

Bar and Theta Hyperoperations

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Abstract

In questionnaires the replacement of the scale of Likert by a bar was suggested in 2008 by Vougiouklis & Vougiouklis. The use of the bar was rapidly accepted in social sciences. The bar is closely related with fuzzy theory and has several advantages during both the filling-in questionnaires and mainly in the research processing. In this paper we relate hyperstructure theory with questionnaires and we study the obtained hyperstructures which are used as an organising device of the problem.

Key words: Hyperstructures, H_v -structures, hopes, ∂ -hopes.

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1. Introduction

Hyperstructures are called the algebraic structures equipped with at least one hyperoperation i.e. a multivalued operation. We have abbreviated the 'hyperoperation' by 'hope' [24]. Therefore, if in a set H at least one hope $\cdot: H \times H \rightarrow P(H) - \{\emptyset\}$ is defined, then (H, \cdot) is called a hypergroupoid. The H_v -structures introduced in 1990 [15], is the largest class of hyperstructures. The H_v -structures satisfy the *weak axioms* where the non-empty intersection replaces the equality. In (H, \cdot) we abbreviate by

WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and by

COW the *weak commutativity*: $xy \cap yx \neq \emptyset, \forall x, y \in H$.

The hyperstructure (H, \cdot) is called H_v -semigroup if it is WASS, and it is called H_v -group if it is reproductive H_v -semigroup, i.e. $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called H_v -ring if both hopes $(+)$ and (\cdot) are WASS, the reproduction axiom is valid for $(+)$ and (\cdot) is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \emptyset, \quad (x+y)z \cap (xz+yz) \neq \emptyset, \quad \forall x, y, z \in R.$$

The main tool to study all hyperstructures are the fundamental relations β^* , γ^* and ε^* , which are defined, in H_V -groups, H_V -rings and H_V -vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space, resp., [17]. An element is called *single* if its fundamental class is singleton.

A way to find the fundamental classes is given by analogous theorems to the following [17],[18],[19],[20],[21],[5]:

Theorem. Let (H, \cdot) be an H_V -group and denote U the set of all finite products of elements of H . We define the relation β in H by setting $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then β^* is the transitive closure of β .

Analogous theorems for the relations γ^* in H_V -rings and ε^* in H_V -modules and H_V -vector spaces, are also proved. These relations were introduced and first studied by T.Vougiouklis, see [17]. One can see generalizations of the classical hyperstructure theory in several papers and books as [3],[4],[6],[17].

Fundamental relations are used for general definitions [17],[20]. Thus, in the general definition of the H_V -field, the γ^* is used: An H_V -ring $(R, +, \cdot)$ is called *H_V -field* if R/γ^* is a field. This definition includes all the well known definitions of hyperfields [15], [17], as special cases.

Motivations. The motivation for H_V -structures is the following: We know that the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group by any subgroup is a hypergroup. Now, the quotient of a group by any partition (or equivalently to any equivalence relation) is an H_V -group. This is the motivation to introduce the H_V -structures [15].

Specifying this motivation we remark: Let (G, \cdot) be a group and R be an equivalence relation (or a partition) in G , then $(G/R, \cdot)$ is an H_V -group, therefore we have the quotient $(G/R, \cdot)/\beta^*$ which is a group, the fundamental one. Remark that the classes of the fundamental group $(G/R, \cdot)/\beta^*$ are a union of some of the R -classes. Otherwise, the $(G/R, \cdot)/\beta^*$ has elements classes of G where they form a partition which classes are larger than the classes of the original partition R .

Let (H, \cdot) , $(H, *)$ be H_V -semigroups on the same set. (\cdot) is called *smaller* than $(*)$, and $(*)$ *greater* than (\cdot) , iff there exists $f \in \text{Aut}(H, *)$ such that $xy \subset f(x*y)$, $\forall x, y \in H$. Then, we write $\cdot \leq^*$ and say that $(H, *)$ *contains* (H, \cdot) . If (H, \cdot) is a structure then it is called *basic structure* and $(H, *)$ is called *H_b -structure*.

Theorem (The Little Theorem), [17],[18]. In all H_V -structures and for all hopes, which are defined on them, greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

This Theorem leads to a partial order on H_v -structures so, to a correspondence between hyperstructures and posets. The determination of all H_v -groups and H_v -rings is hard. In this direction there are many results by R. Bayon and N. Lygeros [1].

Definitions 1 [19],[20]. Let (H, \cdot) be hypergroupoid. We *remove* $h \in H$, if we consider the restriction of (\cdot) in the set $H - \{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by \underline{h} and h does not appear in the structure. $\underline{h} \in H$ *merges* with $h \in H$, if we take as product of any $x \in H$ by \underline{h} , the union of the results of x with both h , \underline{h} , and consider h and \underline{h} as one class with representative \underline{h} , therefore, h does not appear in the hyperstructure.

