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Some algebraic properties of fuzzy S-acts

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Abstract

S-acts, a useful and important algebraic tool, have always been interest to mathematicians, specially to computer scientists. When A. Zadeh introduced the notion of the fuzzy subset in 1965, his idea opened a new direction to reserchers to provide tools in the various fields of mathematics. Here we are going to investigate some algebraic properties of fuzzy S-acts. We first make an S-act from the fuzzy subsets of an S-act A. Then we use this tool to give a characterization for fuzzy S-acts. We then introduce the notion of generated fuzzy S-act by a fuzzy subset of an S-act and give a characterization for the fuzzy actions. And then we define the notion of indecomposable fuzzy S-act and find some indecomposable fuzzy actions.

Key words: Fuzzy set, Fuzzy acts over fuzzy semigroups.

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1 Introduction and Preliminaries

No need to mention the importance of the prominent and well established Fuzzy Set Theory, introduced by Zadeh in 1965 [7], which offered tools and a new approach to model imprecision and uncertainty. Since then, very many researchers have worked on this concept and its applications to logic, set theory, algebra, analysis, topology, computer science, control engineering, information science, etc [1, 2, 3]. Actions of a semigroup (monoid or group) S on a set A have always been interest to mathematicians, specially to computer scientists and logicians. The algebraic structures so obtained are called S-sets, S-acts, and by some other terminologies [4, 6].

In [5] we have used the fuzzy concept and introduced the notion of the actions of a (fuzzy) semigroup on a fuzzy set (fuzzy S-act) and studied the relation between this structure and sheaves. Here we are going to study some of algebraic details of this structure. But first we recall that:

A set X together with a function $\mu : X \to [0,1]$ is called a *fuzzy set* (over X) and is denoted by (X, μ) or $X^{(\mu)}$. We call X the *underlying set* and μ the *membership function* of the fuzzy set $X^{(\mu)}$, and $\mu(x) \in [0,1]$ is the *grade of membership* of x in $X^{(\mu)}$.

If μ is a constant function with value $a \in [0,1]$, $X^{(\mu)}$ is denoted by $X^{(a)}$. The fuzzy set $X^{(1)}$ is called a *crisp set* and may sometimes simply be denoted by X.

For a fuzzy set $X^{(\mu)}$ and $\alpha \in [0,1]$, $X^{(\mu)}_{\alpha} := \{x \in X \mid \mu(x) \geq \alpha\}$ is called the α*-cut* or the α*-level set* of the fuzzy set $X^{(\mu)}$.

A *fuzzy function* from $X^{(\mu)}$ to $Y^{(\eta)}$, written as $f : X^{(\mu)} \to Y^{(\eta)}$, is an ordinary function $f : X \to Y$ such that the following is a **fuzzy triangle**:

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "f"'] & {[0,1]} \\ Y \arrow[ur, "\eta"'] & \end{tikzcd}$$

meaning that $\mu \leq \eta f$ (that is, $\mu(x) \leq \eta f(x)$ for all $x \in X$). The set of all fuzzy sets with a fixed underlying set X is called the *fuzzy power* or the set of *fuzzy subsets* of X and is denoted by $\mathbf{FSub}X$. Clearly fuzzy sets together with fuzzy functions between them form a category denoted by **FSet**.

To define the actions of a (fuzzy) semigroup on a fuzzy set first we note that:

Definition 1.1 *A semigroup S together with a function $\nu : S \to [0,1]$ is called a fuzzy semigroup if its multiplication is a fuzzy function: for every $s, r \in S$, $\nu(s) \wedge \nu(r) \leq \nu(sr)$; that is, the following is a fuzzy triangle:*

$$\begin{tikzcd} S \times S \arrow[r, "\nu\wedge\nu"] \arrow[d, "\lambda_S"'] & {[0,1]} \\ S \arrow[ur, "\nu"'] & \end{tikzcd}$$

If S has an identity 1, one usually add the condition $\nu(1) = 1$.

Now, recall from [5] that, for a (crisp) semigroup S, a (crisp) set A can be made into an (ordinary) S-act in the following two equivalent ways:

Universal algebraic way: The set A together with a family $(\lambda_s : A \to A)_{s\in S}$ of unary operations satisfying $(st)a = s(ta)$ (and $1a = a$, if S has an identity) where $sa = \lambda_s(a)$.

Common way: The set A together with a function $\lambda : S \times A \to A$ satisfying $(st)a = s(ta)$ (and $1a = a$, if S has an identity) where $sa = \lambda(s, a)$.

Now, having these two, so called, universal algebraic and common actions of S on A, we get the following two, not necessarily equivalent, definitions for a fuzzy act over a fuzzy monoid.

