Ratio Mathematica **ISSN**: 1592-7415
Vol. 35, 2018, pp. 5-27 **eISSN**: 2282-8214

Quasi-Uniformity on BL-algebras

R.A. Borzooei [*], N. Kouhestani [†]

Received: 05-06-2018 Accepted: 15-10-2018. Published: 18-12-2018

doi:10.23755/rm.v35i0.423

Abstract

In this paper, by using the notation of filter in a BL-algebra A, we introduce the quasi-uniformity Q and uniformity Q^* on A. Then we make the topologies $T(Q)$ and $T(Q^*)$ on A and show that $(A, \wedge, \vee, \odot, T(Q))$ is a compact connected topological BL-algebra and $(A, T(Q^*))$ is a topological BL-algebra. Also we study Q^*-cauchy filters and minimal Q^*-filters on BL-algebra A and prove that the bi-completion $(\widetilde{A}, \widetilde{Q})$ of quasi-uniform BL-algebra (A, Q) is a topological BL-algebra.

2010 MSC: 06B10, 03G10.
Keywords : BL-algebra, (semi)topological BL-algebra, filter, Quasi-uniforme space, Bicompletion

1 Introduction

BL-algebras have been introduced by Hájek [11] in order to investigate many-valued logic by algebraic means. His motivations for introducing BL-algebras

[*]Department of Mathematics, Shahid Beheshti University, Tehran, Iran; borzooei@sbu.ac.ir
[†]Department of Mathematics, Sistan and Balouchestan University, Zahedan, Iran; Kouhestani@math.usb.ac.ir

were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general" many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0,1]. In 1973, André Weil [24] introduced the concept of a uniform space as a generalization of the concept of a metric space in which many non-topological invariant can be defined. This concept of uniformity fits naturally in the study of topological groups. The study of quasi-uniformities started in 1948 with Nachbin's investigations on uniform preordered spaces. In 1960, Á. Csaszar introduced quasi-uniform spaces and showed that every topological space is quasi-uniformizable. This result established an interesting analogy between metrizable spaces and general topological spaces. Just as a metrizable space can be studied with reference to particular compatible metric(s), a topological space can be studied with reference to particular compatible quasi-uniformity(ies). In this and some other respects, a quasi-uniformity is a more natural generalization of a metric than is a uniformity. Quasi-uniform structures were also studied in algebraic structures. In particular the study of paratopological groups and asymmetrically normed linear spaces with the help of quasi-uniformities is well known. See for example, [17], [18], [19], [20]. In the last ten years many mathematicians have studied properties of BL-algebras endowed with a topology. For example A. Di Nola and L. Leustean [9] studied compact representations of BL-algebras, L. C. Ciungu [7] investigated some concepts of convergence in the class of perfect BL-algebras, J. Mi Ko and Y. C. Kim [21] studied relationships between closure operators and BL-algebras.

In [2] and [4] we study (semi)topological BL-algebras and metrizability on BL-algebras. We showed that continuity the operations $\odot$ and $\rightarrow$ imply continuity $\wedge$ and $\vee$. Also, we found some conditions under which a locally compact topological BL-algebra become metrizable. But in there we can not answer some questions, for example:

(*i*) Is there a topology $\mathcal{U}$ on BL-algebra A such that $(A,\mathcal{U})$ be a (semi)topological BL-algebra?

(*ii*) Is there a topology $\mathcal{U}$ on a BL-algebra A such that $(A,\mathcal{U})$ be a compact connected topological BL-algebra?

(*iii*) Is there a topological BL-algebra $(A,\mathcal{U})$ such that T_0, T_1 and T_2 spaces be equivalent?

(*iv*) If $(A,\rightarrow,\mathcal{U})$ is a semitopological BL-algebra, is there a topology $\mathcal{V}$ coarsere than $\mathcal{U}$ or finer than $\mathcal{U}$ such that $(A,\mathcal{V})$ be a (semi)topological

BL-algebra?

Now in this paper, we answer to some above questions and get some interesting results as mentioned in abstract.

2 Preliminary

Recall that a set X with a family $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of its subsets is called a *topological space*, denoted by $(X,\mathcal{U})$, if $X, \emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of X and the complement of $X \in \mathcal{U}$, that is $X \setminus U$, is said to be a *closed set*. If B is a subset of X, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subset P of X is said to be a *neighborhood* of $x \in X$, if there exists an open set U such that $x \in U \subseteq P$. A subfamily $\{U_\alpha : \alpha \in J\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$ there exists an $\alpha \in J$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is a union of members of $\{U_\alpha\}$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in X. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a $fundamental\ system$ of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists a V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$. $(X,\mathcal{U})$ is said to be *compact*, if each open covering of X is reducible to a finite open covering. Also $(X,\mathcal{U})$ is said to be *disconnected* if there are two nonempty, disjoint, open subsets $U, V \subseteq X$ such that $X = U \cup V$, and connected otherwise. The maximal connected subset containing a point of X is called the *component* of that point. Topological space $(X,\mathcal{U})$ is said to be:
(i) T_0 if for each $x \neq y \in X$, there is one in an open set excluding the other,
(ii) T_1 if for each $x \neq y \in X$, each are in an open set not containing the other,
(iii) T_2 if for each $x \neq y \in X$, both are in two disjoint open set.(See [1])

Definition 2.1. [1] Let $(A, *)$ be an algebra of type 2 and $\mathcal{U}$ be a topology on A. Then $\mathcal{A} = (A, *, \mathcal{U})$ is called a
(i) *left (right) topological algebra* if for all $a \in A$, the map $*_a : A \to A$ is defined by $x \to a * x$ ($x \to x * a$) is continuous, or equivalently, for any x in A and any open set U of $a * x$ $(x * a)$, there exists an open set V of x such that $a * V \subseteq U$ $(V * a \subseteq U)$.
(ii) *semitopological algebra* if $\mathcal{A}$ is a right and left topological algebra.
(iii) *topological algebra* if the operation $*$ is continuous, or equivalently, if for any x, y in A and any open set (neighborhood) W of $x * y$, there exist two open sets (neighborhoods) U and V of x and y, respectively, such that $U * V \subseteq W$.

Proposition 2.2. [1] Let $(A, *)$ be a commutative algebra of type 2 and $\mathcal{U}$ be a topology on A. Then right and left topological algebras are equivalent. Moreover, $(A, *, \mathcal{U})$ is a semitopological algebra if and only if it is right or left topological algebra.

Definition 2.3. [1] Let A be a nonempty set and $\{*_i\}_{i\in I}$ be a family of operations of type 2 on A and $\mathcal{U}$ be a topology on A. Then
(i) $(A, \{*_i\}_{i\in I}, \mathcal{U})$ is a right(left) topological algebra if for any $i \in I$, $(A, *_i, \mathcal{U})$ is a right (left) topological algebra.
(ii) $(A, \{*_i\}_{i\in I}, \mathcal{U})$ is a semitopological (topological) algebra if for all $i \in I$, $(A, *_i, \mathcal{U})$ is a semitopological (topological) algebra.

Definition 2.4. [11] A *BL-algebra* is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid and for any $a, b, c \in A$,

$$c \le a \to b \Leftrightarrow a \odot c \le b, \quad a \wedge b = a \odot (a \to b), \quad (a \to b) \vee (b \to a) = 1.$$

Let A be a BL-algebra. We define $a' = a \to 0$ and denote $(a')'$ by a''. The map $c : A \to A$ by $c(a) = a'$, for any $a \in A$, is called the *negation map*. Also, we define $a^0 = 1$ and $a^n = a^{n-1} \odot a$, for all natural numbers n.

Example 2.5. *[11] (i) Let "$\odot$" and "$\rightarrow$" on the real unit interval $I = [0, 1]$ be defined as follows:*

$$x \odot y = \min\{x, y\} \qquad x \to y = \begin{cases} 1 & , x \le y, \\ y & , \textit{otherwise}. \end{cases}$$

Then $\mathcal{I} = (I, \min, \max, \odot, \rightarrow, 0, 1)$ is a BL-algebra.

(ii) Let $\odot$ be the usual multiplication of real numbers on the unit interval $I = [0, 1]$ and $x \to y = 1$ iff, $x \le y$ and y/x otherwise. Then $\mathcal{I} = (I, \min, \max, \odot, \rightarrow, 0, 1)$ is a BL-algebra.

Proposition 2.6. [11] Let A be a BL-algebra. The following properties hold.
(B_1) $x \odot y \le x, y$ and $x \odot 0 = 0$,
(B_2) $x \le y$ implies $x \odot z \le y \odot z$,
(B_3) $x \le y$ iff $x \to y = 1$,
(B_4) $1 \to x = x$, $1 \odot x = x$,
(B_5) $y \le x \to y$,
(B_6) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,
(B_7) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(B_8) $x \le y \Rightarrow x \to z \ge y \to z, \ z \to x \le z \to y$,

(B_9) $x \to y \leq (z \to x) \to (z \to y)$,
(B_{10}) $x \to y \leq (y \to z) \to (x \to z)$,
(B_{11}) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,
(B_{12}) $(y \wedge z) \to x = (y \to x) \vee (z \to x)$,
(B_{13}) $(y \vee z) \to x = (y \to x) \wedge (z \to x)$,
(B_{14}) $x \to y \leq x \odot z \to y \odot z$,
(B_{15}) $(x \to y) \odot (y \to z) \leq x \to z$,
(B_{16}) $(x \to y) \odot (a \to z) \leq (x \vee a) \to (y \vee z)$,
(B_{17}) $(x \to y) \odot (a \to z) \leq (x \wedge a) \to (y \wedge z)$,
(B_{18}) $(x \to y) \odot (a \to z) \leq (x \odot a) \to (y \odot z)$.

Definition 2.7. [11] A $filter$ of a BL-algebra A is a nonempty set $F \subseteq A$ such that $x, y \in F$ implies $x \odot y \in F$ and if $x \in F$ and $x \leq y$ imply $y \in F$, for any $x, y \in A$.

It is easy to prove that if F is a filter of a BL-algebra A, then for each $x, y \in F$, $x \wedge y$, $x \vee y$ and $x \to y$ are in F

Proposition 2.8. [11] Let F be a subset of BL-algebra A such that $1 \in F$. Then the following conditions are equivalent.
(i) F is a filter.
(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$.
(iii) $x \to y \in F$ and $y \to z \in F$ imply $x \to z \in F$.

Proposition 2.9. [11] Let F be a filter of a BL-algebra A. Define $x \equiv^F y \Leftrightarrow x \to y, y \to x \in F$. Then $\equiv^F$ is a congruence relation on A. Moreover, if $x/F = \{y \in A : y \equiv^F x\}$, then
(i) $x/F = y/F \Leftrightarrow y \equiv^F x$,
(ii) $x/F = 1/F \Leftrightarrow x \in F$.

Definition 2.10. [2] (i) Let A be a BL-algebra and $(A, \{*_i\}, \mathcal{U})$ be a semitopological (topological) algebra, where $\{*_i\} \subseteq \{\wedge, \vee, \odot, \to\}$, then $(A, \{*_i\}, \mathcal{U})$ is called a semitopological (topological) BL-algebra.

Remark 2.11. If $\{*_i\} = \{\wedge, \vee, \odot, \to\}$, we consider $\mathcal{A} = (A, \mathcal{U})$ instead of $(A, \{\wedge, \vee, \odot, \to\}, \mathcal{U})$, for simplicity.

Proposition 2.12. [2] Let $(A, \{\odot, \to\}, \mathcal{U})$ be a topological BL-algebra. Then $(A, \mathcal{U})$ is a topological BL-algebra.