Most of H_v -structures are used in Representation (abbreviate by *rep*) Theory. Reps of H_v -groups can be considered either by generalized permutations or by H_v -matrices [16],[17]. Reps by generalized permutations can be achieved by using translations. In the rep theory the singles are playing a crucial role.

The rep problem by H_v -matrices is the following:

H_v -matrix is called a matrix if has entries from an H_v -ring. The hyperproduct of H_v -matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ H_v -matrices, defined in a usual manner:

$$\mathbf{A} \cdot \mathbf{B} = (a_{ij}) \cdot (b_{ij}) = \{ \mathbf{C} = (c_{ij}) \mid c_{ij} \in \oplus \Sigma a_{ik} \cdot b_{kj} \},$$

where (\oplus) denotes the n -ary circle hope on the hyperaddition [17].

Definition 2. Let (H, \cdot) be H_v -group, $(R, +, \cdot)$ H_v -ring, $\mathbf{M}_R = \{(a_{ij}) \mid a_{ij} \in R\}$, then any map

$$\mathbf{T}: H \rightarrow \mathbf{M}_R: h \rightarrow \mathbf{T}(h) \text{ with } \mathbf{T}(h_1 h_2) \cap \mathbf{T}(h_1) \mathbf{T}(h_2) \neq \emptyset, \quad \forall h_1, h_2 \in H,$$

is called H_v -matrix rep. If $\mathbf{T}(h_1 h_2) \subset \mathbf{T}(h_1) \mathbf{T}(h_2)$, then \mathbf{T} is an *inclusion rep*, if $\mathbf{T}(h_1 h_2) = \mathbf{T}(h_1) \mathbf{T}(h_2)$, then \mathbf{T} is a *good rep*.

Hyperoperations on any type of matrices can be defined:

Definition 3 [13],[8]. Let $A=(a_{ij}) \in \mathbf{M}_{m \times n}$ be matrix and $s, t \in \mathbb{N}$, with $1 \leq s \leq m$, $1 \leq t \leq n$.

Then *helix-projection* is a map $\underline{st}: \mathbf{M}_{m \times n} \rightarrow \mathbf{M}_{s \times t}: A \rightarrow \underline{Ast} = (\underline{a}_{ij})$, where \underline{Ast} has entries

$$\underline{a}_{ij} = \{ a_{i+\kappa s, j+\lambda t} \mid 1 \leq i \leq s, 1 \leq j \leq t \text{ and } \kappa, \lambda \in \mathbb{N}, i+\kappa s \leq m, j+\lambda t \leq n \}$$

Let $A=(a_{ij}) \in \mathbf{M}_{m \times n}$, $B=(b_{ij}) \in \mathbf{M}_{u \times v}$ be matrices and $s = \min(m, u)$, $t = \min(n, v)$. We define a hyper-addition, called *helix-addition*, by

$$\oplus: \mathbf{M}_{m \times n} \times \mathbf{M}_{u \times v} \rightarrow \mathbf{P}(\mathbf{M}_{s \times t}): (A, B) \rightarrow A \oplus B = A \underline{s}t + B \underline{s}t = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset \mathbf{M}_{s \times t}$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$.

Let $A = (a_{ij}) \in \mathbf{M}_{m \times n}$, $B = (b_{ij}) \in \mathbf{M}_{u \times v}$ and $s = \min(n, u)$. We define the *helix-multiplication*, by

$$\otimes: \mathbf{M}_{m \times n} \times \mathbf{M}_{u \times v} \rightarrow \mathbf{P}(\mathbf{M}_{m \times v}): (A, B) \rightarrow A \otimes B = A \underline{m}s \cdot B \underline{s}v = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset \mathbf{M}_{m \times v}$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it} b_{tj}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$.

The helix-addition is commutative, WASS but not associative. The helix-multiplication is WASS, not associative and it is not distributive, not even weak, to the helix-addition. For all matrices of the same type, the inclusion distributivity, is valid.

2. Basic definitions

One can see basic definitions, results, applications and generalizations on hyperstructure theory, not only for H_v -structures, in the books [3],[4],[6],[17] and the survey papers [2],[5],[7],[14],[20],[21]. Here we present some definitions related to our problem.

In a hypergroupoid (H, \cdot) the powers of $h \in H$ are $h^1 = \{h\}, \dots, h^n = h \circ h \circ \dots \circ h$, where (\circ) denotes the *n-ary circle hope*, i.e. take the union of hyperproducts with all possible patterns of parentheses put on them. An H_v -semigroup (H, \cdot) is called *cyclic of period s*, if there exists a *g (generator)* and a number *s*, the minimum, such that $H = h^1 \cup \dots \cup h^s$. The cyclicity for the infinite period is defined in [14]. If there is an *h* and a number *s*, the minimum, such that $H = h^s$, then (H, \cdot) is called *single-power cyclic of period s*.

