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Properties of Quasinormal Groups (PQG)

Behnam Razzaghmanesshi *

Abstract

A subgroup H of a group G is termed permutable (or quasi normal) in G if it satisfies the following equivalent conditions:
For any subgroup K of G, HK (the product of subgroups H and K) is a group.
For any subgroup K of G, HK= KH, i.e., H and K are permuting subgroups. For every g in G, H permutes with the cyclic subgroup generated by g. Also we say that G=AB is the mutually permutable product of the subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A. We say that the product is totally permutable if every subgroup of A permutes with every subgroup of B. In this paper we prove the following theorem.
Let G=AB be the mutually permutable product of the super soluble subgroups A and B. If CoreG(A∩B)=1, then G is super soluble.

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* Assistant Professor of Islamic Azad University Talesh branch. Talesh, Iran.; B_razzagh@yahoo.com.
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1 Introduction

All groups considered in this paper are finite. It is known that a group which is the product of two super soluble groups is not necessarily super soluble, even if the two factors are normal subgroups of the group. Baer proved in [3] that if a group G is the product of two normal supersoluble groups and G′ is nilpotent, then G is super soluble. The search for generalisations of Baer's result has been a fruitful topic of investigation recently (see [5,7]).

Most of the generalisations centre around replacing normality of the factors by different permutability conditions. In [2], Asaad and Shaalan considered products satisfying one of the following conditions. We will follow Carocca [6], and say that G=AB is the mutually permutable product of the subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A. We say that the product is totally permutable if every subgroup of A permutes with every subgroup of B. Essentially, the results by Asaad and Shaalan are devoted to obtaining sufficient conditions for a mutually permutable product of two supersoluble subgroups to be supersoluble. They prove in [2, Theorem 3.8] the following generalisation of Baer's theorem:

Let G be the mutually permutable product of the supersoluble subgroups A and B. If G′ is nilpotent, then G is supersoluble.They also show that the result remains true if "G′ nilpotent" is replaced by "Bnilpotent"[2, Theorem 3.2]. In addition, they prove [2, Theorem 3.1]: If G is the totally permutable product of the supersoluble subgroups A and B, then G is supersoluble. It is well known that if G=AB is a mutually permutable product of two supersoluble subgroups A and B such that A∩B=1, then the product is in fact totally permutable [6,Proposition 3.5], and therefore G is supersoluble. Our main Theorem is a generalisation of this last property.

Theorem 1.

Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. If CoreG(A∩B) =1, then G is supersoluble.

The second aim of the present paper has been to obtain more complete information about the structure of mutually permutable products of two supersoluble groups. As a straightforward consequence of Theorem 1, we have that, in the notation used above, G/CoreG(A∩B) is always supersoluble. Therefore, every mutually permutable product of two supersoluble subgroups is metasupersoluble. It is possible to obtain more precise information about its structure, as our second main theorem shows.

Theorem 2. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. Then G/F (G)is supersoluble and metabelian.This last theorem can not be improved easily, as the following example shows.

Example. Let S3 be the symmetric group of degree 3, given by S3=⟨α, β:α2=β3=1;βα=β2⟩ and let T7 be the non-abelian group of order 73 and exponent 7. Write T7= ⟨a,b⟩ with a7=b7=[a,b]7=1 and let c=[a,b]. We have that S3 acts on T7 in the following way: aα=b, bα=a, cα=c−1, aβ=a2, bβ=b4, cβ=c. Thus, we can consider the semidirect product G=[T7] S3. Take now the subgroups

A= T7 ⟨β⟩ and B=T7 ⟨α⟩ of G. Clearly both A and B are supersoluble, and it is easy to check that G=AB is the mutually permutable product of A and B. Finally, we show that Theorem 1 provides elementary proofs for the results of Asaad and Shaalan about mutually permutable products.2.

Main results: The following four lemmas are needed to prove Theorem 1.

Lemma 1[4, Theorem 2]. If G=AB is the mutually permutable product of the supersoluble subgroups A and B, then G is soluble.

Lemma 2. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. Then, either G is supersoluble or NA < G and NB < G for every minimal normal subgroup N of G.

