Ratio Mathematica Volume 39, 2020, pp. 55-67

New structure of norms on $\mathbb{R}^n$ and their relations with the curvature of the plane curves

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Abstract

Let $f_1, f_2, \ldots, f_n$ be fixed nonzero real-valued functions on $\mathbb{R}$, the real numbers. Let $\varphi_n(X_n) = \left(x_1^2 f_1^2 + x_2^2 f_2^2 + \ldots + x_n^2 f_n^2\right)^{\frac{1}{2}}$, where $X_n = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. We show that φ_n has properties similar to a norm function on the normed linear space. Although φ_n is not a norm on $\mathbb{R}^n$ in general, it induces a norm on $\mathbb{R}^n$. For the nonzero function $F : \mathbb{R}^2 \to \mathbb{R}$, a curvature formula for the implicit curve $G(x, y) = F^2(x, y) = c \neq 0$ at any regular point is given. A similar result is presented when F is a nonzero function from $\mathbb{R}^3$ to $\mathbb{R}$. In continued, we concentrate on $F(x, y) = \int_a^b \varphi_2(x, y)dt$. It is shown that the curvature of $F(x, y) = c$, where $c > 0$ is a positive multiple of c^2. Particularly, we observe that $F(x, y) = \int_0^{\frac{\pi}{2}} \sqrt{x^2 \cos^2 t + y^2 \sin^2 t}dt$ is an elliptic integral of the second kind.

Keywords: norm; curvature; homogeneous function; elliptic integral.
2010 AMS subject classifications: 53A10. **2010 AMS subject classifications**: 53A10. [1]

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[1]Received on October 31st, 2020. Accepted on December 17th, 2020. Published on December 31st, 2020. doi: 10.23755/rm.v39i0.552. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

A *normed linear space* is a real linear space X such that a number $\|x\|$, the *norm* of x, is associated with each $x \in X$, satisfying: $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$; $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and $\|x + y\| \leq \|x\| + \|y\|$.
For example, let X be a Tychonoff space, $C^*(X)$ the ring of all bounded real-valued continuous functions on X. Then $C^*(X)$ is a normed linear space with the norm $\|f\| = \sup\{|f(x)| : x \in X\}$ and pointwise addition and scalar multiplication. This is called the *supremum-norm* on $C^*(X)$. The associated metric is defined by $d(f, g) = \|f - g\|$. A non-empty set $C \subseteq \mathbb{R}^n$ is called a *convex* set if whenever P and Q belong to C, the segment joining P and Q belongs to C. Analytically the definition can be formulated in this way: if P is represented by the vector x, and Q by the vector y, then C is a convex set if with P and Q it contains also every point with a vector of form $\lambda x + (1 - \lambda)y$, where $0 \leq \lambda \leq 1$. A point P is an *interior point* of a set S contained in $\mathbb{R}^n$, if there exists an n-dimensional ball, with center at P, all of whose points lie in S. An *open set* is a set containing only interior points. A subset $C \subseteq \mathbb{R}^n$ is *centrally symmetric* (or 0-*symmetric*) if for every point $Q \in \mathbb{R}^n$ contained in C, $-Q \in C$, where $-Q$ is the reflection of Q through the origin, that is $C = -C$.

Definition 1.1. *([Siegel, 1989, page 5]) A convex body is a bounded, centrally symmetric convex open set in $\mathbb{R}^n$.*

Example 1.1. *The interior of an n-dimensional ball, defined by $x_1^2 + x_2^2 + \cdots + x_n^2 < a^2$ provides an example of a convex body.*

One of the many important ideas introduced by Minkowski into the study of convex bodies was that of gauge function. Roughly, the gauge function is the equation of a convex body. Minkowski showed that the gauge function could be defined in a purely geometric way and that it must have certain properties analogous to those possessed by the distance of a point from the origin. He also showed that conversely given any function possessing these properties, there exists a convex body with the given function as its gauge function.

Definition 1.2. *([Siegel, 1989, page 6]) Given a convex body $\mathcal{B} \subseteq \mathbb{R}^n$ containing the origin O, we define a function $f : \mathbb{R}^n \to [0, \infty)$ as follows.*

$$f(x) = \begin{cases} 1 & \text{if } x \in \partial\mathcal{B}, \\ 0 & \text{if } x = 0, \\ \lambda & \text{if } 0 \neq x = \lambda y, \end{cases}$$

where λ is the unique positive real number such that the ray through O and the point (whose vector is) x intersects the surface $\partial\mathcal{B}$ (the boundary of $\mathcal{B}$) in a point y. The function f so defined is the gauge function of the convex body $\mathcal{B}$.

