Ratio Mathematica, 21, 2011, pp. 75-90

THE TRANSPOSITION AXIOM IN HYPERCOMPOSITIONAL STRUCTURES

Ch. G. Massouros [a,b] and G. G. Massouros [a,b]

[a] *Technological Institute of Chalkis, GR34400, Evia, Greece*
[b] *54, Klious st., GR15561, Cholargos-Athens, Greece*

ABSTRACT. The hypergroup (as defined by F. Marty), being a very general algebraic structure, was subsequently quickly enriched with additional axioms. One of these is the transposition axiom, the utilization of which led to the creation of join spaces (join hypergroups) and of transposition hypergroups. These hypergroups have numerous applications in geometry, formal languages, the theory of automata and graph theory.

This paper deals with transposition hypergroups. It also introduces the transposition axiom to weaker structures, which result from the hypergroup by the removal of certain axioms, thus defining the transposition hypergroupoid, the transposition semi-hypergroup and the transposition quasi-hypergroup. Finally, it presents hypercompositional structures with internal or external compositions and hypercompositions, in which the transposition axiom is valid. Such structures emerged during the study of formal languages and the theory of automata through the use of hypercompositional algebra.

AMS-Classification number: 20N20, 68Q70, 51M05

1. THE TRANSPOSITION AXIOM IN HYPERGROUPS

Hypercompositional structures are algebraic structures equipped with multivalued compositions, which are called hyperoperations or hypercompositions. A hypercomposition in a non-void set H is a function from the Cartesian product $H \times H$ to the powerset $P(H)$ of H. Hypercompositional structures came into being through the notion of the *hypergroup*. The hypergroup was introduced by F. Marty in 1934, during the 8th congress of the Scandinavian

Mathematicians [18]. F. Marty used hypergroups in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups. A hypergroup, which is a generalization of the group, satisfies the following axioms:

i. $(ab)c = a(bc)$ for all $a,b,c \in H$ (associativity),

ii. $aH = Ha = H$ for all $a \in H$ (reproduction).

Note that, if «·» is a hypercomposition in a set H and A, B are subsets of H, then $A \cdot B$ signifies the union $\bigcup_{(a,b)\in A\times B} a \cdot b$. In both cases, aA and Aa have the same meaning as $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified with its member a. In [18], F. Marty also defined the two induced hypercompositions (right and left division) that result from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|\,b} = \{x \in H \mid a \in xb\} \quad \text{and} \quad \frac{a}{b\,|} = \{x \in H \mid a \in bx\}.$$

It is obvious that the two induced hypercompositions coincide, if the hypergroup is commutative. For the sake of notational simplicity, W. Prenowitz [48] denoted division in commutative hypergroups by a / b. Later on, J. Jantosciak used the notation a / b for right division and $b \backslash a$ for left division [14]. Notations $a : b$ and $a..b$ have also been used correspondingly for the above two types of division [21].

In [14] and then in [15], a principle of duality is established in the theory of hypergroups. More precisely, two statements of the theory of hypergroups are dual statements, if each results from the other by interchanging the order of the hypercomposition, i.e. by interchanging any hypercomposition ab with the hypercomposition ba. One can observe that the associativity axiom is self-dual. The left and right divisions have dual definitions, thus they must be interchanged in a construction of a dual statement. Therefore, the following principle of duality holds:

Given a theorem, the dual statement resulting from interchanging the order of hypercomposition "·" (and, necessarily, interchanging of the left and the right divisions), is also a theorem.

This principle is used throughout this paper. The following properties are direct consequences of axioms (i) and (ii) and the principle of duality is used in their proofs [see also 20, 21]:

Property 1.1. *$ab \neq \varnothing$ is valid for all the elements a,b of a hypergroup H.*

P r o o f. Suppose that $ab = \varnothing$ for some $a,b \in H$. Per reproduction, $aH = H$ and $bH = H$. Hence, $H = aH = a(bH) = (ab)H = \varnothing H = \varnothing$, which is absurd.

Property 1.2. *$a/b \neq \varnothing$ and $a \backslash b \neq \varnothing$ for all the elements a,b of a hypergroup H.*

P r o o f. Per reproduction, $Hb = H$ for all $b \in H$. Hence, for every $a \in H$ there exists $x \in H$, such that $a \in xb$. Thus, $x \in a/b$ and, therefore, $a/b \neq \varnothing$. Dually, $a \backslash b \neq \varnothing$.