Notation. From now on, in this paper, we use of *BL-filter* instead of *filter* in BL-algebras.

Definition 2.13. [10] Let X be a non-empty set. A family $\mathcal{F}$ of nonempty subsets of X is called a $filter$ on X if (i) $X \in \mathcal{F}$, (ii) for each F_1, F_2 of elements of $\mathcal{F}$, $F_1 \cap F_2 \in \mathcal{F}$ and, (iii) if $F \in \mathcal{F}$ and $F \subseteq G$, then $G \in \mathcal{F}$.

A subset $\mathcal{B}$ of a filter $\mathcal{F}$ on X is said to be a *base* of $\mathcal{F}$ if every set of $\mathcal{F}$ contains a set of $\mathcal{B}$.
If $\mathcal{F}$ is a family of nonempty subsets of X, then there exists the smallest filter on X containing $\mathcal{F}$, denoted with $fil(\mathcal{F})$ and called generated filter by $\mathcal{F}$.

Definition 2.14. [10] A $quasi\text{-}uniformity$ on a set X is a filter Q on X such that
(i) $\triangle = \{(x,x) \in X \times X : x \in A\} \subseteq q$, for each $q \in Q$,
(ii) for each $q \in Q$, there is a $p \in Q$ such that $p \circ p \subseteq q$, where

$$p \circ p = \{(x,y) \in X \times X : \exists z \in A \ s.t \ (x,z),(z,y) \in p\}.$$

The pair (X, Q) is called a $quasi\text{-}uniform\ space$.

If Q is a quasi-uniformity on a set X, $q \in Q$ and $q^{-1} = \{(x,y) : (y,x) \in q\}$, then $Q^{-1} = \{q^{-1} : q \in Q\}$ is also a quasi-uniformity on X called the $conjugate$ of Q. It is well-known that if Q satisfies condition: $q \in Q$ implies $q^{-1} \in Q$, then Q is a $uniformity$. Furthermore, $Q^* = Q \vee Q^{-1}$ is a uniformity on X. If Q and R are quasi-uniformities on X and $Q \subseteq R$, then Q is called $coarser$ than R. A subfamily $\mathcal{B}$ of quasi-uniformity Q is said to be a base for Q if each $q \in Q$ contains some member of $\mathcal{B}$.(See [10])

Proposition 2.15. [22] Let $\mathcal{B}$ be a family of subsetes of $X \times X$ such that
(i) $\triangle \subseteq q$, for each $q \in \mathcal{B}$,
(ii) for $q_1, q_2 \in \mathcal{B}$, there exists a $q_3 \in \mathcal{B}$ such that $q_3 \subseteq q_1 \cap q_2$,
(iii) for each $q \in \mathcal{B}$, there is a $p \in \mathcal{B}$ such that $p \circ p \subseteq q$.
Then, there is the unique quasiuniformity $Q = \{q \subseteq X \times X : for\ some\ p \in \mathcal{B}, p \subseteq q\}$ on X for which $\mathcal{B}$ is a base.

The topology $T(Q) = \{G \subseteq X : \forall x \in G\ \exists q \in Q\ s.t\ q(x) \subseteq G\}$ is called the topology induced by the quasi-uniformity Q.

Definition 2.16. [10] (i) A filter $\mathcal{G}$ on quasi-uniform space (X, Q) is called Q^*-*cauchy* filter if for each $U \in Q$, there is a $G \in \mathcal{G}$ such that $G \times G \subseteq U$.
(ii) A quasi-uniform space (X, Q) is called *bicomplete* if each Q^*-*cauchy* filter converges with respect to the topology $T(Q^*)$.
(iii) A *bicompletion* of a quasi-uniform space (X, Q) is a bicomplete quasi-uniform space $(Y, \mathcal{V})$ that has a $T(\mathcal{V}^*)$-dense subspace quasi-unimorphic to

(X, Q).
(iv) A Q^*-cauchy filter on a quasi-uniform space (X, Q) is *minimal* provided that it contains no Q^*-cauchy filter other than itself.

Lemma 2.17. *[10] Let $\mathcal{G}$ be a Q^*-cauchy filter on a quasi-uniform space (X, Q). Then, there is exactly one minimal Q^*-cauchy filter coarser than $\mathcal{G}$. Furthermore, if $\mathcal{B}$ is a base for $\mathcal{G}$, then $\{q(B) : B \in \mathcal{B}$ and q is a symetric member of $Q^*\}$ is a base for the minimal Q^*-cauchy filter coarser than $\mathcal{G}$.*

Lemma 2.18. *[10] Let (X, Q) be a T_0 quasi-uniform space and $\widetilde{X}$ be the family of all minimal Q^*-cauchy filters on (A, Q). For each $q \in Q$, let*

$$\widetilde{q} = \{(\mathcal{G}, \mathcal{H}) \in \widetilde{X} \times \widetilde{X} : \exists G \in \mathcal{G} \text{ and } H \in \mathcal{H} \text{ s.t } G \times H \subseteq q\},$$

and $\widetilde{Q} = fil\{\widetilde{q} : q \in Q\}$. Then the following statements hold:
(i) $(\widetilde{X}, \widetilde{Q})$ is a T_0 bicomplete quasi-uniform space and (X, Q) is a quasi-uniformly embedded as a $T(\widetilde{(Q^*)})$-dense subspace of $(\widetilde{X}, \widetilde{Q})$ by the map $i : X \to \widetilde{X}$ such that, for each $x \in X$, $i(x)$ is the $T(Q^*)$-neighborhood filter at x. Furthermore, the uniformities $\widetilde{Q}^*$ and $\widetilde{(Q^*)}$ coincide.

Notation. From now on, in this paper we let A be a BL-algebra and $\mathcal{F}$ be a family of BL-filters in A which is closed under intersection , unless otherwise state.

3 Quasi-uniformity on BL-algebras

In this section, by using of BL-filters we introduce a quasi-uniformity Q on BL-algebra A and stay some properties it. We show that (A, Q) is not a T_1 and T_2 quasi-uniform space but it is a T_0 quasi-uniform space. Also we study Q^*-cauchy filters, minimal Q^*-cauchy filters and we make a quasi-uniform space $(\widetilde{A}, \widetilde{Q})$ of minimal Q^*-cauchy filters of (A, Q) which admits the structure of a BL-algebra.

Lemma 3.1. *Let F be a BL-filter of BL-algebra A and $F_\star(x) = \{y : y \to x \in F\}$, for each $x \in A$. Then for each $x, y \in A$, the following properties hold.*
(i) $x \le y$ implies $F_\star(x) \subseteq F_\star(y)$,
(ii) $F_\star(x) \wedge F_\star(y) = F_\star(x \wedge y) = F_\star(x) \cap F_\star(y)$,
(iii) $F_\star(x) \vee F_\star(y) \subseteq F_\star(x \vee y)$,
(iv) $F_\star(x) \odot F_\star(y) \subseteq F_\star(x \odot y)$,
(v) If for each $a \in A$, $a \odot a = a$, then $F_\star(x) \odot F_\star(y) = F_\star(x \odot y)$,

(vi) $x \in F \Leftrightarrow 1 \in F_\star(x) \Leftrightarrow F_\star(x) = A$,
(vii) *For* $a, b \in A$, *if* $a \vee b \in F_\star(x)$, *then* $a, b \in F_\star(x)$,
$(viii)$ *If* $y \in F_\star(x)$, *then* $F_\star(y) \subseteq F_\star(x)$.

Proof. (i) Let $x, y \in A$, such that $x \leq y$ and $z \in F_\star(x)$. Then by (B_8), $z \to x \leq z \to y$. Since F is a BL-filter and $z \to x \in F$, $z \to y$ is in F and so $z \in F_\star(y)$.
(ii) Let $x, y \in A$, such that $a \in F_\star(x)$ and $b \in F_\star(y)$. Then $a \to x \in F$ and $b \to y \in F$ and so $(a \to x) \odot (b \to y) \in F$. Since by (B_{17}), $(a \to x) \odot (b \to y) \leq (a \wedge b) \to (x \wedge y)$, we get $(a \wedge b) \to (x \wedge y) \in F$. Thus, $a \wedge b \in F_\star(x \wedge y)$. Now, if $a \in F_\star(x \wedge y)$, since $a \to (x \wedge y) \in F$ and by (B_{11}), $a \to (x \wedge y) = (a \to x) \wedge (a \to y)$, we conclude that $a \to x \in F$ and $a \to y \in F$. Hence $a \in F_\star(x) \cap F_\star(y)$. Finally, let $a \in F_\star(x) \cap F_\star(y)$. Since $a = a \wedge a$, then $a \in F_\star(x) \wedge F_\star(y)$.
$(iii), (iv)$ The proof is similar to the proof of (ii), by some modification.
(v) Let $x, y \in A$ such that $z \in F_\star(x \odot y)$. Then $z \to (x \odot y) \in F$. By (B_8), $z \to (x \odot y) \leq z \to x$ and $z \to (x \odot y) \leq z \to y$ which imply that $z \to x, z \to y \in F$. Hence z is in both $F_\star(x)$ and $F_\star(y)$ and so $z = z \odot z \in F_\star(x) \odot F_\star(y)$.
(vi) The proof is clear.
$(vii), (viii)$ The proof come from by (B_{13}) and (B_{15}). $\square$

Lemma 3.2. *Let* F *be a BL-filter of BL-algebra* A. *Define* $F_\star = \{(x, y) \in A \times A : y \in F_\star(x)\}$ *and* $F^*_\star = F_\star \cap F_\star^{-1}$. *Then*
(i) $F_\star^{-1} = \{(x, y) \in A \times A : x \to y \in F\}$,
(ii) $F^*_\star = \{(x, y) \in A \times A : x \equiv^F y\} = {F^*_\star}^{-1}$,
(iii) $F^*_\star(x) = \{y : x \equiv^F y\}$,
(iv) $F_\star^{-1}(x) \to y \subseteq F_\star(x \to y)$,
(v) *If* $\bullet \in \{\wedge, \vee, \odot, \to\}$, *then* $F^*_\star(x) \bullet F^*_\star(y) \subseteq F^*_\star(x \bullet y)$.

Proof. The proof of $(i), (ii)$ and (iii) are clear.
(iv) Let $a \in F_\star^{-1}(x) \to y$. Then there exists a $z \in F_\star^{-1}(x)$ such that $a = z \to y$ and $x \to z \in F$. By (B_{10}), $(z \to y) \to (x \to y) \geq x \to z$. Since F is a filter, $(z \to y) \to (x \to y) \in F$. Hence $a = z \to y \in F_\star(x \to y)$.
(v) Let $a \in F^*_\star(x)$ and $b \in F^*_\star(y)$. Then by (iii), $a \equiv^F x$ and $b \equiv^F y$. By Proposition 2.9, $a \bullet b \equiv^F x \bullet y$. Therefore, $a \bullet b \in F^*_\star(x \bullet y)$. $\square$

Theorem 3.3. *Let* $\mathcal{F}$ *be a family of BL-filters of BL-algebra* A *which is closed under finite intersection. Then the set* $\mathcal{B} = \{F_\star : F \in \mathcal{F}\}$ *is a base for the unique quasi-uniformity* $Q = \{q \subseteq A \times A : \exists F \in \mathcal{F} \text{ s.t } F_\star \subseteq q\}$. *Moreover,* $Q^* = \{q \subseteq A \times A : \exists F \in \mathcal{F} \text{ s.t } F^*_\star \subseteq q\}$.