Example 1.2. *Let $f : \mathbb{R} \to [0, \infty)$ defined by*

$$f(x) = \max\{|x_1|, |x_2|, \dots, |x_n|\},$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then int$\mathcal{B}$, the interior of the cubic $\mathcal{B} = \{(x_1, x_2, \dots, x_n) : |x_i| \leq 1\}$ is a convex body and f is a gauge function of it.

It is shown in [Siegel, 1989, Theorems 4-7] that a function $f : \mathbb{R} \to [0, \infty)$ is a gauge function if and only if the following conditions hold: $f(x) \geq 0$ for $x \neq 0$, $f(0) = 0$; $f(\lambda x) = \lambda f(x)$, for $0 \leq \lambda \in \mathbb{R}$; and $f(x + y) \leq f(x) + f(y)$. Moreover, f is continuous and the convex body of f is $\mathcal{B} = \{x : f(x) < 1\}$.

A brief outline of this paper is as follows. In section 2, we introduce a function φ_n on $\mathbb{R}^n$, by the formula

$$\varphi_n(X_n) = \sqrt{x_1^2 f_1^2 + x_2^2 f_2^2 + \cdots + x_n^2 f_n^2},$$

when n fixed nonzero real-valued functions $f_1, f_2, \dots, f_n$ on $\mathbb{R}$ are given. We show that the mappings φ_n have similar properties such as norm functions within difference the ranges of these functions lie in $\mathbb{R}^{\mathbb{R}}$ while the range of a norm function is in the $[0, \infty)$. This definition allows us to define a norm and hence a gauge function on $\mathbb{R}^n$. So it turns $\mathbb{R}^n$ into a metric space. In Section 3, we focus on $n = 2$, φ_2 and the induced norm on $\mathbb{R}^2$. First, we show that if $F : \mathbb{R}^2 \to \mathbb{R}$ is a nonzero function, then k, the curvature of the implicit $G(x, y) = F^2(x, y) = c \neq 0$ at every regular point is calculated by this formula:

$$k = \frac{|\mathbf{H}G| - 4F^2|\mathbf{H}F|}{4F\big(F_x^2 + F_y^2\big)^{\frac{3}{2}}},$$

where $\mathbf{H}F$ and $\mathbf{H}G$ are the Hessian matrices of F and G respectively. It is also shown if $F(x, y) = \int_a^b \sqrt{x^2 f^2(t) + y^2 g^2(t)}dt$, then $|\mathbf{H}F| = 0$ and the eigenvalues of $\mathbf{H}F$ and $\mathbf{H}G$, where $G = F^2$ are nonnegative. Particularly, when $f(t) = \cos t$ and $g(t) = \sin t$, we prove that $\int_0^{\frac{\pi}{2}} \sqrt{x^2 f^2(t) + y^2 g^2(t)}dt$ is an elliptical integral of the second type.

2 A norm on $\mathbb{R}^n$ made by the real valued functions on $\mathbb{R}$

We begin with the following notation.

Notation 2.1. *Suppose that $f_1, f_2, \dots, f_n$ are nonzero real-valued functions on $\mathbb{R}$ and define $\varphi_n : \mathbb{R}^n \to \mathbb{R}^{\mathbb{R}}$ with*

$$\varphi_n(X_n) = \sqrt{x_1^2 f_1^2 + x_2^2 f_2^2 + \cdots + x_n^2 f_n^2}, \qquad (*)$$

where $X_n = (x_1, x_2, \ldots, x_n)$ *and* $\mathbb{R}^{\mathbb{R}}$ *is the set (in fact, ring) of all real-valued functions on* $\mathbb{R}$.

The following statement is a key lemma. However, its proof is straightforward and elementary, it will be used in the proof of the triangle inequality in the next results.

Lemma 2.1. *Let* a, b, c *and* d *are nonnegative real numbers. Then*

$$\sqrt{ac} + \sqrt{bd} \leq \sqrt{(a+b)(c+d)}.$$

Proposition 2.1. *Let* $X_n, Y_n \in \mathbb{R}^n$, $n = 1, 2$ *or* 3. *Then* $\varphi_n(X_n + Y_n) \leq \varphi_n(X_n) + \varphi_n(Y_n)$.