Definition 1.2 *Let $S^{(\nu)}$ be a fuzzy semigroup and $A^{(\mu)}$ be a fuzzy set such that A is an S-act, as defined above. Then, $A^{(\mu)}$ is called:*

(Universal algebraic) A fuzzy S-act (or fuzzy $S^{(1)}$-act, to emphasize fuzziness) if each λ_s is a fuzzy function; that is $\mu(a) \leq \mu(sa)$, for every $s \in S$ and $a \in A$ (with no mention of ν). That is, for every $s \in S$, the following triangle is fuzzy:

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d, "\lambda_s"'] & {[0,1]} \\ A \arrow[ur, "\mu"'] & \end{tikzcd}$$

(Common) A fuzzy $S^{(\nu)}$-act if $\lambda : S \times A \to A$ is a fuzzy function; that is, $\nu(s) \wedge \mu(a) \leq \mu(sa)$, for every $s \in S$ and $a \in A$. That is, the following triangle is fuzzy:

$$\begin{tikzcd} S \times A \arrow[r, "\nu\wedge\mu"] \arrow[d, "\lambda"'] & {[0,1]} \\ S \arrow[ur, "\mu"'] & \end{tikzcd}$$

Corolary 1.1 *(1) Note that universal algebraic definition implies common definition, and if $S^{(1)} = S$ is a (crisp) semigroup, then $\nu(s) \wedge \mu(a) = 1 \wedge \mu(a) = \mu(a)$, and so the above two definitions are equivalent.*

(2) Every fuzzy semigroup $S^{(\nu)}$ is naturally a fuzzy $S^{(\nu)}$-act and $S^{(1)} = S$ is a fuzzy $S^{(1)} = S$-act (universal algebraicly, and hence commonly). Also, if $S^{(\nu)}$ is a fuzzy left ideal, then it is a fuzzy S-act (universal algebraicly, and hence commonly).

A morphism between fuzzy $S^{(\nu)}$-acts (with both definitions), also called an $S^{(\nu)}$-map is simply an S-map as well as a fuzzy function. The set of all (fuzzy) $S^{(\nu)}$-acts with a fixed A is denoted by $S^{(\nu)}$-$\mathbf{FSub}A$, and the category of all fuzzy $S^{(\nu)}$-acts is denoted by $S^{(\nu)}$-$\mathbf{FAct}$.

Since an S-act A is naturally a (unary) universal algebra, the universal algebraic definition of fuzzy acts, being compatible with the definition of other fuzzy algebraic structures, may be considered to be more natural than the second one. Thus, from now on we consider the universal algebraic definition of fuzzy acts and we recall that:

Theorem 1.1 *[5] An S-act A with $\mu : A \to [0,1]$ is a fuzzy $S = S^{(1)}$-act if and only if for every $\alpha \in [0,1]$, $A_\alpha^{(\mu)}$ is an ordinary S-subact of A.*

2 Fuzzy subsets of an S-act as a fuzzy S-act

In this short section we make an S-action from the fuzzy subsets of an S-act A which is used thorough the paper and give a characterization of fuzzy S-acts defined in preliminary.

Lemma 2.1 *Let S be an commutative monoid and A be an S-act. Then fuzzy subsets of A form an S-act.*

Proof. To prove, for each fuzzy subset $A^{(\mu)}$ and each $m \in S$ we define:

$$\begin{array}{rccc} m\mu : & A & \to & [0,1] \\ & a & \rightsquigarrow & \bigvee\{\mu(x) \mid mx = a\} \end{array}$$

First we note that $m\mu$ is a fuzzy S-act, because $m\mu(na) = \bigvee\{\mu(x) \mid mx = na\}$, for every $n \in S$, and $m\mu(a) = \bigvee\{\mu(x) \mid mx = a\}$. But if $mx = a$, then $nmx = na$, and since S is commutative, $mnx = nmx = na$. Also $\mu(x) \leq \mu(nx)$. So $\bigvee\{\mu(x) \mid mx = a\} \leq \bigvee\{\mu(x) \mid mx = na\}$.