Property 1.3. *In a hypergroup H, the non-empty result of the induced hypercompositions is equivalent to the reproduction axiom.*

P r o o f. Suppose that $x/a \neq \varnothing$ for all $x,a \in H$. Thus, there exists $y \in H$, such that $x \in ya$. Therefore, $x \in Ha$ for all $x \in H$, and so $H \subseteq Ha$. Next, since $Ha \subseteq H$ for all $a \in H$, it follows that $H = Ha$. Per duality, $H = aH$. Conversely now, per Property 1.2, the reproduction axiom implies that $a/b \neq \varnothing$ and $a \backslash b \neq \varnothing$ for all a,b in H.

Property 1.4. *In a hypergroup H equalities (i) $H = H/a = a/H$ and (ii) $H = a \backslash H = H \backslash a$ are valid for all a in H.*

P r o o f. (i) Per Property 1.1, the result of hypercomposition in H is always a non-empty set. Thus, for every $x \in H$ there exists $y \in H$, such that $y \in xa$, which implies that $x \in y/a$. Hence, $H \subseteq H/a$. Moreover, $H/a \subseteq H$. Therefore, $H = H/a$. Next, let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a/y$. Hence, $H \subseteq a/H$. Moreover, $a/H \subseteq H$. Therefore, $H = a/H$. (ii) follows by duality.

The hypergroup (as defined by F. Marty), being a very general algebraic structure, was enriched with additional axioms, some less and some more powerful. These axioms led to the creation of more specific types of hypergroups.

One of these axioms is the transposition axiom. It was introduced by W. Prenowitz, who used it in commutative hypergroups. W. Prenowitz called the resulting hypergroup *join space* [48]. Thus, join space (or join hypergroup) is defined as a commutative hypergroups H, in which
$a/b \cap c/d \neq \varnothing$ implies $ad \cap bc \neq \varnothing$ for all $a,b,c,d \in H$ (transposition axiom) is true. This type of hypergroup has been widely utilized in the study of

Geometry via the use of hypercompositional algebra tools which function without any need of Cartesian or other coordinate-type systems [48, 49]. Later, J. Jantosciak generalized the transposition axiom in an arbitrary hypergroup as follows:

$$b \backslash a \cap c / d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing \text{ for all } a,b,c,d \in H.$$

He named this particular hypergroup *transposition hypergroup* and studied its properties in [14].

The transposition axiom also emerged in the hypercompositional structures which surfaced during the study of formal languages through the use of hypercompositional algebra tools [see, for example, 6, 7, 27, 32, 33, 35, 36, 42, 44; see also 7, 12, 13 for other occurrences of the join space]. The manner in which these structures emerged will be discussed in paragraph 3. In the present paragraph we will only deal with the mathematical description of join space classes which resulted from the theory of formal languages and automata. The basic concept which generated these types of join spaces is the incorporation of a special neutral element e into a transposition hypergroup. This neutral element e possesses the property $ex = xe \subseteq \{e, x\}$ for every element x of the hypergroup and was named *strong*. Thus, the fortification of transposition hypergroups by an identity element came into being.

Therefore a *fortified transposition hypergroup* is a transposition hypergroup H for which the following axioms are valid:

i. $ee = e$,
ii. $x \in ex = xe$ for all $x \in H$,
iii. for every $x \in H - \{e\}$ there exists a unique $y \in H - \{e\}$, such that $e \in xy$ and, furthermore, y satisfies $e \in yx$.

If the commutativity is valid in H, then H is called a *fortified join hypergroup*.

Theorem 1.1. *In a fortified transposition hypergroup H, the identity is strong.*

P r o o f. It must be proven that $ex \subseteq \{e,x\}$ for all x in H. This is true for $x = e$. Let $x \neq e$. Suppose that $y \in ex$. Then, $x \in e \backslash y$. However, $x \in e / x^{-1}$, since $e \in xx^{-1}$. Thus, $e \backslash y \approx e / x^{-1}$ and transposition yields $e = ee \approx yx^{-1}$. Hence, $y \in \{x,e\}$.

Theorem 1.2. *In a fortified transposition hypergroup H, the strong identity is unique.*

P r o o f. Suppose that u is an identity distinct from e. It then follows that there exists z distinct from u, such that $u \in ez$. But, $ez \subseteq \{e,z\}$, so $u \in \{e,z\}$, which is a contradiction.

It is worth noting that a transposition hypergroup H becomes a quasicanonical hypergroup, if it incorporates a scalar identity, i.e. an identity e with the property $ex = xe = x$ for all x in H. Moreover, a join hypergroup is a canonical hypergroup, if it contains a scalar identity [14, 20, 23].