Proof. The inequality clearly holds when $n = 1$. Next, we do the proof for $n = 2$. Take $X_2 = (x_1, y_1)$, $Y_2 = (x_2, y_2) \in \mathbb{R}^2$ and suppose that f and g are nonzero elements of $\mathbb{R}^{\mathbb{R}}$. Then

$$\begin{aligned}
\varphi_2(X_2 + Y_2) &= \sqrt{(x_1 + x_2)^2 f^2 + (y_1 + y_2)^2 g^2} \\
&\leq \sqrt{x_1^2 f^2 + y_1^2 g^2} + \sqrt{x_2^2 f^2 + y_2^2 g^2} \\
&= \varphi_2(X_2) + \varphi_2(Y_2)
\end{aligned}$$

if and only if

$$\begin{aligned}
x_1 x_2 f^2 + y_1 y_2 g^2 &\leq \sqrt{\left[x_1^2 f^2 + y_1^2 g^2\right]\left[x_2^2 f^2 + y_2^2 g^2\right]} \\
&= \varphi_2(X_2)\varphi_2(Y_2). \qquad (\star)
\end{aligned}$$

Now, if we let $B := x_1 x_2 f^2 + y_1 y_2 g^2$ and suppose that $B \geq 0$, then $(\star)$ holds if and only if

$$f^2 g^2 (x_1 y_2 - x_2 y_1)^2 \geq 0,$$

which is always true (note, $(\star)$ trivially holds if $B \leq 0$). Hence, in this case, the proof is complete.
Here, we prove the proposition for $n = 3$. Let $X_3 = (x_1, y_1, z_1) = (X_2, z_1)$ and $Y_3 = (x_2, y_2, z_2) = (Y_2, z_2)$, where $X_2 = (x_1, y_1)$, $Y_2 = (x_2, y_2)$ and let f, g, h be nonzero elements of $\mathbb{R}^{\mathbb{R}}$. Then

$$\begin{aligned}
\varphi_3(X_3 + Y_3) &= \sqrt{(x_1 + x_2)^2 f^2 + (y_1 + y_2)^2 g^2 + (z_1 + z_2)^2 h^2} \\
&\leq \sqrt{x_1^2 f^2 + y_1^2 g^2 + z_1^2 h^2} + \sqrt{x_2^2 f^2 + y_2^2 g^2 + z_2^2 h^2} \\
&= \varphi_3(X_3) + \varphi_3(Y_3)
\end{aligned}$$

if and only if

$$x_1x_2f^2 + y_1y_2g^2 + z_1z_2h^2 \leq \sqrt{[x_1^2f^2 + y_1^2g^2 + z_1^2h^2][x_2^2f^2 + y_2^2g^2 + z_2^2h^2]}$$
$$= \sqrt{\left[\varphi_2^2(X_2) + z_1^2h^2\right]\left[\varphi_2^2(Y_2) + z_2^2h^2\right]}$$

Now, if we let $a = \varphi_2^2(X_2), b = z_1^2h^2, c = \varphi_2^2(Y_2)$ and $d = z_2^2h^2$, then by $(*)$ in Notation 2.1, we have

$$x_1x_2f^2 + y_1y_2g^2 \leq \sqrt{ac}.$$

Moreover, it is clear that $z_1z_2h^2 \leq \sqrt{bd}$. Therefore,

$$x_1x_2f^2 + y_1y_2g^2 + z_1z_2h^2 \leq \sqrt{ac} + \sqrt{bd}.$$

In view of Lemma 2.1, the proof is now complete. □

Next, we state the general case of Proposition 2.1.

Theorem 2.1. *Let $X_n = (x_1, x_2, \ldots, x_n), Y_n = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n, \lambda \in \mathbb{R}$ and φ_n be as defined in Notation 2.1. Then the following statements hold.*

(i) $\varphi_n(X_n) = 0$ *if and only if* $X_n = 0$,

(ii) $\varphi_n(\lambda X_n) = |\lambda|\varphi_n(X_n)$,

(iii) $\varphi_n(X_n + Y_n) \leq \varphi_n(X_n) + \varphi_n(Y_n)$ *(triangle inequality).*

Proof. (i) and (ii) are evident. (iii). The proof is done by induction on n, see Proposition 2.1. If we set $X_{n-1} = (x_1, x_2, \ldots, x_{n-1})$ and $Y_{n-1} = (y_1, y_2, \ldots, y_{n-1})$ then X_n and Y_n can be substituted by (X_{n-1}, x_n) and (Y_{n-1}, y_n) respectively. Therefore,

$$\varphi_n(X_n + Y_n) \leq \varphi_n(X_n) + \varphi_n(Y_n)$$

if and only if

$$x_1y_1f_1^2 + \cdots + x_ny_nf_n^2 \leq \varphi_n(X_n)\varphi_n(Y_n)$$
$$= \sqrt{\left[\varphi_{n-1}^2(X_{n-1}) + x_n^2f_n^2\right]\left[\varphi_{n-1}^2(Y_{n-1}) + y_n^2f_n^2\right]}.$$