In 1989 Corsini & Vougiouklis introduced a method to obtain stricter algebraic structures from given ones through hyperstructure theory. This method was introduced before of the H_v -structures, but in fact the H_v -structures appeared in the procedure.

Definition. *The uniting elements method* is the following: Let G be a structure and d be a property, which is not valid, and it is described by a set of equations. Consider the partition in G for which it is put together, in the same class, every pair of elements that causes the non-validity of d . The quotient G/d is an H_v -structure. Then quotient of G/d by the fundamental relation β^* , is a stricter structure $(G/d)\beta^*$ for which d is valid.

An application of the uniting elements is if more than one property desired. The reason for this is some of the properties lead straighter to the classes: commutativity and the reproductivity are easily applicable. One can do this because the following is valid:

Theorem [17],[21]. Let (G, \cdot) be groupoid, and $F = \{f_1, \dots, f_m, f_{m+1}, \dots, f_{m+n}\}$ a system of equations on G consisting of two subsystems $F_m = \{f_1, \dots, f_m\}$, $F_n = \{f_{m+1}, \dots, f_{m+n}\}$. Let σ and σ_m be the equivalence relations defined by the uniting elements using the F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the grupoid $G_m = (G/\sigma_m)/\beta^*$. Then we have $(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*$.

Definition 4 [17],[21]. Let $(F, +, \cdot)$ be an H_V -field, $(V, +)$ be a COW H_V -group and there exists an external hope

$$\cdot : F \times V \rightarrow P(V): (a, x) \rightarrow ax$$

such that, for all a, b in F and x, y in V we have

$$a(x+y) \cap (ax+ay) \neq \emptyset, \quad (a+b)x \cap (ax+bx) \neq \emptyset, \quad (ab)x \cap a(bx) \neq \emptyset,$$

then V is called an H_V -vector space over F .

In the case of an H_V -ring instead of H_V -field then the H_V -modulo is defined.

In the above cases the fundamental relation ε^* is the smallest equivalence relation such that the quotient V/ε^* is a vector space over the fundamental field F/γ^* .

The general definition of an H_V -Lie algebra over a field F was given in as follows:

Definition 5. Let $(L, +)$ be an H_V -vector space over the field $(F, +, \cdot)$, $\varphi: F \rightarrow F/\gamma^*$ be the canonical map and $\omega_F = \{x \in F: \varphi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^* . Similarly, let ω_L be the core of the canonical map $\varphi': L \rightarrow L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^* . Consider the bracket (commutator) hope:

$$[,] : L \times L \rightarrow P(L): (x, y) \rightarrow [x, y]$$

then L is an H_V -Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset, \quad \forall x, x_1, x_2, y, y_1, y_2 \in L, \quad \lambda_1, \lambda_2 \in F$$

$$(L2) \quad [x,x] \cap \omega_L \neq \emptyset, \forall x \in L$$

$$(L3) \quad ([x,[y,z]]+[y,[z,x]]+[z,[x,y]]) \cap \omega_L \neq \emptyset, \forall x,y \in L.$$

We remark that this is a very general definition therefore one can use special cases in order to face several problems in applied sciences [12]. Moreover, from this definition we can see how the weak properties can be defined as the above weak linearity (L1), anti-commutativity (L2) and the Jacobi identity (L3).

3. ∂ -hopes

In [22] an extremely large class of hopes introduced called theta:

Definition 6. Let H be a set equipped with n operations (or hopes) $\otimes_1, \dots, \otimes_n$ and a map (or multivalued map) $f: H \rightarrow H$ (or $f: H \rightarrow P(H) - \{\emptyset\}$, resp.), then n hopes $\partial_1, \dots, \partial_n$ on H are defined, called *theta-hopes*, (∂ -hopes), by putting

$$x\partial_i y = \{f(x)\otimes_i y, x\otimes_i f(y)\}, \quad \forall x,y \in H \text{ and } i \in \{1,2,\dots,n\}$$

or, in the case where \otimes_i is hope or f is multivalued map, we have

$$x\partial_i y = (f(x)\otimes_i y) \cup (x\otimes_i f(y)), \quad \forall x,y \in H \text{ and } i \in \{1,2,\dots,n\}$$

If \otimes_i is associative, then ∂_i is WASS. Remark that one can use several maps f instead of only one, in a similar way.

In a groupoid (G, \cdot) , or a hypergroupoid, with a ∂ -hope, one can study several properties like the following ones:

Reproductivity. For the reproductivity we must have

$$x\partial G = \cup_{g \in G} \{f(x) \cdot g, x \cdot f(g)\} = G \quad \text{and} \quad G\partial x = \cup_{g \in G} \{f(g) \cdot x, g \cdot f(x)\} = G.$$

If (\cdot) is reproductive, then (∂) is reproductive: $\cup_{g \in G} \{f(x) \cdot g\} = G$.