A hypergroup H with a strong identity e has a natural partition. Let

$A = \{x \in H \mid ex = xe = \{e,x\}\}$ and C=$\{x \in H - \{e\} \mid ex = xe = e\}$.

Then, $H = A \cup C$ and $A \cap C = \varnothing$. A member of A is an *attractive element* and a member of C is a *canonical element*. See [39] for the origin of terminology.

Fortified join hypergroups and fortified transposition hypergroups have been studied in a series of papers [see, for example, 15, 22, 33, 37, 39, 43], in which several very interesting properties of these types of hypergroups were revealed. The following was proven, among others [15]:

Structure Theorem. *A transposition hypergroup H containing a strong identity e is isomorphic to the expansion of a quasicanonical hypergroup $C \cup \{e\}$ by the transposition hypergroup A of all attractive elements through the idempotent e.*

Moreover, from the theory of automata resulted the *transposition polysymmetrical hypergroup* [24, 42, 45], i.e. a transposition hypergroup H, having an identity (or neutral) element e, such that $ee = e$, $x \in ex = xe$ for all $x \in H$ and also, for every $x \in H - \{e\}$ there exists at least one element $x' \in H - \{e\}$, (called symmetric or two-sided inverse of x), such that $e \in xx'$ and $e \in x'x$. The set of the symmetric elements of x is denoted by $S(x)$ and is called the *symmetric set* of x. A commutative transposition polysymmetrical hypergroup is called a *join polysymmetrical hypergroup*.

Theorem 1.3. *If a polysymmetrical transposition hypergroup contains a strong identity e, then this identity is unique.*

Analytical examples of the above hypergoup types are presented in [28]. A thorough study of transposition hypergroups with idempotent identity is presented [30]

2. THE TRANSPOSITION AXIOM IN HYPERGROUPOIDS

In the previous paragraph it was mentioned that the hypergroup was enriched with further axioms, a fact which led to the creation of specific hypergroup families. However, mathematical research also followed the reverse course. Certain axioms were removed from the hypergroup and the resulting weaker structures were studied. Thus, the pair $(H,\cdot)$, where H is a non-empty set and "$\cdot$" a hypercomposition, was named *partial hypegroupoid*, while it was called *hypegroupoid* if $ab \neq \varnothing$ for all $a,b \in H$. A hypergroupoid in which the associativity is valid, was called *semi-hypergroup*, while it was called *quasi-hypergroup*, if only the reproductivity is valid. The quasi-hypergroups in which the weak associativity is valid, i.e. $(ab)c \cap a(bc) \neq \varnothing$ for all $a,b,c \in H$, were named *H_V-groups* [55]. Certain properties of these structures, which are analogous to those of hypergroups, are presented herein.

Property 2.1. *If the weak associativity is valid in a hypergroupoid, then this hypergroupoid is not partial.*

P r o o f. Suppose that $ab = \varnothing$ for some $a,b \in H$. Then, $(ab)c = \varnothing$ for any $c \in H$. Therefore, $(ab)c \cap a(bc) = \varnothing$, which is absurd. Hence, ab is non-void.

The following is a direct consequence of the above property:

Property 2.2. *The result of the hypercomposition in an H_V-group H is always a non-empty set.*

Property 2.3. *A hypergroupoid H is a quasi-hypergroup, if the results of induced hypercompositions in it are non-void.*

P r o o f. Suppose that $x / a \neq \varnothing$ is valid for all $x,a \in H$. Then, there exists $y \in H$, such that $x \in ya$. Therefore, $x \in Ha$ for all $x \in H$ and so $H \subseteq Ha$. But $Ha \subseteq H$ is also valid for all $a \in H$. Hence, $H = Ha$. By duality, $aH = H$. Thus, H is a quasi-hypergroup.

Property 2.4. *$a / b \neq \varnothing$ and $b \backslash a \neq \varnothing$ is valid for all the elements a,b of a quasi-hypergroup H.*

P r o o f. Per equality $H = Hb$, there exists $y \in H$, such that $a \in yb$ for every $a \in H$. Thus, $y \in a / b$ and, therefore, $a / b \neq \varnothing$. $b \backslash a \neq \varnothing$, per the principle of duality.

Property 2.5. *In a quasi-hypergroup* H*, the equalities* $H = a/H = H\backslash a$ *are valid for all* a *in* H.

P r o o f. Let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a/y$. Hence, $H \subseteq a/H$. Moreover, $a/H \subseteq H$. Therefore, $H = a/H$. The other equality follows by duality.

Property 2.6. *In any non-partial hypergroupoid H, the equalities* $H = H/a = a\backslash H$ *are valid for all* a *in* H.