Proof. We prove that $\mathcal{B}$ satisfies in conditions $(i), (ii)$ and (iii) of Proposition 2.15. For (i), it is easy to see that for each $F \in \mathcal{F}$, $\triangle \subseteq F_\star$. Let $F_1, F_2 \in \mathcal{F}$

and $F = F_1 \cap F_2$. If $(x, y) \in F_\star$, then $y \to x \in F = F_1 \cap F_2$. Hence $(x, y) \in F_{1\star} \cap F_{2\star}$. This concludes that $F_\star \subseteq F_{1\star} \cap F_{2\star}$ and so (ii) is true. Finally for (iii), let $F \in \mathcal{F}$ and $(x, y) \in F_\star \circ F_\star$. Then there is a $z \in A$ such that (x, z) and (z, y) are both in $F_\star$. Hence $z \to x$ and $y \to z$ are in F. Since F is a filter and by (B_{15}), $(y \to z) \odot (z \to x) \leq y \to x$, we conclude that $y \to x \in F$. Hence $F_\star \circ F_\star \subseteq F_\star$ and so (iii) is true. Therefore, by Proposition 2.15, Q is a unique quasi-uniformity on A for which $\mathcal{B}$ is a base.

Now, we prove that

$$Q^* = \{q \subseteq A \times A : \exists F \in \mathcal{F} \ s.t \ F_\star^* \subseteq q\}.$$

First we prove that $\mathcal{P} = \{q \subseteq A \times A : \exists F \in \mathcal{F} \ s.t \ F_\star^* \subseteq q\}$ is a uniformity on A. With a similar argument as above, we get $\{F_\star^* : F \in \mathcal{F}\}$ is a base for the quasi-uniformity $\mathcal{P} = \{q \subseteq A \times A : \exists F \in \mathcal{F} \ s.t \ F_\star^* \subseteq q\}$. To prove that $\mathcal{P}$ is a uniformity we have to show that for each $q \in \mathcal{P}$, q^{-1} is in $\mathcal{P}$. Suppose $q \in \mathcal{P}$. Then there exists a $F \in \mathcal{F}$, such that $F_\star^* \subseteq q$. By Lemma 3.2(ii), $F_\star^* = F_\star^{*^{-1}}$. Hence $F_\star^* \subseteq q^{-1}$ and so $q^{-1} \in \mathcal{P}$. Thus $\mathcal{P}$ is a uniformity on A which contains Q. Since $Q^* = Q \vee Q^{-1}$, then $Q^* \subseteq \mathcal{P}$. On the other hand, if $q \in \mathcal{P}$, then there is a $F \in \mathcal{F}$ such that $F_\star^* \subseteq q$. Since $F_\star^* = F_\star \cap F_\star^{-1} \in Q^*$, we get that $q \in Q^*$. Therefore, $Q^* = \mathcal{P}$. $\square$

In Theorem 3.3, we call Q is quasi-uniformity induced by $\mathcal{F}$, the pair (A, Q) is quasi-uniform BL-algebra and the pair (A, Q^*) is uniform BL-algebra.

Notation. From now on, $\mathcal{F}$, Q and Q^* are as in Theorem 3.3.

Example 3.4. *Let $\mathcal{I}$ be the BL-algebra in Example 2.5 (i), and for each $a \in [0, 1)$, $F_a = (a, 1]$. Then F_a is a BL-filter in $\mathcal{I}$ and easily proved that for each $a, b \in [0, 1)$, $F_a \cap F_b = F_{a \wedge b}$. Hence $\mathcal{F} = \{F_a\}_{a \in [0,1)}$ is a family of BL-filters which is closed under intersection. For each $a \in [0, 1)$,*

$$F_{a\star} = (a, 1] \times [0, 1], \ F_{a\star}^{-1} = [0, 1] \times (a, 1] \ and \ F_{a\star}^* = (a, 1] \times (a, 1].$$

By Theorem 3.3, $Q = \{q : \exists a \in [0, 1) \ s.t \ (a, 1] \times [0, 1] \subseteq q\}$ and $Q^* = \{q : \exists a \in [0, 1) \ s.t \ (a, 1] \times (a, 1] \subseteq q\}$.

Recall that a map f from a (quasi)uniform space (X, Q) into a (quasi)uniform space (Y, R) is *(quasi) uniformly continuous*, if for each $V \in R$, there exists a $U \in Q$ such that $(x, y) \in U$ implies $(f(x), f(y)) \in V$. If $f : (X, Q) \hookrightarrow (Y, R)$ is a quasi-uniform continuous map between quasi-uniform spaces, then $f : (X, Q^*) \hookrightarrow (Y, R^*)$ is a uniform continuous map. (See [10])

Proposition 3.5. In BL-algebra A, for each $a \in A$, the mappings $t_a(x) = a \wedge x$, $r_a(x) = a \vee x$, $l_a(x) = a \odot x$ and $L_a(x) = a \rightarrow x$ of quasi-uniform BL-algebra (A, Q) into quasi-uniform BL-algebra (A, Q) are quasi-uniformly continuous. Moreover, they are uniformly continuous mappings of uniform BL-algebra (A, Q^*) into uniform BL-algebra (A, Q^*).

Proof. Let $q \in Q$. Then, there is a $F \in \mathcal{F}$ such that $F_\star \subseteq q$. If $(x, y) \in F_\star$, then $y \rightarrow x \in F$. By (B_{10}) $(a \wedge y) \rightarrow (a \wedge x) \geq y \rightarrow x$ which implies that $(a \wedge y) \rightarrow (a \wedge x) \in F \subseteq q$. Hence t_a is quasi-uniform continuous. Moreover, $t_a : (A, Q^*) \hookrightarrow (A, Q^*)$ is uniform continuous. In a similar fashion and by use of (B_{16}), (B_{14}) and (B_9), we can prove that, respectively, r_a, l_a and L_a are quasi-uniform continuous of $(A, Q) \hookrightarrow (A, Q)$ and are uniform continuous of $(A, Q^*) \hookrightarrow (A, Q^*)$. □

Let (X, Q) be a (quasi)uniform space and $\mathcal{B}$ be a base for it. Recall (X, Q) is
(i) T_0 quasi-uniform if (x, y) and (y, x) are in $\bigcap_{U \in \mathcal{B}} U$, then $x = y$, for each $x, y \in X$,
(ii) T_1 quasi-uniform if $\triangle = \bigcap_{U \in \mathcal{B}} U$,
(iii) T_2 quasi-uniform if $\triangle = \bigcap_{U \in \mathcal{B}} U^{-1} \circ U$. (See [10])

Theorem 3.6. *Quasi-uniform BL-algebra (A, Q) is not T_1 and T_2 quasi-uniform. If $\{1\} \in \mathcal{F}$, then (A, Q) is a T_0 quasi-uniform space and uniform BL-algebra (A, Q^*) is T_0, T_1 and T_2 quasi-uniform space.*

Proof. Let $x, y \in A$ and $F \in \mathcal{F}$. Since $y \rightarrow 1 = 1 \in F$, we get that $(1, y) \in \bigcap_{F \in \mathcal{F}} F_\star$. Hence (A, Q) is not T_0 quasi-uniform. Also since $x \rightarrow 1 = y \rightarrow 1 \in F$, we conclude that $(1, x), (1, y) \in F_\star$. Hence $(x, y) \in F_\star^{-1} \circ F_\star$ which implies that $\triangle \neq \bigcap_{F \in \mathcal{F}} F_\star^{-1} \circ F_\star$. So (A, Q) is not T_2 quasi-unifom
Let $\{1\} \in \mathcal{F}$ and (x, y) and (y, x) be in $\bigcap_{F \in \mathcal{F}} F_\star$. Then for each $F \in \mathcal{F}$, $x \rightarrow y$ and $y \rightarrow x$ are in F. Hence $x \equiv^{\{1\}} y$, which implies that $x = y$. Therefore, (A, Q) is T_0 quasi-uniform. With a similar argument as above, we can prove that (A, Q^*) is a T_0 and T_1 quasi-uniform space. To verify T_2 quasi-uniformity, let $(x, y) \in \bigcap_{F \in \mathcal{F}} F_\star^{*^{-1}} \circ F_\star^*$. Then for each $F \in \mathcal{F}$ there is a $z \in A$ such that $(x, z) \in F_\star^{*^{-1}}$ and $(z, y) \in F_\star^*$. By Lemma 3.2(ii), $x \equiv^F y$. Since $\{1\} \in \mathcal{F}$, we get that $x = y$. Therefore, (A, Q^*) is a T_2 quasi-uniform space. □

Proposition 3.7. Let $\mathcal{B}$ be a base for a Q^*-cauchy filter $\mathcal{G}$ on quasi-uniform BL-algebra (A, Q). Then the set $\{F_\star^*(B) : F \in \mathcal{F},\ B \in \mathcal{B}\}$ is a base for the uniqe minimal Q^*-cauchy filter coarser than $\mathcal{G}$.

Proof. By Lemma 2.17, the set $\{q(B) : B \in \mathcal{B},\ q^{-1} = q \in Q^*\}$ is a base for the unique minimal Q^*-cauchy filter $\mathcal{G}_0$ coarser than $\mathcal{G}$. Let $q^{-1} = q \in Q^*$ and $B \in \mathcal{B}$. Then for some $F \in \mathcal{F}$, $F_\star^* \subseteq q$. So, $F_\star^*(B) \subseteq q(B)$. Now, it is easy to prove that the set $\{F_\star^*(B) : F \in \mathcal{F},\ B \in \mathcal{B}\}$ is a base for $\mathcal{G}_0$. □

Proposition 3.8. $\mathcal{F}$ is a base for a minimal Q^*-cauchy filter on quasi-uniform BL-algebra (A, Q).

Proof. Let $\mathcal{C} = \{S \subseteq A : \exists F \in \mathcal{F}\ s.t\ F \subseteq S\}$. It is easy to prove that $\mathcal{C}$ is a filter and $\mathcal{F}$ is a base for it. We prove that $\mathcal{C}$ is a Q^*-cauchy filter. For this, let $q \in Q$. There is a $F \in \mathcal{F}$ such that $F_\star \subseteq q$. Since F is a filter, clearly $F \times F \subseteq F_\star \subseteq q$. Hence $\mathcal{C}$ is a Q^*-cauchy filter. Now, by Proposition 3.7, the set $\{F_\star^*(F_1) : F, F_1 \in \mathcal{F}\}$ is a base for the unique minimal Q^*-cauchy filter $\mathcal{F}_0$ coarser than $\mathcal{C}$. To complete proof we show that for each $F, F_1 \in \mathcal{F}$, $F_\star^*(F_1) = F_1$. Let $F, F_1 \in \mathcal{F}$. If $y \in F_\star^*(F_1)$, then for some $x \in F_1$, $x \equiv^F y$. By Proposition 2.9, $y \in F_1$. Hence $F_\star^*(F_1) \subseteq F_1$. Clearly, $F_1 \subseteq F_\star^*(F_1)$. Therefore, $F_1 = F_\star^*(F_1)$. Thus proved that $\mathcal{F}$ is a base for $\mathcal{F}_0$. □

Proposition 3.9. The set $\mathcal{B} = \{F_\star^*(0) : F \in \mathcal{F}\}$ is a base for a minimal Q^*-cauchy filter on quasi-uniform BL-algebra (A, Q).