Now, let $a = \varphi_{n-1}^2(X_{n-1}), b = x_n^2f_n^2, c = \varphi_{n-1}^2(Y_{n-1})$ and $d = y_n^2f_n^2$ plus the assumption of induction, we have

$$x_1y_1f_1^2 + \cdots + x_{n-1}y_{n-1}f_{n-1}^2 \leq \sqrt{ac}.$$

Moreover, it is obvious that $x_ny_nf_n^2 \leq \sqrt{bd}$. Thus, $x_1y_1f_1^2 + \cdots + x_ny_nf_n^2 \leq \sqrt{ac} + \sqrt{bd}$. Lemma 2.1 now yields the result. □

Corollary 2.1. *If $f_1, f_2, \ldots, f_n$ are nonzero constant functions, then φ_n is a norm (and hence a gauge function) on $\mathbb{R}^n$.*

By Theorem 2.1, we obtain the following result.

Proposition 2.2. *Let a, b be real numbers, $f_1, f_2, \ldots,$ and f_n the restrictions of some non-zero elements of $\mathbb{R}^{\mathbb{R}}$ on $[a, b]$ such that each of them is nonzero on this set, and let φ_n be as defined in the previous parts (Notation 2.1). Then the mapping $\psi_n : \mathbb{R}^n \to [0, \infty)$ defined by*

$$\psi_n(X_n) = \int_a^b \varphi_n(X_n)dt$$

is a norm on $\mathbb{R}^n$, and hence $d(X_n, Y_n) = \psi(X_n - Y_n)$ turns $\mathbb{R}^n$ into a metric space.

Corollary 2.2. *The mapping ψ_n is a gauge function on $\mathbb{R}^n$ with the convex body $C_n = \{X_n \in \mathbb{R}^n : \psi_n(X_n) < 1\}$.*

3 $F(x, y) = \int_a^b \varphi_2(x, y)dt$ as a norm on $\mathbb{R}^2$ and the curvature in the plane

Proposition 3.1. *([Goldman, 2005, Proposition 3.1]) For a curve defined by the implicit equation $F(x, y) = 0$, the curvature of F (denoted by κ) at a regular point (x_0, y_0) (i.e., the first partial derivatives F_x and F_y at this point are not both equal to 0) is given by the formula*

$$\kappa = \frac{|F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy}|}{\left(F_x^2 + F_y^2\right)^{\frac{3}{2}}},$$

where F_x denotes the first partial derivative with respect to x, F_y, F_{xx} denotes the second partial derivative with respect to x, F_{yy}, and F_{xy} denotes the mixed second partial derivative (for readability of the above formulas, the argument (x_0, y_0) has been omitted).

We recall that the *Hessian matrix* of $z = F(x, y)$ and $w = F(x, y, z)$ are defined to be $\mathbf{H}z = \begin{bmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{bmatrix}$ and $\mathbf{H}w = \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}$ at any point at which all the second partial derivatives of F exist.

Theorem 3.1. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a nonzero function and $(x_0, y_0) \in \mathbb{R}^2$ a regular point. Suppose that the second partial derivatives of F at (x_0, y_0) exist and further $F_{xy} = F_{yx}$ at this point. Let $\mathbf{H}F$ and $\mathbf{H}G$ be the Hessian matrices of F and F^2 respectively (we assume that $G = F^2$) and let k be the curvature of $G(x, y) = F^2(x, y) = c \neq 0$ at (x_0, y_0). Then we have*

$$k = \frac{|\mathbf{H}G| - 4F^2|\mathbf{H}F|}{4F\big(F_x^2 + F_y^2\big)^{\frac{3}{2}}}.$$

Proof. For simplicity, we do the proof without (x_0, y_0). The partial derivatives of $G = F^2$ are as follows:

$$\begin{aligned} G_x &= 2FF_x, & G_{xx} &= 2({F_x}^2 + FF_{xx}), \\ G_y &= 2FF_y, & G_{yy} &= 2(F_y^2 + FF_{yy}), \text{ and } G_{xy}^2 = 4(F_xF_y + FF_{xy})^2. \end{aligned}$$