P r o o f. Since the result of the hypercomposition in a non-partial hypergroupoid is always a non-empty set, there exists $y \in H$ such that $y \in xa$ for every $x \in H$. This implies that $x \in y/a$. Hence, $H \subseteq H/a$. Moreover, $H/a \subseteq H$. Therefore, $H = H/a$. The other equality follows by duality.

The following is a direct consequence of Properties 2.5 and 2.6 above:

Property 2.7. *In any* H_V*-group H, the equalities (i)* $H = H/a = a/H$ *and (ii)* $H = a\backslash H = H\backslash a$ *are valid for all* a *in* H.

Extensive work has been done on the construction of hypergroupoids, on their enumeration and on the study of their structure (see, for example, [3, 4, 5, 6, 9, 10, 11, 29, 50, 51, 52, 54]). As mentioned above, this direction pertained to researching hypercompositional structures resulting from the weakening of the structure of the hypergoup. The opposite direction pertained to researching hypercompositional structures resulting from the reinforcement of the structure of the hypergoup. These two directions are combined in [31], via the introduction of the transposition axiom into the H_V-group, thus leading to the following definition:

Definition 2.1. *An* H_V*-group (H,·) is called* ***transposition*** $\boldsymbol{H_V}$***-group****, if it satisfies the transposition axiom:*

$$b\backslash a \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing \text{ for all } a,b,c,d \in H.$$

A transposition H_V*-group (H,·) is called* ***join*** $\boldsymbol{H_V}$***-group,*** *if H is a commutative* H_V*-group, while it is called* ***weak join*** $\boldsymbol{H_V}$***-group,*** *if H is an* H_V*-commutative group.*

The fortified transposition H_V-group was also defined in [31], in a manner analogous to the definition of the fortified transposition hypergroup, as follows:

Definition 2.2. *A transposition* H_V*-group (H,·) is called* ***fortified****, if H contains an element* e*, which satisfies the axioms:*

i. $ee = e$,

ii. $x \in ex = xe$ *for all* $x \in H$,

iii. *for every* $x \in H - \{e\}$ *there exists a unique* $y \in H - \{e\}$, *such that* $e \in xy$ *and, furthermore, y satisfies* $e \in yx$.

If "·" is commutative, then H is called a ***fortified join H$_V$-group***.

Properties of the structure above, as well as relevant examples are presented in [31]. The elements of the fortified transposition H$_V$-group are partitioned into canonical and attractive, exactly as in hypergroups.

Proposition 2.1. *Let H be a fortified transposition H$_V$-group and suppose that* x, y *are attractive elements with* $y \neq x^{-1}$. *Then,* $x, y \in xy$ *and* $x, y \in yx$.

P r o o f. Since x is an attractive element, $ex = xe = \{e, x\}$ is valid. Therefore, $e / x = x \backslash e = \{e, x^{-1}\}$. Moreover, $y / y = \{z \mid y \in zy\}$. Hence, $e \in y / y$. Thus, $y / y \cap x \backslash e \neq \varnothing$ which, per the transposition axiom, results into $ey \cap yx \neq \varnothing$ or, equivalently, $\{e, y\} \cap yx \neq \varnothing$. Since $y \neq x^{-1}$, it follows that $y \in yx$. Similarly, $x \in yx$ and, per duality, $x, y \in xy$.

Corollary 2.1. *A fortified transposition H$_V$-group containing exclusively attractive elements is weakly commutative.*

As can be observed, the transposition axiom is not dependent on the two hypergroup axioms (asssosiativity and reproduction) and their consequences. Therefore, the transposition axiom can be introduced even into a partial hypergroupoid. Thus, the notions of the *transposition hypergroupoid*, of the *transposition quasi-hypergroup* and of the *transposition semi-hypergroup* emerge. If the commutativity is also valid in the above, the notions of the *join hypergroupoid*, of the *join quasi-hypergroup* and of the *join semi-hypergroup* emerge as well. The following proposition is analogous to the one used in [31] for the construction of transposition H$_V$-groups. The proof of this proposition, as well as of Proposition 2.3 below, is quite straightforward, albeit long, since all the possible cases must be verified.

Proposition 2.2. *Let H be a hypergroupoid (either partial or non- partial) or a quasi-hypergroup. Also, let an arbitrary subset* I_{ab} *of H be associated to each pair of elements* $(a, b) \in H^2$. *If* $\bigcap_{a,b \in H} I_{ab} \neq \varnothing$, *then H endowed with the hypercomposition*: $a * b = ab \cup I_{ab}$, $a, b \in H$ *is a transposition hypergroupoid or a transposition quasi-hypergroup respectively, while it is a join hypergroupoid or* *a join quasi-hypergroup, if the commutativity is valid in* H *and* $I_{ab} = I_{ba}$ *for all* $a, b \in H$.