Commutativity. If (\cdot) is commutative then (∂) is commutative. If f is into the centre, then (∂) is commutative. If (\cdot) is COW then (∂) is COW.

Unit elements. u is right unit if $x\partial u = \{f(x) \cdot u, x \cdot f(u)\} \ni x$. So $f(u) = e$, if e is a unit in (G, \cdot) . The elements of the kernel of f , are the units of (G, ∂) .

Inverse elements. Let (G, \cdot) be a monoid with unit e and u be a unit in (G, ∂) , then $f(u) = e$. For given x , the x' is an inverse with respect to u , if $x\partial x' = \{f(x) \cdot x', x \cdot f(x')\} \ni u$ and $x'\partial x = \{f(x') \cdot x, x' \cdot f(x)\} \ni u$. So, $x' = (f(x))^{-1}u$ and

$x' = u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

Proposition 7. Let (G, \cdot) be a group then, for all maps $f: G \rightarrow G$, the (G, ∂) is an H_V -group.

One can define ∂ -hopes on rings and more complicated structures (or hyper-structures or H_V -structures), where more than one ∂ -hopes can be defined.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions $s(x), t(x)$, we have $s\partial t = \{s't, st'\}$ where $(')$ denotes the derivative.

Example. Let (G, \cdot) be a group and $f(x) = a$ the *constant map* on G , then $x\partial y = \{ay, xa\}$, $\forall x, y \in G$. The $(G, \partial)/\beta^*$ is singleton, indeed, we have $a^{-1}\partial(a^{-1} \cdot x) = \{x, e\}$ $\forall x \in G$, so $x\beta e$, $\forall x \in G$, thus $\beta^*(x) = \beta^*(e)$. For $f(x) = e$ we obtain $x\partial y = \{x, y\}$, the smallest incidence hope.

Propositions 8. Let $g \in G$ is a generator of the group (G, \cdot) . Then,

- (a) for every f, g is a generator in (G, ∂) , with period at most n .
- (b) suppose that there exists an element w such that $f(w) = g$, then the element w is a generator in (G, ∂) , with period at most n .

Definitions 9. Let (G, \cdot) be a groupoid and $f_i: G \rightarrow G$, $i \in I$, be a set of maps. Take the map $f_{\cup}: G \rightarrow \mathcal{P}(G)$ such that $f_{\cup}(x) = \{f_i(x) \mid i \in I\}$, i.e. *the union* of $f_i(x)$. We call *union ∂ -hopes*, if we consider the map $f_{\cup}(x)$. Special case: the union of f with the identity, i.e. $\underline{f} = f_{\cup}(\text{id})$, so $\underline{f}(x) = \{x, f(x)\}$, $\forall x \in G$, which is called *b ∂ -hope*. We denote the b ∂ -hope by $(\underline{\partial})$, so

$$x\underline{\partial}y = \{xy, f(x) \cdot y, x \cdot f(y)\}, \quad \forall x, y \in G.$$

Remark that $\underline{\partial}$ contains the operation (\cdot) so it is a b-hope. If $f: G \rightarrow \mathcal{P}(G) - \{\emptyset\}$, then the b ∂ -hope is defined by using the map $\underline{f}(x) = \{x\} \cup f(x)$, $\forall x \in G$.

A construction between ∂ and $\underline{\partial}$ is the one which obtained by using special maps.

Definition 10. Let (G, \cdot) be a groupoid and $f: G \rightarrow G$ be a map, we call *basic set* of the map f the set $B = B_f = \{x \in G: f(x) = x\}$. Then, if $B \neq \emptyset$, we have

$$\begin{aligned} x\partial y &= xy, \quad \forall x, y \in B, \\ x\partial y &= \{xy, x \cdot f(y)\}, \quad \forall x \in B, y \in G - B, \\ x\partial y &= \{f(x) \cdot y, xy\}, \quad \forall x \in G - B, y \in B, \end{aligned}$$

$$x\partial y = \{ f(x)\cdot y, x\cdot f(y) \}, \quad \forall x,y \in G-B.$$

For (G, \cdot) groups, we obtain the following:

- If B is a subgroup of (G, \cdot) , then (B, ∂) is a sub- H_v -group of (G, ∂) .
- If $e \in B$, then e is a unit of (B, ∂) because it belongs into the kernel of f .
- Inverses: If u is a unit of (B, ∂) , then $x \in G$, has an inverse in (G, ∂) if $f(x)u = uf(x)$. Therefore an element $x \in B$ has an inverse iff $xu = ux$. If $e \in B$ then the element x^{-1} is an inverse of x in (G, \cdot) as well.

Proposition 11. Let $g \in G$ is a generator of the group (G, \cdot) . If $g \in B$ then g is a generator in (G, ∂) , $\forall f$, with period at most n .