Proof. Let $\mathcal{C} = \{S \subseteq A : \exists F \in \mathcal{F}\ s.t\ F_\star^*(0) \subseteq S\}$. It is easy to prove that $\mathcal{C}$ is a filter and the set $\mathcal{B} = \{F_\star^*(0) : F \in \mathcal{F}\}$ is a base for it. To prove that $\mathcal{C}$ is a Q^*-cauchy filter, let $q \in Q$. There is a $F \in \mathcal{F}$ such that $F_\star \subseteq q$. If $x, y \in F_\star^*(0)$, then $x \equiv^F y$ and so $(x, y) \in F_\star^* \subseteq F_\star \subseteq q$. This prove that $F_\star^*(0) \times F_\star^*(0) \subseteq q$. Hence $\mathcal{C}$ is a Q^*-cauchy filter. By Proposition 3.7, the set $\{F_\star^*(F_\star^*(0)) : F \in \mathcal{F}\}$ is a base for the uniqe minimal Q^*-cauchy filter $\mathcal{I}$ coarser than $\mathcal{C}$. But it is easy to pove that fo each $F \in \mathcal{F}$, $F_\star^*(F_\star^*(0)) = F_\star^*(0)$. Therefore, $\mathcal{B}$ is a base for $\mathcal{I}$. □

Lemma 3.10. *Let $\mathcal{G}$ and $\mathcal{H}$ be Q^*-cauchy filters on quasi-uniform BL-algebra (A, Q). If $\bullet \in \{\wedge, \vee, \odot, \rightarrow\}$, then $\mathcal{G} \bullet \mathcal{H} = \{G \bullet H : G \in \mathcal{G},\ H \in \mathcal{H}\}$ is a Q^*-cauchy filter base on quasi-uniform BL-algebra (A, Q).*

Proof. Let $\mathcal{C} = \{S \subseteq A : \exists G,\ H\ s.t\ G \in \mathcal{G},\ H \in \mathcal{H},\ G \bullet H \subseteq S\}$. It is easy to prove that $\mathcal{C}$ is a filter and the set $\mathcal{B} = \{G \bullet H : G \in \mathcal{G},\ H \in \mathcal{H}\}$ is a base for it. We prove that $\mathcal{C}$ is a Q^*-cauchy filter. For this, let $q \in Q$. Then for some a $F \in \mathcal{F}$, $F_\star \subseteq q$. Since $\mathcal{G}, \mathcal{H}$ are Q^*-cauchy filters, there are $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $G \times G \subseteq F_\star$ and $H \times H \subseteq F_\star$. We show that $G \bullet H \times G \bullet H \subseteq F_\star \subseteq q$. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$. Then $(g_1, g_2), (g_2, g_1), (h_1, h_2), (h_2, h_1)$ are in $F_\star$. So $g_1 \equiv^F g_2$ and $h_1 \equiv^F h_2$. By Proposition 2.9, $g_1 \bullet h_1 \equiv^F g_2 \bullet h_2$, which implies that $(g_1 \bullet h_1, g_2 \bullet h_2) \in F_\star$. □

Theorem 3.11. *There is a quasi-uniform space $(\widetilde{A}, \widetilde{Q})$ of minimal Q^*-cauchy filters of quasi-uniform BL-algebra (A, Q) that admits a BL-algebra structure.*

Proof. Let $\widetilde{A}$ be the family of all minimal Q^*-cauchy filters on (A, Q). Let for each $q \in Q$,

$$\widetilde{q} = \{(\mathcal{G}, \mathcal{H}) \in \widetilde{A} \times \widetilde{A} : \exists G \in \mathcal{G}, H \in \mathcal{H} \ s.t \ G \times H \subseteq q\}.$$

If $\widetilde{Q} = fil\{\widetilde{q} : q \in Q\}$, then $(\widetilde{A}, \widetilde{Q})$ is a quasi-uniform space of minimal Q^*-cauchy filters of (A, Q). Let $\mathcal{G}, \mathcal{H} \in \widetilde{A}$. Since $\mathcal{G}, \mathcal{H}$ are minimal Q^*-cauchy filters on A, then by Lemma 3.10, $\mathcal{G} \wedge \mathcal{H}$, $\mathcal{G} \vee \mathcal{H}$, $\mathcal{G} \odot \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ are Q^*-cauchy filter bases on A. Now, we define $\mathcal{G} \curlywedge \mathcal{H}$, $\mathcal{G} \curlyvee \mathcal{H}$, $\mathcal{G} \circledcirc \mathcal{H}$ and $\mathcal{G} \hookrightarrow \mathcal{H}$ as the minimal Q^*-cauchy filters contained $\mathcal{G} \wedge \mathcal{H}$, $\mathcal{G} \vee \mathcal{H}$, $\mathcal{G} \odot \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$, respectively. Thus, $\mathcal{G} \curlywedge \mathcal{H}$, $\mathcal{G} \curlyvee \mathcal{H}$, $\mathcal{G} \circledcirc \mathcal{H}$ and $\mathcal{G} \hookrightarrow \mathcal{H}$ are in $\widetilde{A}$. Now, we will prove that $(\widetilde{A}, \curlywedge, \curlyvee, \circledcirc, \hookrightarrow, \mathcal{I}, \mathcal{F}_0)$ is a BL-algebra, where $\mathcal{I}$ is minimal Q^*-cauchy filter in Proposition 3.9 and $\mathcal{F}_0$ is minimal Q^*-cauchy filter in Proposition 3.8. For this, we consider the following steps:
(1) $(\widetilde{A}, \curlywedge, \curlyvee)$ **is a bounded lattice**.

Let $\mathcal{G}, \mathcal{H}, \mathcal{K} \in \widetilde{A}$. We consider the following cases:
Case 1.1: $\mathcal{G} \curlywedge \mathcal{G} = \mathcal{G}$, $\mathcal{G} \curlyvee \mathcal{G} = \mathcal{G}$
By Proposition 3.7, $S_1 = \{F^*_\star(G) : G \in \mathcal{G}, F \in \mathcal{F}\}$ and $S_2 = \{F^*_\star(G_1 \wedge G_2) : G_1, G_2 \in \mathcal{G}, F \in \mathcal{F}\}$ are bases of the minimal Q^*-cauchy filters $\mathcal{G}$ and $\mathcal{G} \curlywedge \mathcal{G}$, respectively. First, we show that $S_2 \subseteq S_1$. Let $F^*_\star(G_1 \wedge G_2) \in S_2$. Put $G = G_1 \cap G_2$, then $G \in \mathcal{G}$. Let $y \in F^*_\star(G)$. Then there is a $x \in G$ such that $(x, y) \in F^*_\star$. Since $x \wedge x = x$, it follows that $(x \wedge x, y) \in F^*_\star$ and so $y \in F^*_\star(G_1 \wedge G_2)$. Hence $S_2 \subseteq S_1$. Therefore, $\mathcal{G} \curlywedge \mathcal{G} \subseteq \mathcal{G}$. By the minimality of $\mathcal{G}$, $\mathcal{G} \curlywedge \mathcal{G} = \mathcal{G}$. The proof of the other case is similar.
Case 1.2: $\mathcal{G} \curlywedge \mathcal{H} = \mathcal{H} \curlywedge \mathcal{G}$, $\mathcal{G} \curlyvee \mathcal{H} = \mathcal{H} \curlyvee \mathcal{G}$
By Proposition 3.7, $S_1 = \{F^*_\star(G \wedge H) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\}$ and $S_2 = \{F^*_\star(H \wedge G) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\}$ are bases of $\mathcal{G} \curlywedge \mathcal{H}$ and $\mathcal{H} \curlywedge \mathcal{G}$, respectively. For each $G \in \mathcal{G}$ and $H \in \mathcal{H}$, since $G \wedge H = H \wedge G$, for each $F \in \mathcal{F}$, $F^*_\star(G \wedge H) = F^*_\star(H \wedge G)$. Hence $\mathcal{G} \curlywedge \mathcal{H} = \mathcal{H} \curlywedge \mathcal{G}$. The proof of the other case is similar.
Case 1.3: $\mathcal{G} \curlywedge (\mathcal{H} \curlywedge \mathcal{K}) = (\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$, $\mathcal{G} \curlyvee (\mathcal{H} \curlyvee \mathcal{K}) = (\mathcal{G} \curlyvee \mathcal{H}) \curlyvee \mathcal{K}$
By Proposition 3.7, the families

$$S_1 = \{F^*_{1\star}(F^*_{2\star}(G \wedge H) \wedge K) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\},$$

$$S_2 = \{F^*_{1\star}(G \wedge F^*_{2\star}(H \wedge K) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\}$$

are bases for the minimal Q^*-cauchy filters $(\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$ and $\mathcal{G} \curlywedge (\mathcal{H} \curlywedge \mathcal{K})$, respectively. Let $F^*_{1\star}(F^*_{2\star}G \wedge (H \wedge K) \in S_2$ and $F = F_1 \cap F_2$. Then $F \in \mathcal{F}$.

Now, we show that $F^*_\star(F^*_\star(G \wedge H) \wedge K) \subseteq F^*_{1\star}(G \wedge F^*_{2\star}(H \wedge K)$. Let $y \in F^*_\star(F^*_\star(G \wedge H) \wedge K)$. Then there are $x \in F^*_\star(G \wedge H)$, $k \in K$, $g \in G$ and $h \in H$ such that $y \equiv^F x \wedge k$ and $x \equiv^F g \wedge h$. Hence $y \equiv^F (g \wedge h) \wedge k = g \wedge (h \wedge k)$, which implies that $y \in F^*_\star(G \wedge F^*_\star(H \wedge K) \subseteq F^*_{1\star}(G \wedge F^*_{2\star}(H \wedge K)$. Therefore, $\mathcal{G} \curlywedge (\mathcal{H} \curlywedge \mathcal{K}) \subseteq (\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$. By the minimality of $(\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$, $\mathcal{G} \curlywedge (\mathcal{H} \curlywedge \mathcal{K}) = (\mathcal{G} \curlywedge \mathcal{H}) \curlywedge \mathcal{K}$. The proof of the other case is similar.

Case 1.4: $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H}) = \mathcal{G}$, $\mathcal{G} \curlyvee (\mathcal{G} \curlywedge \mathcal{H}) = \mathcal{G}$

It is enough to prove that $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H}) = \mathcal{G}$. The proof of the other case is similar. By Proposition 3.7, the families $S_1 = \{F^*_\star(G) : G \in \mathcal{G}, F \in \mathcal{F}\}$ and $S_2 = \{F^*_{1\star}(G_1 \wedge F^*_{2\star}(G_2 \vee H) : G_1, G_2 \in \mathcal{G}, H \in \mathcal{H}, F_1, F_2 \in \mathcal{F}\}$ are bases for the minimal Q^*-cauchy filters $\mathcal{G}$ and $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H})$, respectively. Let $F^*_{1\star}(G_1 \wedge F^*_{2\star}(G_2 \vee H) \in S_2$. Put $G = G_1 \cap G_2$ and $F = F_1 \cap F_2$. We prove that $F^*_\star(G) \subseteq F^*_{1\star}(G_1 \wedge F^*_{2\star}(G_2 \vee H)$. Let $y \in F^*_\star(G)$. Then there is a $g \in G$ such that $y \equiv^F g$. If $h \in H$, since $g = g \wedge (g \vee h)$, then $y \equiv^F g \wedge (g \vee h)$ and so $y \in F^*_{1\star}(G_1 \wedge F^*_{2\star}(G_2 \vee H)$. Hence $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H}) \subseteq \mathcal{G}$. By the minimality of $\mathcal{G}$, we conclude that $\mathcal{G} \curlywedge (\mathcal{G} \curlyvee \mathcal{H}) = \mathcal{G}$.

Now the cases 1.1,1.2,1.3,1.4 imply that $(\widetilde{A}, \curlywedge, \curlyvee)$ is a lattice.

Case 1.5: The lattice $(\widetilde{A}, \curlywedge, \curlyvee)$ is bounded.