Now we check the S-act properties.

$$\begin{aligned} (m_1m_2)\mu(a) &= \bigvee_{x\in A}\{\mu(x) \mid (m_1m_2)x = a\} \\ &= \bigvee_{x\in A}\{\mu(x) \mid m_2x = y,\ m_1y = a\} \\ &= \bigvee_{x\in A}\{\bigvee_{y\in A}\mu(y) \mid m_2x = y,\ m_1y = a\} \\ &= \bigvee_{y\in A}\{m_2\mu(y) \mid m_1y = a\} \\ &= m_1(m_2\mu(x))(a) \end{aligned}$$

and $1_S\mu(a) = \bigvee\{\mu(x) \mid 1_Sx = a\} = \mu(a)$. Also if $\nu \leq \mu$, then $(m\nu)(a) = \bigvee\{\nu(x) \mid mx = a\} \leq \bigvee\{\mu(x) \mid mx = a\} = (m\mu)(a)$.$\square$

Theorem 2.1 *Let $\mu : A \to [0,1]$ be a fuzzy subset. Then $A^{(\mu)}$ is an fuzzy S-act if and only if $m\mu \leq \mu$ for every $m \in S$.*

Proof. ($\Rightarrow$) Let $A^{(\mu)}$ be a fuzzy S-act and $X_a = \{x \in A \mid mx = a\}$. Then $\mu(x) \leq \mu(mx) = \mu(a)$, for every $x \in X_a$ implies that $\bigvee\{\mu(x) \mid mx = a\} \leq \mu(a)$. That is $m\mu \leq \mu$.

($\Leftarrow$) Let $A^{(\mu)}$ be a fuzzy subset. To prove we show that $\mu(a) \leq \mu(ma)$, for every $m \in S$ and $a \in A$. But we know that $m\mu \leq \mu$ and hence we have $m\mu(ma) \leq \mu(ma)$. Now since $m\mu(ma) = \bigvee_{x \in X_{ma}} \mu(x)$ and $a \in X_{na}$, we have $\mu(a) \leq m\mu(ma) \leq \mu(ma)$. $\square$

3 Cyclic fuzzy S-acts

In this section we define a generated fuzzy S-act by a fuzzy subset of an action and then we characterize the generated fuzzy S-actions by the action introduced in Lemma 2.1. We then define the cyclic fuzzy S-acts which are a useful class of fuzzy S-acts and infact every fuzzy S-act is made of a class of cyclic ones.

Lemma 3.1 *Intersection and union of fuzzy S-acts of an S-set A is an fuzzy S-act.*

Proof. Let $\{A^{(\mu_i)}\}_{i \in I}$ be a family of fuzzy S-act. Then $(\bigcup_{i \in I} \mu_i)(ma) = \bigvee_{i \in I} \mu_i(ma) \geq \bigvee_{i \in I} \mu_i(a) = (\bigcup_{i \in I} \mu_i)(a)$ and $(\bigcap_{i \in I} \mu_i)(ma) = \bigwedge \mu_i(ma) \geq \bigwedge \mu_i(a) = (\bigcap_{i \in I} \mu_i)(a)$. $\square$

Theorem 3.1 *Let $\mu : A \to [0, 1]$ be a fuzzy S-act and $\{\mu_i\}_{i \in I \subseteq [0,1]}$ be family of i-cuts of μ. Then $\bigcup_{i \in I} \mu_i$ and $\bigcap_{i \in I} \mu_i$ are fuzzy S-acts of the form α-cut.*

Proof. By Lemma 3.1, it is enough to show that $\bigcup_{i \in I} \mu_i = \mu_{\bigvee_{i \in I} i}$ and $\bigcap_{i \in I} \mu_i = \mu_{\bigwedge_{i \in I}}$. But since $(\bigcup_{i \in I} \mu_i)(a) = \bigvee \mu_i(a) \geq i$, for every $i \in I$ and $a \in A$, hence $\bigvee \mu_i(a) \geq \bigvee i$. Also $(\bigcap_{i \in I} \mu_i)(a) = \bigwedge \mu_i(a) \geq \bigwedge i$. So $\bigcup_{i \in I} \mu_i$ and $\bigcap_{i \in I} \mu_i$ are fuzzy S-acts of the form α-cut. $\square$

Now by the above Lemma having the following definition is natural.

Definition 3.1 *Let $\mu : A \to [0, 1]$ be a fuzzy S-act. Then we take $< \mu >$ to be $\bigcap\{\nu : A \to [0, 1] \mid \mu \leq \nu$ and ν is a fuzzy S-act$\}$. The fuzzy S-act $< \mu >$ is called the generated fuzzy S-act by μ.*

Theorem 3.2 *Let S be an commutative semigroup and $A^{(\mu)}$ be a fuzzy S-set of A. Then $< \mu >= \bigcup_{m \in S} m\mu$.*

Proof. First we prove that $\bigcup_{m\in S} m\mu$ is an fuzzy S-act. To do we show that $\bigcup_{m\in S} m\mu(a) \leq \bigcup_{m\in S} m\mu(na)$, for each $n \in S$. But $\bigcup_{m\in S} m\mu(a) = \bigvee_{m\in S}(\bigvee_{mx=a} \mu(x))$ and $\bigcup_{m\in S} m\mu(na) = \bigvee_{m\in S}(\bigvee_{mx=na} \mu(x))$. But if $mx = a$, then $mnx = nmx = na$. Since $\mu(x) \leq \mu(nx)$, for every $x \in A$, $\bigcup_{m\in S} m\mu(a) \leq \bigcup_{m\in S} m\mu(na)$.