Proof. Assume that G is not supersoluble. Then both A and B are proper subgroups of G. Let N be a minimal normal subgroup of G and for contradiction assume that NA=G. Then, as N is abelian, N∩A is a normal subgroup of ⟨N,A⟩ = G. Since N is a minimal normal subgroup of G and A<G, we have that N∩A=1 and consequently A is a maximal subgroup of G. Clearly, we can also assume that B is not contained in A. It is not difficult to argue that we can choose an element b of B\A such that bq∈A for some prime q. Since the product G=AB is mutually permutable, A ⟨b⟩ is a subgroup of G and the maximality of A implies that G=A ⟨b⟩ . We now take orders to reach our final contradiction:

|A||N|=|G|=|A|| ⟨b⟩ ||A∩ ⟨b⟩ |=q|A|. Consequently, we have that |N|=q and then G is supersoluble, a contradiction.

Lemma 3. Let G=AB be the mutually permutable product of the subgroups A and B and let N be any minimal normal subgroup of G. Then either

N∩A=N∩B=1 or N=(N∩A)(N∩B).

Proof. Let N be a minimal normal subgroup of G. Clearly A(N∩B) and (N∩A)B are both subgroups of G. Note that A normalizes N∩(A(N∩B))=(N∩A)(N∩B) and B normalizes N∩((A∩N)B)= (N∩A)(N∩B).

Therefore $(N\cap A)(N\cap B)$ is a normal subgroup of G and the minimality of N yields the result.

Lemma 4. Let G be a group, and N a minimal normal subgroup of G such that $|N|=pn$, where p is a prime and $n>1$. Denote $C=CG(N)$ and assume that G/Cis supersoluble. Then, if Q/Cis a subgroup of G/C containing $Op'(G/C)$, we have that Q is normal in G and $N=\prod ti=1Ni$, where Ni are non-cyclic minimal normal subgroups of NQ for i=1,...,t.

Proof. Since by [8, Lemma A.13.6], we have that $Op(G/C)=1$ and the commutator subgroup $(G/C)'$ of G/C is nilpotent because e G/C is supersoluble, it follows that $(G/C)'$is a p'-group. Therefore $(G/C)'$is contained in $Op'(G/C)$ and thus $Op'(G/C)$is a Hall p'-subgroup of G/C. Consequently, Q/Cis a normal subgroup of G/C and hence Q is normal in G. Consider now N as a G-module over GF (p)by conjugation. Then, by Clifford's Theorem [8, Theorem B.7.3], N viewed as a Q-module is a direct sum $N=\prod ti=1Ni$, where Ni are irreducible Q-modules for i=1,...,t. Suppose that there exists $i\in\{1,...,t\}$such that $|Ni|=p$. Then clearly $|Nj|= p$ for all j. Therefore Q/CQ(Ni) is abelian of exponent dividing $p-1$, and the same is true for Q/C. In particular, $Q/C=Op'(G/C)$ is a Hall p'-subgroup of G/C. Since N is not cyclic, it follows that Q = G and thus p divides $|G/C|$. Hence p is the largest prime dividing $|G/C|$. From the supersolubility of G/C, we obtain that 1= Op(G/C) is a Sylow subgroup of G/C, a contradiction. Consequently, Ni is a non-cyclic minimal normal subgroup of NQ for all $i\in\{1,t\}$,as we wanted to prove.

Proof of Theorem 1. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B, with $CoreG(A\cap B)=1$, and suppose that G has been chosen minimal such that its supersoluble residual GU is non-trivial. Let N be a minimal normal subgroup of G contained in GU. Note that N is an elementary abelian p-group for some prime p. Applying Lemma 2, we have that both NA and NB are proper subgroups of G. Moreover, using Lemma 3, we have that either $N=(N\cap A)(N\cap B)$ or $N\cap A=N\cap B=1$. Assume first that $N=(N\cap A)(N\cap B)$.

(i) If $N\cap A=1$, then N is cyclic. Assume that $N\cap A=1$. It follows that N is contained in B. Let N0 be a non-trivial cyclic subgroup of N. Since AN0 is a subgroup of G, we have that $N0 =AN0\cap N$ is anormal subgroup of AN0. Hence every cyclic subgroup of N is normalised by A. Now let N1 be a minimal normal subgroup of B contained in N. Since B is supersoluble, it follows

That N1 is cyclic and thus normalised by A. Hence N1 is a normal subgroup of G. The minimality of N implies that N=N1 and consequently N is cyclic.

(ii) $N\cap A=1$ and $N\cap B=1$.On the contrary, assume that $N\cap A=1$. From (i), we know that N is cyclic. Moreover, Nis contained in B. Hence $AN\cap B= (A\cap B)N$.