There is connection of ∂ -hopes with other hyperstructures:

Example. Merging and ∂ . If (H, \cdot) is a groupoid and $\underline{h} \in H$ merges with the $h \in H$, then h does not appeared and we have for the merge (H, \circ) ,

$$\underline{h} \circ x = \{ \underline{h} \cdot x, h \cdot x \}, \quad x \circ \underline{h} = \{ x \cdot \underline{h}, x \cdot h \}, \quad \underline{h} \circ \underline{h} = \{ \underline{h} \cdot \underline{h}, \underline{h} \cdot h, h \cdot \underline{h}, h \cdot h \}$$

and in rest cases (\circ) coincides with (\cdot) , so we have merge $(H - \{h\}, \circ)$.

Similarly, if (H, \cdot) is a hypergroupoid then we have

$$\underline{h} \circ x = (\underline{h} \cdot x) \cup (h \cdot x), \quad x \circ \underline{h} = (x \cdot \underline{h}) \cup (x \cdot h), \quad \underline{h} \circ \underline{h} = (\underline{h} \cdot \underline{h}) \cup (\underline{h} \cdot h) \cup (h \cdot \underline{h}) \cup (h \cdot h)$$

In order to see a connection of merge with the ∂ -hope, consider the map f such that $f(\underline{h}) = h$ and $f(x) = x$ in the rest cases. Then in $(H - \{h\}, \partial)$ we have, $\forall x, y \in H - \{h\}$

$$\underline{h} \partial x = \{ h \cdot x, \underline{h} \cdot x \}, \quad x \partial \underline{h} = \{ x \cdot \underline{h}, x \cdot h \} \quad \text{and} \quad \underline{h} \partial \underline{h} = \{ h \cdot \underline{h}, \underline{h} \cdot h \}$$

and in the rest cases (∂) coincides with (\cdot) . Therefore, $\partial \leq \circ$, or

$$\underline{h} \partial \underline{h} = \{ h \cdot \underline{h}, \underline{h} \cdot h \} \subset \{ \underline{h} \cdot \underline{h}, \underline{h} \cdot h, h \cdot \underline{h}, h \cdot h \} = \underline{h} \circ \underline{h}$$

and in the remaining cases we have $\partial \equiv \circ$.

Example. P-hopes [14]. Let (G, \cdot) be a commutative semigroup and $P \subset G$. Consider the multivalued map f such that $f(x) = P \cdot x$, $\forall x \in G$.

Then we have $x\partial y = x \cdot y \cdot P$, $\forall x, y \in G$.

So the ∂ -hope coincides with the well known class of P -hopes [20].

One can define theta-hopes on rings and other more complicate structures, where more than one ∂ -hopes can be defined. Moreover, one can replace structures by hyper ones or by H_v -structures, as well [23],[24].

Definition 12. Let $(R, +, \cdot)$ be a ring and $f: R \rightarrow R$, $g: R \rightarrow R$ be two maps. We define two hyperoperations (∂_+) and $(\partial \cdot)$, called both *theta-operations*, on R as follows

$$x\partial_+y = \{f(x)+y, x+f(y)\} \quad \text{and} \quad x\partial \cdot y = \{g(x) \cdot y, x \cdot g(y)\}, \quad \forall x, y \in R.$$

A hyperstructure $(R, +, \cdot)$, where $(+)$, (\cdot) be hyperoperations which satisfy all H_V -ring axioms, except the weak distributivity, will be called *H_V -near-ring*.

Proposition 13. Let $(R, +, \cdot)$ ring and $f: R \rightarrow R$, $g: R \rightarrow R$ maps. The hyperstructure $(R, \partial_+, \partial \cdot)$, called *theta*, is an H_V -near-ring. Moreover $(+)$ is commutative.

Proof. First, one can see that all properties of an H_V -ring, except the distributivity, are valid. For the distributivity we have, $\forall x, y, z \in R$, $x\partial \cdot (y\partial_+z) \cap (x\partial \cdot y)\partial_+(x\partial \cdot z) = \emptyset$. ■

In order more properties to be valid, the ∂ can be replaced by $\underline{\partial}$.

Proposition 14. Let $(R, +, \cdot)$ ring and $f: R \rightarrow R$, $g: R \rightarrow R$ maps, then $(R, \underline{\partial}_+, \underline{\partial} \cdot)$, is an H_V -ring.

Proof. The only one axiom we have to see is the distributivity. So, $\forall x, y, z \in R$,

$$x\underline{\partial} \cdot (y\underline{\partial}_+z) = \{g(x)(y+z), g(x)(f(y)+z), g(x)(y+f(z)), xg(y+z), xg(f(y)+z), xg(y+f(z))\}$$

and $(x\underline{\partial} \cdot y)\underline{\partial}_+(x\underline{\partial} \cdot z) =$

$$\{g(x)(y+z), f(g(x)y)+g(x)z, g(x)y+f(g(x)z), g(x)y+xg(z), f(g(x)y)+xg(z), g(x)y+f(xg(z)), xg(y)+g(x)z, f(xg(y))+g(x)z, xg(y)+f(g(x)z), x(g(y)+g(z)), f(xg(y))+xg(z), xg(y)+f(xg(z))\}.$$

So $x\underline{\partial} \cdot (y\underline{\partial}_+z) \cap (x\underline{\partial} \cdot y)\underline{\partial}_+(x\underline{\partial} \cdot z) = \{g(x)(y+z)\} \neq \emptyset$.

