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On multiplication Γ-modules

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Abstract

In this article, we study some properties of multiplication M_Γ-modules and their prime M_Γ-submodules. We verify the conditions of ACC and DCC on prime M_Γ-submodules of multiplication M_Γ-module.

Key words: Γ-ring, multiplication M_Γ-module, prime M_Γ-submodule, prime ideal.

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1 Introduction

The notion of a Γ-ring was first introduced by Nobusawa [17]. Barnes [5] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. After the Γ-ring was defined by Barnes and Nobusawa, a lot of researchers studied on the Γ-ring. Barnes [5], Kyuno [15] and Luh [16] studied the structure of Γ-rings and obtained various generalizations analogous of corresponding parts in ring theory. Recently, Dumitru, Ersoy, Hoque, *Öztürk*, Paul, Selvaraj, have studied on several aspects in gamma-rings (see [10, 8, 12, 14, 18, 19, 20]).

McCasland and Smith [14] showed that any Noetherian module M contains only finitely many minimal prime submodules. D. D. Anderson [2] generalized the well-known counterpart of this result for commutative rings, i.e., he abandoned the Noetherianness and showed that if every prime ideal minimal over an ideal I is finitely generated, then R contains only finitely many prime ideals minimal over I. Behboodi and Koohy [7] showed that this result of Anderson was true for any associative ring (not necessarily commutative) and also, they extended it to multiplication modules, i.e., if M is a multiplication module such that every prime submodule minimal over a submodule K is finitely generated, then M contains only finitely many prime submodules minimal over K.

In this paper, we study some properties of multiplication left M_Γ-modules and their prime M_Γ-submodules. This paper is organized as follows: In Section 2, we review some basic notions and properties of Γ-rings. In Section 3, the concept of a moltiplication M_Γ-module is introduced and its basic properties are discussed. Also, we show that If L is a left operator ring of the Γ-ring M and A is a multiplication unitary left M_Γ-module, then A is a multiplication left L-module. In Section 4, we proved that in fact this result was true for Γ-rings and M_Γ-modules.

2 Preliminaries

In this section we recall certain definitions needed for our purpose.

Recall that for additive abelian groups M and Γ we say that M is a Γ-ring if there exists a mapping

$$\begin{aligned} \cdot : M \times \Gamma \times M &\longrightarrow M \\ (m, \gamma, m') &\longrightarrow m\gamma m' \end{aligned}$$

such that for every $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following hold:

1. $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$ and $a\alpha(b+c) = a\alpha b + a\alpha c$;

2. $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Note that any ring R, can be regarded as an R-ring. A Γ-ring M is called commutative, if for any $x, y \in M$ and $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$. M is called a Γ-ring with unit, if there exists elements $1 \in M$ and $\gamma_0 \in \Gamma$ such that for any $m \in M$, $1\gamma_0 m = m = m\gamma_0 1$.

If A and B are subsets of a Γ-ring M and $\Theta \subseteq \Gamma$, we denote $A\Theta B$, the subset of M consisting of all finite sums of the form $\sum a_i\gamma_i b_i$, where $(a_i, \gamma_i, b_i) \in A \times \Theta \times B$. For singleton subsets we abbreviate this notation for example, $\{a\}\Theta B = a\Theta B$.

A subset I of a Γ-ring M is said to be a right ideal of R if I is an additive subgroup of M and $I\Gamma M \subseteq I$. A left ideal of M is defined in a similar way. If I is both a right and left ideal, we say that A is an ideal of M.

For each subset S of a Γ-ring M, the smallest right ideal containing S is called the right ideal generated by S and is denoted by $|S\rangle$. Similarly we define $\langle S|$ and $\langle S\rangle$, the left and two-sided (respectively) ideals generated by S. For each a of a Γ-ring M, the smallest right ideal containing a is called the principal right ideal generated by a and is denoted by $|a\rangle$. We similarly define $\langle a|$ and $\langle a\rangle$, the principal left and two-sided (respectively) ideals generated by a. We have $|a\rangle = Za + a\Gamma M$, $\langle a| = Za + M\Gamma a$, and $\langle a\rangle = Za + a\Gamma M + M\Gamma a + M\Gamma a\Gamma M$, where $Za = \{na : n \,\text{is an integer}\}$.

Let I be an ideal of Γ-ring M. If for each $a+I$, $b+I$ in the factor group M/I, and each $\gamma \in \Gamma$, we define $(a+I)\gamma(b+I) = a\gamma b + I$, then M/I is a Γ-ring which we shall call the difference Γ-ring of M with respect to I.