Let L=CoreG(A∩B)N). Clearly, N is contained in Land L=L∩((A∩B)N)=(L∩A∩B)N. It is clear that G/L=(AL/L)(BL/L)is a mutually permutable product of AL/L and BL/L suchthat CoreG/L((AL/L)∩(BL/L))=1. By the minimality of G, it follows that G/L is supersoluble. On the other hand, since N is cyclic, we have that G/CG(N) is abelian. Hence G/CL(N) is supersoluble and GUCL(N)=C. Note that C=N×(C∩A∩B). Therefore C∩A∩B contains a Hall p′-subgroup of C. Since CoreG(A∩B)=1 and Op′(C) is a normal subgroup of G contained in C∩A∩B, we have that Op′(C)=1. Moreover, C′=(C∩A∩B)′ is a normal subgroup of G contained in A∩B. Consequently, C′=1 and C is an abelian p-group. In particular, GU is abelian and thus GU is complemented in G by a supersoluble normalizer D which is also a supersoluble projector of G, by [8, Theorems V.4.2 and V.5.18]. Since N is cyclic, we know that N is central with respect to the saturated formation of all supersoluble groups. By [8,Theorem V.3.2.e], Dcovers N and thus N is contained in D. It follows ND∩GU=1, a contradiction.

(iii) Either N=N∩A or N=N∩B. If we have N=N∩A=N∩B, then N is contained in A∩B, contradicting the factthat CoreG(A∩B)=1. We may assume without loss of generality that N∩A=N.

(iv) AN and BN are both supersoluble. Since N=(N∩A)(N∩B) and N=N∩A, it follows that N∩B is not contained in N∩A. Let n be any element of N∩B such that n/∈N∩A, and write N0 = 〈n〉 . Note that AN0 is a subgroup of G, and AN0∩N=(N∩A)N0. Therefore N0(N∩A) is a normal subgroup of AN0, and consequently A normalizes (A∩N)N0. This yields that A/CA(N/N∩A) acts as a power automorphism group on N/N∩A. This means that AN is supersoluble. If N∩B=N, then BN=B is supersoluble. On the contrary, if N∩B=N, we can argue as above and we obtain that BN is supersoluble. Consequently, ACG(N)/CG(N) and BCG(N)/CG(N) are both abelian groups of exponent dividing p−1. But then G/CG(N)=(ACG(N)/CG(N))(BCG(N)/CG(N)) is a π-group for some set of primes π such that if q∈π, then q divides p−1.

(v) Let B0 be a Hall π-subgroup of B. Then AB0∩N= A∩N.

This follows just by observing that AB0∩Nis contained in each Hall π′-subgroup of AB0 and every Hall π′-subgroup of A is a Hall π′-subgroup of AB0. Note that |G/CG(N)| is a π-number and AB0 contains a Hall π-subgroup of G. Therefore G=(AB0)CG(N). But then A∩N is a normal subgroup of G. The minimality of G yields either A∩N= 1or A∩N= N. This contradicts our assumption 1=N∩A=N, and so we cannot have N=(A∩N)(B∩N). Thus, by Lemma 3, we may assume N∩A=N∩B=1. Let M= CoreG(AN∩BN). Then N∩M= N and G/M is supersoluble by the minimality of G. Again, we reach a contradiction after several steps.

(vi) M=N. Suppose that M=N. Since G/M is supersoluble, we know that N cannot be cyclic. Let us write C=CG(N), and consider the quotient group G/C. It is clear that G/C is supersoluble. Let Q/C=Op(G/C). Since Op(G/C)=1 and

(G/C)′is nilpotent, it follows that Q/C is a normal Hall p′-subgroup of G/C. Let Bp′ be a Hall p′-subgroup of B. Since |N| divides |B:A∩B|, we have that (A∩B)Bp′ is a proper subgroup of B. Let T be a maximal subgroup of B containing (A∩B)Bp′. Then AT is a maximal subgroup of G and |G:AT|= p= |B:T|. If N is not contained in AT, we have G=(AT)N and AT∩N=1. Then |N|=p, a contradiction. Therefore, N is contained in AT. In particular, the family S={X:X is a proper subgroup of B, (A∩B)Bp′X and NAX} is non-empty. Let R be an element of S of minimal order. Observe that AR has p-power index in G and thus ARC/C contains Op′(G/C). Regarding N as a AR-module over GF (p), we know, by Lemma 4, that N is a direct sum N=∏ti=1Ni, where Ni is an irreducible AR-module whose dimension is greater than 1, for all i∈{1,...,t}. Assume that (A∩B)Bp′=R. Then AR=ABp′ and thus N is contained in A, a contradiction. Therefore ABp′∩B=(A∩B)Bp′ is a proper subgroup of R. Let S be a maximal subgroup of R containing (A∩B)Bp′. From the minimality of R, we know that N is not contained in AS. Consequently, there exists some i∈{1,...,t} such that Ni is not contained in AS, which is a maximal subgroup of AR. Hence AR=(AS)Ni. Since Ni is a minimal normal subgroup of AR, it follows that AS∩Ni= 1and |Ni|= |AR:AS|= |R:S|= p, a contradiction.

