\}, \mu_s(y)\}$
$= \min\{\min\{\mu_s(0),\ \mu_s(z)\}, \mu_s(y)\}$
$= \min\{\mu_s(y), \mu_s(z)\}$ [From (ii) of definition 3.1]
We have $\mu_s(x) \geq \min\{\mu_s(y), \mu_s(z)\}$ for all $x, y, z \in A$
And $\gamma_s(x) \leq \max\big\{\gamma_s((x \to y^*), \gamma_s(y)\big)\big\}$ [From (v) of definition 3.1]
$\leq \max\{\max\big\{\gamma_s\big((x \to y^*) \to z)^*\big), \gamma_s(z)\big\}, \gamma_s(y)\}$
$= \max\{\max\{\gamma_s(0), \gamma_s(z)\}, \gamma_s(y)\}$
$= \max\{\gamma_s(y), \gamma_s(z)\}$ [From (iii) of definition 3.1]
Hence, $\gamma_s(x) \leq \max\{\gamma_s(y), \gamma_s(z)\}$ for all $x, y, z \in A$.■

Definition 3.7. An intuitionistic fuzzy set $S = (\mu_s, \gamma_s)$ of residuated lattice Wajsberg algebra A is called an intuitionistic fuzzy lattice ideal of A if it satisfies the following axioms for all $x, y \in A$,

(i) $S = (\mu_s, \gamma_s)$ is intuitionistic monotonic
(ii) $\mu_s(x \vee y) \geq \min\{\mu_s(x), \mu_s(y)\}$
(iii) $\gamma_s(x \vee y) \leq \max\{\gamma_s(x), \gamma_s(y)\}$.

Remark 3.8. In the Definition 3.7(ii) and (iii) can be equivalently replaced by $\mu_s(x \vee y) = \min\{\mu_s(x), \mu_s(y)\}$ and $\gamma_s(x \vee y) = \max\{\gamma_s(x), \gamma_s(y)\}$ respectively by γ.

Example 3.9. Let A be a residuated lattice Wajsberg algebra defined in the Example 3.2 and $S = (\mu_s, \gamma_s)$ be an intuitionistic fuzzy set of A defined by

$$\mu_s(x) = \begin{cases} 1 & \text{if } x \in (0, d) \text{ for all } x \in A \\ m & \text{otherwise for all } x \in A \end{cases},$$
$$\gamma_s(x) = \begin{cases} 0 & \text{if } x \in (0, d) \text{ for all } x \in A \\ n & \text{otherwise for all } x \in A \end{cases}$$

Where $m, n \in [0, 1]$ and $m + n \leq 1$. [From the definition 3.11]
Then, $S = (\mu_s, \gamma_s)$ is an intuitionistic fuzzy lattice ideal of residuated lattice Wajsberg algebra A.

Proposition 3.10. Let A be a residuated lattice Wajsberg algebra. Every intuitionistic FWI-ideal of A is an intuitionistic fuzzy lattice ideal of A.

Proof. Let $S = (\mu_s, \gamma_s)$ be an intuitionistic fuzzy lattice ideal of A.
Then we have $S = (\mu_s, \gamma_s)$ is intuitionistic monotonic. [From proposition 3.6]
Now $((x \vee y) \to y)^* = (((x \to y) \to y)) \to y)^*$ From (ii) of definition 2.3]
$= (x \to y)^* \leq (x^*)^*$ for all $x, y \in A$ [From (ix) of proposition 2.2]
It follows that

$$\mu_s(x \vee y) \geq \min\{\mu_s(x \vee y) \otimes y, \mu_s(y)\}$$
[From definition 3.1 and definition 3.7]
$$\geq \min\{\mu_s(x \to y) \to y) \otimes y, \mu_s(y)\}$$
[From (ii) of definition 2.3]
$$\geq \min\{\mu_s(0), \mu_s(y)\}$$
$$\geq \min\{\mu_s(x), \mu_s(y)\} \text{for all } x, y \in A$$
[From (i) of proposition 2.10]
$$\gamma_s(x) \leq \max\{\gamma_s((x \vee y) \otimes y), \gamma_s(y)\}$$
$$\leq \max\{\gamma_s(((x \to y) \to y) \otimes y), \gamma_s(y)\}$$
[From (ii) of definition 2.3]
$$\leq \max\{\gamma_s(0), \gamma_s(y)\}$$

$\leq \max\{\gamma_s(x), \gamma_s(y)\}$for all $x, y \in A$
[From (ii) of definition 2.10]

And we have

$\mu_s(x \vee y) \geq \min\{\mu_s(x \vee y) \rightarrow y)^*), \mu_s(y)\} \geq \min\{\mu_s(x), \mu_s(y)\}$

$\gamma_s(x) \leq \max\{\gamma_s\big((x \vee y) \rightarrow y)^*\big), \gamma_s(y)\} \leq \max\{\gamma_s(x), \gamma_s(y)\}$

for all $x, y \in A$.

Hence, we have $S = (\mu_s, \gamma_s)$is an intuitionistic fuzzy lattice ideal of residuated lattice Wajsberg algebra A. ■

Proposition 3.11. Let A be a residuated lattice Wajsberg algebra. An intuitionistic fuzzy set $S = (\mu_s, \gamma_s)$ is an intuitionistic *FWI*-ideal of A if and only if the fuzzy subsets μ_s and γ_s^c are FWI-ideal of A, where $\gamma_s^c(x) = 1 - \gamma_s(x)$for all $x \in A$.

Proof. Let $S = (\mu_s, \gamma_s)$ be an intuitionistic *FWI*-ideal of A.
Then μ_s is a *FWI*-ideal of A.
Now, we have $\gamma_s^c = 1 - \gamma_s(0)$

$\geq 1 - \gamma_s(x)$ [From (i) of proposition 2.10]

$\gamma_s^c(0) = \gamma_s^c(x)$ for all $x, y \in A$

And $\gamma_s^c(x) = 1 - \gamma_s(x)$

$\geq 1 - \max\{\gamma_s(x \otimes y), \gamma_s(y)\}$

$= \min\{1 - \gamma_s(x \otimes y), 1 - \gamma_s(y)\}$

$= \min\{\gamma_s^c(x \otimes y), \gamma_s(y)\}$

$\gamma_s^c(x) = 1 - \gamma_s(x)$

$\geq 1 - \max\{\gamma_s((x \rightarrow y)^*), \gamma_s(y)\}$

$= \min\{1 - \gamma_s((x \rightarrow y)^*), 1 - \gamma_s(y)\}$

$\gamma_s^c(x) = \min\{\gamma_s^c((x \rightarrow y)^*), \gamma_s(y)\}$for all $x, y \in A$

Hence, we have γ_s^c is a *FWI*-ideal of A.
Conversely, assume that μ_s and γ_s^c are *FWI*-ideal of A.
Then, we have $\mu_s(0) \geq \mu_s(x)$and $1 - \gamma_s(0) = \gamma_s^c(0) \geq \gamma_s^c(x) = 1 - \gamma_s(x)$

$\gamma_s(0) \leq \gamma_s(x)$ for all $x, y \in A$

Now, $\mu_s(x) \geq \min\{\mu_s^c(x \otimes y), \mu_s^c(y)\}$

$= \min\{1 - \mu_s(x \otimes y), 1 - \mu_s(y)\}$

$= 1 - \max\{\mu_s(x \otimes y), \mu_s(y)\}$

$\gamma_s(x) \leq \max\{\gamma_s(x \otimes y), \gamma_s(y)\}$ for all $x, y \in A$

$\mu_s(x) \geq \min\{\mu_s^c(x \rightarrow y)^*, \mu_s^c(y)\}$

$= \min\{1 - \mu_s((x \rightarrow y)^*), 1 - \mu_s(y)\}$

$= 1 - \max\{\mu_s((x \rightarrow y)^*), \mu_s(y)\}$

$\gamma_s(x) \leq \max\{\gamma((x \rightarrow y)^*), \gamma_s(y)\}$for all $x, y \in A$

Hence, we have $S = (\mu_s, \gamma_s)$ is an intuitionistic FWI-ideal of A.■

Proposition 3.12. Let A be a residuated lattice Wajsberg algebra and $S = (\mu_s, \gamma_s)$ is an intuitionistic FWI-ideal of$A$. Then $S = (\mu_s, \gamma_s)$ is an intuitionistic FWI-ideal of A if and only if (μ_s, μ_s^c) and (γ_s^c, γ_s) are intuitionistic FWI-ideal of A.

Proof. Let $S = (\mu_s, \gamma_s)$ be an intuitionistic FWI-ideal of A.
Then, μ_s and γ_s^c are FWI-ideal of A[From proposition 3.11]
Hence, we have (μ_s, μ_s^c) and (γ_s^c, γ_s) are intuitionistic FWI-ideal of A.
Conversely, if (μ_s, μ_s^c) and (γ_s^c, γ_s) are intuitionistic FWI-idealof A
[From proposition 3.11]
Then, the fuzzy sets μ_s and γ_s^c are FWI-ideal of A
Hence, $S = (\mu_s, \gamma_s)$ is an intuitonistic FWI-ideal of A. ■

Proposition 3.13. Let A be residuated lattice Wajsberg algebra, V a non-empty subset of $[0, 1]$ and $\{I_t / \ t \in V\}$ a collection of FWI -ideal of A such that

(i) $A = \bigcup_{t\in v} I_t$
(ii) $r > t$ if and only if $I_r \subseteq I_t$ for any $r, t \in V$ then the intuitionistic fuzzy set $S = (\mu_s, \gamma_s)$ of A defined by $\mu_s = \sup\{t \in V/x \in I_t\}$ and$\gamma_s = \inf\{t \in V/x \in I_t\}$ for any $x \in A$ is intuitionistic FWI -ideal of A.

Proof. According to proposition 3.10, it is sufficient to show that μ_s and γ_s^c are FWI–idealof A for all$x \in A$.
$\mu_s(0) = \sup\{t \in V/0 \in I_t\} = \sup V \geq \mu_s(x)$ [From (i) of definition 3.1]
If there exists $x, y \in A$ such that $\mu_s(x) < \min\{\mu_s(x \otimes y), \mu_s(y)\}$ and
$\mu_s(x) < \min\{\mu_s((x \to y)^*), \mu_s(y)\}$.
There exists t_1 such that $\mu_s(x) < t_1 < \min\{\mu_s(x \otimes y), \mu_s(y)\}$ and

$$\mu_s(x) < t_1 < \min\{\mu_s((x \to y)^*), \mu_s(y)\}$$

It follows that t_1 such that$t_1 < \mu_s(x \otimes y)$,$t_1 < \mu_s((x \to y)^*)$,$t_1 < \mu_s(y)$ and
Hence, there exist $t_2, t_3 \in V$, $t_2 > t_1$, $t_3 > t_1$, $(x \otimes y) \in I_{t_2}$, $(x \to y)^*) \in I_{t_2}$ and $y \in I_{t_3}$
It follows that $(x \otimes y) \in I_{t_2 \wedge t_3}$, $(x \to y)^*) \in I_{t_2 \wedge t_3}$ and $y \in I_{t_2 \wedge t_3}$
Now, we have $x \in I_{t_2 \wedge t_3}$
That is, $\mu_s(x) = \sup\left\{t \in \frac{V}{x} \in I_t\right\} \geq t_2 \wedge t_3 > t_1$ [From definition 2.16]
Therefore, $\mu_s(x) > t_1$
This is a contradiction.
Hence, we have μ_s is a FWI -ideal of A. $\gamma_s^c$is aFWI -ideal, which can be proved by similar method. ■

4. Conclusions

In this paper, we have introduced the notions of intuitionistic FWI–ideal and intuitionistic fuzzy lattice ideal of residuated Wajsberg algebras. Also, we have shown that every intuitionistic FWI- ideal of residuated lattice Wajsberg algebra is an intuitionistic fuzzy lattice ideal of residuated lattice Wajsberg algebra. Further, we have discussed its converse part.

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Fuzzy homotopy analysis method for solving fuzzy autonomous differential equation

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Abstract

In this paper, we have presented the theory and the applications of the fuzzy homotopy analysis method to find the fuzzy semi-analytical solutions of the second order fuzzy autonomous ordinary differential equation. This method allows for the solution of the fuzzy initial value problems to be calculated in the form of an infinite fuzzy series in which the fuzzy components can be easily calculated. Some numerical results have been given to illustrate the used method. The obtained numerical results have been compared with the fuzzy exact-analytical solutions.

Keywords: fuzzy homotopy analysis method; fuzzy autonomous differential equation; fuzzy series solution.
2010 AMS subject classification: Applied mathematics.

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1. Introduction

The topic of the fuzzy semi-analytical methods(fuzzy series method) for solving the fuzzy initial value problems(FIVPs) has been rapidly growing in recent years, whereas the fuzzy series solutions of FIVP have been studied by several authors during the past few years. Several fuzzy semi-analytical methods have been proposed to obtain the fuzzy series solution of the linear and non-linear FIVB which are mostly first order problems. Some of these methods have been proposed to obtain the fuzzy series solutions of the high order FIVB.

Fuzzy homotopy analysis method was used for the first time to solve the fuzzy differential equations in 2012. Researchers and scientists are continuing to develop this method for solving various types of the fuzzy initial value problems because it represents an efficient and effective technique.

In the following we will review some of the findings of the researchers regarding this method. In 2012, Hashemi, Malekinagad and Marasi[4] suggested and applied the Fuzzy homotopy analysis method for solving a system of fuzzy differential equations with fuzzy initial conditions. In 2013, Abu-Arqub, El-Ajou1 and Momani[6] studied and developed the Fuzzy homotopy analysis method to obtain the analytical solutions of the fuzzy initial value problems. In 2014, Jameel, Ghoreishi and Ismail[8] introduced and applied the Fuzzy homotopy analysis method to obtain the approximate-analytical solutions of the high order fuzzy initial value problems. In 2015, Al-Jassar[9] introduced and presented fuzzy semi-analytical methods (including the fuzzy homotopy analysis method) to obtain the numerical and approximate-analytical solutions of the linear and non-linear fuzzy initial value problems. In 2016, Otadi and Mosleh[12] studied and developed the fuzzy homotopy analysis method to obtain numerical and approximate-analytical solutions of the hybrid fuzzy ordinary differential equations with the fuzzy initial conditions. As well, In 2016, Lee, Kumaresan and Ratnavelu[11] suggested a solution of the fuzzy fractional differential equations with fuzzy initial conditions by using the fuzzy homotopy analysis method. In 2017, Padma and Kaliyappan[14] introduced and presented fuzzy semi-analytical methods(including the fuzzy homotopy analysis transform method) to obtain the numerical and approximate-analytical solutions of the fuzzy fractional initial value problems. In 2018, Sevindir, Cetinkaya and Tabak[16] introduced and presented fuzzy semi-analytical methods(including the fuzzy homotopy analysis method) to obtain the numerical and approximate-analytical solutions of the first order fuzzy initial value problems. Also, In 2018, Jameel, Saaban and Altaie[15] suggested and applied a new concepts for solving the first order non-linear fuzzy initial value problems by using the fuzzy optimal homotopy asymptotic method. In 2020, Nematallah and Najafi[18] introduced and applied the fuzzy homotopy analysis method to obtain the fuzzy semi-analytical solution of the fuzzy fractional initial value problems based on the concepts of generalized Hukuhara differentiability. As well, In 2020, Ali and Ibraheem[17] studied and developed some fuzzy analytical and numerical solutions of the linear first order fuzzy initial value problems by using fuzzy homotopy analysis method based on the Padè Approximate method.

In this work, we have studied and applied the fuzzy homotopy analysis method to find the fuzzy series solution(fuzzy approximate-analytical solution) of the second order fuzzy autonomous ordinary differential equation with real variable coefficients(real-valued function coefficients). The fuzzy semi-analytical solutions that we have obtained during this work are accurate solutions and very close to the fuzzy exact-analytical solutions, based on the comparison that we have introduced between the results that we have obtained and the fuzzy exact-analytical solutions.

2. Basic definitions in fuzzy set theory

In this section, we will present some of the fundamental definitions and the primitive concepts related to the fuzzy set theory, which are very necessary for understanding this subject.

Definition (1), [1] (Fuzzy Set)

The fuzzy set $\widetilde{A}$ can be defined as:

$$\widetilde{A} = \{ (x , \mu_{\widetilde{A}}(x)) \ : \ x \in X ; \ 0 \leq \mu_{\widetilde{A}}(x) \leq 1 \} \qquad (1)$$

where X is the universal set and $\mu_{\widetilde{A}}(x)$ is the grade of membership of x in $\widetilde{A}$.

Definition (2), [7] (α – Level Set)

The α - level (or α - cut) set of a fuzzy set $\widetilde{A}$ can be defined as:

$$A_\alpha = \{ x \in X : \ \mu_{\widetilde{A}}(x) \geq \alpha \ ; \ \alpha \in [0,1] \} . \qquad (2)$$

Definition (3), [9] (Fuzzy Number)

A fuzzy number $\tilde{u}$ is an ordered pair of functions $(\underline{u}(\alpha) , \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$, with the following conditions:

1) $\underline{u}(\alpha)$ is a bounded left continuous and non-decreasing function on $[0,1]$.

2) $\overline{u}(\alpha)$ is a bounded left continuous and non-increasing function on $[0,1]$.

3) $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, $0 \leq \alpha \leq 1$. (3)

Remark (1), [9] :

1) The crisp number u is simply represented by :

$\underline{u}(\alpha) = \overline{u}(\alpha) = u$, $0 \leq \alpha \leq 1$. (4)

2) The set of all the fuzzy numbers is denoted by E^1.

Remark (2), [13]:

The distance between two arbitrary fuzzy numbers $\tilde{u} = (\underline{u}, \overline{u})$ and $\tilde{v} = (\underline{v}, \overline{v})$ can be defined as:

$$D(\tilde{u}, \tilde{v}) = \left[\int_0^1 (\underline{u}(\alpha) - \underline{v}(\alpha))^2 d\alpha + \int_0^1 (\overline{u}(\alpha) - \overline{v}(\alpha))^2 d\alpha \right]^{\frac{1}{2}} \quad (5)$$

Remark (3), [13]:

(E^1, D) is a complete metric space.

Definition (4), [9] (Fuzzy Function)

A mapping $F : T \rightarrow E^1$ for some interval $T \subseteq E^1$ is called a fuzzy function or fuzzy process with non-fuzzy variable (crisp variable), and we denote α - level sets by:

$$[F(t)]_\alpha = \left[\underline{F}(t ; \alpha) , \overline{F}(t ; \alpha) \right] \quad (6)$$

Where $t \in T$, $\alpha \in [0,1]$. we refer to $\underline{F}$ and $\overline{F}$ as the lower and upper branches on F.

Definition (5), [9] (H-Difference)

Let $u , v \in E^1$. If there exist $w \in E^1$ such that $u = v + w$ then w is called the H-difference (Hukuhara-difference) of u and v and it is denoted by $w = u \ominus v$, where $u \ominus v \neq u + (-1) v$.

Definition (6), [13] (Fuzzy Derivative)

Let $F : T \rightarrow E^1$ for some interval $T \subseteq E^1$ and $t_0 \in T$. We say that F is H-differential(Hukuhara-differential) at t_0 , if there exists an element $F'(t_0) \in E^1$ such that for all $h > 0$ (sufficiently small) , $\exists\ F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits(in the metric D)

$$\lim_{h\to 0}\frac{F(t_0+h)\ominus F(t_0)}{h}=\lim_{h\to 0}\frac{F(t_0)\ominus F(t_0-h)}{h}=F'(t_0) \qquad (7)$$

Then $F'(t_0)$ is called the fuzzy derivative(H-derivative) of F at t_0 . where D is the distance between two fuzzy numbers.

Definition (7), [9] (Nth Order Fuzzy Derivative)

Let $F' : T \to E^1$ for some interval $T \subseteq E^1$ and $t_0 \in T$. We say that F' is H-differential(Hukuhara-differential) at t_0, if there exists an element $F^{(n)}(t_0) \in E^1$ such that for all $h > 0$ (sufficientlysmall), $\exists\, F^{(n-1)}(t_0+h) \ominus F^{(n-1)}(t_0)$, $F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0 - h)$ and the limits(in the metric D)

$$\lim_{h\to 0}\frac{F^{(n-1)}(t_0+h)\ominus F^{(n-1)}(t_0)}{h}=$$
$$\lim_{h\to 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0-h)}{h}=F^{(n)}(t_0) \qquad (8)$$

Then $F^{(n)}(t_0)$ is called the nth order fuzzy derivative (H-derivative of order n) of F at t_0.