Therefore,

$$\begin{aligned} |\mathbf{H}G| &= G_{xx}G_{yy} - G_{xy}^2 = 4\big({F_x}^2 + FF_{xx}\big)\big(F_y^2 + FF_{yy}\big) - 4\big(F_xF_y + FF_{xy}\big)^2 \\ &= 4\Big[F_x^2F_y^2 + FF_x^2F_{yy} + FF_y^2F_{xx} + F^2F_{xx}F_{yy} - F_x^2F_y^2 - 2FF_xF_yF_{xy} \\ &\quad - F^2F_{xy}^2\Big] \\ &= 4\Big[F^2\big(F_{xx}F_{yy} - F_{xy}^2\big) + F\big(F_x^2F_{yy} - 2F_xF_yF_{xy} + F^2yF_{xx}\big)\Big] \\ &= 4\Big[F^2|\mathbf{H}F| + F\big(F_x^2F_{yy} - 2F_xF_yF_{xy} + F^2yF_{xx}\big)\Big]. \end{aligned}$$

In view of Proposition 3.1, we have

$$\begin{aligned} |\mathbf{H}G| &= 4\Big[F^2|\mathbf{H}F| + F\big(F_x^2F_{yy} - 2F_xF_yF_{xy} + F^2yF_{xx}\big)\Big] \\ &= 4\Big[F^2|\mathbf{H}F| + Fk\big(F_x^2 + F_y^2\big)^{\frac{3}{2}}\Big] \end{aligned}$$

Therefore,

$$k = \frac{|\mathbf{H}G| - 4F^2|\mathbf{H}F|}{4F\big(F_x^2 + F_y^2\big)^{\frac{3}{2}}},$$

and we are done. □

The next result is a similar consequence for the implicit surface.

Theorem 3.2. *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a nonzero function and $(x_0, y_0, z_0) \in \mathbb{R}^3$ a regular point. Suppose that the second partial derivatives of F at (x_0, y_0, z_0) exist* *and further the mixed partial derivatives at this point are equivalent. If k is the curvature of $G(x,y,z) = F^2(x,y,z) = c \neq 0$ at (x_0, y_0, z_0), then we have*

$$k = \frac{|\mathbf{H}G| - 8F^3|\mathbf{H}F|}{8F^2\big(F_x^2 + F_y^2 + F_z^2\big)^{\frac{3}{2}}},$$

where $\mathbf{H}F$ and $\mathbf{H}G$ are the Hessian matrices of F and F^2 respectively (we assume that $G = F^2$).

Proof. As we did in the previous theorem, the proof is done without (x_0, y_0, z_0). Let $K = \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{xy} & F_{yy} & F_{yz} & F_y \\ F_{xz} & F_{yz} & F_{zz} & F_z \\ F_x & F_y & F_z & 0 \end{bmatrix}$. It is known that the curvature k of the implicit surface $F(x,y,z) = 0$ is $k = |K|$ at every regular point in which the second partial derivatives of F exist. We first calculate the partial derivatives of G and in continued we obtain determinant of $\mathbf{H}G$.

$$\begin{aligned} G_x &= 2FF_x, & G_{xx} &= 2({F_x}^2 + FF_{xx}), & G_{xy}^2 &= 4(F_xF_y + FF_{xy})^2 \\ G_y &= 2FF_y, & G_{yy} &= 2(F_y^2 + FF_{yy}), & G_{xz}^2 &= 4(F_xF_z + FF_{xz})^2 \\ G_z &= 2FF_z, & G_{zz} &= 2(F_z^2 + FF_{zz}), & G_{yz}^2 &= 4(F_yF_z + FF_{yz})^2. \end{aligned}$$

Recall that the Hessian matrices of F and G are

$$\mathbf{H}F = \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{xy} & F_{yy} & F_{yz} \\ F_{xz} & F_{yz} & F_{zz} \end{bmatrix}, \text{and } \mathbf{H}G = \begin{bmatrix} G_{xx} & G_{xy} & G_{xz} \\ G_{xy} & G_{yy} & G_{yz} \\ G_{xz} & G_{yz} & G_{zz} \end{bmatrix}.$$

Here, we compute the determinant of $\mathbf{H}G$.