Let M be a Γ-ring and F the free abelian group generated by $\Gamma \times M$. Then $A = \{\sum_i n_i(\gamma_i, x_i) \in F : a \in M \Rightarrow \sum_l n_i a\gamma_i x_i = 0\}$ is a subgroup of F. Let $R = F/A$, the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. It can be verified easily that $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ and $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$ for all α, $\beta \in \Gamma$ and x, $y \in M$. We define a multiplication in R by $\sum_i[\alpha_i, x_i]\sum_J[\beta_j, y_j] = \sum_{iJ}[\alpha_i, x_i\beta_j y_j]$. Then R forms a ring. If we define a composition on $M \times R$ into M by $a\sum_l[\alpha_i, x_i] = \sum_i a\alpha_i x_i$ for $a \in M$, $\sum_i[\alpha_i, x_i] \in R$, then M is a right R-module, and we call R the right operator ring of the Γ -ring M. Similarly, we may construct a left operator ring L of M so that M is a left L-module. Clearly I is a right (left) ideal of M if and only if I is a right R-module (left L- module) of M. Also if A is a right (left) ideal of $R(L)$, then MA(AM) is an ideal of M. For subsets $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i[\gamma_i, x_i]$ in R, where $\gamma_i \in \Phi$, $x_i \in N$, and we denote by $[(\Phi, N)]$ the set of all elements $[\varphi, x]$ in R, where $\varphi \in \Phi$, $x \in N$. Thus, in particular, $R = [\Gamma, M]$.

An ideal P of M is prime if, for any ideals U and V of M, $U\Gamma U \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$. A subset S of M is an m-system in M if $S = \emptyset$ or if $a, b \in S$ implies $< a > \Gamma < b > \cap S \neq \emptyset$. The prime radical $\mathcal{P}(A)$ is the set of x in M such that every m-system containing x meets A. The prime radical of the zero ideal in a Γ-ring M is called the prime radical of the Γ-ring M which we denote by $\mathcal{P}(M)$.

An ideal Q of M is semi-prime if, for any ideals U of M, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$.

Proposition 2.1. *[15] If Q is an ideal in a commutative Γ-ring with unit M, then $P(Q)$ is the smallest semi-prime ideal in M which contains Q, i.e.*

$$\mathcal{P}(Q) = \bigcap P$$

where P runs over all the semi-prime ideals of M such that $Q \subseteq P$.

Let P be a proper ideal in a commutative Γ-ring with unit M. It is clear that the following conditions are equivallent.

1. P is semi-prime.

2. For any $a \in M$, if $a\gamma_0 a \in P$, then $a \in P$.

3. For any $a \in M$ and $n \in \mathbb{N}$, if $(a\gamma_0)^n a \in P$, then $a \in P$.

Proposition 2.2. *[13] Let Q be an ideal in a commutative Γ-ring with unit M and A be the set of all $x \in M$ such that $(x\gamma_0)^n x \in Q$ for some $n \in \mathbb{N} \cup \{0\}$, where $(x\gamma_0)^0 x = x$. Then $A = \mathcal{P}(Q)$.*

3 M_Γ-module

Let M be a Γ-ring. A left M_Γ-module is an additive abelian group A together with a mapping $\cdot : M \times \Gamma \times A \longrightarrow A$ (the image of (m, γ, a) being denoted by $m\gamma a$), such that for all $a, a_1, a_2 \in A$, $\gamma, \gamma_1, \gamma_2 \in \Gamma$, and $m, m_1, m_2 \in M$ the following hold:

1. $m\gamma(a_1 + a_2) = m\gamma a_1 + m\gamma a_2$;

2. $(m_1 + m_2)\gamma a = m_1\gamma m + m_2\gamma a$;

3. $m_1\gamma_1(m_2\gamma_2 a) = (m_1\gamma_1 m_2)\gamma_2 a$.

A right M_Γ-module is defined in analogous manner. If I is a left ideal of a Γ-ring M, then I is a left M_Γ-module with $r\gamma a$ ($r \in M$,$\gamma \in \Gamma$,$a \in I$) being the ordinary product in M. In particular, $\{0\}$ and M are M_Γ-modules.

Let A be a left M_Γ-module and B a nonempty subset of A. B is a M_Γ-submodule of A, which we denote by $B \leq A$, provided that B is an additive subgroup of A and $m\gamma b \in B$, for all $(m, \gamma, b) \in M \times \Gamma \times B$.

Definition 3.1. *Let A be a left M_Γ-module and X a subset of A. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be the family of all M_Γ-submodule of A which contain X. Then $\bigcap_{\lambda \in \Lambda} A_\lambda$ is called the M_Γ-submodule of A generated by the set X and denoted $\langle X|$.*

If $B \subseteq A$, $N \subseteq M$ and $\Theta \subseteq \Gamma$, we denote $N\Theta B$, the subset of A consisting of all finite sums of the form $\sum n_i\gamma_i b_i$ where $(n_i, \gamma_i, b_i) \in N \times \Theta \times B$. For singleton subsets we abbreviate this notation for example, $\{n\}\Theta B = n\Theta B$.