Theorem(1), [9]:

Let $F : T \to E^1$ for some interval $T \subseteq E^1$ be an nth order Hukuhara differentiable functions at $t \in T$ and denote

$$[\,F(t)\,]_\alpha = \left[\underline{F}(t\,;\,\alpha),\ \overline{F}(t\,;\,\alpha)\right], \forall\, \alpha \in [0,1].$$

Then the boundary functions $\underline{F}(t\,;\,\alpha)$, $\overline{F}(t\,;\,\alpha)$ are both nth order Hukuhara differentiable functions and

$$[\,F^{(n)}(t)\,]_\alpha = \left[\underline{F}^{(n)}(t\,;\,\alpha),\ \overline{F}^{(n)}(t\,;\,\alpha)\right], \forall\, \alpha \in [0,1]. \qquad (9)$$

3. Fuzzy autonomous differential equation

A fuzzy ordinary differential equation is said to be autonomous if it is independent of it's independent crisp variable t. This is to say an explicit nth order fuzzy autonomous differential equation is of the following form[13] :

$$x^{(n)}(t) = f\left(x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t)\right), \ t \in [t_0, h] \qquad (10)$$

with the fuzzy initial conditions :

$$x(t_0) = x_0\ ,\ x'(t_0) = x_0'\ ,\ x''(t_0) = x_0''\ , \ldots ,\ x^{(n-1)}(t_0) = x_0^{(n-1)}$$

where :

x is a fuzzy function of the crisp variable t ,

$f\left(x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t)\right)$ is a fuzzy function of the crisp variable t and the fuzzy variable x ,

$x^{(n)}(t)$ is the fuzzy derivative of the $x(t)$, $x'(t)$, $x''(t), \ldots,$ $x^{(n-1)}(t)$,

and $x(t_0)$, $x'(t_0)$, $x''(t_0), \ldots,$ $x^{(n-1)}(t_0)$ are fuzzy numbers.

The fuzzy differential equations that are dependent on t are called non-autonomous, and a system of fuzzy autonomous differential equations is called a fuzzy autonomous system.

The main idea in solving the fuzzy autonomous differential equation is to convert it into a system of non-fuzzy(crisp) differential equations, and then solve this system by the known and commonly used methods of solving the non-fuzzy differential equations.

Now it is possible to replace (10) by the following equivalent system of the nth order crisp ordinary differential equations:

$$\underline{x}^{(n)}(t) = \underline{f}\left(x, x', x'', \ldots, x^{(n-1)}\right)$$

$$= F(\underline{x}, \underline{x}', \underline{x}'', \ldots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \ldots, \overline{x}^{(n-1)});$$

$$\underline{x}(t_0) = \underline{x}_0, \ \underline{x}'(t_0) = \underline{x}'_0, \ \underline{x}''(t_0) = \underline{x}''_0, \ \ldots, \ \underline{x}^{(n-1)}(t_0) = \underline{x}_0^{(n-1)}, \quad (11)$$

$$\overline{x}^{(n)}(t) = \overline{f}\left(x, x', x'', \ldots, x^{(n-1)}\right)$$

$$= G(\underline{x}, \underline{x}', \underline{x}'', \ldots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \ldots, \overline{x}^{(n-1)});$$

$$\overline{x}(t_0) = \overline{x}_0, \ \overline{x}'(t_0) = \overline{x}'_0, \ \overline{x}''(t_0) = \overline{x}''_0, \ldots, \ \overline{x}^{(n-1)}(t_0) = \overline{x}_0^{(n-1)} \quad (12)$$

Where

$$F(\underline{x}, \underline{x}', \underline{x}'', \ldots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \ldots, \overline{x}^{(n-1)}) =$$

$$\mathrm{Min}\left\{ f(t, u) : u \in \left[\underline{x}, \underline{x}', \underline{x}'', \ldots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \ldots, \overline{x}^{(n-1)}\right]\right\}, \quad (13)$$

$$G(\underline{x}, \underline{x}', \underline{x}'', \ldots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \ldots, \overline{x}^{(n-1)}) =$$

$$\mathrm{Max}\left\{ f(t, u) : u \in \left[\underline{x}, \underline{x}', \underline{x}'', \ldots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \ldots, \overline{x}^{(n-1)}\right]\right\}. \quad (14)$$

The parametric form of system (13-14) is given by:

$\underline{x}^{(n)}(t,\alpha) = F(\underline{x}(t,\alpha), \underline{x}'(t,\alpha), \underline{x}''(t,\alpha), \dots, \underline{x}^{(n-1)}(t,\alpha), \overline{x}(t,\alpha), \overline{x}'(t,\alpha), \overline{x}''(t,\alpha), \dots, \overline{x}^{(n-1)}(t,\alpha))$

$\underline{x}(t_0,\alpha) = \underline{x}_0(\alpha)$, $\underline{x}'(t_0,\alpha) = \underline{x}'_0(\alpha)$, $\underline{x}''(t_0,\alpha) = \underline{x}''_0(\alpha)$, … , $\underline{x}^{(n-1)}(t_0,\alpha) = \underline{x}_0^{(n-1)}(\alpha)$ (15)

$\overline{x}^{(n)}(t,\alpha) = G(\underline{x}(t,\alpha), \underline{x}'(t,\alpha), \underline{x}''(t,\alpha), \dots, \underline{x}^{(n-1)}(t,\alpha), \overline{x}(t,\alpha), \overline{x}'(t,\alpha), \overline{x}''(t,\alpha), \dots, \overline{x}^{(n-1)}(t,\alpha))$

$\overline{x}(t_0,\alpha) = \overline{x}_0(\alpha)$, $\overline{x}'(t_0,\alpha) = \overline{x}'_0(\alpha)$, $\overline{x}''(t_0,\alpha) = \overline{x}''_0(\alpha)$, … , $\overline{x}^{(n-1)}(t_0,\alpha) = \overline{x}_0^{(n-1)}(\alpha)$ (16)

Where $t \in [t_0, h]$ and $\alpha \in [0,1]$.

The following theorem ensures the existence and uniqueness of the fuzzy solution of the nth order fuzzy autonomous differential equation.

Theorem(2), [13] :

If we return to problem (10),

$x^{(n)}(t) = f\left(x(t), x'(t), x''(t), \dots, x^{(n-1)}(t)\right)$, $t \in [t_0, h]$

Let $f_i : T \to E^1$, $1 \le i \le n$ be a continuous fuzzy functions, T= $[t_0, h]$ and assume that there exist a real numbers $k_i > 0$ such that

$D\left(f_i(t, z_i), f_i(t, w_i)\right) \le k_i D(z_i, w_i)$ (17)

For all $t \in T$ and all $z_i, w_i \in E^1$.

Then the above nth order FIVB has a unique fuzzy solution on T in each case.

4. Fuzzy homotopy analysis method

A fuzzy homotopy analysis method is one of the fuzzy semi- analytical methods used to obtain the fuzzy series solution(fuzzy approximate-analytical solution) of the FIVBs. This technique utilizes homotopy in order to generate a convergent fuzzy series of fuzzy linear equations from fuzzy non-linear ones. This means that this technique is based on generating a convergent fuzzy series of fuzzy solutions to approximate the fuzzy analytical solution of the FIVB.

The basic mathematical concepts of the fuzzy homotopy analysis method are the same as the basic mathematical concepts of the homotopy analysis method, but with the use of the concepts of the fuzzy set theory. This means that solving any FIVB by using fuzzy homotopy analysis method is based on converting the FIVB into a system of non-fuzzy(crisp) initial value problems by using the steps that we explained in section(3), and then using the homotopy analyss method to solve this system.

The fuzzy homotopy analysis method provides us with both the freedom to choose proper base fuzzy functions for approximating a non-linear fuzzy problem and a simple way to ensure the convergence of the fuzzy series solution.

5. Description of the method

To describe the basic mathematical ideas of the fuzzy homotopy analysis method, we consider the following nth order FIVB :

$$[\mathrm{N}(\mathrm{x}(\mathrm{t})]_\alpha = 0 \quad , \tag{18}$$

Where N is the fuzzy non-linear operator , t denotes the independent crisp variable , $\mathrm{x}(\mathrm{t})$ is an unknown fuzzy function .

By the concepts of section(3), We can conclude that:

$$[\mathrm{N}(\mathrm{x}(\mathrm{t})]_\alpha = [\,[\mathrm{N}(\mathrm{x}(\mathrm{t})]_\alpha^{\mathrm{L}} , [\mathrm{N}(\mathrm{x}(\mathrm{t})]_\alpha^{\mathrm{U}}\,] \tag{19}$$

Since $0 = [\,0 , 0]$, we can get:

$$[\mathrm{N}(\mathrm{x}(\mathrm{t})]_\alpha^{\mathrm{L}} = 0 \tag{20i}$$

$$[\mathrm{N}(\mathrm{x}(\mathrm{t})]_\alpha^{\mathrm{U}} = 0 \tag{20ii}$$

Now, we construct the zero-order fuzzy deformation equation:

$$[(1-\mathrm{w})\mathrm{L}\,(\theta(\mathrm{t}\,;\,\mathrm{w}) - \mathrm{x}_0(\mathrm{t}))]_\alpha = [\mathrm{w}\,\mathrm{h}\,\mathrm{N}(\theta(\mathrm{t}\,;\,\mathrm{w}))]_\alpha\,, \tag{21}$$

Where $\mathrm{w} \in [0 , 1]$ is the homotopy embedding parameter, $\mathrm{h} \in [-1 , 0)$ is the convergence control parameter , L is the fuzzy linear operator , $\mathrm{x}_0(\mathrm{t})$ is the fuzzy initial guess of $\mathrm{x}(\mathrm{t})$ and $\theta(\mathrm{t};\mathrm{w})$ is a fuzzy function.

By the concepts of section(3), We can get:

$$(1-\mathrm{w})\mathrm{L}(\,[\theta(\mathrm{t}\,;\,\mathrm{w})]_\alpha^{\mathrm{L}} - [\mathrm{x}_0(\mathrm{t})]_\alpha^{\mathrm{L}}\,) = \mathrm{w}\,\mathrm{h}(\,[\mathrm{N}(\theta(\mathrm{t}\,;\,\mathrm{w}))]_\alpha^{\mathrm{L}}\,) \tag{22i}$$

$$(1-\mathrm{w})\mathrm{L}(\,[\theta(\mathrm{t}\,;\,\mathrm{w})]_\alpha^{\mathrm{U}} - [\mathrm{x}_0(\mathrm{t})]_\alpha^{\mathrm{U}}\,) = \mathrm{w}\,\mathrm{h}(\,[\mathrm{N}(\theta(\mathrm{t}\,;\,\mathrm{w}))]_\alpha^{\mathrm{U}}\,) \tag{22ii}$$

Obviously , when $w = 0$ and $w = 1$, both

$$[\theta(t\,;0)]_\alpha = [x_0(t)]_\alpha\ , \tag{23}$$

$$[\theta(t\,;1)]_\alpha = [x(t)]_\alpha \tag{24}$$

Hold, therefore when w is increasing from 0 to 1 , the fuzzy solutions $[\theta(t\,;\,w)]_\alpha^L$ and $[\theta(t\,;\,w)]_\alpha^U$ varies from the fuzzy initial guess $[x_0(t)]_\alpha$ to the fuzzy solution$[x(t)]_\alpha$.

Thus, we have:

$$[\theta(t\,;0)]_\alpha^L = [x_0(t)]_\alpha^L \tag{25i}$$

$$[\theta(t\,;0)]_\alpha^U = [x_0(t)]_\alpha^U \tag{25ii}$$

$$[\theta(t\,;1)]_\alpha^L = [x(t)]_\alpha^L \tag{26i}$$

$$[\theta(t\,;1)]_\alpha^U = [x(t)]_\alpha^U \tag{26ii}$$

By expanding $[\theta(t\,;w)\,]_\alpha$ in Taylor series with respect to w , one has:

$$[\theta(t\,;w)]_\alpha = [x_0(t)]_\alpha + \sum_{m=1}^{\infty}[x_m(t)w^m]_\alpha \tag{27}$$

Where

$$[x_m(t)]_\alpha = \frac{1}{m!}\,\frac{\partial^m\,[\theta(t\,;w)]_\alpha}{\partial w^m}\bigg|_{w=0} \tag{28}$$

By the concepts of parametric form in section(3), We can conclude that:

$$[\theta(t\,;w)]_\alpha^L = [x_0(t)]_\alpha^L + \sum_{m=1}^{\infty}[x_m(t)]_\alpha^L\ w^m \tag{29i}$$

$$[\theta(t\,;w)]_\alpha^U = [x_0(t)]_\alpha^U + \sum_{m=1}^{\infty}[x_m(t)]_\alpha^U\ w^m \tag{29ii}$$

Where

$$[x_0(t)]_\alpha^L = \frac{1}{m!}\,\frac{\partial^m\,[\theta(t\,;\,w)]_\alpha^L}{\partial w^m}\bigg|_{w=0} \tag{30i}$$

$$[x_0(t)]_\alpha^U = \frac{1}{m!}\,\frac{\partial^m\,[\theta(t\,;\,w)]_\alpha^U}{\partial w^m}\bigg|_{w=0} \tag{30ii}$$

If the fuzzy linear operator , the fuzzy initial guess , the auxiliary parameter h , and the auxiliary fuzzy function are so properly chosen , then the fuzzy series (27) converges at $w = 1$, and one has:

$$[\theta(t\,;1)]_\alpha = [x(t)]_\alpha = [x_0(t)]_\alpha + \sum_{m=1}^{\infty}[x_m(t)]_\alpha \tag{31}$$

Where

$$[\theta(t\,;1)]^{L}_{\alpha} = [x(t)]^{L}_{\alpha} = [x_0(t)]^{L}_{\alpha} + \sum_{m=1}^{\infty}[x_m(t)]^{L}_{\alpha} \tag{32i}$$

$$[\theta(t\,;1)]^{U}_{\alpha} = [x(t)]^{U}_{\alpha} = [x_0(t)]^{U}_{\alpha} + \sum_{m=1}^{\infty}[x_m(t)]^{U}_{\alpha} \tag{32ii}$$

Which must be one of the fuzzy solutions of the problem(18).

If $h = -1$, (21) becomes

$$[(1-w)L\,(\theta(t\,;\,w) - x_0(t))]_{\alpha} + [w\,N(\theta(t\,;\,w))]_{\alpha} = 0\,, \tag{33}$$

Where

$$(1-w)L(\,[\theta(t\,;\,w)]^{L}_{\alpha} - [x_0(t)]^{L}_{\alpha}) + w\,(\,[N(\theta(t\,;\,w))]^{L}_{\alpha}\,) = 0 \tag{34i}$$

$$(1-w)L(\,[\theta(t\,;\,w)]^{U}_{\alpha} - [x_0(t)]^{U}_{\alpha}) + w\,(\,[N(\theta(t\,;\,w))]^{U}_{\alpha}\,) = 0 \tag{34ii}$$

Which is used mostly in the fuzzy homotopy analysis method.

We define the fuzzy vectors

$$[\,\vec{x}_i\,]_{\alpha} = \{\,[\,x_0(t)]_{\alpha}\,,\ [x_1(t)]_{\alpha}\,, [\,x_2(t)]_{\alpha}\,, \dots, [\,x_i(t)]_{\alpha}\,\} \tag{35}$$

Where

$$[\,\vec{x}_i\,]^{L}_{\alpha} = \{\,[\,x_0(t)]^{L}_{\alpha}\,, [\,x_1(t)]^{L}_{\alpha}\,,\ [\,x_2(t)]^{L}_{\alpha}\,, \dots, [\,x_i(t)]^{L}_{\alpha}\,\} \tag{36i}$$

$$[\,\vec{x}_i\,]^{U}_{\alpha} = \{\,[\,x_0(t)]^{U}_{\alpha}\,, [\,x_1(t)]^{U}_{\alpha}\,,\ [\,x_2(t)]^{U}_{\alpha}\,, \dots, [\,x_i(t)]^{U}_{\alpha}\,\} \tag{36ii}$$

Now, by differentiating (21) m- times with respect to the parameter w and then setting $w = 0$ and finally dividing them by $m!$, we have the mth-order fuzzy deformation equation:

$$L(\,[x_m(t)]_{\alpha} - \chi_m[x_{m-1}(t)]_{\alpha}\,) = h(\,[R_m(\vec{x}_{m-1})\,]_{\alpha}\,) \tag{37}$$

Where $[\,R_m(\vec{x}_{m-1})\,]_{\alpha} =$

$$\frac{1}{(m-1)!}\,\frac{\partial^{m-1}[N(\theta(t\,;w))]_{\alpha}}{\partial w^{m-1}}\bigg|_{w=0} \tag{38}$$

By the concepts of parametric form in section(3), We get:

$$L(\,[x_m(t)]^{L}_{\alpha} - \chi_m[x_{m-1}(t)]^{L}_{\alpha}) = h\,(\,[R_m(\vec{x}_{m-1})]^{L}_{\alpha}) \tag{39i}$$

$$L(\,[x_m(t)]^{U}_{\alpha} - \chi_m[x_{m-1}(t)]^{U}_{\alpha}) = h\,(\,[R_m(\vec{x}_{m-1})]^{U}_{\alpha}) \tag{39ii}$$

Where

$$[R_m(\vec{x}_{m-1})]_\alpha^L = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1}([N(\theta(t\,;w))]_\alpha^L)}{\partial w^{m-1}}\right|_{w=0} \quad (40i)$$

$$[R_m(\vec{x}_{m-1})]_\alpha^U = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1}([N(\theta(t\,;w))]_\alpha^U)}{\partial w^{m-1}}\right|_{w=0} \quad (40ii)$$

$$\chi_m = \begin{cases} 0\,, & m \le 1\,, \\ 1\,, & m > 1\,. \end{cases} \quad (41)$$

6. Applied example

In this section, one fuzzy problem has been solved in order to clarify the efficiency and the accuracy of the method. According to Liao's book [3], the optimal value of h was found to be approximately $-1 \le h < 0$. In addition, the practical examples in[3,5,10] showed that the optimal value of h can be determined while solving the problem by experimenting with a number of different values of h. The optimal value of h depends greatly on the nature of the problem, but still h = -1 is an optimal value and achieves a rapid convergence.