$$\begin{aligned} 1/8|\mathbf{H}G| &= F_{xx}\big(F_{yy}F_{zz} - F_{yz}^2\big) - F_{xy}\big(F_{xy}F_{zz} - F_{xz}F_{yz}\big) \\ &+ F_{xz}\big(F_{xy}F_{yz} - F_{xz}F_{yy}\big) \\ &= F_{xx}F_{yy}F_{zz} - F_{xx}F_{yz}^2 - F_{yy}F_{xz}^2 - F_{zz}F_{xy}^2 + 2F_{xy}F_{yz}F_{xz} \\ &= \big(F_x^2 + FF_{xx}\big)\big(F_y^2 + FF_{yy}\big)\big(F_z^2 + FF_{zz}\big) \\ &- \big(F_x^2 + FF_{xx}\big)\big(F_yF_z + FF_{yz}\big)^2 \\ &- \big(F_y^2 + FF_{yy}\big)\big(F_xF_z + FF_{xz}\big)^2 - \big(F_z^2 + FF_{zz}\big)\big(F_xF_y + FF_{xy}\big)^2 \\ &+ \big(F_xF_z + FF_{xz}\big)\big(F_yF_z + FF_{yz}\big)\big(F_xF_y + FF_{xy}\big) \\ &= F^3\Big[F_{xx}F_{yy}F_{zz} - F_{xx}F_{yz}^2 - F_{yy}F_{xz}^2 - F_{xx}F_{xy}^2 + 2F_{xy}F_{yz}F_{xz}\Big] \\ &+ F^2\Big[F_{xx}F_{yy}F_z^2 + F_{xx}F_{zz}F_y^2 + F_{yy}F_{zz}F_x^2 - 2F_{xy}F_{xz}F_yF_z \\ &- 2F_{xy}F_{yz}F_xF_z - 2F_{xz}F_{yz}F_xF_y + F_{xy}^2F_z^2 + F_{xz}^2F_y^2 + F_{yz}^2F_x^2\Big] + F\big[0\big]. \end{aligned}$$

Therefore, we have $1/8|\mathbf{H}G| = F^3|\mathbf{H}F| + F^2k(F_x^2 + F_y^2 + F_z^2)^{\frac{3}{2}}$. So the result is obtained, i.e.,

$$k = \frac{|\mathbf{H}G| - 8F^3|\mathbf{H}F|}{8F^2\big(F_x^2 + F_y^2 + F_z^2\big)^{\frac{3}{2}}}.$$

□

Theorem 3.3. *Let f, g be nonzero real-valued functions on $\mathbb{R}$, $a, b \in \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,y) = \int_a^b \sqrt{x^2f^2(t) + y^2g^2(t)}d(t)$. Then*

(i) The curvature of $F(x,y) = c$, where $c > 0$ at any point of the curve is positive multiple of c^2.

(ii) tr$(\mathbf{H}F) = F_{xx} + F_{yy} \geq 0$.

Proof. (i). First, we note that $F \geq 0$. The surface F meets the plane $z = 0$ at the origin only. But the intersection of F with the plane $z = c$ (where $c > 0$) is the curve $F(x,y) = c$. Here the partial derivatives of F are calculated (see [Rudin, 1976, Theorem 9.42]).

$$F_x = \int_a^b \frac{xf^2(t)}{\sqrt{x^2f^2(t) + y^2g^2(t)}}d(t), \qquad F_y = \int_a^b \frac{yg^2(t)}{\sqrt{x^2f^2(t) + y^2g^2(t)}}d(t),$$

$$F_{xx} = \int_a^b \frac{y^2f^2(t)g^2(t)}{\big(x^2f^2(t) + y^2g^2(t)\big)^{\frac{3}{2}}}d(t), \qquad F_{yy} = \int_a^b \frac{x^2f^2(t)g^2(t)}{\big(x^2f^2(t) + y^2g^2(t)\big)^{\frac{3}{2}}}d(t),$$

and

$$F_{xy} = -\int_a^b \frac{xyf^2(t)g^2(t)}{\big(x^2f^2(t) + y^2g^2(t)\big)^{\frac{3}{2}}}d(t) = F_{yx}.$$

Let us put $\varphi := \sqrt{x^2f^2(t) + y^2g^2(t)}$. For the simplicity, we set

$$F_x = \int \frac{xf^2}{\varphi}, \quad F_y = \int \frac{yg^2}{\varphi}, \text{ and so on} \ldots$$