(vii) M is an elementary abelian p-group. Note that M=N(M∩A)=N(M∩B) and |M∩A|=|M∩B|=|M|/|N|. Moreover, A(M∩B)is a subgroup of G such that A(M∩B)∩M= (M∩A)(M∩B). Hence (M∩A)(M∩B) is also a subgroup of G. If M∩A= M∩B, then M∩A is a normal subgroup of G contained in A∩B. This implies that M∩A=1 and consequently M=N, a contradiction. It yields that M∩A=M∩B. Next we see that (M∩A)(M∩B) is a normal subgroup of G. Since (M∩A)(M∩B)= M∩A(M∩B), we have that A normalizes (M∩A)(M∩B). Similarly, B normalises

(M∩A)(M∩B) since (M∩A)(M∩B)= M∩B(M∩A). This implies normality of (M∩A)(M∩B) in G. Let X=(M∩A)(M∩B). Since we cannot have M∩A= M∩B, M∩A must be strictly contained in X. Thus X=X∩M=(X∩N)(M∩A) > M∩A gives us X∩N=1. But then X∩N=N, giving NX. Suppose that Q is a Hall p′-subgroup of M∩B. Then QA is a subgroup and so QA∩M=Q(M∩A) is also a subgroup which contains Q. Hence, as |M:M∩A|=pk for some k, we have that QM∩A∩B. Thus QB∩MM∩A∩B and similarly QA∩MM∩A∩B. Consequently, QM is contained in M∩A∩B. Since QM=Op(M), it follows that Op(M) is a normal subgroup of G contained in A∩B. Hence Op(M)=1, a contradiction, and consequently Q=1andMis a p-group. Hence N is contained in Z(M) and M=N×(M∩A)=N×(M∩B). Thus φ(M)=φ(M∩A)=φ(M∩B) is a normal subgroup of G contained in A∩B. This implies that φ(M)=1 and M is an elementary abelian p-group, as claimed. (viii) Final contradiction. We have from the previous steps that M∩A is not contained in M∩B and that M∩B is not contained in M∩A because otherwise, since |M∩A|=|M∩B|, it follows that

M∩A=M∩B is a normal subgroup of G contained in A∩B. This would imply M∩A=M∩B=1, and M=(M∩A)N=N. This fact contradicts step (vi).

Let x be an element of M∩B such that x/∈M∩A. Then A 〈x〉 is a subgroup of G, and so is M0=A 〈x〉 ∩M=(A∩M) 〈x〉 . Therefore, M0 is an A-invariant subgroup of G. In particular, since M=(M∩A)(M∩B), we have that every subgroup of M/M∩A is A-invariant; that is, A/CA(M/M∩A) acts as a group of power automorphisms on M/M∩A. It is clear that M/M∩A is A-isomorphic to N. Consequently, A/CA(N) acts as a group of power automorphisms on N. This implies that A normalises each subgroup of N. A nalogously, B normalises each subgroup of N. It follows that N is a cyclic group. We argue as in step (ii) above to reach a final contradiction. We have that G/M is supersoluble and M is abelian. Therefore GUM and thus GU is abelian and complemented in G by a supersoluble normaliser, D say, by [8, Theorem V.5.18]. Since N is cyclic, we know that D covers N and thus NGU∩D=1, a contradiction. Proof of Theorem 2.