Example 1: Consider the second order fuzzy autonomous differential equation

$$x''(t) + x(t) = 0\ , \quad (42)$$

Subject to the fuzzy initial conditions :

$$[x(0)]_\alpha = [0,0]\ ,\ [x'(0)]_\alpha = [0.01\alpha + 0.02\,, -0.01\alpha + 0.04]\ ,\ \alpha \in [0,1]\ .$$

Solution:

The fuzzy linear operator is :

$$[L(\theta(t;w))]_\alpha = [\,[L(\theta(t;w))]_\alpha^L\,, [L(\theta(t;w))]_\alpha^U\,] \quad (43)$$

Where

$$[L(\theta(t;w))]_\alpha^L = \left[\frac{\partial^2\theta(t;w)}{\partial t^2}\right]_\alpha^L \quad (44i)$$

$$[L(\theta(t;w))]_\alpha^U = \left[\frac{\partial^2\theta(t;w)}{\partial t^2}\right]_\alpha^U \quad (44ii)$$

We define the fuzzy non-linear operator as :

$$[N(\theta(x;w))]_\alpha = [\,[N(\theta(x;w))]_\alpha^L\,, [N(\theta(x;w))]_\alpha^U\,] \quad (45)$$

Where

$$[N(\theta(x;w))]_\alpha^L = \frac{\partial^2[\theta(t;w)]_\alpha^L}{\partial t^2} + [\theta(t;w)]_\alpha^L \tag{46i}$$

$$[N(\theta(x;w))]_\alpha^U = \frac{\partial^2[\theta(t;w)]_\alpha^U}{\partial t^2} + [\theta(t;w)]_\alpha^U \tag{46ii}$$

The fuzzy series solution is :

$$[x(t)]_\alpha = [\, [x(t)]_\alpha^L , [x(t)]_\alpha^U] \tag{47}$$

Where

$$[x(t)]_\alpha^L = [x_0(t)]_\alpha^L + [x_1(t)]_\alpha^L + [x_2(t)]_\alpha^L + [x_3(t)]_\alpha^L + \cdots \tag{48i}$$

$$[x(t)]_\alpha^U = [x_0(t)]_\alpha^U + [x_1(t)]_\alpha^U + [x_2(t)]_\alpha^U + [x_3(t)]_\alpha^U + \cdots \tag{48ii}$$

By Taylor series expansion , the fuzzy initial approximation is:

$$[x_0(t)]_\alpha = [\, [x_0(t)]_\alpha^L , [x_0(t)]_\alpha^U] \tag{49}$$

Where

$$[x_0(t)]_\alpha^L = 0.01\,\alpha t + 0.02\,t \tag{50i}$$

$$[x_0(t)]_\alpha^U = -0.01\,\alpha t + 0.04\,t \tag{50ii}$$

To find $[x_1(t)]_\alpha = [\, [x_1(t)]_\alpha^L , [x_1(t)]_\alpha^U]$

From(29), we can find

$$[\theta(t;w)]_\alpha^L = [x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L \tag{51i}$$

$$[\theta(t;w)]_\alpha^U = [x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U \tag{51ii}$$

From(37), we can find

$$L([x_1(t)]_\alpha^L - 0) = h[R_1]_\alpha^L \tag{52i}$$

$$L([x_1(t)]_\alpha^U - 0) = h[R_1]_\alpha^U \tag{52ii}$$

Then from(38), we can get :

$$[R_1]_\alpha^L = [N(\theta(t;w))]_\alpha^L|_{w=0} \tag{53i}$$

$$[R_1]_\alpha^U = [N(\theta(t;w))]_\alpha^U|_{w=0} \tag{53ii}$$

Then , we apply the following steps :

$$\frac{\partial[\theta(t;w)]_\alpha^L}{\partial t} = \frac{\partial[x_0(t)]_\alpha^L}{\partial t} + w\frac{\partial[x_1(t)]_\alpha^L}{\partial t} \quad (54i)$$

$$\frac{\partial[\theta(t;w)]_\alpha^U}{\partial t} = \frac{\partial[x_0(t)]_\alpha^U}{\partial t} + w\frac{\partial[x_1(t)]_\alpha^U}{\partial t} \quad (54ii)$$

$$\frac{\partial^2[\theta(t;w)]_\alpha^L}{\partial t^2} = \frac{\partial^2[x_0(t)]_\alpha^L}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^L}{\partial t^2} \quad (55i)$$

$$\frac{\partial^2[\theta(t;w)]_\alpha^U}{\partial t^2} = \frac{\partial^2[x_0(t)]_\alpha^U}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^U}{\partial t^2} \quad (55ii)$$

$$[N(\theta(x;w))]_\alpha^L = \frac{\partial^2[x_0(t)]_\alpha^L}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^L}{\partial t^2} + [x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L \quad (56i)$$

$$[N(\theta(x;w))]_\alpha^U = \frac{\partial^2[x_0(t)]_\alpha^U}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^U}{\partial t^2} + [x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U \quad (56ii)$$

$$[R_1]_\alpha^L = \frac{\partial^2[x_0(t)]_\alpha^L}{\partial t^2} + [x_0(t)]_\alpha^L \quad (57i)$$

$$[R_1]_\alpha^U = \frac{\partial^2[x_0(t)]_\alpha^U}{\partial t^2} + [x_0(t)]_\alpha^U \quad (57ii)$$

$$[R_1]_\alpha^L = 0.01\,\alpha t + 0.02\,t \quad (58i)$$

$$[R_1]_\alpha^U = -0.01\,\alpha t + 0.04\,t \quad (58ii)$$

$$L([x_1(t)]_\alpha^L) = 0.01\,\alpha ht + 0.02\,ht \quad (59i)$$

$$L([x_1(t)]_\alpha^U) = -0.01\,\alpha ht + 0.04\,ht \quad (59ii)$$

$$[x_1(t)]_\alpha^L = \iint (0.01\,\alpha ht + 0.02\,ht)\,dt\,dt \quad (60i)$$

$$[x_1(t)]_\alpha^U = \iint (-0.01\,\alpha ht + 0.04\,ht)\,dt\,dt \quad (60ii)$$

$$[x_1(t)]_\alpha^L = 0.001667\,\alpha ht^3 + 0.003333\,ht^3 \quad (61i)$$

$$[x_1(t)]_\alpha^U = -0.001667\,\alpha ht^3 + 0.006667\,ht^3 \quad (61ii)$$

Now, to find $[x_2(t)]_\alpha = [\,[x_2(t)]_\alpha^L, [x_2(t)]_\alpha^U\,]$

From(29), we can find

$$[\theta(t;w)]_\alpha^L = [x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L + w^2[x_2(t)]_\alpha^L \quad (62i)$$

$$[\theta(t;w)]_\alpha^U = [x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U + w^2[x_2(t)]_\alpha^U \quad (62ii)$$

From(37), we can find

$$L([x_2(t)]^L_\alpha - [x_1(t)]^L_\alpha) = h[R_2]^L_\alpha \quad (63i)$$

$$L([x_2(t)]^U_\alpha - [x_1(t)]^U_\alpha) = h[R_2]^U_\alpha \quad (63ii)$$

Then from (38) , we can get :

$$[R_2]^L_\alpha = \frac{\partial[N(\theta(t;w))]^L_\alpha}{\partial w}\Big|_{w=0} \quad (64i)$$

$$[R_2]^U_\alpha = \frac{\partial[N(\theta(t;w))]^U_\alpha}{\partial w}\Big|_{w=0} \quad (64ii)$$

Then , we apply the following steps :

$$\frac{\partial[\theta(t;w)]^L_\alpha}{\partial t} = \frac{\partial[x_0(t)]^L_\alpha}{\partial t} + w\frac{\partial[x_1(t)]^L_\alpha}{\partial t} + w^2\frac{\partial[x_2(t)]^L_\alpha}{\partial t} \quad (65i)$$

$$\frac{\partial[\theta(t;w)]^U_\alpha}{\partial t} = \frac{\partial[x_0(t)]^U_\alpha}{\partial t} + w\frac{\partial[x_1(t)]^U_\alpha}{\partial t} + w^2\frac{\partial[x_2(t)]^U_\alpha}{\partial t} \quad (65ii)$$

$$\frac{\partial^2[\theta(t;w)]^L_\alpha}{\partial t^2} = \frac{\partial^2[x_0(t)]^L_\alpha}{\partial t^2} + w\frac{\partial^2[x_1(t)]^L_\alpha}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]^L_\alpha}{\partial t^2} \quad (66i)$$

$$\frac{\partial^2[\theta(t;w)]^U_\alpha}{\partial t^2} = \frac{\partial^2[x_0(t)]^U_\alpha}{\partial t^2} + w\frac{\partial^2[x_1(t)]^U_\alpha}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]^U_\alpha}{\partial t^2} \quad (66ii)$$

$$[N(\theta(x;w))]^L_\alpha = \frac{\partial^2[x_0(t)]^L_\alpha}{\partial t^2} + w\frac{\partial^2[x_1(t)]^L_\alpha}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]^L_\alpha}{\partial t^2} + [x_0(t)]^L_\alpha + w[x_1(t)]^L_\alpha + w^2[x_2(t)]^L_\alpha \quad (67i)$$

$$[N(\theta(x;w))]^U_\alpha = \frac{\partial^2[x_0(t)]^U_\alpha}{\partial t^2} + w\frac{\partial^2[x_1(t)]^U_\alpha}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]^U_\alpha}{\partial t^2} + [x_0(t)]^U_\alpha + w[x_1(t)]^U_\alpha + w^2[x_2(t)]^U_\alpha \quad (67ii)$$

$$\frac{\partial[N(\theta(t;w))]^L_\alpha}{\partial w} = \frac{\partial^2[x_1(t)]^L_\alpha}{\partial t^2} + 2w\frac{\partial^2[x_2(t)]^L_\alpha}{\partial t^2} + [x_1(t)]^L_\alpha + 2w[x_2(t)]^L_\alpha \quad (68i)$$

$$\frac{\partial[N(\theta(t;w))]^U_\alpha}{\partial w} = \frac{\partial^2[x_1(t)]^U_\alpha}{\partial t^2} + 2w\frac{\partial^2[x_2(t)]^U_\alpha}{\partial t^2} + [x_1(t)]^U_\alpha + 2w[x_2(t)]^U_\alpha \quad (68ii)$$

$$[R_2]^L_\alpha = \frac{\partial^2[x_1(t)]^L_\alpha}{\partial t^2} + [x_1(t)]^L_\alpha \quad (69i)$$

$$[R_2]^U_\alpha = \frac{\partial^2[x_1(t)]^U_\alpha}{\partial t^2} + [x_1(t)]^U_\alpha \quad (69ii)$$

$$[R_2]^L_\alpha = 0.010002\ \alpha ht + 0.019998\ ht + 0.001667\ \alpha ht^3 + 0.003333\ ht^3 , \quad (70i)$$

$$[R_2]^U_\alpha = -0.010002\ \alpha ht + 0.040002\ ht - 0.001667\ \alpha ht^3 + 0.006667\ ht^3 . \quad (70ii)$$

$L([x_2(t)]_\alpha^L - [x_1(t)]_\alpha^L) = 0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3$ (71i)

$L([x_2(t)]_\alpha^U - [x_1(t)]_\alpha^U) = -0.010002\ \alpha h^2 t + 0.040002\ h^2 t - 0.001667\ \alpha h^2 t^3 + 0.006667\ h^2 t^3$ (71ii)

$[x_2(t)]_\alpha^L - [x_1(t)]_\alpha^L = \iint (0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3)\ dtdt$,
(72i)

$[x_2(t)]_\alpha^U - [x_1(t)]_\alpha^U = \iint (-0.010002\alpha h^2 t + 0.040002 h^2 t - 0.001667\alpha h^2 t^3 + 0.006667 h^2 t^3)\ dtdt$.
(72ii)

$[x_2(t)]_\alpha^L - [x_1(t)]_\alpha^L =$

$0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3 + 0.000083\ \alpha h^2 t^5 + 0.000167\ h^2 t^5$, (73i)

$[x_2(t)]_\alpha^U - [x_1(t)]_\alpha^U = -0.001667\ \alpha h^2 t^3 + 0.006667\ h^2 t^3 - 0.000083\ \alpha h^2 t^5 + 0.000333\ h^2 t^5$.
(73ii)

$[x_2(t)]_\alpha^L = 0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3 + 0.000083\ \alpha h^2 t^5 + 0.000167\ h^2 t^5 + 0.001667\ \alpha h t^3 + 0.003333\ h t^3$ (74i)

$[x_2(t)]_\alpha^U = -0.001667\ \alpha h^2 t^3 + 0.006667\ h^2 t^3 - 0.000083\ \alpha h^2 t^5 + 0.000333\ h^2 t^5 - 0.001667\ \alpha h t^3 + 0.006667\ h t^3$ (74ii)

Now, to find $[x_3(t)]_\alpha = [\ [x_3(t)]_\alpha^L , [x_3(t)]_\alpha^U]$:

From(29), we can find

$[\theta(t;w)]_\alpha^L = [x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L + w^2[x_2(t)]_\alpha^L + w^3[x_3(t)]_\alpha^L$ (75i)

$[\theta(t;w)]_\alpha^U = [x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U + w^2[x_2(t)]_\alpha^U + w^3[x_3(t)]_\alpha^U$ (75ii)

From(37), we can find

$L([x_3(t)]_\alpha^L - [x_2(t)]_\alpha^L) = h[R_3]_\alpha^L$ (76i)

$L([x_3(t)]_\alpha^U - [x_2(t)]_\alpha^U) = h[R_3]_\alpha^U$ (76ii)

Then from(38), we can get :

$$[R_3]_\alpha^L = \frac{1}{2}\frac{\partial^2[N(\theta(t;w))]_\alpha^L}{\partial w^2}\bigg|_{w=0} \quad (77i)$$

$$[R_3]_\alpha^U = \frac{1}{2}\frac{\partial^2[N(\theta(t;w))]_\alpha^U}{\partial w^2}\bigg|_{w=0} \quad (77ii)$$

Then , we apply the following steps :

$$\frac{\partial[\theta(t;w)]_\alpha^L}{\partial t} = \frac{\partial[x_0(t)]_\alpha^L}{\partial t} + w\frac{\partial[x_1(t)]_\alpha^L}{\partial t} + w^2\frac{\partial[x_2(t)]_\alpha^L}{\partial t} + w^3\frac{\partial[x_3(t)]_\alpha^L}{\partial t} \quad (78i)$$

$$\frac{\partial[\theta(t;w)]_\alpha^U}{\partial t} = \frac{\partial[x_0(t)]_\alpha^U}{\partial t} + w\frac{\partial[x_1(t)]_\alpha^U}{\partial t} + w^2\frac{\partial[x_2(t)]_\alpha^U}{\partial t} + w^3\frac{\partial[x_3(t)]_\alpha^U}{\partial t} \quad (78ii)$$

$$\frac{\partial^2[\theta(t;w)]_\alpha^L}{\partial t^2} = \frac{\partial^2[x_0(t)]_\alpha^L}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^L}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + w^3\frac{\partial^2[x_3(t)]_\alpha^L}{\partial t^2} \quad (79i)$$

$$\frac{\partial^2[\theta(t;w)]_\alpha^U}{\partial t^2} = \frac{\partial^2[x_0(t)]_\alpha^U}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^U}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + w^3\frac{\partial^2[x_3(t)]_\alpha^U}{\partial t^2} \quad (79ii)$$

$$[N(\theta(x;w))]_\alpha^L = \frac{\partial^2[x_0(t)]_\alpha^L}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^L}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + w^3\frac{\partial^3[x_3(t)]_\alpha^L}{\partial t^2} + [x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L + w^2[x_2(t)]_\alpha^L + w^3[x_3(t)]_\alpha^L \quad (80i)$$

$$[N(\theta(x;w))]_\alpha^U = \frac{\partial^2[x_0(t)]_\alpha^U}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^U}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + w^3\frac{\partial^3[x_3(t)]_\alpha^U}{\partial t^2} + [x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U + w^2[x_2(t)]_\alpha^U + w^3[x_3(t)]_\alpha^U \quad (80ii)$$

$$\frac{\partial[N(\theta(t;w))]_\alpha^L}{\partial w} = \frac{\partial^2[x_1(t)]_\alpha^L}{\partial t^2} + 2w\frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + 3w^2\frac{\partial^3[x_3(t)]_\alpha^L}{\partial t^2} + [x_1(t)]_\alpha^L + 2w[x_2(t)]_\alpha^L + 3w^2[x_3(t)]_\alpha^L \quad (81i)$$

$$\frac{\partial[N(\theta(t;w))]_\alpha^U}{\partial w} = \frac{\partial^2[x_1(t)]_\alpha^U}{\partial t^2} + 2w\frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + 3w^2\frac{\partial^3[x_3(t)]_\alpha^U}{\partial t^2} + [x_1(t)]_\alpha^U + 2w[x_2(t)]_\alpha^U + 3w^2[x_3(t)]_\alpha^U \quad (81ii)$$

$$\frac{\partial^2[N(\theta(t;w))]_\alpha^L}{\partial w^2} = 2\frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + 6w\frac{\partial^3[x_3(t)]_\alpha^L}{\partial t^2} + 2[x_2(t)]_\alpha^L + 6w[x_3(t)]_\alpha^L \quad (82i)$$

$$\frac{\partial^2[N(\theta(t;w))]_\alpha^U}{\partial w^2} = 2\frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + 6w\frac{\partial^3[x_3(t)]_\alpha^U}{\partial t^2} + 2[x_2(t)]_\alpha^U + 6w[x_3(t)]_\alpha^U \quad (82ii)$$

$$[R_3]_\alpha^L = \frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + [x_2(t)]_\alpha^L \quad (83i)$$

$$[R_3]_\alpha^U = \frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + [x_2(t)]_\alpha^U \quad (83ii)$$

$$[R_3]_\alpha^L = 0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.003327\ \alpha h^2 t^3 + 0.006673\ h^2 t^3 + 0.010002\ \alpha h t + 0.019998\ h t + 0.000083\ \alpha h^2 t^5 +$$

$0.000167\ h^2t^5 + 0.001667\ \alpha ht^3 + 0.003333\ ht^3$
(84i)

$[R_3]_\alpha^U = -0.010002\ \alpha h^2 t + 0.040002\ h^2 t - 0.003327\ \alpha h^2 t^3 + 0.013327\ h^2t^3 - 0.010002\ \alpha ht + 0.040002\ ht - 0.000083\ \alpha h^2 t^5 + 0.000333\ h^2t^5 - 0.001667\ \alpha ht^3 + 0.006667\ ht^3$ (84ii)

$L([x_3(t)]_\alpha^L - [x_2(t)]_\alpha^L) = 0.010002\ \alpha h^3 t + 0.019998\ h^3 t + 0.003327\ \alpha h^3t^3 + 0.006673\ h^3t^3 + 0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.000083\ \alpha h^3 t^5 + 0.000167\ h^3t^5 + 0.001667\ \alpha h^2t^3 + 0.003333\ h^2t^3$
(85i)

$L([x_3(t)]_\alpha^U - [x_2(t)]_\alpha^U) = -0.010002\ \alpha h^3 t + 0.040002\ h^3 t - 0.003327\ \alpha h^3t^3 + 0.013327\ h^3t^3 - 0.010002\ \alpha h^2 t + 040002\ h^2 t - 0.000083\ \alpha h^3 t^5 + 0.000333\ h^3t^5 - 0.001667\ \alpha h^2t^3 + 0.006667\ h^2t^3$
(85ii)

$[x_3(t)]_\alpha^L - [x_2(t)]_\alpha^L = \iint (0.010002\ \alpha h^3 t + 0.019998\ h^3 t + 0.003327\ \alpha h^3t^3 + 0.00667\ h^3t^3 + 0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.000083\ \alpha h^3 t^5 + 0.000167\ h^3t^5 + 0.001667\ \alpha h^2t^3 + 0.003333\ h^2t^3)\ dt\,dt$ (86i)

$[x_3(t)]_\alpha^U - [x_2(t)]_\alpha^U = \iint (-0.010002\ \alpha h^3 t + 0.040002\ h^3 t - 0.003327\ \alpha h^3t^3 + 0.013327 h^3t^3 - 0.010002\ \alpha h^2 t + 0.040002\ h^2 t - 0.000083\ \alpha h^3 t^5 + 0.000333\ h^3t^5 - 0.001667\ \alpha h^2t^3 + 0.006667\ h^2t^3)\ dt\,dt$ (86ii)

$[x_3(t)]_\alpha^L - [x_2(t)]_\alpha^L = 0.001667\ \alpha h^3t^3 + 0.000333\ h^3t^3 + 0.000166\ \alpha h^3t^5 + 0.000334\ h^3t^5 + 0.001667\ \alpha h^2t^3 + 0.000333\ h^2t^3 + 0.000002\ \alpha h^3t^7 + 0.000004\ h^3t^7 + 0.000083\ \alpha h^2t^5 + 0.000167\ h^2t^5$
(87i)

$[x_3(t)]_\alpha^U - [x_2(t)]_\alpha^U = -0.001667\ \alpha h^3t^3 + 0.006667\ h^3t^3 - 0.000166\ \alpha h^3t^5 + 0.000666\ h^3t^5 - 0.001667\ \alpha h^2t^3 + 0.006667\ h^2t^3 - 0.000002\ \alpha h^3t^7 + 0.000008\ h^3t^7 - 0.000083\ \alpha h^2t^5 + 0.000333\ h^2t^5$
(87ii)

$[x_3(t)]_\alpha^L = 0.001667\ \alpha h^3t^3 + 0.000333\ h^3t^3 + 0.000166\ \alpha h^3t^5 + 0.000334\ h^3t^5 + 0.003334\ \alpha h^2t^3 + 0.003666\ h^2t^3 + 0.000002\ \alpha h^3t^7 + 0.000004\ h^3t^7 + 0.000334\ h^2t^5 + 0.000166\ \alpha h^2t^5 + 0.001667\ \alpha ht^3 + 0.003333\ ht^3$ (88i)

$[x_3(t)]_\alpha^U = -0.001667\ \alpha h^3t^3 + 0.006667\ h^3t^3 - 0.000166\ \alpha h^3t^5 + 0.000666\ h^3t^5 - 0.003334\ \alpha h^2t^3 + 0.013334\ h^2t^3 - 0.000002\ \alpha h^3t^7 +$

$0.000008\ h^3t^7 - 0.000166\ \alpha h^2t^5 + 0.000666\ h^2t^5 - 0.001667\ \alpha ht^3 + 0.006667\ ht^3$ (88ii)

Then, the fuzzy series solution is:

$$[x(t)]_\alpha = [\ [x(t)]_\alpha^L , [x(t)]_\alpha^U] \quad (89)$$

Where

$$[x(t)]_\alpha^L = 0.01\ \alpha t + 0.02\ t + 0.005001\ \alpha ht^3 + 0.009999\ ht^3 + 0.005001\ \alpha h^2t^3 + 0.006999\ h^2t^3 + 0.000249\ \alpha h^2t^5 + 0.000501\ h^2t^5 + 0.001667\ \alpha h^3t^3 + 0.000333\ h^3t^3 + 0.000166\ \alpha h^3t^5 + 0.000334\ h^3t^5 + 0.000002\ \alpha h^3t^7 + 0.000004\ h^3t^7 + \cdots \quad (90i)$$

$$[x(t)]_\alpha^U = -0.01\ \alpha t + 0.04\ t - 0.005001\ \alpha ht^3 + 0.020001\ ht^3 - 0.005001\ \alpha h^2t^3 + 0.020001\ h^2t^3 - 0.000249\ \alpha h^2t^5 + 0.000999\ h^2t^5 - 0.001667\ \alpha h^3t^3 + 0.006667\ h^3t^3 - 0.000166\ \alpha h^3t^5 + 0.000666\ h^3t^5 - 0.000002\ \alpha h^3t^7 + 0.000008\ h^3t^7 + \cdots \quad (90ii)$$

The fuzzy series solution at $h = -1$, will be

$$[x(t)]_\alpha = [\ [x(t)]_\alpha^L , [x(t)]_\alpha^U] \quad (91)$$

Where

$$[x(t)]_\alpha^L =$$

$$0.01\ \alpha t + 0.02\ t - 0.001667\ \alpha t^3 - 0.003333\ t^3 + 0.000083\ \alpha t^5 + 0.000167\ t^5 - 0.000002\ \alpha t^7 - 0.000004\ t^7 + \cdots \quad (92i)$$

$$[x(t)]_\alpha^U = -0.01\ \alpha t + 0.04\ t + 0.001667\ \alpha t^3 - 0.006667\ t^3 - 0.000083\ \alpha t^5 + 0.000333\ t^5 + 0.000002\ \alpha t^7 - 0.000008\ t^7 + \cdots$$
(92ii)

7. Discussion

When solving a fuzzy autonomous differential equation by using the fuzzy homotopy analysis method , the accuracy of the results depends greatly on the value of the parameter h, other factors also affect, including : the number of terms of the solution series, the value of the constant α and the period to which the variable t belongs. The fuzzy semi-analytical solutions that we obtained during this work are accurate solutions and very close to the fuzzy exact-analytical solutions, based on the comparison that we will make between the results that we obtained and the fuzzy exact-analytical solutions to the chosen problem.