By formula of the curvature k in Proposition 3.1, we obtain

$$\begin{aligned}
k &= \frac{1}{\big(F_x^2+F_y^2\big)^{\frac{3}{2}}}\Big[\big(y^2\int\frac{f^2g^2}{\varphi^3}\big)\big(y\int\frac{g^2}{\varphi}\big)^2+2\int\frac{xyf^2g^2}{\varphi^3}\int\frac{xf^2}{\varphi}\int\frac{yg^2}{\varphi}\\
&\quad+\big(x^2\int\frac{f^2g^2}{\varphi^3}\big)\big(x\int\frac{f^2}{\varphi}\big)^2\Big]\\
&= \frac{\int\frac{f^2g^2}{\varphi^3}}{\big(F_x^2+F_y^2\big)^{\frac{3}{2}}}\Big[y^4\big(\int\frac{g^2}{\varphi}\big)^2+2x^2y^2\int\frac{f^2}{\varphi}\int\frac{g^2}{\varphi}+x^4\big(\int\frac{f^2}{\varphi}\big)^2\Big]\\
&= \frac{\int\frac{f^2g^2}{\varphi^3}}{\big(F_x^2+F_y^2\big)^{\frac{3}{2}}}\Big[\int\frac{x^2f^2}{\varphi}+\int\frac{y^2g^2}{\varphi}\Big]^2\\
&= \frac{\int\frac{f^2g^2}{\varphi^3}}{\big(F_x^2+F_y^2\big)^{\frac{3}{2}}}\Big[\int\frac{x^2f^2+y^2g^2}{\varphi}\Big]^2\\
&= \frac{\int\frac{f^2g^2}{\varphi^3}}{\big(F_x^2+F_y^2\big)^{\frac{3}{2}}}\Big[\int\varphi\Big]^2\\
&= \frac{\int\frac{f^2g^2}{\varphi^3}}{\big(F_x^2+F_y^2\big)^{\frac{3}{2}}}F^2(x,y).
\end{aligned}$$

Hence, we observe that the curvature of $F(x,y)=c$ at (x_0,y_0) is a positive multiple of $F^2(x_0,y_0)=c^2$, and we are done.
(ii). Since

$$\frac{f^2g^2(x^2+y^2)}{\varphi^3}\geq 0,$$

it is clear that $F_{xx}+F_{yy}\geq 0$. So the result holds. □

Lemma 3.1. *Let $F:\mathbb{R}^2\to\mathbb{R}$ be a homogeneous function of degree one. Suppose that the second derivatives of F at $(a,b)\in\mathbb{R}^2$ exist. Moreover, $F_{xy}=F_{yx}$ at this point. Then*

(i) $|\mathbf{H}F|_{(a,b)}=0$.

(ii) The eigenvalues of $\mathbf{H}F$ *are* 0 *and tr*$(\mathbf{H}F)$ *at* (a,b).

Proof. (i). First, we note that $F(\lambda x,\lambda y)=\lambda F(x,y)$, for all $(x,y)\in\mathbb{R}^2$ and $\lambda\in\mathbb{R}$. Also, we remind the reader of the following fact, which is known as *Euler's property*,

$$xF_x+yF_y=F(x,y).$$

Therefore,

$$xF_{xx} + F_x + yF_{xy} = F_x, \text{ and } xF_{xy} + F_y + yF_{yy} = F_y.$$

Consequently, $xF_{xx} = -yF_{xy}$ and $xF_{xy} = -yF_{yy}$. Now, consider the Hessian matrix $\mathbf{H}F = \begin{bmatrix} Fxx & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix}$ of F. For the point $(0,b)$, where $b \neq 0$, we have $F_{yy}(0,b) = 0 = F_{xy}(0,b)$. This implies that $|\mathbf{H}F| = 0$. Also, considering the point $(a,0)$, where $a \neq 0$ gives $F_{xy}(a,0) = 0 = F_{xx}(a,0)$, this again yields $|\mathbf{H}F| = 0$. Now, let (a,b) such that $a \neq 0$ and $b \neq 0$. Then $F_{xx}(a,b) = \frac{-b}{a}F_{xy}(a,b)$ and $F_{yy}(a,b) = \frac{-a}{b}F_{xy}(a,b)$. Hence, $|\mathbf{H}F| = 0$. So we always have $|\mathbf{H}F| = 0$. The proof of (i) is now complete. (ii). Recall that the characteristic equation of $\mathbf{H}F$ is

$$\lambda^2 - (tr(\mathbf{H}F) = F_{xx} + F_{yy})\lambda + (|\mathbf{H}F| = F_{xx}F_{yy} - F_{xy}^2) = 0.$$

So $\lambda^2 - (F_{xx} + F_{yy})\lambda = 0$. Therefore, $\lambda = 0$ or $\lambda = \text{tr}(\mathbf{H}F)$, and we are done. □

Proposition 3.2. *Let f, g be nonzero real-valued functions on $\mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,y) = \int_a^b \sqrt{x^2 f^2(t) + y^2 g^2(t)}dt$ and let $G(x,y) = F^2(x,y)$. Then the eigenvalues of $\mathbf{H}F$ and $\mathbf{H}G$ at any point except the origin are nonnegative. (In fact, the eigenvalues of $\mathbf{H}F$ are zero and tr$(\mathbf{H}F)$ at that point).*