Let M=GU denote the supersoluble residual of G. Theorem 1yields that G/CoreG(A∩B) is supersoluble. Therefore, M is contained in CoreG(A∩B). In particular, M is supersoluble. Let F(M) be the Fitting subgroup of M. Since A and Bare supersoluble, we have that [M,A]F(A)∩MF(M) and [M,B]F(B)∩MF(M). Consequently, [M,G] is contained in F(M). Note now that the chief factors of G between F(M) and Mare cyclic,and recall that G/M is supersoluble. Therefore, we have that G/F (M) is supersoluble. This implies that M=F(M) and thus M is nilpotent. Consequently, G/F (G) is supersoluble. We now show that G/F (G) is metabelian. We prove first that A′ and B′ both centralise every chief factor of G. Let H/K be a chief factor of G. If H/K is cyclic, then as G′ centralizes H/K, so do A′ an dB′. Hence we may assume that H/K is a non-cyclic p-chief factor of G for some prime p. Note that we may assume that H is contained in M because G/M is supersoluble and H/K is non-cyclic. To simplify notation, we can consider K=1. Since F(G) centralizes H [8, Theorem A.13.8.b], G/CG(H) is supersoluble. Let Ap′ be a Hall p′-subgroup of A. By Maschke's theorem [8, Theorem A.11.5],H is a completely reducible Ap′-module and HAp′ is supersoluble because H is contained in A. Therefore Ap′/CAp′(H) is abelian of exponent dividing p−1. This implies that the primes involved in |A/CA(H)| can only be p or divisors of p−1.The same is true for |B/CB(H)|. This implies that if p divides |G/CG(H)|, then p is the largest prime dividing |G/CG(H)|. But since Op(G/CG(H))=1 and G/CG(H) is supersoluble, it follows that G/CG(H) must be a p′-group. Consider H as A-module over GF (p). Since ACG(H)/CG(H) is a p′-group, we have that H is a completely reducible A-module and every irreducible A-submodule of H is cyclic. Consequently A′ centralizes H, and the same is true for B′. Let now U/V be a chief factor of G. Then G/CG(U/V)is the product of the abelian subgroups ACG(U/V)/CG(U/V) and BCG(U/V)/CG(U/V). By Itô's theorem [9], we have that $G/C_G(U/V)$ is metabelian. Since $F(G)$ is the intersection of the centralisers of all chief factors (again by [8, Theorem A.13.8.b]), we can conclude that $G/F(G)$ is metabelian.3. Final remarks Finally, Theorem 1 enables us to give succinct proofs of earlier results on mutually permutable products.

Corollary 1[2, Theorem 3.2]. Let $G=AB$ be the mutually permutable product of the subgroups A and B. If A is supersoluble and B is nilpotent, then G is supersoluble.

Proof. Assume that the assertion is false, and let G be a minimal counterexample. We have that G is a primitive group, and so G has a unique minimal normal subgroup, N say, with $N=C_G(N)$ a p-group for some prime p. Since G is not supersoluble, applying Theorem 1, we know that $Core_G(A\cap B)=1$. This yields that N is contained in $A\cap B$. Now, since N is contained in B, which is nilpotent, it follows that any p'-element of B must centralize N. Since $C_G(N)=N$, we have that B itself is a p-group. Consequently, A must contain a Hall p'-subgroup of G. Now let $T/N=O_{p'}(G/N)$. The previous argument yields that T/N is contained in A/N. Note that if $B=N$, then $G=AN=A$ is supersoluble, a contradiction. Thus, N is a proper subgroup of B. This implies that p must divide $|G:T|$. Since G/N is supersoluble, p must divide $q-1$ for some prime $q\in\pi(T/N)$. It is clear then that q can not divide $p-1$. Therefore, there exists a Sylow q-subgroup A_q of A which centralizes N. Using that $C_G(N)=N$, it yields that $A_q=1$ and thus q does not divide $|G|$, a contradiction.

Corollary 2[2, Theorem 3.8]. Let $G=AB$ be the mutually permutable product of thes upersoluble subgroups A and B. If G' is nilpotent, then G is supersoluble.

Proof. We assume the result to be false, and choose a minimal counterexample G. Thus, G is a primitive group with unique minimal normal subgroup N. We also have that $G=NM$, where M is a maximal subgroup of G, $N\cap M=1$ and $N=F(G)=O_p(G)$ for some prime p. Now G' is nilpotent and thus $G'=F(G)=N$. Therefore, M is an abelian group. Since N is self-centralising, arguing as we did in the previous corollary, we have that N is contained in $A\cap B$. Note that $M\cong G/N$, and thus $O_p(M)=1$. Since M is abelian, this yields that M is a p'-group. Thus M is in fact a Hall p'-subgroup of G. Applying [1, Theorem 1.3.2], wehave that there exist a Hall p'-subgroup $A_{p'}$ of A and a Hall p'-subgroup $B_{p'}$ of B suchthat $M=A_{p'}B_{p'}$. Since $N A\cap B$, it follows that both $A_{p'}$ and $B_{p'}$ must have exponent dividing $p-1$.Regarding N as a M-module, it is easy to see that M must be a cyclic group. Now, since $M=A_{p'}B_{p'}$ has exponent dividing $p-1$, it follows that N is a cyclic group as well. This implies that G is supersoluble, a contradiction.

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