If we go back to example(1) :

$x''(t) + x(t) = 0\ ,\ \ t \in [0\ ,\ \ 0.5]$ (93)

The fuzzy exact-analytical solution for this problem is :

$[x(t)]_\alpha = [\ [x(t)]_\alpha^L\ , [x(t)]_\alpha^U\ ]$ (94)

Where

$[x(t)]_\alpha^L = (\ 0.02 + 0.01\alpha\ )\sin t$ (95i)

$[x(t)]_\alpha^U = (\ 0.04 - 0.01\alpha\ )\sin t$ (95ii)

While the fuzzy semi-analytical solution that we got(at $h = -1$, $\alpha = 0.3$) is :

$[x(t)]_\alpha = [\ [x(t)]_\alpha^L\ , [x(t)]_\alpha^U\ ]$ (96)

Where

$[x(t)]_\alpha^L = 0.023\ t - \ 0.003833\ t^3\ +\ 0.000122\ t^5\ -\ 0.000005\ t^7\ \ + \cdots$, (97i)

$[x(t)]_\alpha^U =\ 0.037\ t - \ 0.006167\ t^3\ +\ 0.000308\ t^5\ -\ 0.000007\ t^7\ \ + \cdots$. (97ii)

Also, the fuzzy semi-analytical solution that we got(at $h = -1$, $\alpha = 0.4$) is :

$[x(t)]_\alpha = [\ [x(t)]_\alpha^L\ , [x(t)]_\alpha^U\ ]$ (98)

Where

$[x(t)]_\alpha^L = 0.024\ t - \ 0.004\ t^3\ +\ 0.0002\ t^5\ -\ 0.000005\ t^7\ \ + \cdots$ (99i)

$[x(t)]_\alpha^U =\ \ 0.036\ t - \ 0.006\ t^3\ +\ 0.0003\ t^5\ -\ 0.000007\ t^7\ \ + \cdots$ (99ii)

We test the accuracy of the obtained solutions by computing the absolute errors

$[error]_\alpha^L\ = |\ [x_{exact}(t)]_\alpha^L\ -\ [x_{series}(t)]_\alpha^L\ |$ (100i)

$[error]_\alpha^U\ = |\ [x_{exact}(t)]_\alpha^U\ -\ [x_{series}(t)]_\alpha^U\ |$ (100ii)

The following tables provides a comparison between the fuzzy exact-analytical solution and the fuzzy semi- analytical solution for this problem.

t	$[x_{series}(t)]_\alpha^L$	$[error]_r^L$	$[x_{series}(t)]_\alpha^U$	$[error]_r^U$

0	0	0	0	0
0.05	0.001149520	1.99 e-11	0.001849229	4.18 e-11
0.10	0.002296168	3.63 e-10	0.003693836	3.37 e-10
0.15	0.003437072	4.00 e-9	0.005529209	1.00 e-9
0.20	0.004569374	1.90 e-8	0.007350762	2.00 e-9
0.25	0.005690228	6.20 e-8	0.009153940	5.00 e-9
0.30	0.006796804	1.60 e-7	0.010934237	9.00 e-9
0.35	0.007886297	3.51 e-7	0.012687203	1.50 e-8
0.40	0.008955929	6.92 e-7	0.014408454	2.40 e-8
0.45	0.010002950	1.26 e-6	0.016093689	3.50 e-8
0.50	0.011024648	2.14 e-6	0.017738695	4.90 e-8

Table 1. Comparison of the results of example(1), $\alpha = 0.3$.

t	$[x_{series}(t)]_{\alpha}^{L}$	$[error]_{r}^{L}$	$[x_{series}(t)]_{\alpha}^{U}$	$[error]_{r}^{U}$
0	0	0	0	0
0.05	0.001199500	0	0.001799250	0
0.10	0.002396002	0	0.003594002	0
0.15	0.003586515	4.20 e-13	0.005379772	2.30 e-13
0.20	0.004768063	3.08 e-12	0.007152095	1.78 e-12
0.25	0.005937695	1.48 e-11	0.008906542	8.32 e-12
0.30	0.007092484	5.34 e-11	0.010638727	2.93 e-11
0.35	0.008229547	1.58 e-10	0.012344321	8.41 e-11
0.40	0.009346039	4.07 e-10	0.014019060	2.08 e-10
0.45	0.010439171	9.40 e-10	0.015658759	4.59 e-10
0.50	0.011506210	1.00 e-9	0.017259320	9.23 e-10

Table 2. Comparison of the results of example(1), $\alpha = 0.4$.

8. Conclusion

In this work, we have studied the fuzzy approximate-analytical solutions of the second order fuzzy autonomous differential equation. Obviously the accuracy of the results that can be obtained when solving using the fuzzy homotopy analysis method, these results may improve further when increasing the number of terms of the solution series or using another value for the parameter h. The value of the variable t greatly affects the accuracy of the results, if the value of the variable t is close to the initial value, the results will be more accurate. Also, the value of the constant α greatly affects the accuracy of the results. Certainly, the best value of the constant α cannot be determined, as it changes from one problem to another.

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On δ-preregular e*-open sets in topological spaces

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Abstract

In this paper, we introduce a new class of sets called, δ-preregular e^*-open sets and investigate their properties and characterizations. By using δ-preregular e^*-open sets, we obtain decompositions of complete continuity and decompositions of perfect continuity.

Keywords: δ-preopen;e^*-open;e^*-closed;δpe^*-open;δpe^*-closed;δpe^*-continuity.

2020 AMS subject classifications: $54A05$,$54C08$. [1]

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1 Introduction

The study of δ-open sets was initiated by Veličko[Velicko, 1968] in 1968. Following this Raychaudhuri and Mukherjee[Raychaudhuri and Mukherjee, 1993] established the concept of δ-preopen sets. Later, Ekici[Ekici, 2009] introduced the concept of e^*-open sets as a generalization of e-open sets. The aim of this paper is to introduce and study a new class of sets called, δ-preregular e^*-open sets using δ-preinterior and e^*-closure operators. The notion of δpe*-continuity is also introduced which is stronger than δ-precontinuity.Finally, we obtain decompositions of complete continuity and decompositions of perfect continuity.

Throughout this paper, (U,τ) and (V,η)(or simply U and V) represent topological spaces on which no separation axioms are assumed unless explicitly stated and f:(U,τ)$\rightarrow$(V,η) or simply f:U $\rightarrow$ V denotes a function f of a topological space U into a topological space V. Let N $\subseteq$ U, then
cl(N) = $\cap\{$F: N $\subseteq$ F and $F^c \in \tau\}$ is the closure of N and
int(N) = $\cup\{$O: O $\subseteq$ N and O $\in \tau\}$ is the interior of N.

2 Preliminaries

Definition 2.1. *A set M $\subseteq$ U is called δ-closed[Velicko, 1968] if M = δ-cl(M) where δ-cl(M)=$\{$p$\in$U:int(cl(G))$\cap$M$\neq\phi$,G$\in\tau$ and p$\in$G $\}$.*

Definition 2.2. *A set M $\subseteq$ U is called*
(1) e-open[Ekici, 2008c] if M $\subseteq$ cl(δ-int(M))$\cup$int(δ-cl(M)) and e-closed if cl(δ-int(M))$\cap$int(δ-cl(M))$\subseteq$M.
(2) a-open[Ekici, 2008d] if M $\subseteq$ int(cl(δ-int(M))) and a-closed if cl(int(δ-cl(M)))$\subseteq$M.
(3)e^*-open[Ekici, 2009] if M $\subseteq$ cl(int(δ-cl(M))) and e^*-closed if int(cl(δ-int(M)))$\subseteq$M .
(4)δ-semiopen[Park et al., 1997] if M$\subseteq$cl(δ-int(M))) and δ-semiclosed if int(δ-cl(M))$\subseteq$M).
(5)δ-preopen[Raychaudhuri and Mukherjee, 1993] if M$\subseteq$int(δ-cl(M)) and δ-preclosed if cl(δ-int(M))$\subseteq$M.
(6)regular-open[Stone, 1937] if M = int(cl(M)) and regular-closed if M=cl(int(M)).

Definition 2.3. *[Ekici, 2008b] A subet M of a space U is said to be a δ-dense set if δ-cl(M)=U.*

The class of open(resp,closed, regular open,δ-preopen, δ-semiopen, e^*-open and clopen) sets of (U,τ) is denoted by O(U) (resp,C(U), RO(U), δPO(U), δSO(U), e^*O(U) and CO(U)).

Theorem 2.1. *[Raychaudhuri and Mukherjee, 1993] Let M be a subset of a space (U,τ), then δ-pcl(M)=M$\cup$cl(δ-int(M)) and δ-pint(M)=M$\cap$int(δ-cl(M)).*

Theorem 2.2. *[Ekici, 2009]Let M be a subset of a space (U,τ),then:*
(i) e^*-cl(M) = M$\cup$ int(cl(δ-int(M)) and e^*-int(M) = M$\cap$cl(int(δ-cl(M))
(ii) int(cl(δ-int(M))=e^*-cl(δ-int(M))=δ-int(e^*-cl(M)).

Theorem 2.3. *Let M be a subset of a space (U,τ),then:*
(i)δ-pint(e^*-cl(M))=e^*-cl(M)$\cap$ int(δ-cl(M)).
(ii)δ-pint(e^*-cl(M))=δ-pint(M)$\cup$ int(cl(δ-int(M)).
(iii)δ-pint(e^*-cl(M))=δ-pint(M)$\cup$ e^*-cl(δ-int(M))
(iv)δ-pint(e^*-cl(M))=δ-pint(M)$\cup$ δ-int(e^*-cl(M)).
(v) δ-pint(e^*-cl(M)) = (M$\cap$int(δ-cl(M))$\cup$ int(cl(δ-int(M))

Lemma 2.1. *[Benchalli et al., 2017]For a subset M of a space (U,τ),the following are equivalent:*
(a)M is clopen;
(b)M is δ-open and δ-closed;
(c)M is regular-open and regular-closed.

Definition 2.4. *[Kohli and Singh, 2009] A space (U,τ) is called δ-partition if δO(U)=C(U).*

Definition 2.5. *[Caldas and Jafari, 2016] A space (U,τ) is a δ-door space if every subset of U is δ-open or δ-closed.*

Theorem 2.4. *[Caldas and Jafari, 2016] If (U,τ) is a δ-door space, then every δ-preopen set in (U,τ) is δ-open.*

3 δ-preregular e^*-open sets in topological spaces

Definition 3.1. *A subset N of a space (U,τ) is said to be δ-preregular e^*-open(briefly δpe^*-open) if N =δ-pint(e^*-cl(N)). The complement of a δ-preregular e^*-open is called a δ-preregular e^*-closed(briefly δpe^*-closed) set.*
Clearly, N is δpe^*-closed if and only if N = δ-pcl(e^*-int(N))

The class of δpe^*-open (resp,δpe^*-closed) sets of (U,τ) will be denoted by δPE^*O(U)(resp,δPE^*C(U)).

Theorem 3.1. *Let (U,τ) be a topological space and M, N $\subseteq$ U. Then the following hold:*
(i) If M $\subseteq$ N, then δ-pint(e^*-cl(M) $\subseteq$ δ-pint(e^*-cl(N)).
(ii) If M $\in\delta$PO(U), then M $\subseteq$ δ-pint(e^*-cl(M)).
(iii) If M $\in e^*$C(U), then e^*-cl(δ-pint(M)) $\subseteq$ M.

(iv) δ-pint(e^*-cl(N)) is δpe^*-open
(v) If M $\in e^*$C(U), then δ-pint(M) is δpe^*-open..
Proof:*(i)Obvious.*
(ii) Let M $\in$ δPO(U). As M $\subseteq$ e^*-cl(M),then M $\subseteq$ δ-pint(e^*-pcl(M).
(iii) Let M $\in$ e^*C(U). Since δ-pint(M) $\subseteq$ M, then e^*-cl(δ-pint(M)) $\subseteq$ M.
(iv) We have
δ-pint(e^*-cl(δ-pint(e^*-cl(M)) $\subseteq$ δ-pint(e^*-cl(e^*-cl(M)) = δ-pint(e^*-cl(M) and
δ-pint(e^*-cl(δ-pint(e^*-cl(M))) $\supseteq$ δ-pint(δ-pint(e^*-cl(M)) = δ-pint(e^*-cl(M).
Hence δ-pint(e^*-cl(δ-pint(e^*-cl(M))) = δ-pint(e^*-cl(M).
(v) Suppose that M $\in$ e^*C(U). By (i),
δ-pint(e^*-cl(δ-pint(M))$\subseteq$ δ-pint(e^*-cl(M)=δ-pint(M).
On the other hand, we have
δ-pint(M) $\subseteq$ e^*-cl(δ-pint(M) so that
δ-pint(M) $\subseteq$ δ-pint(e^*-cl(δ-pint(M)).
Therefore δ-pint(e^*-cl(δ-pint(M))=δ-pint(M).
This shows that δ-pint(M) is δpe^*-open.

Theorem 3.2. *(i)Every δpe^*-open set is δ-preopen(hence e-open,e^*-open).*
(ii)Every δpe^*-open set is e^*-closed..
Proof: *(i)Let M be δpe^*-open,then by Theorem 2.3(i),*
δ-pint(e^*-cl(M))=e^*-cl(M)$\cap$int(δ-cl(M).
Therefore, M $\subseteq$int(δ-cl(M), M is δ-preopen.
(ii)Let N be δpe^*-open.By Theorem 2.3(ii), N =δ-pint(N)$\cup$int(cl(δ-int(N))).
Therefore,int(cl(δ-int(N)))$\subseteq$N. Thus N is e^*-closed.

Remark 3.1. *By the following example,we show that every δ-preopen(resp,e^*-closed) set need not be a δpe^*-open set*

Example 3.1. *Let U = $\{a,b,c,d\}$ and τ = $\{U, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. Then $\{a,b,c\}$ is a δ-preopen set but $\{a,b,c\} \notin \delta PE^*O(U)$ and $\{d\}$ is e^*-closed but $\{d\} \notin \delta PE^*O(U)$it is not δpe^*-open*

Corolary 3.1. *For a topological space (U,τ), we have*
δ-PO(U)$\cap\delta$-PC(U) $\subseteq$ $\delta PE^*O(U) \subseteq$ e^*O(U)$\cap e^*$C(U).
Proof: *Obvious.*
The converse inclusions in the above corollary need not be true as seen from the following example

Example 3.2. *Let (U,τ) as in Example 3.1,then $\{b\}$ is δpe^*-open but it is not δ-preclopen. Moreover, $\{a,d\}$ is e^*-clopen but not δpe^*-open*

Remark 3.2. *The notions of δpe^*-open sets and δ-open sets (hence a-open sets, δ-semiopen sets, δ^*-sets) are independent of each other.*

Example 3.3. *Consider (U,τ) as in Example 3.1.The set $\{a\}$ is δpe^*-open but it is not δ^*-set. Moreover, $\{a,b,c\}$ is δ-open but not δpe^*-open*

Theorem 3.3. *In a δ-partition space (U,τ), a subset M of U is δpe^*-open if and only if it is δ-preopen.*
Proof: *Necessity:It follows from Theorem 3.2(i) .*
Sufficiency:Let N be δ-preopen. Since (U,τ) is δ-partition and by Theorem 2.3(ii), we have δ-pint(e^*-cl(M)) = δ-pint(M)$\cup$ int(cl(δ-int(M))

$$= M \cup int(cl(cl(M))$$
$$= M \cup int(cl(M)$$
$$= M \cup \delta\text{-}int(cl(M)$$
$$= M \cup \delta\text{-}int(\delta\text{-}int(M)$$
$$= M \cup \delta\text{-}int(M) = M$$

Therefore, δ-pint(e^*-cl(M)) = M.Hence M is δpe^*-open.

Theorem 3.4. *A subset $N \subseteq U$ is δpe^*-open if and only if N is e^*-closed and δ-preopen.*
Proof: *Necessity:It follows from Theorem 3.2.*
Sufficiency:Let N be both e^*-closed and δ-preopen. Then N = e^*-cl(N) and N = δ-pint(N). Therefore, δ-pint(e^*-cl(N)) = δ-pint(N) = N. Hence N is δpe^*-open.

Remark 3.3. *The class of δpe^*-open sets is not closed under finite union as well as finite intersection. It will be shown in the following example.*

Example 3.4. *Consider (U,τ) as in Example 3.1. Let A = $\{a,c\}$ and B = $\{b,c\}$,the A and B are δpe^*-open sets but $A \cup B = \{a,b,c\} \notin \delta PE^*O(U)$.*
Moreover,C = $\{a,b,d\}$ and D = $\{b,c,d\}$ are δpe^*-open sets but
$C \cap D = \{b,d\} \notin \delta PE^*O(U)$.

Theorem 3.5. *For a subset M of a space (U,τ),the following are equivalent:*
(i) M is δpe^*-open.
(ii) M = e^*-cl(M)$\cap$ int(δ-cl(M)).
(iii) M = δ-pint(M)$\cup$ int(cl(δ-int(M)).
(iv)M = δ-pint(M)$\cup$ e^*-cl(δ-int(M))
(v) M = δ-pint(M)$\cup$ δ-int(e^*-cl(M)).
(vi) M = (M$\cap$int(δ-cl(M))$\cup$ int(cl(δ-int(M)).
Proof:*It follows from Theorem 2.3*

Theorem 3.6. *In any space (U,τ) , the empty set is the only subset which is nowhere δ-dense and δpe^*-open.*
Proof: *Suppose M is nowhere δ-dense and δpe^*-open. Then by Theorem 2.3(i), M = δ-pint(e^*-cl(M)) =e^*-cl(M)$\cap$ int(δ-cl(M)= e^*-cl(M)$\cap\phi$ = ϕ.*

Lemma 3.1. *If (U,τ) is a δ-door space, then any finite intersection of δ-preopen sets is δ-preopen.*
Proof:*Obvious since δO(X) is closed under finite intersection.*

Theorem 3.7. *If (U,τ) is a δ-door space, then any finite intersection of δpe^*-open sets is δpe^*-open.*
Proof:*Let $\{A_i$:i=1,2,...,n$\}$ be a finite family of δpe^*-open. Since the space (U,τ) is δ-door, then by Lemma 3.1, we have $\bigcap_{i=n}^{n} A_i \in \delta PO(U)$.*
By Theorem 3.1(ii), $\bigcap_{i=n}^{n} A_i \subseteq$ δ-pint(e^*-cl($\bigcap_{i=n}^{n} A_i$)).
For each i, we have $\bigcap_{i=n}^{n} A_i \subseteq A_i$ and thus δ-pint(e^*-cl($\bigcap_{i=n}^{n} A_i$)$\subseteq$ δ-pint(e^*-cl(A_i) = A_i. Therefore, δ-pint(e^*-cl($\bigcap_{i=n}^{n} A_i$)$\subseteq \bigcap_{i=n}^{n} A_i$.