Corollary 2.1. *If H is a hypergroupoid (either partial or non- partial) or a quasi-hypergroup and w is an arbitrary element of H, then H endowed with the hypercomposition*

$$x * y = xy \cup \{x, y, w\}$$

is a transposition hypergroupoid or a transposition quasi-hypergroup respectively, while it is a join hypergroupoid or a join quasi-hypergroup, if the commutatity is valid in H.

Proposition 2.3. *Let* H *be a set with more that two elements and let* w *be an arbitrary element in* H. *Two hypercompositions are defined in* H *as follows:*

$a \circ_l b = \{a, w\}$ *for all* $a, b \in H$ *and* $a \circ_r b = \{b, w\}$ *for all* $a, b \in H$.

Then, $(H, \circ_l)$ *and* $(H, \circ_r)$ *are transposition semi-hypergroups.*

3. THE TRANSPOSITION AXIOM IN HYPERCOMPOSITIONAL STRUCTURES WITH INTERNAL COMPOSITIONS

M. Krasner was the first to expand hypercompositional structures via the creation of structures containing composition and hypercompositions. Thus, in 1956, he replaced the additive group of a field with a special hypergroup, thereby introducing the *hyperfield*. He then used the hyperfield as the proper algebraic tool, in order to define a certain approximation of complete valued fields by sequences of such fields [16, 17]. Later, he introduced a more general structure, which relates to hyperfields in the same way rings relate to fields. He called this structure *hyperring*. Additional hypercompositional structures, similar to the above, introduced by various researchers, soon followed. Examples of those are the ***superring*** and the ***superfield***, in which both the addition and the multiplication are hypercompositions [47]. Additionally, the study of formal languages introduced structures in which the hypercompositional component is a join hypergroup.

Indeed, let A be an alphabet, let A^* denote the set of the words defined over A and let λ be the empty word. Then, set A^* is a semigroup with regard to the concatenation of the worlds. This semigroup has λ as its neutral element, since $\lambda a = a\lambda = a$ for all a in A^*. In addition, the expression $a + b$, where a and b are words over A, is used in formal languages theory to denote «either a or

b ». Based on the fact that $a+b$ is in essence a biset, hypercomposition $a+b=\{a,b\}$ appears in the word set A^*. It has been proven that A^* is a join hypergroup [32, 33] with regard to this hypercomposition. This hypergroup was named *B(iset)-hypergroup*. However, since A^* is a semigroup with regard to world concatenation and since it has been proven that world concatenation is distributive with regard to the hypercomposition, a new hypercompositional structure thus emerged. This structure was named hyperringoid.

Definition 3.1. *A **hyperringoid** is a non-empty set Y equipped with an operation "·" and a hyperoperation "+", such that:*

i) (Y,+) is a hypergroup,

ii) (Y, ·) is a semigroup,

iii) the operation "·" is distributive on both sides of the hyperoperation "+ ".

If (Y,+) is a join hypergroup, (Y,+,·) is called *join hyperringoid*. The join hyperringoid that results from a B-hypergroup is called *B-hyperringoid* and the special B-hyperringoid that appears in the theory of formal languages is the *linguistic hyperringoid*. Join hyperringoids are studied in [38, 40, 41].

Another notion in the theory of formal languages is the null word, the introduction of which resulted from the theory of automata. The null word is symbolized with 0 and is bilaterally absorbing with regard to word concatenation. Therefore, the extension of the composition and of the hypercomposition onto $A^*\cup\{0\}$ results into the following:

$$0a=a0=0,\ \ 0+a=a+0=\{0,a\} \quad \text{for all } a\in A^*.$$

With these extensions, structure $(A^*\cup\{0\},+,\cdot)$ continues to be a hyperringoid, which, however now also has an absorbing element. The additive structure of these hyperringoid comprises a fortified join hypergroup. Thus, a new hypercompositional structure appeared:

Definition 3.2. *If the additive part of a hyperringoid is a fortified join hypergroup whose zero element is bilaterally absorbing with respect to the multiplication, then, this hyperringoid is named **join hyperring**. A **join hyperdomain** is a join hyperring which has no divisors of zero. A **proper join hyperring** is a join hyperring which is not a Krasner hyperring. A join hyperring K is called **join hyperfield** if $K^*=K-\{0\}$ is a multiplicative group.*

Join hyperrings are studied in [25, 41].