For this, for each $\mathcal{G}, \mathcal{H} \in \widetilde{A}$, define $\mathcal{G} \leq \mathcal{H} \Leftrightarrow \mathcal{G} \curlywedge \mathcal{H} = \mathcal{G}$. It is clear that $(\widetilde{A}, \leq)$ is a partial ordered. Now, we prove that for each $\mathcal{G} \in \widetilde{A}$, $\mathcal{I} \leq \mathcal{G} \leq \mathcal{F}_0$. First, we show that $\mathcal{I} \leq \mathcal{G}$. Let $S \in \mathcal{I}$. Then for some a $F \in \mathcal{F}$, $F^*_\star(0) \subseteq S$. Since $\mathcal{G}$ is a minimal Q^*-cauchy filter, there is a $G \in \mathcal{G}$ such that $G \times G \subseteq F_\star$. We show that $F^*_\star(G \wedge F^*_\star(0)) \subseteq S$. Let $y \in F^*_\star(G \wedge F^*_\star(0))$. Then there are $g \in G$ and $x \in F^*_\star(0)$ such that $y \equiv^F g \wedge x$. On the other hand, since $x \equiv^F 0$, we get $g \wedge x \equiv^F 0$. Hence $y \equiv^F 0$ which implies that $y \in F^*_\star(0) \subseteq S$. Since $F^*_\star(G \wedge F^*_\star(0)) \in \mathcal{G} \curlywedge \mathcal{I}$, then $S \in \mathcal{G} \curlywedge \mathcal{I}$. By the minimality of $\mathcal{G} \curlywedge \mathcal{I}$, $\mathcal{G} \curlywedge \mathcal{I} = \mathcal{I}$. Now, we prove that $\mathcal{G} \leq \mathcal{F}_0$. By Proposition 3.7, the set $S_1 = \{F^*_\star(G \wedge F_1) : G \in \mathcal{G},\ F, F_1 \in \mathcal{F}\}$ is a base for $\mathcal{G} \curlywedge \mathcal{F}_0$. Let $F^*_\star(G \wedge F_1) \in S_1$. We prove that $F^*_\star(G) \subseteq F^*_\star(G \wedge F_1)$. Let $y \in F^*_\star(G)$. Then, there is a $g \in G$ such that $y \equiv^F g = g \wedge 1$. Hence $y \in F^*_\star(G \wedge F_1)$. By the minimality of $\mathcal{G}$, $\mathcal{G} \curlywedge \mathcal{F}_0 = \mathcal{G}$.

(2) $(\widetilde{A}, \circledcirc)$ **is a commutative monoid**

Case 2.1: $(\widetilde{A}, \circledcirc)$ is a commutative semigroup.

We will prove that $\mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K}) = (\mathcal{G} \circledcirc \mathcal{H}) \circledcirc \mathcal{K}$. By Proposition 3.7, the sets

$$S_1 = \{F^*_{1\star}(G \odot F^*_{2\star}(H \odot K)) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\},$$

$$S_2 = \{F^*_{1\star}(F^*_{2\star}(G \odot H) \odot K)) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\}$$

are bases from $\mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K})$ and $(\mathcal{G} \circledcirc \mathcal{H}) \circledcirc \mathcal{K}$, respectively. Let $F^*_{1\star}(F^*_{2\star}(G \odot H) \odot K)) \in S_2$, $F = F_1 \cap F_2$ and $y \in F^*_\star(G \odot F^*_\star(H \odot K)$. Then there are $g \in G$, $x \in F^*_\star(H \odot K)$, $h \in H$ and $k \in K$ such that $y \overset{F}{\equiv} g \odot x$ and $x \overset{F}{\equiv} h \odot k$. Hence

$y \overset{F}{\equiv} g\odot(h\odot k) = (g\odot h)\odot k$ and so $y \in F^*_\star(F^*_\star(G\odot H)\odot K) \subseteq F^*_{1\star}(F^*_{2\star}(G\odot H)\odot K))$. Therefore, $S_2 \subseteq S_1$ which implies that $(\mathcal{G} \circledcirc \mathcal{H}) \circledcirc \mathcal{K} \subseteq \mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K})$. Now, by the minimality of $\mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K})$, $\mathcal{G} \circledcirc (\mathcal{H} \circledcirc \mathcal{K}) = (\mathcal{G} \circledcirc \mathcal{H}) \circledcirc \mathcal{K}$. Finally, it is easy to prove that $\mathcal{G} \circledcirc \mathcal{H} = \mathcal{H} \circledcirc \mathcal{G}$.

Case 2.2: $(\widetilde{A}, \circledcirc)$ is a monoid

We prove that $\mathcal{G}\circledcirc\mathcal{F}_0 = \mathcal{G}$. By Proposition 3.7, the set $S_2 = \{F^*_\star(G\odot F_1) : G \in \mathcal{G}, F, F_1 \in \mathcal{F}\}$ is a base for $\mathcal{G} \circledcirc \mathcal{F}_0$. It is clear that for each $F^*_\star(G \odot F_1) \in S_2$, $F^*_\star(G) \subseteq F^*_\star(G \odot F_1)$ and this implies that $\mathcal{G} \circledcirc \mathcal{F}_0 \subseteq \mathcal{G}$. By the minimality of $\mathcal{G}$, $\mathcal{G} \circledcirc \mathcal{F}_0 = \mathcal{G}$.

(3) $\mathcal{G} \circledcirc (\mathcal{G} \hookrightarrow \mathcal{H}) = \mathcal{G} \curlywedge \mathcal{H}$

By Proposition 3.7, the families

$$S_1 = \{F^*_\star(G \wedge H) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\},$$

$$S_2 = \{F^*_{1\star}(G_1 \odot F^*_{2\star}(G_2 \to H)) : G_1, G_2 \in \mathcal{G}, H \in \mathcal{H}, F_1, F_2 \in \mathcal{F}\}$$

are bases for $\mathcal{G} \curlywedge \mathcal{H}$ and $\mathcal{G} \circledcirc (\mathcal{G} \hookrightarrow \mathcal{H})$, respectively. Let $F^*_{1\star}(G_1 \odot F^*_{2\star}(G_2 \to H)) \in S_2$, $G = G_1 \cap G_2$ and $F = F_1 \cap F_2$. We will prove that $F^*_\star(G \wedge H) \subseteq F^*_{1\star}(G_1 \odot F^*_{2\star}(G_2 \to H))$. Let $y \in F^*_\star(G \wedge H)$. Then there are $g \in G$ and $h \in H$ such that $y \equiv^F g \wedge h$. It follows from $g \wedge h = g \odot (g \to h)$ which $y \in F^*_{1\star}(G_1 \odot F^*_{2\star}(G_2 \to H))$. Hence $F^*_\star(G \wedge H) \subseteq F^*_{1\star}(G_1 \odot F^*_{2\star}(G_2 \to H))$ which implies that $\mathcal{G}\circledcirc(\mathcal{G} \hookrightarrow \mathcal{H}) \subseteq \mathcal{G} \curlywedge \mathcal{H}$. Now, by the minimality of $\mathcal{G} \curlywedge \mathcal{H}$, we get $\mathcal{G} \circledcirc (\mathcal{G} \hookrightarrow \mathcal{H}) = \mathcal{G} \curlywedge \mathcal{H}$.

(4) $\mathcal{G} \leq \mathcal{H} \hookrightarrow \mathcal{K} \Leftrightarrow \mathcal{G} \circledcirc \mathcal{H} \leq \mathcal{K}$

First, we prove the following statements:

(a) $\mathcal{G} \leq \mathcal{H} \Leftrightarrow \mathcal{G} \hookrightarrow \mathcal{H} = \mathcal{F}_0$

(b) $\mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K}) = \mathcal{G} \circledcirc \mathcal{H} \hookrightarrow \mathcal{K}$.

(a) To prove it, let $\mathcal{G} \hookrightarrow \mathcal{H} = \mathcal{F}_0$. Then $\mathcal{G} \circledcirc (\mathcal{G} \hookrightarrow \mathcal{H}) = \mathcal{G} \circledcirc \mathcal{F}_0 = \mathcal{G}$. By (3), $\mathcal{G} \curlywedge \mathcal{H} = \mathcal{G}$ and so $\mathcal{G} \leq \mathcal{H}$.

Conversely, let $\mathcal{G} \leq \mathcal{H}$. By Proposition 3.7, the set $S = \{F^*_\star(G \to H) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\}$ is a base for $\mathcal{G} \hookrightarrow \mathcal{H}$. Let $F^*_\star(G \to H) \in S$. We prove that $1 \in F^*_\star(G \to H)$. Since by Lemma 3.10, $G \to H$ is a Q^*-cauchy filter base, there are $G_1 \in \mathcal{G}$ and $H_1 \in \mathcal{H}$ such that $(G_1 \to H_1) \times (G_1 \to H_1) \subseteq F_\star$. Put $G_2 = G_1 \cap G$ and $H_2 = H_1 \cap H$. It is easy to see that $G_2 \wedge H_2 \subseteq F^*_\star(G_2 \wedge H_2) \in \mathcal{G} \curlywedge \mathcal{H}$. Since $\mathcal{G} \curlywedge \mathcal{H} = \mathcal{G}$, there is a $G_3 \in \mathcal{G}$ such that $G_3 \subseteq G_1$ and $G_3 \subseteq G_2 \wedge H_2$. Since $G_3 \neq \phi$, there are $g_3 \in G_3$, $g \in G_2$ and $h \in H_2$ such that $g_3 = g \wedge h$. Since $(g_3 \to h, g \to h)$ and $(g \to h, g_3 \to h)$ both are in $(G_1 \to H_1) \times (G_1 \to H_1) \subseteq F_\star$, we get $g \to h \equiv^F g_3 \to h = 1$ and so $1 \in F^*_\star(G \to H)$. Hence $F^*_\star(1) \subseteq F^*_\star(G \to H)$. This implies that $\mathcal{G} \hookrightarrow \mathcal{H} \subseteq \mathcal{F}_0$. By the minimality of $\mathcal{F}_0$, $\mathcal{G} \hookrightarrow \mathcal{H} = \mathcal{F}_0$. Therefore, we have (a).