Lemma 3.2. *If a subset M of a space (U,τ) is regular open,then M = int(cl(M)=int(δ-cl(M)).*

Theorem 3.8. *Every regular open set is δpe^*-open.*
Proof: *Let M be regular open. Then M=int(cl(M))=int(δ-cl(M)). By Theorem 2.6(i), δ-pint(e^*-cl(M)) = e^*-cl(M)$\cap$int(δ-cl(M))=e^*-cl(M)$\cap$M=M. This shows that M is δpe^*-open.*

Definition 3.2. *A subset M of a space (U,τ) is called δ^*-set if int(δ-cl(M))$\subseteq$cl(δ-int(M))*

Theorem 3.9. *(i) Every δ-semiopen set is δ^*-set.*
(ii)Every δ-semiclosed set is δ^*-set.
Proof:*Clear*

Definition 3.3. *A subset M of a space (U,τ) is called*
b^*-open if M = cl(δ-int(M))$\cup$ int(δ-cl(M)).
b^*-closed if M = cl(δ-int(M))$\cap$ int(δ-cl(M))

Theorem 3.10. *A subset M of a space (U,τ) is regular open if and only if it is b^*-closed.*
Proof:*Let M be regular open. Then by Lemma 3.2, M = int(cl(M)=int(δ-cl(M)). Since every regular open set is δ-open, we have cl(δ-int(M))$\cap$ int(δ-cl(M)) = cl(M)$\cap$ M = M. Hence A is b^*-closed.*
Conversely, let M be b^*-closed.Then int(cl(δ-int(M))$\subseteq$int(δ-cl(δ-int(M))$\subseteq$ cl(δ-int(M))$\cap$ int(δ-cl(M))=M. By Definition 3.3, we have M $\subseteq$ int(δ-cl(M)) $\subseteq$ int(δ-cl(cl(δ-int(M))) = int(cl(cl(δ-int(M))) = int(cl(δ-int(M))).
Therefore, M = int(cl(δ-int(M)). Now, int(cl(M)) = int(cl(int(cl(δ-int(M))) = int(cl(δ-int(M)) = M. Hence M is regular open.

Theorem 3.11. *(i) Every b^*-closed set is δ-preopen.*
(ii)Every b^*-closed set is δ-semiopen.
(iii)Every b^*-closed set is δpe^*-open.
Proof:*(i) and (ii) are obvious*
(iii)Let M be b^*-closed,then we have M = int(cl(δ-int(M)). Then δ-pint(e^*-cl(M)) = δ-pint(M) $\cup$ int(cl(δ-int(M)) = δ-pint(M) $\cup$ M = M.Hence M is δpe^*-open

Remark 3.4. *The above discussions can be summarized in the following diagram:*

DIAGRAM

$$\begin{array}{ccccccccc}
\text{regular open} & \longrightarrow & \delta\text{-open} & \longrightarrow & a\text{-open} & \longrightarrow & \delta\text{-semiopen} & \longrightarrow & \delta^*\text{-set} \\
\updownarrow & & & & & & \downarrow & & \downarrow \\
b^*\text{-closed} & \longrightarrow & \delta pe^*\text{-open} & \longrightarrow & \delta\text{-preopen} & \longrightarrow & e\text{-open} & \longrightarrow & e^*\text{-open}
\end{array}$$

Theorem 3.12. *For a subset M of a space (U,τ), the following are equivalent:*
(i) M is regular open;
(ii) M is δpe^*-open and δ-open;
(iii) M is δpe^*-open and a-open;
(iv) M is δpe^*-open and δ-semiopen;
(v) M is δpe^*-open and δ^*-set.
Proof: *(i) $\longrightarrow$(ii)$\longrightarrow$(iii)$\longrightarrow$(iv)$\longrightarrow$(v):Follows from the above diagram*
(v)$\longrightarrow$(i):Let M be δpe^*-open and δ^*-set.Then int(δ-cl(M))$\subseteq$cl(δ-int(M)) and int(δ-cl(M))$\subseteq$ int(cl(δ-int(M))$\subseteq$int(δ-cl(δ-int(M))$\subseteq$int(δ-cl(M)).
Therefore we have int(δ-cl(M))=int(cl(δ-int(M)).
Since M is δpe^*-open, M = δ-pint(δ-pcl(M))
=(M$\cup$int(cl(δ-int(M))$\cap$int(δ-cl(M))
=int(δ-cl(M)$\cap$int(δ-cl(M))
=int(δ-cl(M)).
Therefore M =int(δ-cl(M))=int(cl(M)) and hence M is regular open.

Theorem 3.13. *For a subset M of a space (U,τ), the following are equivalent:*
(i) M is regular open.
(ii) M is δpe^*-open and δ-semiclosed.
(iii) M is e^*-closed and a-open.
Proof: *(i)$\longrightarrow$(ii):It follows from Theorem 3.8*
(ii)$\longrightarrow$(i):Let M be δpe^*-open and δ-semiclosed. Since every δ-semiclosed set is δ^*-set. Hence by Theorem 3.12(v), M is regular open.
(ii) $\longrightarrow$(iii):Clear
(i)$\longleftrightarrow$(iii):It is shown in Theorem 3 [Ekici, 2008b]

Corolary 3.2. *For a subset M of a space (U,τ), the following are equivalent:*
(i) M is regular open;
(ii) M is δpe^*-open and δ-open;

(iii) M is δpe^*-open and a-open;
(iv) M is δpe^*-open and δ-semiopen;
(v) M is δpe^*-open and δ^*-set;.
(vi) M is δpe^*-open and δ-semiclosed;
(vii) M is e^*-closed and a-open;
(viii) M is b^*-closed.

Theorem 3.14. *For a subset M of a space (U,τ), the following are equivalent:*
(i) M is clopen;
(ii) M is δ-open and δ-closed;
(iii) M is regular open and regular closed;
(iv) M is δpe^*-open and δ-closed.
Proof: *(i) $\longleftrightarrow$(ii)$\longleftrightarrow$(iii):Follows from Lemma 2.1*
(iii)$\longrightarrow$(iv). It follows from Theorem 3.8
(iv)$\longrightarrow$(ii)Let M be δpe^*-open and δ-closed.By Theorem 2.3(i) , we have N = e^*-cl(N) $\cap$ int(δ-cl(N)) = e^*-cl(N) $\cap\delta$-int(δ-cl(N))=δ-pcl(N)$\cap\delta$-int(N)=δ-int(N). Therefore M is δ-open.

4 Decompositions of complete continuity

In this section, the notion of regular δ-preopen continuity is introduced and the decompositions of complete continuity are discussed.

Definition 4.1. *A function f:(U,τ)$\rightarrow$(V,σ) is said to be*
(i) δpe^*-continuous if the inverse image of every open subset of (V,σ) is δpe^*-open set in (U,τ).
(ii)perfectly continuous[Noiri, 1984] (resp,e-continuous[Ekici, 2008c], e^*-continuous[Ekici, 2009], δ-almost continuous[Raychaudhuri and Mukherjee, 1993], δ^*-continuous, contra-super-continuous[Jafari and Noiri, 1999], completely continuous[Arya and Gupta, 1974], RC-continuous[Dontchev and Noiri, 1998], super-continuous[Munshi and Bassan, 1982], contra continuous[Dontchev, 1996], a-continuous[Ekici, 2008d], δ-semicontinuous[Noiri, 2003], contra e^*-continuous[Ekici, 2008a], contra δ-semicontinuous[Ekici, 2004], contra b^*-continuous) if the inverse image of every open subset of (V,σ) is clopen (resp,e-open,e^*-open,δ-preopen,δ^*-set, δ-closed, regular open, regular closed, δ-open, closed, a-open, δ-semiopen, e^*-closed, δ-semiclosed, b^*-closed) set in (U,τ)

By Theorems 3.9 and 3.11, we obtain the following theorem.

Theorem 4.1. *(i) Every contra b^*-continuous set is δ-almost continuous.*
(ii)Every contra b^*-continuous set is δ-semicontinuous

(iii)Every contra b^*-continuous set is δpe^*-continuous.
(iv) Every δ-semicontinuous set is δ^*-continuous.
(v)Every contra δ-semicontinuous is δ^*-continuous.

Remark 4.1. *By Diagram I, we have the following diagram:*

DIAGRAM II

$$\begin{array}{ccccccccc}
c.cont. & \longrightarrow & s.cont. & \longrightarrow & a.cont. & \longrightarrow & \delta s.cont. & \longrightarrow & \delta^*.cont. \\
\updownarrow & & & & & & \downarrow & & \downarrow \\
cb^*.cont. & \longrightarrow & \delta pe^*.cont. & \longrightarrow & \delta p.cont. & \longrightarrow & e.cont. & \longrightarrow & e^*.cont.
\end{array}$$

where c.cont.=completely continuity, s.cont.=super continuity, a.cont.=a-continuity, δs.cont.=δ-semicontinuity, δ^*.cont.=δ^*-continuity, cb^*.cont.=contra b^*-continuity, δpe^*.cont.=δ-preregular e^*-continuity, δp.cont.=δ-precontinuity, e.cont.=e-continuity, e^*.cont.=e^*-continuity

Theorem 4.2. *For a function f:$(U,\tau)\to(V,\eta)$, the following are equivalent:*
(i) f is completely continuous;
(ii)f is δpe^*-continuous and super continuous;
(iii)f is δpe^*-continuous and a-continuous;
(iv) f is contra e^*-continuous and a-continuous;
(v)f is δpe^*-continuous and δ-semicontinuous;
(vi)f is δpe^*-continuous and contra δ-semicontinuous;
(vii)f is δpe^*-continuous and δ^*-continuous;
(viii) f is contra b^*-continuous.

Remark 4.2. *(i) δpe^*-continuity and super-continuity(hence a-continuity,δ-semicontinuity, δ^{**}-continuity) are independent notions.*
(ii) δpe^*-continuity and contra δ-semicontinuity are independent notions.

Example 4.1. *Let (U,τ) be a space as in Example 3.1 and let $\eta = \{U, \phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$*
(i) Define f:$(U,\tau) \to (U,\eta)$ by f(a) = f(c) = a , f(b) = b and f(d) = d. Clearly f is super-continuous but for $\{a,b\}\in O(V)$, $f^{-1}(\{a,b\}) = \{a,b,c\} \notin \delta PE^*O(U)$. Therefore f is not δpe^*-continuous.
Define g:$(U,\tau) \to (U,\eta)$ by g(a) = b, g(b) = g(c) = g(d) = a.Then g is δpe^*-continuous but for $\{a\} \in O(V)$, $g^{-1}(\{a\}) = \{b,c,d\} \notin q^*O(U)$. Therefore g is not q^*-continuous.
(ii)Define f:$(U,\tau) \to (U,\eta)$ by f(a) = f(c) = f(d) = b and f(b) = a. Clearly f is δ-semiregular-continuous but for $\{b\}\in O(V)$, $f^{-1}(\{b\}) = \{a,c,d\} \notin \delta PE^*PO(U)$. Therefore f is not δpe^*-continuous.
Define g:$(U,\tau) \to (U,\eta)$ by g(a) = g(b)=g(d)=a,g(c) = b.Then g is δpe^*-continuous

but for $\{a\} \in O(V)$, $g^{-1}(\{a\}) = \{a,b,d\} \notin \delta SC(U)$*.Therefore g is not contra* δ*-semicontinuous.*

5 Decompositions of perfectly continuity

In this section, the decompositions of perfectly continuity are obtained.

Theorem 5.1. *For a function* $f:(U,\tau) \rightarrow (U,\eta)$*, the following are equivalent:*
(i) f is perfectly continuous;
(ii) f is super continuous and contra super continuous;
(iii) f is completely continuous and RC-continuous;
(iv) f is δpe^**-continuous and contra super continuous.*
Proof: *It is a direct consequence of Theorem 3.14*

Remark 5.1. *As shown by the following examples,*δpe^**-continuity and contra super continuity are independent of each other.*

Example 5.1. *Consider* (U,τ) *as in Example 3.1 and* (U,η) *as in Example 4.1. Define* $f: (U,\tau) \rightarrow (U,\eta)$ *by* $f(a) = f(c) = f(d) = a$ *and* $f(b) = c$*. Then f is contra super continuous but it is not* δpe^**-continuous since* $\{a\} \in O(V)$, $f^{-1}(\{a\}) = \{a,c,d\} \notin \delta PE^* O(U)$*. Define* $g: (U,\tau) \rightarrow (U,\eta)$ *by* $g(a) = b$, $g(b) = g(c) = g(d) = a$*.Then g is* δpe^**-continuous but it is not contra super continuous since* $\{a\} \in O(V)$, $g^{-1}(\{a\}) = \{b,c,d\} \notin \delta C(U)$*.*

6 Conclusions:

The notions of sets and functions in topological spaces and fuzzy topological spaces are extensively developed and used in many engineering problems, information systems, particle physics, computational topology and mathematical sciences. By researching generalizations of closed sets, some new continuity have been founded and they turn out to be useful in the study of digital topology. Therefore, δpe*-continuous functions defined by δpe*-open sets will have many possibilities of applications in digital topology and computer graphics.

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About two countable families in the finite sets of the Collatz Conjecture

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Abstract

With $\boldsymbol{t} \in \mathbb{N}$ we define the sets $\boldsymbol{K_t}$ and $\boldsymbol{K_t^*}$ containing all positive integers that converge to 1 in *t* iterations in the form of Collatz algorithm. The following are the properties of the $\{\boldsymbol{K_t}\}_{\boldsymbol{t}\in\mathbb{N}}$ and $\{\boldsymbol{K_t^*}\}_{\boldsymbol{t}\in\mathbb{N}}$: countability, empty intersection between the elements of the same family, and - at the end of the work - we conjecture that both of the two families are a partition of $\mathbb{N}_{\boldsymbol{0}}$. We demonstrate also that each set $\boldsymbol{K_t}$ and $\boldsymbol{K_t^*}$ is the union of two sets, a set includes even positive integers, the other, if it is non-empty, includes odd positive integers different from 1 and we go on proving that the maximum of each set $\boldsymbol{K_t}$ and $\boldsymbol{K_t^*}$ is 2^t and that $\boldsymbol{K_t} \cap \boldsymbol{K_t^*} = \{\boldsymbol{2^t}\}$.

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1 Introduction

Let us consider the Collatz conjecture (Leggerini, 2004), also known as the $3n + 1$ problem. We start from a positive integer *n*, if it is even we divide it by two, if it is odd we multiply it by three and add one to it, then we start over by applying the same rules on the number obtained. For example, starting from 3 the sequence is generated: 3, 10, 5, 16, 8, 4, 2, 1. In the second form of the algorithm of $3n + 1$ we calculate $\frac{3n+1}{2}$ if *n* is odd. With 3 we obtain the sequence 3, 5, 8, 4, 2, 1. It is conjectured that, from any positive integer we start, the sequences always arrive at 1 in a finite number of steps. It seems that all trajectories fall into the banal cycle 4, 2, 1 if $n > 2$. The conjecture has not yet been proven and many mathematicians believe the question be undecidable (Conway, J. H, 1972). By applying the algorithm to a positive integer *n*, a sequence of integers is generated which we will call a **sequence** or **trajectory** of n which we will denote with $T(n)$ (or $T^*(n)$ with the second form of the algorithm). For example $T(5) = \{5,16,8,4,2,1\}$ and $T^*(3) = \{3,5,8,4,2,1\}$. Let $\mathbb{N} = \{0, 1, 2 \dots\}$ and $\mathbb{N}_0 = \{1, 2, 3 \dots\}$. If $i \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we denote by $T_i(n)$ the element of place *i* in the trajectory $T(n)$. If $i = 0$ we set $T_0(n) = n$. The same meaning will have $T_i^*(n)$. For example $T_0(5) = 5, T_3(5) = 4,\ T_2^*(3) = 8$. We define convergent a trajectory that contains the number 1. In any trajectory containing 1 we will ignore the terms subsequent. If the trajectory generated by the integer *n* converges we will say that the number *n* converges. Any number of a trajectory will be treated as a positive integer. The term "*t-convergent*" will be equivalent to "convergent in *t* iterations". We will call the number *t* the convergence time. The notation k_t will indicate that the positive integer *k* is *t-convergent*. In the following TC will be the set of convergence times of the converging positive integers.

2 The two forms of Collatz Conjecture

First form. With $n \in \mathbb{N}_0$ e $i \in \mathbb{N}_0$ the algorithm is the iteration of the function:

$$T_i(n) = \begin{cases} \frac{T_{i-1}(n)}{2} & \text{if } T_{i-1}(n) \equiv 0(mod2) \\ 3 \cdot T_{i-1}(n) + 1 & \text{if } T_{i-1}(n) \equiv 1(mod2) \end{cases} \tag{2.1}$$

with $T_0(n) = n$ if $i = 0$.

Second form. With $n \in \mathbb{N}_0$ and $i \in \mathbb{N}_0$, the algorithm is the iteration of the function:

$$T_i^*(n) = \begin{cases} \frac{T_{i-1}^*(n)}{2} & \text{if } T_{i-1}^*(n) \equiv 0(mod\ 2) \\ \frac{3T_{i-1}^*(n)+1}{2} & \text{if } T_{i-1}^*(n) \equiv 1(mod\ 2) \end{cases} \qquad (2.2)$$

with $T_0^*(n) = n\ if\ i = 0$.

3 Construction of the sets K

Let us put in the same set K_t the totality of positive integers *t-convergent* with the algorithm in the first form:

$$\forall t \in \mathbb{N}, K_t = \{k \in N_0 : k = k_t\}. \qquad (3.1)$$

For example, applying the algorithm in the first form:

$K_0 = \{1\}$ because 1 converges to 1 in zero iterations;
$K_1 = \{2\}$ because 2 converges to 1 in an iteration;
$K_2 = \{4\}$ because 4 converges to 1 in two iterations;
… … … …

If the Collatz algorithm is used in the second form, in (3.1) we will add to K_t and its elements the symbol $*$, that is:

$$\forall t \in \mathbb{N}, K_t^* = \{k^* \in \mathbb{N}_0 : k^* = k_t^*\}. \qquad (3.2)$$

Proposition 3.1. (Basic)
$\forall t \in \mathbb{N}, K_t$ e K_t^* are non-empty.

Proof. Trivially: whatever $t \in \mathbb{N}$, the number 2^t converges to 1 in *t* iterations, hence in K_t there is at least 2^t. For the same reason k_t^* is also non-empty. □

We consider the set *TC* of all times of convergence. Since each $t \in \mathbb{N}$ can be associated with a K_t and a K_t^* by means of 2^t and vice versa, we can state that $TC = \mathbb{N}$ and that the families $\{K_t\}_{t\in\mathbb{N}}$ and $\{K_t^*\}_{t\in\mathbb{N}}$ are countable.

The following corollaries then hold.

Corollary 3.2.
Any positive or null integer is a time of convergence.

Corollary 3.3.
Each of the families $\{K_t\}_{t\in\mathbb{N}}$ *and* $\{K_t^*\}_{t\in\mathbb{N}}$ *is countable.*

Proposition 3.4.
If t_1 *and* t_2*, with* $t_1 \neq t_2$*, are in the set TC, then it results:*

i) $K_{t_1} \cap K_{t_2} = \emptyset$
*i**) $K_{t_1}^* \cap K_{t_2}^* = \emptyset$.

Proof. *i*) **Algorithm in the first form.** By Proposition 3.1, $K_{t_1}$and K_{t_2} are non-empty. Assume that $K_{t_1} \cap K_{t_2} \neq \emptyset$, with $t_1 \neq t_2$. If $k \in K_{t_1} \cap K_{t_2}$ then k must converge in the same number of iterations, so $t_1 = t_2$, against the hypothesis. Therefore $K_{t_1} \cap K_{t_2} = \emptyset$. •

*i**) **Algorithm in the second form**. The proof is similar to the previous one: just insert the asterisk to the sets K_t . □

Each family $\{K_t\}_{t\in\mathbb{N}}$ and $\{K_t^*\}_{t\in\mathbb{N}}$ divides $\mathbb{N}_0$ into classes that we cannot consider at the moment of equivalence.