Moreover, hypercompositional structures having external operations and hyperoperations on hypergroups appeared [see, for example, 1, 2, 19, 56]. The notions of the set of operators and hyperoperators from a hyperringoid Y over an arbitrary non-void set M were introduced in [33, 34], in order to describe the action of the state transition function in the theory of Automata. Y is a set of *operators* over M, if there exists an external operation from $M \times Y$ to M, such that $(s\kappa)\lambda = s(\kappa\lambda)$ for all $s \in \mathrm{M}$ and $\kappa, \lambda \in Y$ and, moreover, $s1 = s$ for all $s \in \mathrm{M}$, when Y is a unitary hyperringoid. If there exists an external hyperoperation from $M \times Y$ to $P(M)$ which satisfies the above axiom, with the variation that $s \in s1$ when Y is a unitary hyperringoid, then Y is a set of *hyperoperators* over M. If M is a hypergroup and Y is a hyperringoid of operators over M, such that, for each $\kappa, \lambda \in Y$ and $s, t \in M$, the axioms: (i) $(s+t)\lambda = s\lambda + t\lambda$, (ii) $s(\kappa + \lambda) \subseteq s\kappa + s\lambda$ hold, then M is called *right hypermoduloid* over Y. If Y is a set of hyperoperators, then M is called *right supermoduloid*. If the second of the above axioms holds as an equality, then the hypermoduloid is called *strongly distributive*. There is a similar definition of the *left hypermoduloid* and the *left supermoduloid* over Y, in which the elements of Y operate from the left side. When M is both right and left hypermoduloid (resp. supermoduloid) over Y, it is called *Y-hypermoduloid* (resp. *Y-supermoduloid*) [33, 34]. If M is a canonical hypergroup, the set of operators Y is a hyperring and, if $s1 = s$, $s0 = 0$ for all $s \in M$, then M is named *right hypermodule,* while it is named *right supermodule* if Y is a set of hyperoperators [26]. A study of external operations and hyperoperations on hypergroups is carried out in [26].

BIBLIOGRAPHY.

1. R. Ameri, M. M. Zahedi: «Fuzzy Subhypermodules over Fuzzy Hyperrings», *Proceedings of the 6th International Congress in Algebraic Hyperstructures and Applications*, Publ. Democritus Univ. of Thrace (1997) pp. 1-14
2. R. Ameri: «On categories of hypergroups and hypermodules», *Journal of Discrete Mathematical Sciences & Cryptography* Vol. 6, No 2-3 (2003) pp. 121-123.
3. P. Corsini: «Binary relations and hypergroupoids», *Italian J. Pure & Appl. Math.* 7, (2000) 11-18.
4. P. Corsini: «On the hypergroups associated with binary relations», *Mult. Valued Log.*, 5 (2000), 407-419.
5. J. Chvalina: «Commutative hypergroups in the sense of Marty and ordered sets», *Proceedings of the Summer School on General Algebra and Ordered Sets*, 1994, Olomouc, Czech Republic, pp. 1930.
6. J. Chvalina: «Relational Product of Join Spaces determined by quasi-orderings», *Proceedings of the 6th International Congress on Algebraic Hyperstructures and Applications*, Publ. Democritus Univ. of Thrace (1997), pp. 15-23.
7. J. Chvalina, L. Chvalinova: «State hypergroups of Automata» *Acta Mathematica et Informatica Univ. Ostraviensis 4* (1996) pp. 105-120.
8. Chvalina, Jan, Hošková, Šárka. «Modelling of join spaces with proximities by first-order linear partial differential operators», *Ital. J. Pure Appl. Math.* No. 21 (2007), pp. 177-190.
9. I. Cristea, M. Stefanescu: «Binary relations and reduced hypergroups», *Discrete Math*. 308 (2008), 3537{3544, doi:10.1016/j.disc.2007.07.01
10. I. Cristea, M. Ştefãnescu, C. Angheluţă: «About the fundamental relations defined on the hypergroupoids associated with binary relations», *European Journal of Combinatorics* Volume 32, Issue 1, (2011) pp. 72-81.
11. I. Cristea, M. Jafarpourd, S. S. Mousavic, A. Soleymanid: «Enumeration of Rosenberg hypergroups», Computers and Mathematics with Applications, 32 (2011), pp. 72-81.
12. S. Hoskova, J. Chvalina: «Discrete transformation hypergroups and transformation hypergroups with phase tolerance space, *Discrete Math.*, 308 (2008), 4133{4143, doi:10.1016/j.disc.2007.08.005.
13. S. Hoskova: «Abelization of join spaces of affine transformation of ordered field with proximity», *Appl. Gener. Topology* 6, No 1 (2005), 57-65.
14. J. Jantosciak: «Transposition hypergroups, Noncommutative Join Spaces», *Journal of Algebra* 187(1997), 97-119.