Therefore, $(R, \underline{\partial}_+, \underline{\partial} \cdot)$ is an H_V -ring. ■

Remark. If $(R, +, \cdot)$ ring and $f: R \rightarrow R$, $g: R \rightarrow R$ maps, then $(R, \partial_+, \underline{\partial} \cdot)$ is still an H_V -near-ring.

Theorems 15. (a) Consider the group of integers $(\mathbf{Z}, +)$ and let $n \neq 0$ be a natural number. Take the map f such that $f(0)=n$ and $f(x)=x$, $\forall x \in \mathbf{Z} - \{0\}$. Then $(\mathbf{Z}, \partial)/\beta^* \cong (\mathbf{Z}_n, +)$.

(b) Consider the ring of integers $(\mathbf{Z}, +, \cdot)$ and let $n \neq 0$ be a natural. Consider the map f such that $f(0)=n$ and $f(x)=x$, $\forall x \in \mathbf{Z} - \{0\}$. Then $(\mathbf{Z}, \partial_+, \partial \cdot)$ is an H_V -near-ring, with

$$(\mathbf{Z}, \partial_+, \partial \cdot)/\gamma^* \cong \mathbf{Z}_n.$$

Special case of the above is for $n=p$, prime, then $(\mathbf{Z}, \partial_+, \partial \cdot)$ is an H_v -field.

Proposition 16. Let $(V, +, \cdot)$ be an algebra over the field $(F, +, \cdot)$ and $f: V \rightarrow V$ be a map. Consider the ∂ -hope defined only on the multiplication of the vectors (\cdot) , then $(V, +, \partial)$ is an H_v -algebra over F , where the related properties are weak. If, moreover f is linear then we have more strong properties.

Definition 17. Let L be a Lie algebra, defined on an algebra $(V, +, \cdot)$ over the field $(F, +, \cdot)$ with Lie bracket $[x, y] = xy - yx$. Consider a map $f: L \rightarrow L$, then the ∂ -hope is defined by

$$x\partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}$$

Proposition 18. Let $(V, +, \cdot)$ be an algebra over the field $(F, +, \cdot)$ and $f: V \rightarrow V$ be a linear map. Consider the ∂ -hope defined only on the multiplication of the vectors (\cdot) , then $(V, +, \partial)$ is an H_v -algebra over F , with respect to Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

If (G, \cdot) is a semigroup then, for every f , the $b\partial$ -operation (∂) is WASS.

4. Hyprestructures in questionnaires

During last decades hyperstructures seem to have a variety of applications not only in other branches of mathematics but also in many other sciences including the social ones. These applications range from biomathematics and hadronic physics to automata theory, to mention but a few. This theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too.

In several papers, such as [2],[4],[11],[12] one can find numerous applications; similarly, in the books [4], [6] a wide variety of applications is also presented.

An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis. The suggestion is the following [10]:

Definition 19. “In every question substitute the Likert scale with ‘the bar’ whose poles are defined with ‘0’ on the left end, and ‘1’ on the right end:

$$0 \text{ ————— } 1$$

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question”.

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm, see [9], [25].

Now we state our main problem for this paper by using this bar and we can describe in mathematical model using theta-hopes.

Problem 20. In the research processing suppose that we want to use Likert scale dividing the continuum [01] both by, first, into equal steps (segments) and, second, into equal-area spaces according to Gauss distribution [9], [25]. If we consider both types of divisions into n segments, then the continuum [01] is divided into $2n-1$ segments, if n is odd number and into $2(n-1)$ segments, if n is even number. We can number the segments and we can consider as an organized devise the group $(Z_{k,\oplus})$ where $k=2n-1$ or $2(n-1)$. Then we can obtain several hyperstructures using ∂ -hopes as the following way: We can have two partitions of the final segments, into n classes either using the division into equal steps or the Gauss distribution by putting in the same class all segments that belong (a) to the equal step or (b) to equal-area spaces according to Gauss distribution. Then we can consider two kinds of maps (i) a multi-map where every element corresponds to the hole class or (ii) a map where every element corresponds to one special fixed element of the same class. Using these maps we define the ∂ -hopes and we obtain the corresponding H_v -structure.