(b) By Proposition 3.7, the families

$$S_1 = \{F^*_{1\star}(G \to F^*_{2\star}(H \to K)) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\},$$

$$S_2 = \{F^*_{1\star}(F^*_{2\star}(G \odot H) \to K) : G \in \mathcal{G}, H \in \mathcal{H}, K \in \mathcal{K}, F_1, F_2 \in \mathcal{F}\}$$

are bases of $\mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K})$ and $(\mathcal{G} \circledcirc \mathcal{H}) \hookrightarrow \mathcal{K}$, respectively. Let $F^*_{1\star}(F^*_{2\star}(G \odot H) \to K) \in S_2$, $F = F_1 \cap F_2$ and $y \in F^*_\star(G \to F^*_\star(H \to K))$. Then there are $g \in G$ and $x \in F^*_\star(H \to K)$ such that $y \equiv^F g \to x$. Also there are $h \in H$ and $k \in K$ such that $x \equiv^F h \to k$. Hence $y \equiv^F g \to x \equiv^F g \to (h \to k) = (g \odot h) \to k$. Therefore, $y \in F^*_{1\star}(F^*_{2\star}(G \odot H) \to K)$. This implies that $(\mathcal{G} \circledcirc \mathcal{H}) \hookrightarrow \mathcal{K} \subseteq \mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K})$. By the minimality of $\mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K})$, we get $\mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K}) = \mathcal{G} \circledcirc \mathcal{H} \hookrightarrow \mathcal{K}$. Hence we have (b).
Now, by (a) and (b), we have

$$\mathcal{G} \le \mathcal{H} \hookrightarrow \mathcal{K} \Leftrightarrow \mathcal{G} \hookrightarrow (\mathcal{H} \hookrightarrow \mathcal{K}) = \mathcal{F}_0 \Leftrightarrow (\mathcal{G} \circledcirc \mathcal{H}) \hookrightarrow \mathcal{K} = \mathcal{F}_0 \Leftrightarrow \mathcal{G} \circledcirc \mathcal{H} \le \mathcal{K}.$$

So $\mathcal{G} \le \mathcal{H} \hookrightarrow \mathcal{K} \Leftrightarrow \mathcal{G} \circledcirc \mathcal{H} \le \mathcal{K}$.
(5) $(\mathcal{G} \hookrightarrow \mathcal{H}) \curlyvee (\mathcal{H} \hookrightarrow \mathcal{G}) = \mathcal{F}_0$
By Proposition 3.7, the set

$$S = \{F^*_{1\star}(F^*_{2\star}(G_1 \to H_1) \vee F^*_{3\star}(H_2 \to G_2)) : G_1, G_2 \in \mathcal{G}, H_1, H_2 \in \mathcal{H}, F_1, F_2, F_3 \in \mathcal{F}\}$$

is a base for $(\mathcal{G} \hookrightarrow \mathcal{H}) \curlyvee (\mathcal{H} \hookrightarrow \mathcal{G})$. Let $F^*_{1\star}(F^*_{2\star}(G_1 \to H_1) \vee F^*_{3\star}(H_2 \to G_2)) \in S$, $G = G_1 \cap G_2$, $H = H_1 \cap H_2$ and $F = F_1 \cap F_2 \cap F_3$. We show that $1 \in F^*_\star(F^*_\star(G \to H) \vee F^*_\star(H \to G))$. Let $g \in G$ and $h \in H$. Since A is a BL-algebra, we have $(g \to h) \vee (h \to g) = 1$. Since $g \to h \in F^*_\star(G \to H)$ and $h \to g \in F^*_\star(H \to G)$, we have $(g \to h) \vee (h \to g) \in F^*_\star(F^*_\star(G \to H) \vee F^*_\star(H \to G))$ and so $1 \in F^*_\star(F^*_\star(G \to H) \vee F^*_\star(H \to G))$. Hence $F^*_\star(1) \subseteq F^*_\star(F^*_\star(G \to H) \vee F^*_\star(H \to G))$ which implies that $(\mathcal{G} \hookrightarrow \mathcal{H}) \curlyvee (\mathcal{H} \hookrightarrow \mathcal{G}) \subseteq \mathcal{F}_0$. By the minimality of $\mathcal{F}_0$, $(\mathcal{G} \hookrightarrow \mathcal{H}) \curlyvee (\mathcal{H} \hookrightarrow \mathcal{G}) = \mathcal{F}_0$.

□

4 Some topological properties on quasi-unifom BL-algebra (A, Q)

Let $T(Q)$ and $T(Q^*)$ be topologies induced by Q and Q^*, respectively. Our goal in this section is to study (semi)topological BL-algebras $(A, T(Q))$ and $(A, T(Q^*))$. We prove that $(A, \wedge, \vee, \odot, T(Q))$ is a compact connected topological BL-algebra and $(A, T(Q^*))$ is a regular topological BL-algebra. We study separation axioms on $(A, T(Q))$ and $(A, T(Q^*))$. Also we stay conditions under which (A, Q) becomes totally bounded. Finally, we show that if

(A,Q) is a T_0 quasi-uniform space, then the BL-algebra $(\widetilde{A},\widetilde{Q})$ in Theorem 3.11 is the bicomplition topological BL-algebra of (A,Q).

Theorem 4.1. *The set $T(Q)=\{G\subseteq A : \forall x\in G\ \exists F\in\mathcal{F}\ s.t\ F_\star(x)\subseteq G\}$ is the topology induced by Q on A such that $(A,\{\wedge,\vee,\odot\},T(Q))$ is a topological BL-algebras. Also $(A,\rightarrow,T(Q))$ is a left topological BL-algebra. Furthermore, if the negation map $c(x)=x'$ is one to one, then $(A,T(Q))$ is a topological BL-algebra.*

Proof. First we prove that $T(Q)$ is a nonempty set. For this, we prove that for each $F\in\mathcal{F}$ and each $x\in A$, $F_\star(x)\in T(Q)$. Let $F\in\mathcal{F}$, $x\in A$ and $y\in F_\star(x)$. If z is an arbitrary element of $F_\star(y)$, then $z\rightarrow y\in F$. Since $y\rightarrow x\in F$, by (B_{15}), we get $z\rightarrow x\in F$. Hence $F_\star(y)\subseteq F_\star(x)$ which implies that $F_\star(x)\in T(Q)$. Now we prove that $T(Q)$ is a topology on A. Clearly, $\phi, A\in T(Q)$. Also it is easy to prove that the arbitrary union of members of $T(Q)$ is in $T(Q)$. Let $G_1,...,G_n$ be in $T(Q)$ and $x\in\bigcap_{i=1}^{i=n}G_i$. There are $F_1,...,F_n\in\mathcal{F}$ such that $F_{i\star}(x)\subseteq G_i$, for $1\leq i\leq n$. Let $F=F_1\cap...\cap F_n$. Then $F\in\mathcal{F}$ and $F_\star(x)\subseteq F_{1\star}(x)\cap...\cap F_{n\star}(x)\subseteq\bigcap_{i=1}^{i=n}G_i$. Hence $T(Q)$ is a topology. Since for each $F\in\mathcal{F}$, $F_\star$ belongs to Q, then $T(Q)$ is the topology induced by Q. Now, by Lemmas 3.1, it is clear that $(A,\{\wedge,\vee,\odot\},T(Q))$ is a topological BL-algebra. In continue, we prove that $(A,\rightarrow,T(Q))$ is a left topological BL-algebra. Let $x,y,z\in A$, and $z\in F_\star(y)$. By (B_9), $(x\rightarrow z)\rightarrow(x\rightarrow y)\geq z\rightarrow y$ which implies that $(x\rightarrow z)\rightarrow(x\rightarrow y)\in F$. So $x\rightarrow z\in F_\star(x\rightarrow y)$. Hence $x\rightarrow F_\star(y)\subseteq F_\star(x\rightarrow y)$ and so $(A,\rightarrow,T(Q))$ is a left topological BL-algebra.
To complete the proof, suppose that the negation map c is one to one. Since $(A,\rightarrow,T(Q))$ is a topological BL-algebra, c is continuous. Now by [[2], Theorem(3.15)], $(A,T(Q))$ is a topological BL-algebra. □

Theorem 4.2. *BL-algebra $(A,T(Q))$ is a connected and compact space and each $F\in\mathcal{F}$, is a closed compact set in $(A,T(Q))$.*

Proof. First we prove that if $\{G_i : i\in I\}$ is an open cover of A in $T(Q)$, then for some $i\in I$, $A=G_i$. Let $A=\bigcup_{i\in I}G_i$, where $G_i\in T(Q)$. Then, there are $i\in I$ and $F\in\mathcal{F}$ such that $1\in G_i$ and $F_\star(1)\subseteq G_i$. By Lemma 3.1 (vi), $A=F_\star(1)$. Hence $A=G_i$. Now, it is easy to show that $(A,T(Q))$ is connected and compact. In continue we prove that each $F\in\mathcal{F}$, is a closed, compact set in $(A,T(Q))$. For this, let $F\in\mathcal{F}$ and $x\in\overline{F}$. Then, there is a $y\in F_\star(x)\cap F$. Since $y\in F$ and $y\rightarrow x\in F$, we get $x\in F$. Hence $\overline{F}=F$. Now, Since $(A,T(Q))$ is compact, F is compact. □

Theorem 4.3. *(i) BL-algebra $(A,T(Q))$ is not a T_1 and T_2 topological space.*
(ii) BL-algebra $(A,T(Q))$ is a T_0 topological space iff, for each $1\neq x\in A$, there is a $F\in\mathcal{F}$ such that $x\notin F$.

Proof. (i) $(A, T(Q))$ is not a T_1 and T_2 topological space because for each $G \in T(Q)$, $1 \in G$ if and only if $G = A$.
(ii) Suppose for each $1 \neq x \in A$, there is a $F \in \mathcal{F}$ such that $x \notin F$. We prove that $(A, T(Q))$ is a T_0 topological space. For this, let $1 \neq x \in A$. Then for some $F \in \mathcal{F}$, $x \notin F$. Since $1 \to x = x$, then $1 \notin F_\star(x)$. Moreover, since $(A, \to, T(Q))$ is a left topological BL-algebra, by [[2], Proposition(4.2)], $(A, T(Q))$ is a T_0 topological space. Conversely, let $(A, T(Q))$ is a T_0 topological space and $1 \neq x \in A$. Then for some $F \in \mathcal{F}$, $1 \notin F_\star(x)$. Hence $x = 1 \to x \notin F$. $\square$

Theorem 4.4. *The set $T(Q^*) = \{G \subseteq A : \forall x \in G\ \exists F \in \mathcal{F}\ s.t\ F^*_\star(x) \subseteq G\}$ is the topology induced by Q^* on BL-algebra A such that $(A, T(Q^*))$ is a topological BL-algebras.*

Proof. By the similar argument as Theorem 4.1, we can prove that $T(Q^*)$ is the topology induced by Q^* on A. By Lemma 3.2(v), $(A, T(Q^*))$ is a topological BL-algebra. $\square$

Theorem 4.5. (i) *BL-algebra $(A, T(Q^*))$ is connected iff, $\mathcal{F} = \{A\}$,*
(ii) *$\mathcal{F}$ has only a proper filter iff, each $F \in \mathcal{F}$ is a component.*

Proof. (i) Let $\mathcal{F} = \{A\}$. Then it is easy to prove that $T(Q^*) = \{\phi, A\}$. Hence $(A, T(Q^*))$ is connected.
Conversely, let $\mathcal{F} \neq \{A\}$. Then, there is a filter $F \in \mathcal{F}$ such that $F \neq A$. Since for each $x \in F$, $F^*_\star(x) \subseteq F$, we conclude that $F \in T(Q^*)$. Let $y \in \overline{F}$. Then there is a $z \in F^*_\star(y) \cap F$. This proves that $y \in F$. Hence F is closed. Now, since F is a closed and open subset of A, then A is not connected.
(ii) Let $\mathcal{F}$ has a proper filter F. By the similar argument as (i), we get that F is closed and open. We show that F is connected. Let G_1 and G_2 be in $T(Q^*)$ and $F = (F \cap G_1) \cup (F \cap G_2)$. Without loss of generality, Suppose that $1 \in F \cap G_1$, then $F \subseteq F^*_\star(1) \subseteq G_1$. Hence $F \cap G_1 = F$, which implies that F is connected. Therefore, F is a component.
Conversely, suppose each $F \in \mathcal{F}$ is a component. If F_1 and F_2 are in $\mathcal{F}$, then $F_1 \cap F_2$ is in $\mathcal{F}$ and is component. Hence $F_1 = F_1 \cap F_2 = F_2$. $\square$

Recall that a topological space $(X, \mathcal{U})$ is regular if for each $x \in G \in \mathcal{U}$ there is a $U \in \mathcal{U}$ such that $x \in U \subseteq \overline{U} \subseteq G$.