4 Decomposition of sets K

Let $t \in \mathbb{N}_0$. Applying the first form of the Collatz algorithm we will prove that each set K_t is formed by a set A_t and a set B_t, that is $K_t = A_t \cup B_t$ with A_t containing only even numbers and B_t empty or containing only odd numbers different from 1. Applying the second form of the Collatz algorithm we will prove that $K_t^* = A_t^* \cup B_t^*$ with A_t^* containing only even numbers and B_t^* empty or containing only odd numbers different from 1. We will also prove that the elements of K_t can be obtained from all the elements of K_{t-1} and the elements of K_t^* can be obtained from all elements of K_{t-1}^*. If t = 0 it is $K_0 = K_0^* = \{1\}$ and therefore $2K_0 = 2K_0^* = \{2\} = K_1 = K_1^*$. Some B sets are empty such as sets $B_{1,}\ B_1^*, B_2,\ B_2^*,\ B_3,\ B_3^*, B_4, B_4^*, B_6, B_8, B_{10}$. I don't know if there are other empty B sets. In this study $A_0 = \{1\}$, $A_0^* = \{1\}$, $B_0 = \emptyset$ and $B_0^* = \emptyset$.

Let $t \in \mathbb{N}$. Here we will assume that K_{t+1} is made up of two sets of numbers:

1) by the doubles of the numbers of K_t;
2) from the integers $b \neq 1$ which are odd solutions in $\mathbb{N}_0$ of the equation $3b + 1 = k_t$, with $k_t \in K_t$, k_t even and $k_t \neq 4$;

and that K_{t+1}^* is formed by two sets of numbers:

1*) by the doubles of the numbers of K_t^*;
2*) from the integers $b^* \neq 1$ which are odd solutions in $\mathbb{N}_0$ of the equation $\frac{3b^*+1}{2} = k_t^*$, with $k_t^* \in K_t^*$ and $k_t^* \neq 2$.

Called P the set of even positive integers, we denote by $2K_t$ (*set of even derivatives of the first type or set of even derivatives of* K_t *or set of doubles of the first type*) the set obtained by doubling all the numbers of K_t:

$$\forall t \in \mathbb{N}, 2K_t = \{a \in P: a = 2k_t, k_t \in K_t\}. \tag{4.1}$$

We denote by $2K_t^*$ (*set of even derivatives of the second type or set of even derivatives of* K_t^* *or set of doubles of the second type*) the set obtained by doubling all the numbers of K_t^*:

$$\forall t \in \mathbb{N}, 2K_t^* = \{a^* \in P: a^* = 2k_t^*, k_t^* \in K_t^*\}\,. \tag{4.2}$$

We denote by B_{t+1} (*set of odd derivatives of* K_t *or set of odd derivatives of the first type*) the numbers with the property 2) and by B_{t+1}^* (*set of odd derivatives of* K_t^* *to set of odd derivatives of second type*) numbers with the property 2*). Called D the set of integers odd positive, the set of odd derivatives of K_t we have:

$$\forall t \in \mathbb{N}, B_{t+1} = \{b \in D - \{1\}: 3b + 1 = k_t, k_t \in K_t \cap P, k_t \neq 4\} \tag{4.3}$$

while the set of odd derivatives of K_t^* is

$$\forall t \in \mathbb{N}, B_{t+1}^* = \left\{b^* \in D - \{1\}: \frac{3b^*+1}{2} = k_t^*, k_t^* \in K_t^*, k_t^* \neq 2\right\}. \tag{4.4}$$

Theorem 4.1. (Theorem of the inclusion of doubles)
The even derivative of K_t *(*K_t^**) is contained in* $K_{t+1}(K_{t+1}^*)$, *that is*:

$$\text{a) } \forall t \in \mathbb{N}, 2K_t \subseteq K_{t+1} \qquad \text{b) } \forall t \in \mathbb{N}, 2K_t^* \subseteq K_{t+1}^*. \tag{4.5}$$

Proof. By Corollary 3.2 every t is a time of convergence. Given $t \in \mathbb{N}$, we consider K_t (*which is non-empty by Proposition 3.1*). Trivially: $\forall k_t \in K_t$, the trajectory $T(k_t) = \{k_t, \ldots, 4,2,1\}$ is contained in the trajectory $T(2k_t) = \{2k_t, k_t, \ldots, 4,2,1\}$. This means that $2k_t$ is *(t+1)-convergent*, so $2k_t \in K_{t+1}$. •

If K_{t+1} is devoid of odd numbers, only the sign of equality holds. To prove it, let's suppose that K_{t+1} is devoid of odd numbers and that, absurdly, it contains an even number a_{t+1} which does not is double of any number of K_t. Since the even a_{t+1} is also is *(t+1)-convergent*, the trajectory $T(a_{t+1}) = \left\{a_{t+1}, \frac{a_{t+1}}{2}, \dots, 4,2,1\right\}$ will contain the trajectory $T\left(\frac{a_{t+1}}{2},\right) = \left\{\frac{a_{t+1}}{2}, \dots, 4,2,1\right\}$ so $\frac{a_{t+1}}{2}$ is *t-convergent*, that is $\frac{a_{t+1}}{2} \in K_t$, against our hypothesis. It follows that $2K_t$ coincides with K_{t+1} if this is devoid of odd. Then the relation a) of (4.5) holds for the arbitrariness of *t*. •

In the case of $2K_t^*$ proceed in the same way, *mutatis mutandis*. □

Theorem 4.2. (Odd derivative theorem of the first type)
Let k_t be even and $k_t \neq 4$. If there is a positive integer b satisfying the equation

$$3b + 1 = k_t \tag{4.6}$$

then

$$b = \frac{k_t - 1}{3} \tag{4.7}$$

belongs to B_{t+1}.

Proof. Let *b* and K_t satisfy the hypotheses. Since *b* is odd and different from 1, its successor is k_t, because to *b* is applied (2.1), so the trajectory $T(b) = T\left(\frac{k_t - 1}{3}\right) = \left\{\frac{k_t - 1}{3}, k_t, \dots, 4,2,1\right\}$ contains the trajectory $T(k_t) = \{k_t, \dots, 4,2,1\}$. This means that *b* converges in *t + 1* iterations, that is $b \in B_{t+1}$. □

Recall that an odd derivative B_t either is empty or is formed only by odd positive different from 1.

Theorem 4.3. (Theorem of strict inclusion of odd derivatives of the first type)
The odd derivative of $K_t(K_t^*)$ is strictly contained in $K_{t+1}(K_{t+1}^*)$, that is:

$$\text{a) } \forall t \in \mathbb{N}, B_{t+1} \subset K_{t+1} \qquad \text{b) } \forall t \in \mathbb{N}, B_{t+1}^* \subset K_{t+1}^*. \tag{4.8}$$

Proof. By Proposition 3.1 every $K_{t+1}(K_{t+1}^*)$ is non-empty because it contains at least the even number 2^{t+1}, therefore B_{t+1} even if it were empty could not coincide with $K_{t+1}(K_{t+1}^*)$. □

Theorem 4.4. (Theorem of the union of even and odd derivatives of the first type) *The set K_{t+1} is the union of the set of doubles of K_t and of the odd derivative of K_t, that is:*

$$\forall t \in \mathbb{N}, K_{t+1} = 2K_t \cup B_{t+1}. \tag{4.9}$$

(remarkable equality, algorithm in first form)

Proof. Let us consider K_t, with $t \in \mathbb{N}$. It is necessary to demonstrate that

1) there are no other even integers *(t+1)-convergent* beyond those of $2K_t$;
2) the odd numbers *(t+1)-convergent* are only those of B_{t+1}.

We prove 1). We denote by A_{t+1} the totality of even positive integers converging in *t + 1* iterations that we know to be non-empty (each A_t contains at least 2^t). It immediately turns out that $\forall t \in \mathbb{N}, 2K_t \subseteq A_{t+1}$.
We show that

$$\forall t \in \mathbb{N}, 2K_t = A_{t+1}. \tag{4.10}$$

If for a fixed $t \in \mathbb{N}$ there were an even $a_{t+1} \in A_{t+1}$ that was not double of any positive integer of K_t, it would be absurd because the trajectory $T(a_{t+1}) = \left\{a_{t+1}, \frac{a_{t+1}}{2}, \dots 4,2,1\right\}$ would contain the trajectory $T\left(\frac{a_{t+1}}{2}\right) = \left\{\frac{a_{t+1}}{2}, \dots ,4,2,1\right\}$ whose seed at $\frac{a_{t+1}}{2} \in K_t$ and whose double a_{t+1} is in A_{t+1}, against the hypothesis. Hence the strict inclusion cannot hold and, by the arbitrariness of *t*, (4.10) is true. •

We prove 2). With the same fixed $t \in \mathbb{N}$, we denote by β_{t+1} the totality of the odd positive integers converging in *t + 1* iterations . Obviously we have $B_{t+1} \subseteq \beta_{t+1}$.
We show that

$$\forall t \in \mathbb{N}, B_{t+1} = \beta_{t+1} . \tag{4.11}$$

If for the fixed *t*, $\beta_{t+1} = \emptyset$, then also $B_{t+1} = \emptyset$ and therefore $K_{t+1} = 2K_t$, that is (4.9) for the arbitrariness of *t*.
Otherwise, for fixed *t*, let $\beta_{t+1} \neq \emptyset$. If there was an $b_{t+1} \in \beta_{t+1}$ not coming by any even of K_t, that is such that $b_{t+1} \notin B_{t+1}$, then an absurdity would follow because the trajectory $T(b_{t+1}) = \{b_{t+1}, 3b_{t+1} + 1, \dots ,4,2,1\}$ would contain the trajectory $T(3b_{t+1} + 1) = \{3b_{t+1} + 1, \dots ,4,2,1\}$ whose even seed $3b_{t+1} + 1 = k_t \in K_t$, therefore, by Theorem 4.2, $b_{t+1} \in B_{t+1}$ against the hypothesis. For this reason strict inclusion cannot be valid and, due to the arbitrariness of *t* (4.11) is true. •

From **1)** and **2)** follows the remarkable equality (4.9). □

By (4.10), (4.9) becomes

$$\forall t \in \mathbb{N}, K_{t+1} = A_{t+1} \cup B_{t+1}\ . \tag{4.12}$$

(*remarkable equality, algorithm in the first form*)

If for a given *t* the derivative B_{t+1} of K_t is empty, we have

$$K_{t+1} = A_{t+1}. \tag{4.13}$$

We now find the numbers of B^*_{t+1}.

Theorem 4.5. (Theorem of the odd derivative of the second type)
Let $k^*_t \in \mathbb{N}_0$, $k^*_t \neq 2$. *If there exists the positive integer b satisfying the equation*

$$3b^* + 1 = 2k^*_t \tag{4.14}$$

then

$$b^* = \frac{2k^*_t - 1}{3} \tag{4.15}$$

belongs to B^*_{t+1}.

Proof. Let k^*_t and b^* satisfy the hypotheses. Since b^* is odd and different from 1, its successor is k^*_t, because (2.2) is applied to b^*, so the trajectory $T(b^*) = T\left(\frac{2k^*_t-1}{3}\right) = \left\{\frac{2k^*_t-1}{3}, k^*_t, \dots, 4,2,1\right\}$ contains the trajectory $T(k^*_t) = \{k^*_t, \dots, 4,2,1\}$. This means that b^* converges in *t + 1* iterations, that is $b^* \in B^*_{t+1}$. □

Recall that an odd derivative B^*_t o is either empty or is formed only by odd positive integers different from 1.

As shown for (4.10) it results

$$\forall t \in \mathbb{N}, 2K^*_t = A^*_{t+1} \tag{4.16}$$

where A_{t+1}^{*} is the totality of the even positive integers *(t+1)-convergent*, that is of the doubles of the numbers of K_t^{*}. Equation (4.16) is demonstrated how it is done for the first part of the proof of the Theorem 4.4 by adding the asterisk $*$ to the $2K_t$ and A_t sets. Equation (4.16) will occur in the proof of first part of Theorem 4.6.

Theorem 4.6. (Theorem of the union of even and odd derivatives of the second type) *The set K_{t+1}^{*} is the union of the set of doubles of K_t^{*} and the odd derivative of K_t^{*}, that is:*

$$\forall t \in \mathbb{N}, K_{t+1}^{*} = 2K_t^{*} \cup B_{t+1}^{*} \, . \tag{4.17}$$

(*remarkable equality, algorithm in the second form*)

Proof. We proceed as in the proof of Theorem 4.4 adding the asterisk $*$ to all the sets and considering, in the second part, $\left(\frac{3b_{t+1}^{*}+1}{2}\right)$ as successor of $b_{t+1}^{*} \in \beta_{t+1}^{*}$. □

By (4.16), (4.17) can be written

$$\forall t \in \mathbb{N}, K_{t+1}^{*} = A_{t+1}^{*} \cup B_{t+1}^{*} \tag{4.18}$$

(*remarkable equality, algorithm in the second form*)

and if, for a certain *t*, the derivative B_{t+1}^{*} of K_{t+1}^{*} it is empty, then

$$K_{t+1}^{*} = A_{t+1}^{*} \, . \tag{4.19}$$

5 Examples

To obtain the set K_{t+1} it will be necessary to double all the numbers k_t of K_t in order to have A_{t+1} and it will be necessary to determine all the numbers $b \in B_{t+1}$ starting from the even numbers k_t of K_t, that is, it will be necessary to verify if $k_t - 1$ is divisible by three when k_t is even with $k_t \neq 4$ (Theorem 4.2 and definition of B_{t+1} in (4.3)).

► We determine the sets K_8 and K_9.

K_8

We use the set $K_7 = \{3,20,21,128\}$. We have $A_8 = 2K_7 = \{6,40,42,256\}$. It turns out $B_8 = \emptyset$ since none of the equations

(1) 3b + 1 = 20

(2) 3b + 1 = 128

has solutions in $\mathbb{N}_0$. Hence $K_8 = A_8 \cup \emptyset = \{6,40,42,256\}$.

K_9
We use the set $K_8 = \{6,40,42,256\}$. We have A_9= $2K_8$ = {12, 80, 84, 512}. We solve in $\mathbb{N}_0$ the following equations:

(1) 3b + 1 = 6

(2) 3b + 1 = 40

(3) 3b + 1 = 42

(4) 3b + 1 = 256.

The first and third equations have no solutions in $\mathbb{N}_0$. The second and fourth equations have as solutions in $\mathbb{N}_0$ 13 and 85 respectively, therefore B_9 = {13, 85}.
Thus $K_9 = A_9 \cup B_9 = \{12,80,84,512\} \cup \{13,85\} = \{12,13,80,84,85,512\}$.

In the same way they are obtained

$K_{10} = \{4 - 26 - 160 - 168 - 170 - 1024\}$

$K_{11} = \left\{48 - 52 - \underline{53} - 320 - 336 - 340 - \underline{341} - 2048\right\}$

$K_{12} = \left\{\underline{17} - 96 - 104 - 106 - \underline{113} - 640 - 672 - 680 - 682 - 4096\right\}$

…

The underlined numbers are the odd derivatives of the previous set.

To obtain the set K^*_{t+1} it will be necessary to double all the numbers k^*_t of K^*_t in order to have A^*_{t+1} and it will be necessary to determine all the numbers $b^* \in B^*_{t+1}$ starting from each k^*_t of K^*_t, that is, it will be necessary to verify whether $2k^*_t - 1$ is divisible by three when $k^*_t \neq 2$ (Theorem 4.5 and definition of B^*_{t+1} in (4.4)).

► We determine the sets K^*_5 and K^*_6.

K^*_5
We consider $K^*_4 = \{5,16\}$. Its even derivative is $A^*_5 = \{10,32\}$. Of the two equations

(1) $\frac{3b^{*}+1}{2} = 5$

(2) $\frac{3b^{*}+1}{2} = 16$

only the first admits in $\mathbb{N}_0$ the solution b^*= 3 therefore $B_5^* = \{3\}$ e $K_5^* = A_5^* \cup B_5^* = \{3,10,32\}$.

K_6^*
We consider $K_5^* = \{3,10,32\}$. Its even derivative is $A_6^* = \{6,20,64\}$. Of the three equations

(1) $\frac{3b^{*}+1}{2} = 3$

(2) $\frac{3b^{*}+1}{2} = 10$

(3) $\frac{3b^{*}+1}{2} = 32$

only the third has solution $b^* = 21$ in $\mathbb{N}_0$. Hence $B_6^* = \{21\}$ e $K_6^* = A_6^* \cup B_6^* = \{6,20,21,64\}$.

In the same way they are obtained

$K_7^* = \{12 - 13^* - 40 - 42 - 128\}$

$K_8^* = \{24 - 26 - 80 - 84 - 85^* - 256\}$

$K_9^* = \{17^* - 48 - 52 - 53^* - 160 - 168 - 170 - 512\}$

$K_{10}^* = \{11^* - 34 - 35^* - 96 - 104 - 106 - 113^* - 320 - 336 - 340 - 341^* - 1024\}$

...

The numbers with an asterisk are the odd derivatives of the previous set.

6 The maxima of K_t and K_t^*

By examining the sets K, we can suppose that the number 2^t is the maximum of every set K_t and of every K_t^*. This is confirmed by the subsequent Theorem 6.2. The following Lemma 6.1 contains some obvious conclusions.

Lemma 6.1.
i) If $k_t \in K_t$ then $2k_t \in A_{t+1}, \forall t \in \mathbb{N}$
ii) If $a_t \in A_t$ then $2a_t \in A_{t+1}, \forall t \in \mathbb{N}_0$
i)* If $k_t^* \in K_t^*$ then $2k_t^* \in A_{t+1}^*, \forall t \in \mathbb{N}$
ii)* If $a_t^* \in A_t^*$ then $2a_t^* \in A_{t+1}^*, \forall t \in \mathbb{N}_0$.

Proof. Recall that (4.10) and (4.16) hold.

i) Let $k_t \in K_t$, with $t \in \mathbb{N}$. The trajectory $T(k_t)$ is contained in the trajectory $T(2k_t) = \{2k_t, k_t, \dots, 4,2,1\}$ because $2k_t$ is an even that converges in t + *1* iterations, that is $2k_t \in A_{t+1}$. •

ii) Let $a_t \in A_t$, with t $\in \mathbb{N}_0$. Since $A_t \subseteq K_t$ is also $a_t \in K_t$. Applying *i)* it follows that $2a_t \in A_{t+1}$ $\forall t \in \mathbb{N}_0$. •

The *i*)* and *ii*)* prove to be the *i)* and *ii)* respectively, just asterisking the sets K_t, A_t and their elements. □

Theorem 6.2. (Maxima theorem of K_t and K_t^*)

$i)$ $\forall t \in \mathbb{N}_0, max(K_t) = 2^t$

i)* $\forall t \in \mathbb{N}_0, max(K_t^*) = 2^t$.

Proof. *i)* We will proceed by induction using the remarkable equality (4.12). If $t = 1$ then max $max(K_1) = 2^1 = 2$. Let us fix a $t > 1$ and let, by inductive hypothesis

$$max(K_t) = 2^t. \tag{6.1}$$

We will prove that it is also $max(K_{t+1}) = 2^{t+1}$. To do this, it will be necessary to prove that

1) max $max(A_{t+1}) = 2^{t+1}$

and

2) every number of B_{t+1} is less than 2^{t+1}.