15. J. Jantosciak, Ch. G. Massouros: «Strong Identities and fortification in Transposition hypergroups», *Journal of Discrete Mathematical Sciences and Cryptography* 6, No 2-3(2003), 169-193.
16. M. Krasner: «Approximation des corps valués complets de caractéristique p≠0 par ceux de caractéristique 0» *Colloque d' Algèbre Supérieure* (Bruxelles, Decembre 1956), Centre Belge de Recherches Mathématiques, Établissements Ceuterick, Louvain, Librairie Gauthier-Villars, Paris, 1957, pp. 129-206.
17. M. Krasner: «A class of hyperrings and hyperfields» *Internat. J. Math. & Math. Sci.* vol. 6, no. 2, 1983, pp. 307-312.
18. F. Marty: «Sur un généralisation de la notion de groupe». *Huitième Congrès des mathématiciens Scand*., Stockholm 1934, pp. 45-49.
19. Ch. G. Massouros: «Free and cyclic hypermodules», *Annali Di Mathematica Pura ed Applicata*, Vol. CL. 1988, pp. 153-166.
20. Ch. G. Massouros: «Hypergroups and convexity», *Riv. di Mat. pura ed applicata* 4, (1989) pp. 7-26.
21. Ch. G. Massouros, «On the semi-subhypergroups of a hypergroup», *Internat. J. Math. and Math. Sci*. 14, No 2 (1991), pp. 293-304.
22. Ch. G. Massouros: «Normal homomorphisms of Fortified Join Hypergroups» *Proceedings of the 5th International Congress on Algebraic Hyperstructures and Applications*, Hadronic Press (1994), pp. 133-142.
23. Ch. G. Massouros: «Canonical and Join Hypergroups», *Analele Stiintigice Ale Universitatii "AL.I.CUZA"*, Tom. XLII Matematica, fasc. 1 (1996) pp. 175-186.
24. Ch. G. Massouros, G. G. Massouros: «Transposition Polysymmetrical Hypergroups with Strong Identity», *Journal of Basic Science* Vol. 4, No. 1 (2008) pp. 85-93.
25. Ch. G. Massouros, G. G. Massouros: «On Join Hyperrings», *Proceedings of the 10th International Congress on Algebraic Hyperstructures and Applications*, Brno, Czech Republic 2008, pp. 203-216.
26. Ch. G. Massouros, G. G. Massouros: «Operators and Hyperoperators acting on Hypergroups», *Proceedings of the International Conference on Numerical Analysis and Applied Mathematics*, ICNAAM 2008, American Institute of Physics (AIP) Conference Proceedings, pp. 380-383.
27. Ch. G. Massouros, G. G. Massouros: «Hypergroups Associated with Graphs and Automata», *Proceedings of the International Conference on Numerical Analysis and Applied Mathematics, ICNAAM 2009*, American Institute of Physics (AIP) Conference Proceedings, pp. 164-167.
28. Ch. G. Massouros, G. G. Massouros: «Identities in Multivalued Algebraic Structures», *Proceedings of the International Conference on Numerical*

Analysis and Applied Mathematics, ICNAAM 2010, American Institute of Physics (AIP) Conference Proceedings, pp. 2065-2068.