An example for the case (i) is the following:

Example 21. Suppose that we take the case of the Likert scale with 5 equal steps: [0-1.24-2.48-3.72-4.96-6.2] and the Gauss 5 equal areas: [0-2.4-2.9-3.3-3.8-6.2] we have 9 segments as follows

$$[0 - 1.24 - 2.4 - 2.48 - 2.9 - 3.3 - 3.72 - 3.8 - 4.96 - 6.2]$$

Therefore, if we consider the set Z_9 and if we name the segments by 1, 2, ..., 8, 0, then if we consider the equal steps partition: {1}, {2,3}, {4,5,6}, {7,8}, {0} we take, according to the above construction the multi-map f such that $f(1)=\{1\}$, $f(2)=\{2,3\}$, $f(3)=\{2,3\}$, $f(4)=\{4,5,6\}$, $f(5)=\{4,5,6\}$, $f(6)=\{4,5,6\}$, $f(7)=\{7,8\}$, $f(8)=\{7,8\}$, $f(0)=\{0\}$, then we obtain the following table:

| \oplus | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 0 |
|----------|-------|---------|---------|-----------|---------|-----------|---------|---------|-------|
| 1 | 2 | 3,4 | 3,4 | 5,6,7 | 5,6,7 | 5,6,7 | 0,8 | 0,8 | 1 |
| 2 | 3,4 | 4,5 | 4,5,6 | 6,7,8 | 6,7,8 | 6,7,8,9 | 0,1 | 0,1,2 | 2,3 |
| 3 | 3,4 | 4,5,6 | 5,6 | 0,6,7,8 | 0,6,7,8 | 0,6,7,8 | 0,1,2 | 0,1,2 | 2,3 |
| 4 | 5,6,7 | 6,7,8 | 0,6,7,8 | 0,1,8 | 0,1,2,8 | 0,1,2,3,8 | 2,3,4 | 2,3,4,5 | 4,5,6 |
| 5 | 5,6,7 | 6,7,8 | 0,6,7,8 | 0,1,2,8 | 0,1,2 | 0,1,2,3 | 2,3,4 | 3,4,5 | 4,5,6 |
| 6 | 5,6,7 | 6,7,8,9 | 0,6,7,8 | 0,1,2,3,8 | 0,1,2,3 | 1,2,3 | 2,3,4,5 | 3,4,5 | 4,5,6 |
| 7 | 0,8 | 0,1 | 0,1,2 | 2,3,4 | 2,3,4 | 2,3,4,5 | 5,6 | 5,6,7 | 7,8 |
| 8 | 0,8 | 0,1,2 | 0,1,2 | 2,3,4,5 | 3,4,5 | 3,4,5 | 5,6,7 | 6,7 | 7,8 |
| 0 | 1 | 2,3 | 2,3 | 4,5,6 | 4,5,6 | 4,5,6 | 7,8 | 7,8 | 0 |

5. Hyperstructures in several scales obtained from the bar

Now we represent a mathematic model on obtained from the Problem 20:

Construction 22. Consider a group (\mathbf{G}, \cdot) and suppose take a partition $\mathbf{G}_i, i \in I$ of the \mathbf{G} . Select and fix an element g_i of each partition class \mathbf{G}_i , and consider the map

$$f: \mathbf{G} \rightarrow \mathbf{G} \text{ such that } f(x) = g_i, \quad \forall x \in \mathbf{G}_i,$$

then (\mathbf{G}, ∂) is an H_v -group. Moreover the fundamental group $(\mathbf{G}/R, \cdot)/\beta^*$ is (up to isomorphism) a subgroup of the corresponding fundamental group $(\mathbf{G}, \partial)/\beta^*$.

Proof. First, we remark that the ∂ - H_v -group (\mathbf{G}, ∂) is an H_v -group because this is true for all given maps. Now, let us call R the given partition. For all $x \in \mathbf{G}_i$ and $y \in \mathbf{G}_j$ we have $x \partial y = \{g_i y, x g_j\}$, thus we remark that the elements $g_i y$ and $x g_j$ belong to the same R class. Therefore, the β^* -classes with respect to ∂ , are subsets of the β^* -classes with respect to the R -classes. The fundamental group $(\mathbf{G}/R, \cdot)/\beta^*$ is (up to isomorphism) a subgroup of the corresponding fundamental group $(\mathbf{G}, \partial)/\beta^*$. ■

Theorem 23. In the above construction, if one of the selected elements is the unit element e of the group (\mathbf{G}, \cdot) , otherwise, if there exist an element $z \in \mathbf{G}$ such that $f(z) = e$, then we have

$$(\mathbf{G}/R, \cdot)/\beta^* = (\mathbf{G}, \partial)/\beta^*.$$

Proof. Since there exist $z \in \mathbf{G}$ such that $f(z) = e$, then for all $x \in \mathbf{G}_i$, we have $f(x) = g_i$, consequently, $f(e) = e$. Moreover, for all $x \in \mathbf{G}_i$, we have

$$x \partial e = \{g_i \cdot e, x \cdot e\} = \{g_i, x\},$$

thus, x belongs to the fundamental class to g_i with respect to ∂ -hope. So $\mathbf{G}_i \subset \beta^*(g_i)$ and from the above theorem we obtain that

$$(G/R, \cdot) / \beta^* = (G, \partial) / \beta^*. \blacksquare$$

In hypergroups does not necessarily exist any unit element and if there exists a unit this is not necessarily unique. Moreover the ∂ -hopes do not have always the unit element of the group as unit for the corresponding ∂ -hope. This is so because

$$e\partial e = \{f(e)e, ef(e)\} = \{f(e)\}.$$

However for the above hyperstructure we have the following:

Proposition 24. Suppose (\mathbf{G}, \cdot) be a group and $\mathbf{G}_i, i \in I$ be a partition of \mathbf{G} . For any class we fix a $g_i \in \mathbf{G}_i$, and take the map $f: \mathbf{G} \rightarrow \mathbf{G}: f(x) = g_i, \forall x \in \mathbf{G}_i$. If for the unit element e , in (\mathbf{G}, \cdot) , we have $f(e) = e$, i.e. e is any fixed element, then e is also a unit element of the H_V -group (\mathbf{G}, ∂) . Moreover $(f(x))^{-1}$ is an inverse element in the ∂ - H_V -group (\mathbf{G}, ∂) , of x .

Proof. For all $x \in \mathbf{G}$ we have

$$x\partial e = \{f(x)e, xe\} = \{f(x), x\} \ni x.$$

Thus, e is a unit element in (\mathbf{G}, ∂) .

Moreover, $\forall x \in \mathbf{G}$, denoting $(f(x))^{-1}$ the inverse of $f(x)$ in (\mathbf{G}, \cdot) , we have

$$x\partial(f(x))^{-1} = \{f(x) \cdot (f(x))^{-1}, x \cdot f((f(x))^{-1})\} = \{e, x \cdot f((f(x))^{-1})\} \ni e.$$

Therefore the element $(f(x))^{-1}$ is an inverse of x with respect to ∂ .

This theorem states that the inverse g_i^{-1} in (\mathbf{G}, \cdot) , of every fixed element g_i , is also an inverse in (\mathbf{G}, ∂) of all elements which belong to their partition class \mathbf{G}_i . Finally, remark that some of the elements of \mathbf{G} may have more than one inverse in (\mathbf{G}, ∂) . \blacksquare

Now we conclude with an example of the above Construction 22 on our main Problem 20:

Example 11. In the case of the Likert scale with 6 equal steps: [0-1-2.1-3.1-4.1-5.1-6.2] and the Gauss 6 equal areas: [0-2.23-2.73-3.1-3.47-3.97-6.2] we have 10 segments as follows

$$[0 - 1 - 2.1 - 2.23 - 2.73 - 3.1 - 3.47 - 3.97 - 4.1 - 5.1 - 6.2]$$

Therefore, if we consider the set \mathbf{Z}_{10} and if we name the segments by 1, 2, ..., 9, 0, then if we consider the Gauss partition: $\{1,2,3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$,

{8,9,0} we take, according to the above Theorem, the map f such that $f(1)=\{1\}$, $f(2)=\{1\}$, $f(3)=\{1\}$, $f(4)=\{4\}$, $f(5)=\{5\}$, $f(6)=\{6\}$, $f(7)=\{7\}$, $f(8)=\{0\}$, $f(9)=\{0\}$, $f(0)=\{0\}$, then we obtain the following table:

| \oplus | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1 | 2 | 2,3 | 2,4 | 5 | 6 | 7 | 8 | 1,9 | 0,1 | 1 |
| 2 | 2,3 | 3 | 3,4 | 5,6 | 6,7 | 7,8 | 8,9 | 2,9 | 0,2 | 1,9 |
| 3 | 2,4 | 3,4 | 4 | 5,7 | 6,8 | 7,9 | 0,8 | 3,9 | 0,3 | 1,3 |
| 4 | 5 | 5,6 | 5,7 | 8 | 9 | 0 | 1 | 2,4 | 3,4 | 4 |
| 5 | 6 | 6,7 | 6,8 | 9 | 0 | 1 | 2 | 3,5 | 4,5 | 5 |
| 6 | 7 | 7,8 | 7,9 | 0 | 1 | 2 | 3 | 4,6 | 5,6 | 6 |
| 7 | 8 | 8,9 | 0,8 | 1 | 2 | 3 | 4 | 5,7 | 6,7 | 7 |
| 8 | 1,9 | 2,9 | 3,9 | 2,4 | 3,5 | 4,6 | 5,7 | 8 | 8,9 | 0,8 |
| 9 | 0,1 | 0,2 | 0,3 | 3,4 | 4,5 | 5,6 | 6,7 | 8,9 | 9 | 0,9 |
| 0 | 1 | 1,9 | 1,3 | 4 | 5 | 6 | 7 | 0,8 | 0,9 | 0 |

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