First part

1) We show that every number of A_{t+1} is less than or equal to 2^{t+1} and that $2^{t+1}$is in A_{t+1}. Let $k_t \in K_t$. Then, by hypothesis (6.1)

$$\forall k_t \in K_t, k_t \leq 2^t. \tag{6.2}$$

By the *i)* of Lemma 6.1

$$2k_t \in A_{t+1}. \tag{6.3}$$

From (6.2) it follows that

$$\forall k_t \in K_t, 2k_t \leq 2^{t+1}. \tag{6.4}$$

Since for the inductive hypothesis (6.1) it is $2^t \in K_t$, then, for the remarkable equality (4.12), we have $2^t \in A_t$, from which, for the *ii)* of Lemma 6.1, it follows that

$$2^{t+1} \in A_{t+1}. \tag{6.5}$$

From (6.3), (6.4) and (6.5) we obtain that $max(A_{t+1}) = 2^{t+1}$. •

Second part

2) If $B_{t+1} = \emptyset$ from (4.12) it follows that $K_{t+1} = A_{t+1}$ and from $max(A_{t+1}) = 2^{t+1}$ (***First part***) it follows that $max(K_{t+1}) = 2^{t+1}$. Let $B_{t+1} \neq \emptyset$. We show that every element b_{t+1} of B_{t+1} is less than 2^{t+1}. The numbers of B_{t+1} are the odd numbers of the form (4.7):

$$b_{t+1} = \frac{k_t - 1}{3} con\, k_t \in K_t \; and \; k_t \; even \tag{6.6}$$

but, from $k_t - 1 < k_t$ we get that

$$\frac{k_t - 1}{3} < k_t \tag{6.7}$$

then from (6.6), (6.7) and (6.1) it follows that

$$b_{t+1} < k_t \leq 2^t \tag{6.8}$$

and therefore: $\forall b_{t+1} \in B_{t+1}$, $b_{t+1} < 2^{t+1}$, that is **2**). From the first and the second part it follows that all the numbers of K_{t+1} are less than or equal to $2^{t+1}$and this proves the *i).* •

*i**) We will proceed by induction using the remarkable equality (4.18). If $t = 1$ then $max(K_1^*) = 2^1$=2. Let, by inductive hypothesis, be

$$max(K_t^*) = 2^t \text{ con } t > 1 . \qquad (6.9)$$

We will prove that it is also $max(K_{t+1}^*) = 2^{t+1}$. To do this, it will be necessary to prove that

1*) $max(A_{t+1}^*) = 2^{t+1}$

and

2*) *every number of B_{t+1}^* is less than* 2^{t+1} .

First part $*$

1*) The proof is similar to that of the first part of *i)*, just adding the asterisk $*$ to the sets A_{t+1}, K_{t+1} and their elements. Therefore 2^{t+1} is the maximum of A_{t+1}^* and **1***) is proved. •

Second part $*$

2*) If $B_{t+1}^* \neq \emptyset$ from 4.18) it follows that $K_{t+1}^* = A_{t+1}^*$ and from $max(A_{t+1}^*) = 2^{t+1}$ (***First part****) it follows that $max(K_{t+1}^*) = 2^{t+1}$.

Let $B_{t+1}^* \neq \emptyset$. We show that every b_{t+1}^* of B_{t+1}^* is less than 2^{t+1}. The numbers of B_{t+1}^* are the odd numbers of the form (4.15):

$$b_{t+1}^* = \frac{2k_t^*-1}{3}, \text{ with } k_t^* \in K_t^* \qquad (6.10)$$

but, from $2k_t^* - 1 < 2k_t^*$ we get that

$$\frac{2k_t^*-1}{3} < 2k_t^* \qquad (6.11)$$

and for the inductive hypothesis (6.9) we also have that

$$\forall k_t^* \in K_t^*, 2k_t^* \leq 2^{t+1} . \qquad (6.12)$$

Finally for (6.10), (6.11), (6.12) we can write that $b_{t+1}^* < 2k_t^* \leq 2^{t+1}$, then $\forall t > 1$ all numbers b_{t+1}^* di B_{t+1}^* are less than 2^{t+1}. The **2***) is thus proved. •

From **1***) and from **2***) it follows that all integers of K_{t+1}^* are less than or equal to 2^{t+1} and so i*) is also proved. □

The following corollaries immediately follow from Theorem 6.2.

Corollary 6.3.

i) $\forall t \in \mathbb{N}, max(2K_t) = 2^{t+1}$

i)* $\forall t \in \mathbb{N}, max(2K_t^*) = 2^{t+1}$.

Corollary 6.4.

i) $\forall t \in \mathbb{N}_0, max(A_t) = 2^t$

i)* $\forall t \in \mathbb{N}_0, max(A_t^*) = 2^t$.

Theorem 6.2 provides indications on the type of numbers contained in the sets K: either there is only 2^t or there are positive integers less than or equal to 2^t and this means that each set K is finite. Therefore, the following corollary can also be stated.

Corollary 6.5.

$\forall t \in \mathbb{N}, K_t$ *and* K_t^* *are finite.*

Each set K is formed by the finite numerical sets A and B. It follows that if B is non-empty then it has a maximum. Therefore the following corollary holds.

Corollary 6.6.

i) If for $t \in \mathbb{N}_0$ *is* $B_t \neq \emptyset$ *then* $\exists\, max(B_t)$

i) If for* $t \in \mathbb{N}_0$ *is* $B_t^* \neq \emptyset$ *then* $\exists\, max(B_t^*)$.

In the following paragraph 7 we will investigate the maxima of the sets B.

7 On the maxima of the sets B

We give a strict increase of the maxima of the sets B.

Proposition 7.1.

i) If $B_{t+1} \neq \emptyset$ *then* $\exists k_t \in K_t, k_t \neq 4, k_t$ *even* $: max(B_{t+1}) < \frac{k_t-1}{2}$

i) If* $B^*_{t+1} \neq \emptyset$ *then* $\exists k^*_t \in K^*_t, k^*_t \neq 2 : max(B^*_{t+1}) < \frac{2k^*_t-1}{2}$.

Proof. i) If $B_{t+1} \neq \emptyset$, then by definition of B_{t+1} in correspondence of every odd $b_{t+1} \in B_{t+1}$ will exist an even number $k_t \in K_t$ with $k_t \neq 4$ such that $b_{t+1} = \frac{k_t-1}{3}$ but $\frac{k_t-1}{3} < \frac{k_t-1}{2}$, then $b_{t+1} < \frac{k_t-1}{2}$. Then, in particular, *i*) holds also for the maximum of B_{t+1}. •

i)* If $B^*_{t+1} \neq \emptyset$, then by definition of B^*_{t+1} in correspondence of every odd $b^*_{t+1} \in B^*_{t+1}$ will exist an even number $k^*_t \in K^*_t$ with $k^*_t \neq 2$ such that $b^*_{t+1} = \frac{2k^*_t-1}{3}$ but $\frac{2k^*_t-1}{3} < \frac{2\,k^*_t-1}{2}$, then $b^*_{t+1} < \frac{2\,k^*_t-1}{2}$. Then, in particular, also for the maximum of B^*_{t+1} holds *i*)*. □

From Proposition 7.1 follows the following corollary which gives a plus a bit more large of the maxima of the sets B.

Corollary 7.2.

i) If $B_{t+1} \neq \emptyset$, *then* $max(B_{t+1}) < 2^t$

i) If* $B^*_{t+1} \neq \emptyset$, *then* $max(B^*_{t+1}) < 2^t$.

Proof. *i)* If $B_{t+1} \neq \emptyset$, then the inequality *i)* of Proposition 7.1 holds and also $\frac{k_t-1}{2} < k_t$ but, by Theorem 6.2, the maximum of K_t is 2^t, so $max(B^*_{t+1}) < 2^t$. •

i)* If $B^*_{t+1} \neq \emptyset$, then the inequality *i*)* of Proposition 7.1 holds and also $\frac{2k^*_t-1}{2} < k^*_t$ but, by Theorem 6.2, the maximum of K^*_t is 2^t, so $max(B^*_{t+1}) < 2^t$. □

In some cases it is possible to determine the maximum of the sets B. Let's see how. The numbers of $B_{t+1}$and of B_{t+1}^* come from the integer solutions, if they exist, of the equations

$$b_{t+1} = \frac{k_t - 1}{3} \text{ with } k_t \in K_t, k_t \text{ even and } k_t \neq 4 \tag{7.1}$$

$$b_{t+1}^* = \frac{2k_t^* - 1}{3} \text{ with } k_t^* \in K_t^* \text{ and } k_t^* \neq 2 \tag{7.2}$$

by the Theorems, respectively, 4.2 and 4.5. In fact, the largest odd integer that can be obtained from (7.1), if we substitute the maximum of K_t for k_t, is $\frac{2^t-1}{3}$, which is integer if $2^t - 1$ is divisible by three. Likewise, the largest odd integer which can be obtained from (7.2), if we replace k_t^* by the maximum of K_t^*, is $\frac{2^{t+1}-1}{3}$, which is integer if $2^{t+1} - 1$ is divisible by 3. We can therefore state the following theorem.

Theorem 7.3.
i) If $2^t - 1 \equiv 0 (mod\ 3)$*, with* $t \in \mathbb{N}_0$ *and* $t > 2$*, then* $max(B_{t+1}) = \frac{2^t-1}{3}$

i) If* $2^{t+1} - 1 \equiv 0 (mod\ 3)$*, with* $t \in \mathbb{N}_0$ *and* $t > 1$*, then* $max(B_{t+1}^*) = \frac{2^{t+1}-1}{3}$.

SECOND DEMONSTRATION OF THE THEOREM 7.3

Proof. i) By hypothesis the number 2^t is *t-convergent* and the equation $3b + 1 = 2^t$ is satisfied by $b = \frac{2^t-1}{3}$ which is different from 1 because $t > 2$, therefore, by theorem 4.2 it is $b \in B_{t+1}$. Assume that $\exists \beta \in B_{t+1}: b < \beta$ that is, taking into account the form of b and β, we suppose that it is $\frac{2^t-1}{3} < \frac{k_t-1}{3}$ with $k_t \in K_t$ e k_t even; from this it follows that $2^t < k_t$, absurd thing because the maximum of K_t is 2^t. Then it must turn out $\forall \beta \in B_{t+1}: \beta \leq b$, that is the thesis. •

i)* By hypothesis the number 2^t is *(t+1)-convergent* and the equation $3b+1=2^{t+1}$ is satisfied by $b=\frac{2^{t+1}-1}{3}$ which is different from 1 because $t>1$, therefore, by theorem 4.5 it is $b^* \in B^*_{t+1}$. Assume that $\exists \beta^* \in B^*_{t+1}: b^* < \beta^*$ that is, taking into account the form of b^* and β^*, supposing it is $\frac{2^{t+1}-1}{3} < \frac{2k^*_t-1}{3}$ with $k^*_t \in K^*_t$, from this it follows that $2^t < k^*_t$, which is absurd because the maximum of K^*_t is 2^t. Then it must turn out $\forall \beta^* \in B^*_{t+1}: \beta^* \le b^*$ that is the thesis. □

For example:

- ...
- for $t=14$ risults $2^{14}-1 \equiv 0(mod3)$, then $max(B_{15}) = max(B^*_{14}) = 5461$
- for $t=16$ risults $2^{16}-1 \equiv 0(mod3)$, then $max(B_{17}) = max(B^*_{16}) = 21845$
- for $t=18$ risults $2^{18}-1 \equiv 0(mod3)$, then $max(B_{19}) = max(B^*_{18}) = 87381$
- ....

8 On the intersection of K_t and K^*_t

In this paragraph we will prove that the intersection of the sets K_t and K^*_t is $\{2^t\}$.

Lemma 8.1.
The intersection of the odd derivatives of the first type t-convergent and of the even derivatives of the second type t-convergent is empty, that is

$$\forall t \in \mathbb{N}_0,\ B_t \cap A^*_t = \emptyset. \tag{8.1}$$

Proof. Obviously, because an odd derivative either is empty or is made up of odd integers different from 1 and an even derivative contains only even numbers. □

Lemma 8.2.
The intersection of the odd derivatives of the second type t-convergent and of the even derivatives of the first type t-convergent is empty, that is

$$\forall t \in \mathbb{N}_0, B^*_t \cap A_t = \emptyset. \tag{8.2}$$

Proof. Obviously, because an odd derivative either is empty or is made up of odd integers different from 1 and an even derivative contains only even numbers. □

Lemma 8.3.
The intersection of the odd derivatives of the first type t-convergent and of the odd derivatives of the second type t-convergent is empty, that is

$$\forall t \in \mathbb{N}_0, B_t \cap B_t^* = \emptyset. \tag{8.3}$$

Proof. Trivially, if $t = 1$ the sets B_1 and B_1^* are both empty. Assume absurdly that for $t > 1$ it results $B_t \cap B_t^* \neq \emptyset$ and consider every $n_t \in B_t \cap B_t^*$.

From

$$n_t \in B_t = \{n_t \in D - \{1\}: 3n_t + 1 = k_{t-1}, k_{t-1} \in K_{t-1} \cap P, k_{t-1} \neq 4, k_{t-1} - 1 \equiv 0(mod\ 3)\}$$

follows that n_t is an odd integer of the form (4.7), that is $n_t = \frac{k_{t-1}-1}{3}$.

From

$$n_t \in B_t^* = \left\{n_t \in D - \{1\}: \frac{3n_t + 1}{2} = k_{t-1}^*, k_{t-1}^* \in K_{t-1}^*, k_{t-1}^* \neq 2, 2k_{t-1}^* - 1 \equiv 0(mod\ 3)\right\}$$

it follows that n_t is an odd integer of the form (4.15), that is $n_t = \frac{2k_{t-1}^*-1}{3}$.

By equating the two expressions of n_t we have $\frac{k_{t-1}-1}{3} = \frac{2k_{t-1}^*-1}{3}$ and therefore

$$k_{t-1} = 2k_{t-1}^*. \tag{8.4}$$

Equality (8.4) is manifestly absurd because k_{t-1} is *(t-1)-convergent* and $2k_{t-1}^*$ is *t-convergent*. Therefore it makes no sense to suppose that the intersection $B_t \cap B_t^*$ for $t > 1$ is non-empty and (8.3) is proved. □

Lemma 8.4.
The intersection of K_t and K_t^* is equal to the intersection of the even derivatives of K_{t-1} and of the derivatives even of K_t^*, that is

$$\forall t \in \mathbb{N}_0, K_t \cap K_t^* = A_t \cap A_t^*. \tag{8.5}$$

Proof. We will use the notable equalities 4.12) and 4.18). We have $\forall t \in \mathbb{N}_0$:

$$K_t \cap K_t^* = (A_t \cup B_t) \cap (A_t^* \cup B_t^*) =$$

$$= \big((A_t \cup B_t) \cap A_t^*\big) \cup \big((A_t \cup B_t) \cap B_t^*\big) =$$

$$= (A_t \cap A_t^*) \cup (B_t \cap A_t^*) \cup (A_t \cap B_t^*) \cup (B_t \cap B_t^*). \tag{8.6}$$

The thesis follows by applying, in order, Lemmas 8.1, 8.2 and 8.3 to the second, third and fourth intersection in the last line of (8.6). □

Lemma 8.5.
The intersection of the even derivatives of the first and second type t-convergent is $\{2^t\}$*, that is:*

$$\forall t \in \mathbb{N}_0, A_t \cap A_t^* = \{2^t\}. \tag{8.7}$$

Proof. Applying the equalities (4.10) and (4.16) to the intersection $A_t \cap A_t^*$ we have:

$$\forall t \in \mathbb{N}_0, A_t \cap A_t^* = 2K_{t-1} \cap 2K_{t-1}^* = 2(K_{t-1} \cap K_{t-1}^*). \tag{8.8}$$

Applying Lemma 8.4 to the intersection in the last parenthesis of (8.8) we have

$$\forall t \in \mathbb{N}_0, 2(K_{t-1} \cap K_{t-1}^*) = 2(A_{t-1} \cap A_{t-1}^*) =$$
$$= 2(2K_{t-2} \cap 2K_{t-2}^*) = 2^2(K_{t-2} \cap K_{t-2}^*). \tag{8.9}$$

Applying Lemma 8.4 again to the intersection in the last parenthesis of (8.9) and iterating, we obtain

$$\forall t \in \mathbb{N}_0, 2^2(K_{t-2} \cap K_{t-2}^*) = 2^2(A_{t-2} \cap A_{t-2}^*) = \cdots = 2^{t-1}(K_1 \cap K_1^*). \tag{8.10}$$

Finally, applying Lemma 8.4 again to the intersection in the last parenthesis of (8.10), we have

$$\forall t \in \mathbb{N}_0, 2^{t-1}(K_1 \cap K_1^*) = 2^{t-1}(A_1 \cap A_1^*) = 2^{t-1}(2K_0 \cap 2K_0^*) =$$

$$= 2^t(K_0 \cap K_0^*) = 2^t(\{1\} \cap \{1\}) = \{2^t\}\,. \quad \square \tag{8.11}$$

Theorem 8.6.
The intersection between K_t *and* K_t^* *is equal to* $\{2^t\}$*, that is*

$$\forall t \in \mathbb{N}_0, K_t \cap K_t^* = \{2^t\}. \tag{8.12}$$

Proof. Applying Lemma 8.4 to the intersection $K_t \cap K_t^*$, we have (8.5). Applying the Lemma 8.5 at the intersection $A_t \cap A_t^*$ we obtain (8.12). □

9 Conclusions

Collatz's conjecture can be re-proposed using the sets K and their first properties. We have seen that the sets K_t and K_t^* are non-empty (Basic 3.1) and they are also two by two disjoint (Corollary 3.4). So, if the following coverage equalities of $\mathbb{N}_0$ were also true:

a) $\bigcup_{t=0}^{+\infty} K_t = \mathbb{N}_0\,, \quad t \in \mathbb{N}$ b) $\bigcup_{t=0}^{+\infty} K_t^* = \mathbb{N}_0\,, \quad t \in \mathbb{N}$

we could say that each of the families $\{K_t\}_{t\in\mathbb{N}}$ and $\{K_t^*\}_{t\in\mathbb{N}}$ is a partition of $\mathbb{N}_0$. In this case the Collatz conjecture would be proved.

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Mixed and Non-mixed Normal Subgroups of Dihedral Groups Using Conjugacy classes

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Abstract

In this paper, we characterize and compute the mixed and non-mixed basis of Dihedral groups. Also, by computing the conjugacy classes, we describe all the mixed and non-mixed normal subgroups of Dihedral Groups.
Keywords: group; Dihedral group; mixed and non-mixed basis; normal subgroups; conjugacy classes;
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1 Introduction

There are many interesting functions from the family of Dihedral groups to set of natural numbers. For the Dihedral group D_n of order $2n$, Cavior [1975] proved that the number of subgroups is $d(n) + \sigma(n)$ where $\sigma(n)$ is the sum of positive divisors of n and $d(n)$ denote number of positive divisors of n. For elementary facts about dihedral groups see Conrad [Retrieveda]. Conrad [Retrievedb] describes the subgroups of D_n, including the normal subgroups. using characterization of dihedral groups in terms of generators and relations. Calugareanu [2004] presents a formula for the total number of subgroups of a finite abelian group. In Tărnăuceanu [2010] an arithmetic method is developed to count the number of some types of subgroups of finite abelian groups.

Subgroups of groups of smaller sizes are widely studied because their group properties can be easily verified and larger groups are usually studied in terms of their subgroups (see Miller [1940]). In this paper we characterize and compute the different basis of Dihedral groups. Also we describe all mixed and non-mixed normal subgroups of Dihedral groups via conjugacy classes.

2 Notations and Basic Results

Most of the notations, definitions and results we mentioned here are standard and can be found in Gallian [1994] and Dummit and Foote [2003]. For any given natural number n denote:

$$\begin{aligned} d(n) &= \text{ the number of positive divisors of } n. \\ \sigma(n) &= \text{ the sum of positive divisors of } n. \\ \varphi(n) &= \text{ the number of non- negative integers less than } n \text{ and relatively} \\ &\quad\ \text{prime to } n. \end{aligned}$$

Also, the greatest common divisor of m and n is denoted by (m, n). Let G be a group and $a_1, a_2, \ldots, a_p \in G$. Then the subgroup generated by $a_1, a_2, \ldots, a_p$ is denoted by $< a_1, a_2, \ldots, a_p >$.

Definition 2.1. *A group generated by two elements r and s with orders n and 2 such that $srs^{-1} = r^{-1}$ is said to be the n^{th} dihedral group and is denoted by D_n.*

Theorem 2.1. *For each divisor d of n, the group $\mathbb{Z}_n$ has a unique subgroup of order d, namely $\langle \frac{n}{d} \rangle$.*

Theorem 2.2. *For each divisor d of n, the group $\mathbb{Z}_n$ has exactly $\varphi(d)$ elements of order d, namely $\{k\frac{n}{d} : 0 \leq k \leq d-1, (k, d) = 1\}$.*

Theorem 2.3. *The number of subgroups of $\mathbb{Z}_n$ is $d(n)$, namely $\langle \frac{n}{d} \rangle$ where d is a divisor of n.*

Theorem 2.4. *Let G be a group generated by a and b such that $a^n = e$, $b^2 = e$ and $bab^{-1} = a^{-1}$. If the size of G is $2n$ then G is isomorphic to D_n.*

By theorem 2.4, we make an abstract definition for dihedral groups.