29. Ch. G. Massouros, Ch. G. Tsitouras: «Enumeration of hypercompositional structures defined by binary relations», *Italian J. Pure & Appl. Maths.*, to appear
30. Ch. G. Massouros: «Transposition hypergroups with idempotent identity» Submitted for publication.
31. Ch. G. Massouros, A. Dramalidis: «Transposition H_v-Groups» Submitted for publication.
32. G. G. Massouros, J. Mittas: «Languages, Automata and hypercompositional structures», *Proceedings of the 4th International Congress in Algebraic Hyperstructures and Applications,* World Scientific (1991), pp. 137-147.
33. G. G. Massouros: «Automata, Languages and hypercompositional structures» Doctoral Thesis, National Technical University of Athens, 1993.
34. G. G. Massouros: «Automata and Hypermoduloids», *Proceedings of the 5th International Congress in Algebraic Hyperstructures and Applications*, Hadronic Press (1994), pp. 251-266.
35. G. G. Massouros: «Hypercompositional Structures in the Theory of the Languages and Automata», *An. stiintifice Univ. Al. I. Cuza, Iasi, Informatica*, t. iii,(1994), pp. 65-73.
36. G. G. Massouros: «A new approach to the theory of Languages and Automata», *Proceedings of the 26th Annual Iranian Mathematics Conference*, Kerman, Iran, 2,(1995) pp. 237-239.
37. G. G. Massouros, Ch. G. Massouros, J. D. Mittas: «Fortified Join Hypergroups», *Annales Matematiques Blaise Pascal*, Vol 3, no 2 (1996) pp. 155-169.
38. G.G. Massouros: «Solving equations and systems in the environment of a Hyperringoid», *Proceedings of the 6th International Congress on Algebraic Hyperstructures and Applications*, Publ. Democritus Univ. of Thrace (1997), pp. 103-113.
39. G. G. Massouros: «The subhypergroups of the Fortified Join Hypergroup», *Italian J. Pure & Appl. Maths.*, no 2, (1997) pp. 51-63.
40. G. G. Massouros: «The Hyperringoid», *Multiple Valued Logic*, Vol. 3 (1998) pp. 217-234
41. G. G. Massouros, Ch. G. Massouros: «Homomorphic Relations on Hyperringoids and Join Hyperrings», *Ratio Mathematica* No. 13 (1999) pp. 61-70.
42. G. G. Massouros: «Hypercompositional Structures from the Computer Theory», *Ratio Matematica*, No 13, (1999) pp. 37-42.

43. G. G. Massouros: «Monogene Symmetrical Subhypergroups of the Fortified Join Hypergroup», *Advances in Generalized Structures Approximate Reasoning and Applications*, published by the cooperation of: University of Teramo, Romanian Society for Fuzzy Systems and BMFSA Japan (2001) pp. 45 – 53.
44. G. G. Massouros: «On the attached hypergroups of the order of an automaton», *Journal of Discrete Mathematical Sciences & Cryptography* Vol. 6, No 2-3 (2003) pp. 207-215.
45. G. G. Massouros, F. A. Zafiropoulos, Ch. G. Massouros: «Transposition Polysymmetrical hypergroups», *Proceedings of the 8^{th} International Congress on Algebraic Hyperstructures and Applications*, Spanidis Press (2003), pp. 191-202.
46. J. D. Mittas: «Hypergroupes canoniques», *Mathematica Balkanica*, 2 (1972) pp. 165-179.
47. J. D. Mittas: «Sur certaines classes de structures hypercompositionnelles», *Proceedings of the Academy of Athens*, 48, (1973), pp. 298 - 318.
48. W. Prenowitz: «A Contemporary Approach to Classical Geometry». *Amer. Math. Month.* 68, no 1, part II (1961) pp. 1-67.
49. W. Prenowitz , J. Jantosciak: «Join Geometries. A Theory of convex Sets and Linear Geometry», *Springer - Verlag*, 1979.
50. I. Rosenberg: «Hypergroups and join spaces determined by relations», *Italian J. Pure & Appl. Math.*, 4, (1998) pp. 93 - 101.
51. Mario De Salvo, Giovanni Lo Faro: «Hypergroups and binary relations» *Mult.-Valued Log.* 8, No.5-6, (2002) pp. 645-657.
52. Mario De Salvo, Giovanni Lo Faro: «A new class of hypergroupoids associated to binary relations» *J. Mult.-Val. Log. Soft Comput.* 9, No. 4, (2003) pp. 361-375.
53. Ch. G. Tsitouras, Ch. G. Massouros: «On enumeration of hypergroups of order 3», *Computers and Mathematics with Applications* Vol. 59 (2010) pp. 519-523.
54. Ch. G. Tsitouras, Ch. G. Massouros: «Enumeration of Rosenberg type Hypercompositional structures defined by binary relations» Submitted for publication.
55. T. Vougiouklis: «The fundamental relation in hyperrings. The general hyperfield», *Proceedings of the 4^{th} International Congress on Algebraic Hyperstructures and Applications*, World Scientific (1991), pp. 209-217.
56. D. Zofota: «Feeble Hypermodules over a Feeble Hyperring», *Proceedings of the 5^{th} International Congress on Algebraic Hyperstructures and Applications*, Hadronic Press (1994), pp.207-214.

Ch. G. Massouros,
e mail: masouros@teihal.gr
URL: http://www.teihal.gr/gen/profesors/massouros/index.htm

G. G. Massouros,
e mail: germasouros@gmail.com
URL: https://sites.google.com/site/gerasimosgmassouros/