Theorem 4.6. *BL-algebra $(A, T(Q^*))$ is a regular space.*

Proof. First we prove that for each $F \in \mathcal{F}$ and $x \in A$, $\overline{F^*_\star(x)} = F^*_\star(x)$. Let $y \in \overline{F^*_\star(x)}$. Then there is a $z \in F^*_\star(y) \cap F^*_\star(x)$. Hence $y \equiv^F z \equiv^F x$ which implies that $y \in F^*_\star(x)$. Therefore, $\overline{F^*_\star(x)} = F^*_\star(x)$. Now if $x \in G \in T(Q^*)$, then for some a $F \in \mathcal{F}$, $x \in \overline{F^*_\star(x)} = F^*_\star(x) \subseteq G$. Hence $(A, T(Q^*))$ is a regular space. $\square$

Theorem 4.7. *On BL-algebra $(A, T(Q^*))$ the follwing statements are equivalent.*
(i) $(A, T(Q^*))$ *is a* T_0 *space,*
(ii) $\bigcap_{F\in\mathcal{F}} F^*_\star(1) = \{1\}$,
(iii) $(A, T(Q^*))$ *is a* T_1 *space,*
(iv) $(A, T(Q^*))$ *is a* T_2 *space.*

Proof. $(i \Rightarrow ii)$ Let $(A, T(Q^*))$ be a T_0 space and $1 \neq x \in A$. By [[2], Proposition(4.2)], there is a $F \in \mathcal{F}$ such that $1 \notin F^*_\star(x)$. Hence $x \notin F$. This implies that $x \notin F^*_\star(1)$. Therefore, $x \notin \bigcap_{F\in\mathcal{F}} F^*_\star(1)$.
$(ii \Rightarrow i)$ Let $\bigcap_{F\in\mathcal{F}} F^*_\star(1) = \{1\}$ and $1 \neq x \in A$. Then for some a $F \in \mathcal{F}$, $x \notin F$. Hence $1 \notin F^*_\star(x)$. Now by [[2], Proposition(4.2)], $(A, T(Q^*))$ is a T_0 space.
By Theorems 4.4 and 4.6, $(A, T(Q^*))$ is a regular topological BL-algebra. Hence by [[2], Theorem(4.7)], the statements (ii), (iii) and (iv) are equivalent. □

Example 4.8. *In Example 3.4, For each* $a \in [0, 1)$ *and* $x \in [0, 1]$

$$F_{a*}(x) = \begin{cases} [0,x] & , x \leq a, \\ [0,1] & , x > a. \end{cases} \quad F^{-1}_{a*}(x) = \begin{cases} [x,1] & , x \leq a, \\ (a,1] & , x > a. \end{cases}$$

$$F^*_{a*}(x) = \begin{cases} x & , x < a, \\ a & , x = a \\ (a,1] & , x > a. \end{cases}$$

If $T(Q)$ *is the induced topology by* Q *and* $G \in T(Q)$, *then for each* $x \in G$, *there is a* $a \in [0, 1)$ *such that* $F^*_{a\star}(x) \subseteq G$. *Hence* $[0, x] \subseteq G$ *or* $G = [0, 1]$. *If* $G \in T(Q)$ *and* $G \neq [0, 1]$, *then for each* $x \in G$, $[0, x] \subseteq G$. *If* $g = supG$, *then* $G = [0, g]$ *or* $[0, g)$. *Therefore* $T(Q) = \{[0, x] : x \in [0, 1]\} \cup \{[0, x) : x \in [0, 1]\}$. *Also if* $T(Q^*)$ *is topology induced by* Q^* *and* $G \in T(Q^*)$, *then for each* $x \in G$, *there is a* $a \in [0, 1)$ *such that* $F^*_{a\star}(x) \subseteq G$. *Hence if* $G \in T(Q^*)$, *then for some* $a \in [0, 1)$, $a \in G$ *or* $(a, 1] \subseteq G$.
Now since for each $a \in [0, 1)$, $F^*_{a\star}(1) = (a, 1]$, *we get that* $\bigcap_{a\in[0,1)} F^*_{a\star}(1) = \{1\}$. *Hence by Theorems 4.4, 4.6 and 4.7,* $(A, T(Q^*))$ *is a* T_i *regular topological BL-algebra, when* $0 \leq i \leq 2$.

Theorem 4.9. *Let* $(A, \rightarrow, \mathcal{U})$ *be a semitopological BL-algebra and* F_0 *be an open proper BL-filter in* A. *Then, there exists a nontrivial topology* $\mathcal{V}$ *on* A *such that* $\mathcal{V} \subseteq \mathcal{U}$ *and* $(A, \mathcal{V})$ *is a topological BL-algebra.*

Proof. Let $\mathcal{F}$ be a collection of BL-open filters in A which closed under finite intersection and $F_0 \in \mathcal{F}$. Let Q be the quasi-uniformity induced by $\mathcal{F}$. Since

$F_0 \neq A$, by Lemma 3.1(vi), there is a $x \in A$ such that $F^*_{0\star}(x) \neq A$. So $T(Q^*)$ is a nontrivial topology. We prove that $T(Q^*) \subseteq \mathcal{U}$. Let $x \in G \in T(Q^*)$. Then, there is a $F \in \mathcal{F}$ such that $F^*_\star(x) \subseteq G$. Since $x \to x = 1 \in F \in \mathcal{U}$, there is a $U \in \mathcal{U}$ such that $x \in U$ and $U \to x \subseteq F$ and $x \to U \subseteq F$. If $z \in U$, then $z \to x, x \to z \in F$ and so $z \in F^*_\star(x)$. Hence $x \in U \subseteq G$. Therefore, $T(Q^*)$ is a nontrivial topology coaser than $\mathcal{U}$ and so by Theorem 4.4, $(A, T(Q^*))$ is a topological BL-algebra. □

Example 4.10. *Let $\mathcal{I}$ be the BL-algebra in Example 2.5(ii), and $\mathcal{U}$ be a topology on $\mathcal{I}$ with the base $S = \{(a, b] \cap \mathcal{I} : a, b \in \mathbb{R}\}$. We prove that $(\mathcal{I}, \to, \mathcal{U})$ is a semitopological BL-algebra. Let $x, y \in I$, and $x \to y \in (a, b]$. If $x \leq y$, then $[0, x]$ and $(ax, y]$ are two open neighborhoods of x and y, respectively, such that $(0, x] \to y \subseteq (a, 1]$ and $x \to (ax, y] \subseteq (a, 1]$. If $x > y$ and $y = 0$, then $(0, x]$ and $\{0\}$ are two open neighborhoods of x and 0, respectively, such that $(0, x] \to 0 \subseteq [0, b]$ and $x \to \{0\} \subseteq [0, b]$. If $x > y$ and $y \neq 0$, then $(y/b, y/a]$ and $(ax, bx]$ are two open sets of x, y, respectively, such that $(y/b, y/a] \to y \subseteq (a, b]$ and $x \to (ax, bx] \subseteq (a, b]$. It is easy to prove that $\mathcal{F} = \{(0, 1], A\}$ is a collection of BL-filters which is closed under intersection. Now since for each $x \in A$, $A^*_\star(x) = A$ and $(0, 1]^*_\star(x) = (0, 1]$, we conclude $T(Q^*) = \{\phi, (0, 1], A\}$. By Theorem 4.9, $(A, T(Q^*))$ is a topological BL-algebra.*

Recall a quasi-uniform space (X, Q) is *totally-bounded* if for each $q \in Q$, there exist sets $A_1, ..., A_n$ such that $X = \bigcup_{i=1}^{i=n} A_i$ and for each $1 \leq i \leq n$, $A_i \times A_i \subseteq q$.(See [10])

Theorem 4.11. *The following conditions on BL-algebra $(A, T(Q^*))$ are equivalent.*
(i) *For each $F \in \mathcal{F}$, A/F is finite,*
(ii) *(A, Q) is totally bounded,*
(iii) *$(A, T(Q^*))$ is compact.*

Proof. ($i \Rightarrow ii$) Let for each $F \in \mathcal{F}$, A/F be finite. We prove that (A, Q) is totally bounded. For this it is enough to prove that, for each $F \in \mathcal{F}$, there are $a_1, ..., a_n \in A$, such that for each $1 \leq i \leq n$, $a_i/F \times a_i/F \subseteq F_\star$. Let $F \in \mathcal{F}$. Since A/F is finite, there are $a_1, ..., a_n \in A$, such that $A = \cup_{i=1}^{n} a_i/F$. For each $1 \leq i \leq n$, $a_i/F \times a_i/F \subseteq F_\star$ because if $(x, y) \in a_i/F \times a_i/F$, then $x \equiv^F a_i \equiv^F y$ and so $(x, y) \in F_\star$. This proves that (A, Q) is totally bounded.
($ii \Rightarrow iii$) Let (A, Q) be totally bounded and $F \in \mathcal{F}$. There exist sets $A_1, ..., A_n$, such that $\bigcup_{i=1}^{i=n} A_i = A$ and for each $1 \leq i \leq n$, $A_i \times A_i \subseteq F_\star$. Let $1 \leq i \leq n$ and $x, y \in A_i$. Since (x, y) and (y, x) are in $F_\star$, we get $x \equiv^F y$. This proves that $A_i = a_i/F$, for some $a_i \in A_i$.

Now to prove that $(A, T(Q^*))$ is compact let $A = \bigcup_{i\in I} G_i$, where each G_i is in $T(Q^*)$. Then there are $H_1, ..., H_n \in \{G_i : i \in I\}$, such that $a_i \in H_i$, for each $1 \leq i \leq n$. Now suppose $x \in A$, then $x \in a_i/F$, for some $1 \leq i \leq n$, and so $x \in F^*_\star(a_i) \subseteq H_i$. Therefore, $A \subseteq \bigcup_{i=1}^n H_i$, which shows that $(A, T(Q^*))$ is compact.
$(iii \Rightarrow i)$ Let $F \in \mathcal{F}$. Since $\{F^*_\star(x) : x \in A\}$ is an open cover of A in $T(Q^*)$, then there are $a_1, ..., a_n \in A$, such that $A \subseteq \bigcup_{i=1}^n F^*_\star(a_i)$. Now, it is easy to see that $A/F = \{a_1/F, ..., a_n/F\}$.

□

In the end, we prove that the quasi-uniform Bl-algeba $(\widetilde{A}, \widetilde{Q})$ in Theorem 3.11, is T_0 bicomplition quasi-uniform of BL-algebra (A, Q).

Theorem 4.12. *If quasi-uniform BL-algebra (A, Q) is T_0, then*
(i) $(\widetilde{A}, \widetilde{Q})$ is the bicompletion of (A, Q).
(ii) $(\widetilde{A}, T(\widetilde{Q}))$ is a topological BL-algebra.
(iii) A is a sub BL-algebra of $\widetilde{A}$.
(iv) $(\widetilde{A}, T(\widetilde{Q^*}))$ is a topological BL-algebra.