Definition 2.2. *For $n \geq 3$, let $R_n = \{r_0, r_1, \ldots, r_{n-1}\}$ and $S_n = \{s_0, s_1, \ldots, s_{n-1}\}$. Define a binary operation on $G_n = R_n \cup S_n$ by the following relations:*

$$\begin{array}{lll} r_i \cdot r_j = r_{i+j\,mod(n)} & r_i \cdot s_j = s_{i+j\,mod(n)} & \\ s_i \cdot s_j = r_{i-j\,mod(n)} & s_i \cdot r_j = s_{i-j\,mod(n)} & \text{for all} \quad 0 \leq i, j \leq n-1. \end{array}$$

Then $(G_n, \cdot)$ is a group of order $2n$.

Note that in the group $(G_n, \cdot)$, the identity element is r_0, $r_i = r_j$ if and only if $i = j\,mod(n)$, $s_i = s_j$ if and only if $i = j\,mod(n)$, the inverse of r_i is r_{n-i} and the inverse of s_i is s_i for all $0 \leq i, j \leq n-1$. It is also clear that $r_1^i = r_i$ and $r_j \cdot s_0 = s_j$ for all $0 \leq i, j \leq n-1$. Since G_n is a group of order $2n$ and can be generated by r_1 and s_0 such that:

$$r_1^n = r_n = r_0,\ s_0^2 = r_0 \text{ and } s_0 r_1 s_0^{-1} = s_0 r_1 s_0 = s_{-1} s_0 = r_{-1} = r_{n-1} = r_1^{-1}.$$

Therefore the group G_n is isomorphic to $D_n =< r_1, s_0 >$. The elements of R_n are called rotations and that of S_n are called reflections. A subgroup of D_n which contain both rotations and reflections is called a mixed subgroup and subgroups contain rotations only is called non-mixed subgroup. From the group D_n, we have the following.

Theorem 2.5. *R_n is a subgroup of D_n and is isomorphic to $\mathbb{Z}_n$.*

Theorem 2.6. *If n is even, the number of elements of order 2 in D_n is $n+1$, namely $\{r_{n/2}, s_j : \ 0 \leq j \leq n-1\}$.*

Theorem 2.7. *If n is odd, the number of elements of order 2 in D_n is n, namely $\{s_j : \ 0 \leq j \leq n-1\}$.*

Theorem 2.8. *If d divide n and $d \neq 2$, the number of elements of order d in D_n is $\varphi(d)$ namely $\{r_{kn/d} : \ 0 \leq k \leq d-1,\ (k, d) = 1\}$.*

Theorem 2.9. *If a and b are two elements in D_n, then $< a, b >= \{a^k b^m \ : 0 \leq k, m \leq n-1\}$*

Definition 2.3. *Let G be a finite group. An element $y \in G$ is said to be a conjugate of $x \in G$ iff $y = gxg^{-1}$, for some g in G.*

This relation conjugacy in a group G is an equivalence relation on G. The equivalence class determined by the element x is denoted by $cl(x)$. Thus $cl(x) = \{gxg^{-1} \ : g \in G\}$. The summation, $\sum_{x \in G} |cl(x)|$, where summation runs over one element from each conjugacy class of x is called the class equation of G.

Definition 2.4. *A subgroup H of the group G is said to be a normal subgroup if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.*

A normal subgroup which contain rotations alone is called a non- mixed normal subgroup and normal subgroups which contains both reflections and rotations is called mixed normal subgroup.

Theorem 2.10. *Every normal subgroup is a union of conjugacy classes.*

Theorem 2.11. *Every subgroup of a cyclic normal subgroup of the group G is also normal in G.*

3 Subgroups of D_n

Theorem 3.1. *The number of non-mixed subgroups of D_n is $d(n)$, namely $\{< r_{n/d} >: \ d \ \text{is a divisor of} \ n\}$.*

Proof. The non-mixed subgroups of D_n are subgroups of R_n. Since R_n is isomorphic to $\mathbb{Z}_n$, for each divisor d of n, the group R_n has a unique subgroup of order d, namely $< r_{n/d} >$. Hence the number of non-mixed subgroups of D_n is $d(n)$, namely $\{< r_{n/d} >: \ d \ \text{is a divisor of} \ n\}$. □

Theorem 3.2. *Every mixed subgroup of D_n has even order of which half of them are rotation and half of them are reflection.*

Proof. Let H be a mixed subgroup of D_n containing a reflection s. Let A denote the set of rotations of H and B denote the set of all reflections of H. Define a map $\psi : A \to B$ by $\psi(r) = r \cdot s$ for all $r \in A$. If s_j is an element in B then $s_j \cdot s$ is an element of A and $\psi(s_j \cdot s) = s_j ss = s_j$. Hence ψ is onto. Also $\psi(r) = \psi(r') \implies rs = r's \implies r = r'$ and hence ψ is one-one. □

Theorem 3.3. *Every mixed subgroup of D_n is Dihedral.*

Proof. Let H be a mixed subgroup of D_n. By theorem 3.2 , $|H| = 2d$ for some d and $H \cap R_n =< r_{n/d} >$. Since order of H is $2d$ and $< r_{n/d} >$ is its subgroup of order d, we have $H =< r_{n/d} > \cup < r_{n/d} > s =< r_{n/d}, s >$, for some s in H. Since $(r_{n/d})^d = r_o, s^2 = r_0$ and $sr_{n/d}s^{-1} = (r_{n/d})^{-1}$, we have $H \equiv D_d$ and hence the proof. □

Corolary 3.1. *If H is a mixed subgroup of D_n then,*

1. $|H| = 2d$*, for some d which divides n.*
2. $H \equiv D_n =< r_{n/d}, s >$ *for some $s \in H$.*

Here we have a usual question: If d divides n, does there exist a subgroup of order $2d$? If it exists, how many?

Theorem 3.4. *If d divides n, the number of mixed subgroups of order $2d$ is $\frac{n}{d}$.*

Proof. By the corollary 3.1, it is clear that the mixed subgroups D_n of order $2d$ are $\{< r_{n/d}, s_j >: \ 0 \le j \le n-1\}$, all of them need not be distinct. Suppose $< r_{n/d}, s_i >=< r_{n/d}, s_j >$ for some $0 \le i, j \le n-1$.

$$\begin{aligned}
& < r_{n/d}, s_i > =< r_{n/d}, s_j > \\
\Longleftrightarrow & < r_{n/d} > \cup < r_{n/d} > s_i =< r_{n/d} > \cup < r_{n/d} > s_j \\
\Longleftrightarrow & < r_{n/d} > s_i =< r_{n/d} > s_j \\
\Longleftrightarrow & \ s_i s_j^{-1} \in < r_{n/d} > \\
\Longleftrightarrow & \ s_i s_j^{-1} = r_{kn/d} \ \text{ for some } 0 \le k \le d-1 \\
\Longleftrightarrow & \ s_i s_j = r_{kn/d} \\
\Longleftrightarrow & \ r_{i-j} = r_{kn/d} \\
\Longleftrightarrow & \ i - j \equiv \frac{kn}{d}\, mod(n) \ \text{ for some } 0 \le k \le d-1 \\
\Longleftrightarrow & \ d(i-j) \equiv 0\, mod(n) \\
\Longleftrightarrow & \ i - j \equiv 0\, mod(\frac{n}{d}) \\
\Longleftrightarrow & \ i \equiv j\, mod(\frac{n}{d})
\end{aligned}$$

Hence the number of distinct mixed subgroups of order $2d$ in D_n is $\frac{n}{d}$, namely $\{< r_{n/d}, s_i >: \ 0 \le i < \frac{n}{d}\}$. □

Theorem 3.5. *The number of mixed subgroups of D_n is $\sigma(n)$.*

Proof. By theorem 3.4, the mixed subgroups of D_n is $\sum_{d/n} \frac{n}{d} = \sum_{d/n} d = \sigma(n)$. They are $\cup_{d/n}\{< r_{n/d}, s_i >: \ 0 \le i \le \frac{n}{d} - 1\}$. □

From theorem 3.1 and theorem3.5 we have,

Theorem 3.6. *The number of subgroups of D_n is $\sigma(n) + d(n)$.*

Theorem 3.7. *The number of abelian subgroups of D_n is $d(n)+n$ if n is odd and $d(n)+n+\frac{n}{2}$ if n is even.*

Proof. All non-mixed subgroups of D_n are cyclic and hence abelian. So by theorem 3.1, there are $d(n)$ non- mixed abelian subgroups for D_n. If n is odd, by theorem 3.3 and corollary 3.1, the mixed abelian subgroups of D_n are of order 2 and hence there are n such subgroups. Thus if n is odd, the number of abelian subgroups of D_n is $d(n)+n$. If n is even, by theorem 3.3 and corollary 3.1, the mixed abelian subgroups of D_n are of order 2 and 4, and hence there are $n+\frac{n}{2}$ such subgroups. Thus if n is even, the number of abelian subgroups of D_n is $d(n)+n+\frac{n}{2}$. □

Theorem 3.8. *The number of cyclic subgroups of D_n is $d(n)+n$.*

Proof. By theorem 3.1, the number of non-mixed cyclic subgroups of D_n is $d(n)$. Also by theorem 3.3 and corollary 3.1,the mixed cyclic subgroups of D_n is n. Hence the number of cyclic subgroups of D_n is $d(n)+n$. □

4 Basis of D_n

A basis of D_n which contain both rotation and reflection is called a mixed basis and other basis is called non-mixed basis. By the definition 2.2, it is obvious that two rotations cannot generate D_n. Hence non-mixed basis of D_n are basis consisting of two reflections.

Theorem 4.1. *For $n \geq 3$, the number of mixed basis of D_n is $n\varphi(n)$.*

Proof. Let $s_j (0 \leq j \leq n-1)$ be a reflection in D_n. Then for any $0 \leq i \leq n-1$,

$$\begin{aligned}
< r_i, s_j > &= \{r_i^m s_j^t \ : 0 \leq m, t \leq n-1\} \quad ; \text{by theorem 2.9} \\
&= \{r_i^m s_j,\, r_i^m r_0 \ : 0 \leq m \leq n-1\} \quad ; \text{since } s_j^t = s_j \text{ or } r_0 \\
&= \{r_i^m s_j,\, r_i^m \ : 0 \leq m \leq n-1\} \\
&= \{r_i^m s_j \ : 0 \leq m \leq n-1\} \cup \{r_i^m \ : 0 \leq m \leq n-1\} \\
&=< r_i > s_j \cup < r_i >= D_n \ \text{ if and only if } (i,n)=1
\end{aligned}$$

Hence corresponding to each reflection $s_j (0 \leq j \leq n-1)$ there are $\varphi(n)$ mixed bases, namely $\{\{s_j, r_i\} : 0 \leq i \leq n-1 \text{ and } (i,n)=1\}$. So the number of mixed basis for $D_n (n \geq 3)$ is $n\varphi(n)$. □

Theorem 4.2. *For $n \geq 3$, the number of non-mixed basis of D_n is $\frac{n\varphi(n)}{2}$.*

Proof. Since the dimension of D_n is 2, any basis of D_n contain exactly two elements. The subgroup generated by two rotations always lies in R_n and hence cannot form a basis. Therefore any non- mixed basis of D_n contain exactly two reflections. : Let $s_j(0 \leq j \leq n-1)$ be a reflection in D_n. Then for any $0 \leq i \leq n-1$,

$$\begin{aligned} < s_i, s_j > &=< r_{i-j}s_j, s_j >=< r_{i-j}, s_j > \\ &\cong D_n \text{ if and only if } i-j \equiv k \; mod(n) \text{ and } (k,n)=1 \end{aligned}$$

Hence corresponding to each reflection $s_j(0 \leq j \leq n-1)$ there are $\varphi(n)$ non-mixed basis for D_n namely $\{\{s_{i+j}, s_j\} : 0 \leq i \leq n-1 \text{ and } (i,n)=1\}$. If $\{s_i, s_j\}$ is a mixed basis corresponding to the reflection s_i, then it is also a basis corresponding to the reflection s_j. Hence the number of non-mixed basis for $D_n(n \geq 3)$ is $\dfrac{n\varphi(n)}{2}$. □

Theorem 4.3. *For $n \geq 3$, the number of different basis for D_n is $\dfrac{3n}{2}\varphi(n)$.*

Proof. The collection of all different bases of $D_n(n \geq 3)$ is the union of all mixed and non-mixed bases. Hence the different bases of $D_n(n \geq 3)$ is $\dfrac{n\varphi(n)}{2} + n\varphi(n) = \dfrac{3n}{2}\varphi(n)$. □

5 Congugacy classes of D_n

In this section we will compute all conjugacy classes and class equation of Dihedral groups.

Theorem 5.1. *If n is odd, the number of conjugacy classes in D_n is $\dfrac{n+3}{2}$.*

Proof. Let $r_i(0 \leq i \leq n-1)$ be a rotation in D_n. Then

$$\begin{aligned} cl(r_i) &= \{r_j r_i r_j^{-1}, s_j r_i s_j^{-1} \; : \; 0 \leq j \leq n-1\} \\ &= \{r_j r_i r_{-j}, s_j r_i s_j \; : \; 0 \leq j \leq n-1\} \\ &= \{r_i, s_j r_i s_j \; : \; 0 \leq j \leq n-1\} \\ &= \{r_i, s_{j-i} s_j \; : \; 0 \leq j \leq n-1\} \\ &= \{r_i, r_{-i}\} \end{aligned}$$

Since n is odd, $r_i = r_{-i}$ if and only if $i = 0$. Therefore

$cl(r_0) = \{r_0\}$ and $cl(r_i) = \{r_i, r_{-i}\}$, a two element set, for all $1 \leq i \leq n-1$.

Also,

$$\begin{aligned} cl(s_0) &= \{r_j s_0 r_j^{-1}, s_j s_0 s_j^{-1} \ : \ 0 \le j \le n-1\} \\ &= \{r_j s_0 r_j^{-1}, s_j s_0 s_j \ : \ 0 \le j \le n-1\} \\ &= \{r_j s_0 r_{-j}, s_j s_0 s_j \ : \ 0 \le j \le n-1\} \\ &= \{s_{2j} \ : \ 0 \le j \le n-1\} \\ &= \{s_j \ : \ 0 \le j \le n-1\}, \text{ since } n \text{ odd.} \end{aligned}$$

Hence, if n is odd, $\{\{s_j \ : \ 0 \le j \le n-1\}, \{r_0\}, \{r_i, r_{-i} \ \} : \ 1 \le i \le (n-1)/2\}$ are the conjugacy classes of D_n. Thus if n is odd, the number of conjugacy class of D_n is $\dfrac{(n-1)}{2} + 2 = \dfrac{(n+3)}{2}$. □

Corolary 5.1. *The class equation of* D_n*(*n *odd) is* $1+2+2+\ldots+2+n = 2n$*, the summation runs over* $(n-1)/2$ *times.*

Theorem 5.2. *If* n *is even, the number of conjugacy classes in* D_n *is* $\dfrac{n+6}{2}$.

Proof. Let $r_i(0 \le i \le n-1)$ be a rotation in D_n. Then

$$\begin{aligned} cl(r_i) &= \{r_j r_i r_j^{-1}, s_j r_i s_j^{-1} \ : \ 0 \le j \le n-1\} \quad = \{r_j r_i r_{-j}, s_j r_i s_j \ : \ 0 \le j \le n-1\} \\ &= \{r_i, s_j r_i s_j \ : \ 0 \le j \le n-1\} \\ &= \{r_i, s_{j-i} s_j \ : \ 0 \le j \le n-1\} \\ &= \{r_i, r_{-i}\} \end{aligned}$$

Since n is even $r_i = r_{-i}$ if and only if $i = 0$ or $\dfrac{n}{2}$. Therefore

$cl(r_0) = \{r_0\}, cl(r_{n/2}) = \{r_{n/2}\}$ and $cl(r_i) = \{r_i, r_{-i}\},$ a two element set, for all $1 \le i \le n-1$ and $i \ne \dfrac{n}{2}$.

Also,

$$\begin{aligned} cl(s_0) &= \{r_j s_0 r_j^{-1}, s_j s_0 s_j^{-1} \ : \ 0 \le j \le n-1\} \\ &= \{r_j s_0 r_j^{-1}, s_j s_0 s_j \ : \ 0 \le j \le n-1\} \\ &= \{r_j s_0 r_{-j}, s_j s_0 s_j \ : \ 0 \le j \le n-1\} \\ &= \{s_{2j} \ : \ 0 \le j \le n/2-1\} \end{aligned}$$

Again,

$$\begin{aligned} cl(s_1) &= \{r_j s_1 r_j^{-1}, s_j s_1 s_j^{-1} \ : \ 0 \le j \le n-1\} \\ &= \{r_j s_1 r_j^{-1}, s_j s_1 s_j \ : \ 0 \le j \le n-1\} \\ &= \{r_j s_1 r_{-j}, s_j s_1 s_j \ : \ 0 \le j \le n-1\} \\ &= \{s_{2j+1} \ : \ 0 \le j \le n-1\} \\ &= \{s_{2j+1} \ : \ 0 \le j \le n/2-1\} \end{aligned}$$

Hence, if n is even,

$$\Big\{\{s_{2j} \ : \ 0 \le j < n/2\}, \{s_{2j+1} \ : \ 0 \le j < n/2\}, \{r_0\}, \{r_{n/2}\}, \\ \{r_i, r_{-i}\ \} : \ 1 \le i \le (n-2)/2\Big\}$$

are the conjugacy classes of D_n. Thus if n is even, the number of conjugacy class of D_n is $\dfrac{(n-2)}{2} + 4 = \dfrac{(n+6)}{2}$. □

Corolary 5.2. *The class equation of $D_n(n$ even $)$ is $1+1+2+2+\ldots+2+n/2+n/2 = 2n$, the summation runs over $(n-2)/2$ times.*

Corolary 5.3. *Each conjugacy class of D_n contains either rotations alone or reflections alone.*

Corolary 5.4. *The number of conjugacy classes of D_n which contain rotations alone is $\dfrac{(n+1)}{2}$ if n is odd and $\dfrac{(n+2)}{2}$ if n is even.*

Corolary 5.5. *The number of conjugacy classes of D_n which contain reflections alone is 1, namely D_n, if n is odd and is 2, namely $\Big\{\{s_{2j} \ : \ 0 \le j < n/2\}, \{s_{2j+1} \ : \ 0 \le j < n/2\}\Big\}$, if n is even.*

6 Normal subgroups of D_n

In this section we will describe all mixed and non-mixed normal subgroups of D_n.

Theorem 6.1. *The number of non-mixed normal subgroups of D_n is $d(n)$.*

Proof. Since R_n is a cyclic normal subgroup of D_n, by theorem 2.11, the non-mixed subgroups and non-mixed normal subgroup of D_n are same. Hence the number of non-mixed normal subgroups of D_n is $d(n)$. □

Theorem 6.2. *The number of mixed normal subgroups of D_n is 1 if n odd and 3 if n even.*

Proof. Since normal subgroups are union of conjugacy classes, a mixed normal subgroup contain at least one conjugacy class having reflection. If n is odd, there is only one conjugacy class having reflection, namely $\{s_j : 0 \leq j \leq n-1\}$. Therefore D_n is the only mixed normal subgroup of D_n if n is odd. If n even, $\{s_{2j} : 0 \leq j < n/2\}$ and $\{s_{2j+1} : 0 \leq j < n/2\}$ are the only conjugacy classes having reflection. Therefore $\{s_{2j}, r_{2j} : 0 \leq j < n/2\}, \{s_{2j+1}, r_{2j} : 0 \leq j < n/2\}$ and D_n are the only mixed normal subgroups of D_n if n is even. Therefore the number of mixed normal subgroups of D_n is 3 if n is even.

□

Corolary 6.1. *The number of normal subgroups of D_n is $d(n) + 1$ if n odd and $d(n) + 3$ if n even.*

7 Conclusion

In this paper, it is proved that the number of mixed basis and non-mixed basis for $D_n(n \geq 3)$ are $n\varphi(n)$ and $\frac{n\varphi(n)}{2}$ respectively, where $\varphi(n)$is the number of non- negative integers less than n and relatively prime to n. Also it is shown that the number of different bases for $D_n(n \geq 3)$ is $\frac{3n}{2}\varphi(n)$. If n is odd, the number of conjugacy classes in D_n is $\frac{n+3}{2}$ and if n is even, the number of conjugacy classes in D_n is $\frac{n+6}{2}$. Finally we have shown that the number of non-mixed normal subgroups of D_n is $d(n)$ and the number of mixed normal subgroups of D_n is 1 if n odd and 3 if n even.

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