Proof. We observe that F is a homogeneous function of degree one. So Lemma 3.1 and Theorem 3.3 (ii) yield the result. For the matrix $\mathbf{H}G$, we look to the Theorem 3.1. Since, $F^2|\mathbf{H}F| = 0$, we have

$$|\mathbf{H}G| = 4Fk\big(F_x^2 + F_y^2\big)^{\frac{3}{2}}.$$

We notice that $F, k \geq 0$ gives $|\mathbf{H}G| \geq 0$. On the other hand, tr$(\mathbf{H}G) = G_{xx} + G_{yy} \geq 0$. Therefore, the roots of $\lambda^2 - \text{tr}(\mathbf{H}G)\lambda + |\mathbf{H}G| = 0$, which are the eigenvalues of $\mathbf{H}G$, are nonnegative. The proof is finished. □

In the following result, we present a norm on $\mathbb{R}^2$ which is an elliptic integral of the second kind.

Corollary 3.1. *Let $f(t) = \cos t$, $g(t) = \sin t$ and let $F : \mathbb{R}^2 \to \mathbb{R}$ is given by*

$$F(x,y) = \int_0^{\frac{\pi}{2}} \sqrt{x^2 \cos^2 t + y^2 \sin^2 t}dt.$$

Then the following statements hold.

(i) The eigenvalues of $\mathbf{H}F$ and $\mathbf{H}G$, where $G = F^2$ at every point except the origin are nonnegative.

(ii) $F(x,y)$ *is an elliptic integral of the second kind.*

Proof. (i). It follows from Proposition 3.2. (ii). Notice that

$$F(x,y) = \int_0^{\frac{\pi}{2}} \sqrt{x^2(1-\sin^2\theta) + y^2\sin^2\theta}d\theta = |x| \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}d\theta,$$

where $k = \frac{\sqrt{x^2-y^2}}{|x|}$ and $|x| \geq |y|$. So this gives $F(x,y)$ is an elliptic integral of the second kind and we are done. $\square$

Corollary 3.2. *There are ordered pairs* (x,y) *with rational coordinates (other than the origin) which satisfy the inequality* $\int_0^{\frac{\pi}{2}} \sqrt{x^2\cos^2\theta + y^2\sin^2\theta}d\theta \leq r$, *when* $0 < r \in \mathbb{Q}$. *Also, if* $r \notin \mathbb{Q}$ *then* (x,y) *has irrational coordinates.*

Proof. It is sufficient to take the pairs $(r,0), (0,r), (-r,0)$ and $(0,-r)$. $\square$

We end this article with the next results.

Proposition 3.3. *Let* $0 \leq x, y \in \mathbb{R}$. *Then*

$$\int_0^{\frac{\pi}{2}} \sqrt{x^2\cos^2 t + y^2\sin^2 t}dt \leq x+y.$$

Proof. First, note that

$$x^2\cos^2 t + y^2\sin^2 t = (x\cos t + y\sin t)^2 - 2xy\sin t\cos t,$$

and take $0 \leq \phi \leq \frac{\pi}{2}$ such that $\tan\phi = \frac{y}{x}$ (if $x > 0$). Now,

$$\begin{aligned}(x\cos t + y\sin t)^2 &= x^2(\cos t + \frac{y}{x}\sin t)^2 = x^2(\cos t + \frac{\sin\phi}{\cos\phi}\sin t)^2 \\ &= \frac{x^2(\cos t\cos\phi + \sin t\sin\phi)^2}{\cos^2\phi} = \frac{x^2\cos^2(t-\phi)}{\cos^2\phi} \\ &= (x^2+y^2)\cos^2(t-\phi) \text{ (note, } \cos^2\phi = \frac{x^2}{x^2+y^2}).\end{aligned}$$

Hence, $x^2\cos^2 t + y^2\sin^2 t \leq (x^2+y^2)\cos^2(t-\phi)$. Therefore,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sqrt{x^2\cos^2 t + y^2\sin^2 t}dt &\leq \int_0^{\frac{\pi}{2}} \sqrt{(x^2+y^2)\cos^2(t-\phi)}dt \\ &= \sqrt{x^2+y^2}\int_0^{\frac{\pi}{2}} |\cos(t-\phi)|dt \\ &= \sqrt{x^2+y^2}\int_{-\phi}^{\frac{\pi}{2}-\phi} \cos T dt \quad (T = t-\phi) \\ &= x+y.\end{aligned}$$

$\square$

Remark 3.1. *We find* $4\int_0^{\frac{\pi}{2}}\sqrt{x^2\cos^2 t + y^2\sin^2 t}dt \leq 2(2x+2y)$. *The left phrase is the length of the ellipse* $x' = x\cos t$ *and* $y' = y\sin t$, *while* $2x$ *and* $2y$ *are the major axis and minor axis of this ellipse.*

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