Proof. (i) By Theorem 3.11 and Lemma 2.18, $(\widetilde{A}, \widetilde{Q})$ is an unique T_0-bicompletion quasi-uniform of (A, Q) and the mapping $i : A \to \widetilde{A}$ by $i(x) = \{W \subseteq A : W \ is \ a \ T(Q^*) - neighborhood \ of \ x\}$ is a quasi-uniform embedded and $cl_{T(Q^*)} i(A) = \widetilde{A}$.
(ii) It is clear that

$$T(\widetilde{Q}) = \{S \subseteq \widetilde{A} : \forall \mathcal{G} \in S \ \exists F \in \mathcal{F} \ s.t \ \widetilde{F_\star}(\mathcal{G}) \subseteq S\}.$$

Let $\bullet \in \{\wedge, \vee, \odot\}$ and $\widetilde{\bullet} \in \{\curlywedge, \curlyvee, \circledcirc\}$. We have to prove that for each $\mathcal{G}, \mathcal{H} \in \widetilde{A}$, $\widetilde{F_\star}(\mathcal{G}) \widetilde{\bullet} \widetilde{F_\star}(\mathcal{H}) \subseteq \widetilde{F_\star}(\mathcal{G} \widetilde{\bullet} \mathcal{H})$. Let $\mathcal{G}_1 \in \widetilde{F_\star}(\mathcal{G})$ and $\mathcal{H}_1 \in \widetilde{F_\star}(\mathcal{H})$. Then, there are $G \in \mathcal{G}$, $G_1 \in \mathcal{G}_1$, $H \in \mathcal{H}$ and $H_1 \in \mathcal{H}_1$ such that $G \times G_1 \subseteq F_\star$, $H \times H_1 \subseteq F_\star$. By Proposition 3.7, $S_1 = \{F^*_\star(G \bullet H) : G \in \mathcal{G}, H \in \mathcal{H}, F \in \mathcal{F}\}$ and $S_2 = \{F^*_\star(G_1 \bullet H_1) : G_1 \in \mathcal{G}_1, H_1 \in \mathcal{H}_1, F \in \mathcal{F}\}$ are bases of $\mathcal{G} \widetilde{\bullet} \mathcal{H}$ and $\mathcal{G}_1 \widetilde{\bullet} \mathcal{H}_1$, respectively. We show that $\mathcal{G}_1 \widetilde{\bullet} \mathcal{H}_1 \in \widetilde{F_\star}(\mathcal{G} \widetilde{\bullet} \mathcal{H})$. For this, it is enough to show that $F^*_\star(G \bullet H) \times F^*_\star(G_1 \bullet H_1) \subseteq F_\star$. Let $(y, y_1) \in F^*_\star(G \bullet H) \times F^*_\star(G_1 \bullet H_1) \subseteq F_\star$. Then, there are $g \in G$, $g_1 \in G_1$, $h \in H$ and $h_1 \in H_1$ such that $y \equiv^F g \bullet h$ and $y_1 \equiv^F g_1 \bullet h_1$. By $(B_{17}), (B_{18})$ and (B_{19}), we have $(g_1 \to g) \odot (h_1 \to h) \leq (g_1 \bullet h_1) \to (g \bullet h)$. It follows from $(g, g_1) \in G \times G_1 \subseteq F_\star$ and $(h, h_1) \in H \times H_1 \subseteq F_\star$ that $g_1 \to g$ and $h_1 \to h$ are in F. Hence $g_1 \bullet h_1 \to g \bullet h \in F$. Therefore, $y_1 \to y \in F$ and so $(y, y_1) \in F_\star$. Thus we proved that $\widetilde{F_\star}(\mathcal{G}) \widetilde{\bullet} \widetilde{F_\star}(\mathcal{H}) \subseteq \widetilde{F_\star}(\mathcal{G} \widetilde{\bullet} \mathcal{H})$.
(iii) Let $\bullet \in \{\wedge, \vee, \odot, \to\}$, $\widetilde{\bullet} \in \{\curlywedge, \curlyvee, \circledcirc, \hookrightarrow\}$ and $a, b \in A$. We shall prove

that $i(a)\widetilde{\bullet}i(b) = i(a \bullet b)$. By Proposition 3.7, the set $S = \{F_\star^*(W_a \bullet W_b) : F \in \mathcal{F},\ W_a, W_b\ are\ T(Q^*) - neighborhoods\ of\ a, b\}$ is a base for $i(a)\widetilde{\bullet}i(b)$. Since $F_\star^*(a \bullet b) \subseteq F_\star^*(W_a \bullet W_b)$ and $F_\star^*(a \bullet b) \in i(a \bullet b)$, we deduce that filter $i(a)\widetilde{\bullet}i(b)$ is contained in the filter $i(a \bullet b)$. Since they are minimal Q^*-cauchy filters, $i(a)\widetilde{\bullet}i(b) = i(a \bullet b)$. Hence A is a sub-BL-algebra of $\widetilde{A}$.
(iv) By Lemma 2.18, $\widetilde{Q^*} = (\widetilde{Q})^*$. Hence

$$T(\widetilde{Q^*}) = \{S \subseteq \widetilde{A} : \forall \mathcal{G} \in S\ \exists F \in \mathcal{F}\ s.t\ \widetilde{F_\star^*}(\mathcal{G}) \subseteq S\}.$$

We prove that $(\widetilde{A}, T(\widetilde{Q^*}))$ is a topological BL-algebra. Let $\bullet \in \{\wedge, \vee, \odot, \rightarrow\}$ and $\widetilde{\bullet} \in \{\curlywedge, \curlyvee, \circledcirc, \hookrightarrow\}$ and let $\mathcal{G}\widetilde{\bullet}\mathcal{H} \in \widetilde{F_\star^*}(\mathcal{G}\widetilde{\bullet}\mathcal{H})$. We show that $\widetilde{F_\star^*}(\mathcal{G})\widetilde{\bullet}\widetilde{F_\star^*}(\mathcal{H}) \subseteq \widetilde{F_\star^*}(\mathcal{G}\widetilde{\bullet}\mathcal{H})$. Let $\mathcal{G}_1 \in \widetilde{F_\star^*}(\mathcal{G})$ and $\mathcal{H}_1 \in \widetilde{F_\star^*}(\mathcal{H})$. Then, there are $G \in \mathcal{G}$, $G_1 \in \mathcal{G}_1$, $H \in \mathcal{H}$ and $H_1 \in \mathcal{H}_1$ such that $G \times G_1 \subseteq F_\star^*$ and $H \times H_1 \subseteq F_\star^*$. By Proposition 3.7, $F_\star^*(G_1 \bullet H_1) \in \mathcal{G}_1\widetilde{\bullet}\mathcal{H}_1$ and $F_\star^*(G \bullet H) \in \mathcal{G}\widetilde{\bullet}\mathcal{H}$. We have to prove that $\mathcal{G}_1\widetilde{\bullet}\mathcal{H}_1 \in \widetilde{F_\star^*}(\mathcal{G}\widetilde{\bullet}\mathcal{H})$. For this, it is enough to show that $F_\star^*(G \bullet H) \times F_\star^*(G_1 \bullet H_1) \subseteq F_\star^*$. Let $y \in F_\star^*(G \bullet H)$ and $y_1 \in F_\star^*(G_1 \bullet H_1)$. Then $y \equiv^F g \bullet h$ and $y_1 \equiv^F g_1 \bullet h_1$ for some $g \in G$, $g_1 \in G_1$, $h \in H$ and $h_1 \in H_1$. Since $(g, g_1), (h, h_1)$ are in $F_\star^*$, we get $g \bullet h \equiv^F g_1 \bullet h_1$. Hence $(y, y_1) \in F_\star^*$. $\square$

5 Conclusions

The aim of this paper is to study In [2] and [4] we study (semi)topological BL-algebras and metrizability on BL-algebras. We showed that continuity the operations $\odot$ and $\rightarrow$ imply continuity $\wedge$ and $\vee$. Also, we found some conditions under which a locally compact topological BL-algebra become metrizable. But in there we can not answer some questions, for example:
(i) Is there a topology $\mathcal{U}$ on BL-algera A such that $(A, \mathcal{U})$ be a (semi)topological BL-algebra?
(ii) Is there a topology $\mathcal{U}$ on a BL-algebra A such that $(A, \mathcal{U})$ be a compact connected topological BL-algebra?
(iii) Is there a topological BL-algebra $(A, \mathcal{U})$ such that T_0, T_1 and T_2 spaces be equivalent?
(iv) If $(A, \rightarrow, \mathcal{U})$ is a semitopological BL-algebra, is there a topology $\mathcal{V}$ coarsere than $\mathcal{U}$ or finer than $\mathcal{U}$ such that $(A, \mathcal{V})$ be a (semi)topological BL-algebra?

Now in this paper, we answered to some above questions and got some interesting results as mentioned in abstract.

References

[1] A. Arhangel'skii, M. Tkachenko, *Topological groups and related structures*, Atlantis press, 2008.

[2] R. A. Borzooei, G. R. Rezaei, N. Kouhestani, *On (semi)topological BL-algebra,* Iranian Journal of Mathematical Sciences and Informatics 6(1) (2011), 59-77.

[3] R. A. Borzooei, G. R. Rezaei, N. Kouhestani, *Metrizability on (semi)topological BL-algebra*, Soft Computing, 16 (2012), 1681-1690.

[4] R. A. Borzooei, G. R. Rezaei, N. Kouhestani, *Separation axioms in (semi)topological quotient BL-algebras,* Soft Computing, 16 (2012), 1219-1227.

[5] N. Bourbaki, *Elements of mathematics general topology*, Addison-Wesley Publishing Company, 1966.

[6] D. Busneag, D. Piciu, *Boolean BL-algebra of fractions*, Annals of University of Craiova, Math. Comp. Soi. Ser., 31 (2004), 1-19.

[7] L. C. Ciungu, *Convergences in Perfect BL-algebras*, Mathware Soft Computing, 14 (2007) 67-80.

[8] J. Chvalina, B. Smetana, *Algebraic Spaces and Set Decompositions*, Ratio Mathematica, 34 (2018), 67-76.

[9] A. Dinola, L. Leustean,*Compact Representations of BL-algebra*, Arch. Math. Logic, 42 (2003), 737-761.

[10] P. Fletcher, W.F.Lindgren, *Quasi-uniform Spaces*, Lecture Notes in Pure and Applied Mathematics, Marcel dekker, New York, 77, 1982.

[11] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer academic publishers, 1998.

[12] M. Haveshki, A. Borumand Saeid, E. Eslami, *Some Types of Filters in BL-algebras*, Soft computing, 10 (2006), 657-664.

[13] M. Haveshki, E. Eslami and A. Borumand Saeid, *A topology induced by uniformity on BL-algebras*, Mathematical logic Quarterly, 53(2) (2007), 162-169.

[14] S. Hoskova-Mayerova, *An Overview of Topological and Fuzzy Topological Hypergroupoids*, Ratio Mathematica, 33 (2017), 21-38.

[15] T. Husain, *Introduction to topological groups*, W. B. Sunders Company, 1966.

[16] H.P.A. Künzi, *Quasi-uniform spaces-eleven years later*, Top. Proc. 18, (1993), 143-171.

[17] H.P.A. Künzi, J.Marin, S.Romaguera, *Quasi-Uniformities on Topological semigroups and Bicompletion*, Semigroups Forum, 62 (2001), 403-422.

[18] H.P.A. Künzi, S. Romaguera, O. Sipacheva, *The Doitchinov completion of a regular paratopological group*, Serdica math. J., 24 (1998), 73-88.

[19] J. Marin, S. Romaguera, *Bicompleting the left quasi-uniformity of a paratopological group*, Archiv Math. (Basel), 70 (1998), 104-110.

[20] D. Marxen, *Uniform semigroups*, Math. Ann., 202 (1973), 27-36.

[21] J. Mi Ko, Y. C. Kim, *Closure operators on BL-algebras*, Commun. Korean Math. Soc., 19(2) (2004), 219-232.

[22] M.G. Murdeshwar, S.A. Naimpally, *Quasi-Uniform Topological Spaces*, 1966

[23] A. Rezaei, A. Borumand Saeid, R. A. Borzooei, *Relation between Hilbert Algebras and BE-Algebras*, Applications and Applied Mathematics, 8(2) (2013), 573 - 584.

[24] A. Weil, *sur les espaces a structure uniforme et sur la topologiebgeneral*, Gauthier-Villars, Paris, 1973.