Now let $A^{(\nu)}$ be a fuzzy S-act such that $\mu \leq \nu$. Then for every $m \in S$ we have $m\mu \leq \mu \leq \nu$, see Theorem 2.1 for the first inequality, and hence $\bigcup_{m\in S} m\mu(a) \leq \nu(a)$, for every $a \in A$. $\Box$

In the following we have some morte propeties about generated fuzzy S acts.

Theorem 3.3 *(1)* $<<\mu>>=<\mu>$.

(2) $<\bigcup_{i\in I}\mu_i>=\bigcup_{i\in I}<\mu_i>$.

Proof. (1) It is trivial by definition of generated fuzzy S-act.

(2) By Theorem 3.2 we have

$$\begin{aligned} <\bigcup_{i\in I}\mu_i>(a) &= \bigvee\{\bigcup \mu_i(x) \mid mx = a \text{ for some } m \in M\} \\ &= \bigvee_{i\in I}\bigvee_{mx=a} \mu_i(x) \\ &= \bigcup_{i\in I} <\mu_i>(a) \end{aligned} \tag{1}$$

for every $a \in A$. $\Box$

Definition 3.2 *Let A be an S-act and $\alpha \in [0,1]$ and $x \in A$. Then by cyclic fuzzy S-act $<x_\alpha>$ we mean:* $<x_\alpha>(a) = \begin{cases} \alpha & \text{if } a \in Sx \\ 0 & \text{othrewise} \end{cases}$ *for every $a \in A$.*

Corolary 3.1 *Let μ be a fuzzy S-act of an S-act A and $x \in A$. Then* $<x_{\mu(x)}> \leq \mu$.

Theorem 3.4 *Let S be a monoid and μ be a fuzzy S-act of an S-act A. Then $\mu = \bigcup_{x\in A} <x_{\mu(x)}>$.*

Proof. $\bigcup_{x\in A} <x_{\mu(x)}>(a) = \bigvee_{x\in A} <x_{\mu(x)}>(a) = \bigvee\{\mu(x) \mid a = mx \text{ for some } m \in S\}$. But since $1_S a = a$, $\mu(a) \leq \bigvee\{\mu(x) \mid a = mx \text{ for some } m \in M\}$. Also since μ is a fuzzy S-act, $\mu(x) \leq \mu(mx)$. Hence we have $\mu(a) \leq \bigvee_{mx=a} \mu(x) \leq \mu(a)$, for every $a \in A$. that is $\bigcup_{x\in A} <x_{\mu(x)}>= \mu$ $\Box$

Theorem 3.5 *For every $m \in S$ and every cyclic fuzzy S-act $< x_\alpha >$ of A, $m < x_\alpha > = < x_\alpha >$.*

Proof.

$$\begin{aligned} m < x_\alpha > (a) &= \bigvee \{< x_\alpha > (y) \mid my = a\} \\ &= \begin{cases} \alpha & \text{if } a \in Sx \\ 0 & otherwise \end{cases} \\ &= < x_\alpha > (a). \end{aligned}$$ □

4 Decomposable and Indecomposable Fuzzy S-act

Here we give a definition of indecomposable Fuzzy S-act and show that the cyclic fuzzy S-acts are indecomposable. We also see some properties of indecomposable fuzzy S-acts in this section.

Definition 4.1 *An fuzzy S-act $\mu \neq 0$ of A is called decomposable whenever there exist two fuzzy S-acts $\nu, \eta \neq 0$ of A such that $\nu, \eta \leq \mu$ and $\eta \vee \nu = \mu$, and $\eta \wedge \nu = 0$. Otherwise μ is called indecomposable.*

Theorem 4.1 *Let S be a commutative monoid. Then every cyclic fuzzy S-act $< x_i >$ of A is indecomposable.*

Proof. Let $< x_i >$ be decomposable. Then there are fuzzy S-acts ν and η of A such that $\nu, \eta \leq < x_i >$ and $\eta \vee \nu = < x_i >$, and $\eta \wedge \nu = 0$. So for every $a \in A$, $\eta(a) \wedge \nu(a) = 0$ and $\eta(a) \vee \nu(a) = \begin{cases} i & \text{if} a = mx \\ 0 & \text{otherwise} \end{cases}$. Now let $\nu(m_0 x) = i$. Then we claim that for every $m \in S$, $\nu(mx) = i$ and $\eta(mx) = 0$. Because if there exists $m_1 \in S$ such that $\eta(m_1 x) = i$, then $\eta(m_1 m_0 x) = i$ and $\nu(m_1 m_0 x) = i$, so $\eta(m_1 m_0 x) \wedge \nu(m_1 m_0 x) = i \neq 0$. □

Definition 4.2 *A fuzzy S-act $A^{(\mu)}$ is called finitely generated whenever $\mu = < \bigcup_{i=1}^{n} (x_i)_{\alpha_i} >$, where $\alpha_i \in [0, 1]$.*

Theorem 4.2 *Let $f : A \to B$ be an S-act homomorphism and $A^{(\mu)}$ be an fuzzy S-act. Then $B^{f(\mu)}$ is finitely generated, if so is μ.*

Proof. To proof, we show that $f(\mu) =< \bigcup_{i=1}^n f((x_i)_{\alpha_i}) >$, where $\mu =< \bigcup_{i=1}^n (x_i)_{\alpha_i} >$. For

$$\begin{aligned} f(\mu)(b) &= \bigvee \{\mu(a) \mid f(a) = b\} \\ &= \begin{cases} \bigvee_{j \in J \subseteq \{1,\ldots,n\}} \alpha_j & f(x_j) = b \\ 0 & \text{otherwise} \end{cases} \\ &= \bigcup_{i \in I} f(< (x_i)_{\alpha_i} >)(b). \end{aligned}$$ □

Lemma 4.1 *Let G be a group. Then for every finitely generated fuzzy G-act $A^{(\mu)}$, there exists a finite subsets $\{a_1, \ldots, a_n\} \subseteq [0,1]$ and $\{x_1, \ldots, x_n\} \subseteq A$ such that $\mu(O(x_i)) = a_i$ and $O(x_i) \cap O(x_j) = \emptyset$, if $i \neq j$, where $O(x_i)$ is a notation for orbit of x_i that is the set $\{gx_i \mid g \in G$.*

Proof. Since μ is finitely generated, $\mu =< \bigcup_{i=1}^n (x_i)_{a_i}) >$ and since G is a group, $O(x_i) \cap O(x_j) = \emptyset$, if $i \neq j$. So

$$\mu(x) = \begin{cases} a_i & \text{if } x \in O(x_i) \\ 0 & \text{otherwise.} \end{cases}$$ □

Theorem 4.3 *Let $f : A \to B$ be an S-act homomorphism with commutative S, and $A^{(\mu)}$ be an fuzzy S-act. Then $f(\mu) =< f(\nu) >$, if $\mu =< \nu >$.*

Proof.

$$\begin{aligned} f(\mu)(b) &= \bigvee \{\mu(a) \mid f(a) = b\} \\ &= \bigvee \{(m\nu)(a) \mid m \in S,\ f(a) = b\} \quad \text{by Theorem 3.2} \\ &= \bigvee \{\nu(x) \mid m \in M, mx = a,\ f(a) = b\} \\ &= \bigvee \{f(\nu)(y) \mid m \in M,\ my = b\} \\ &= \bigvee_{m \in M} mf(\nu)(b) \\ &=< f(\nu) > (b). \end{aligned}$$ □

Theorem 4.4 *Let $\{A^{(\nu_i)}\}_{i \in I}$ be a family of fuzzy S-act in which there is $i_0 \in I$ such that μ_{i_0} is indecomposable. Then $\bigvee \mu_i$ is indecomposable.*

Proof. Let $\bigvee_{i \in I} \mu_i$ be decomposable. So there are $\nu_1, \nu_2 \leq \bigvee_{i \in I} \mu_i$ such that $\bigvee_{i \in I} \mu_i = \nu_1 \vee \nu_2$ and $\nu_1 \wedge \nu_2 = 0$. Then $\mu_{i_0} = \mu_{i_0} \wedge (\bigvee_{i \in I} \mu_i) = (\mu_{i_0} \wedge \nu_1) \vee (\mu_{i_0} \wedge \nu_2)$ Also $(\mu_{i_0} \wedge \nu_1) \wedge (\mu_{i_0} \wedge \nu_2) = \mu_{i_0} \wedge (\nu_1 \wedge \nu_2) = 0$. □

Corolary 4.1 *The union of indecomposable fuzzy S-acts is indecomposable.*

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