If $X = \{a_1, \ldots, a_n\}$, we write $\langle a_1, \ldots, a_n|$ in place of $\langle X|$. If $A = \langle a_1, \ldots, a_n|$, ($a_i \in A$), A is said to be finitely generated. If $a \in A$, the M_Γ-submodule $\langle a|$ of A is called the cyclic M_Γ-submodule generated by a. We have $\langle X| = ZX + M\Gamma X$, where $ZS = \{\sum_{i=1}^k n_i x_i : n_i \in Z, x_i \in S \text{ and } k \text{ is an integer}\}$.

Finally, if $\{B_\lambda\}_{\lambda \in \Lambda}$ is a family of M_Γ-submodules of A, then the M_Γ-submodule generated by $X = \bigcup_{\lambda \in \Lambda} B_\lambda$ is called the sum of the M_Γ-modules

B_λ and usually denoted $\langle X| = \sum_{\lambda\in\Lambda} B_\lambda$. If the index set Λ is finite, the sum of $B_1, \ldots, B_k$ is denoted $B_1 + B_2 + ... + B_k$. It is clear that if $\{B_\lambda\}_{\lambda\in\Lambda}$ is a family of M_Γ-submodules of A, then $\sum_{\lambda\in\Lambda} B_\lambda$ consists of all finite sums $b_{\lambda_1} + \ldots + b_{\lambda_k}$ with $b_{\lambda_j} \in B_{\lambda_l}$.

Proposition 3.1. *Let M be a Γ-ring and $\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of left ideals of M. If A is a left M_Γ-module, then*

$$(\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A = \sum_{\lambda\in\Lambda}(I_\lambda \Gamma A).$$

Proof. Let $x \in (\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A$. Then there exists $a_1, \ldots, a_k \in A$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$ and $x_1, \ldots, x_k \in \sum_{\lambda\in\Lambda} I_\lambda$ such that $x = \sum_{t=1}^{k} x_t\gamma_t a_t$, it follows that for $1 \leq t \leq k$, $x_t = \sum_{j=1}^{k_t} i_{\lambda_{jt}}$ with $i_{\lambda_{jt}} \in I_{\lambda_{jt}}$. Hence $x = \sum_{t=1}^{k}\sum_{j=1}^{k_t} i_{\lambda_{jt}}\gamma_t a_t \in \sum_{\lambda\in\Lambda}(I_\lambda\Gamma A)$. Therefore $(\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A \subseteq \sum_{\lambda\in\Lambda}(I_\lambda\Gamma A)$. Also, Since for every $\lambda \in \Lambda$, $I_\lambda\Gamma A \subseteq (\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A$, we conclude that $\sum_{\lambda\in\Lambda}(I_\lambda\Gamma A) \subseteq (\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A$. Hence $(\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A = \sum_{\lambda\in\Lambda}(I_\lambda\Gamma A)$. □

Definition 3.2. *If A is a left M_Γ-module and $\mathcal{S}$ is the set of all M_Γ-submodules B of A such that $B \neq A$, then $\mathcal{S}$ is partially ordered by set-theoretic inclusion. B is a maximal M_Γ-submodule if and only if B is a maximal element in the partially ordered set $\mathcal{S}$.*

Proposition 3.2. *If A is a non-zero finitely generated left M_Γ-module, then the following statements are hold.*

1. *If K is a proper M_Γ-submodule of A, then there exists a maximal M_Γ-submodule of A such that contain K.*
2. *A has a maximal M_Γ-submodule.*

Proof. (1) Let $A = \langle a_1, \ldots, a_n|$ and

$$\mathcal{S} = \{L : K \subseteq L \text{ and } L \text{ is a proper } M_\Gamma\text{-submodule of } A\}.$$

$\mathcal{S}$ is partially ordered by inclusion and note that $\mathcal{S} \neq \emptyset$, since $K \in \mathcal{S}$. If $\{L_\lambda\}_{\lambda\in\Lambda}$ is a chain in $\mathcal{S}$, then $L = \bigcup_{\lambda\in\Lambda} L_\lambda$ is a M_Γ-submodule of A. We show that $L \neq A$. If $L = A$, then for every $1 \leq i \leq n$, there exists $\lambda_i \in \Lambda$ such that $a_i \in L_{\lambda_i}$. Since $\{L_\lambda\}_{\lambda\in\Lambda}$ is a chain in $\mathcal{S}$, we conclude that there exists $1 \leq j \leq n$ such that $a_1, \ldots, a_n \in L_{\lambda_j}$. Therefore $A = L_{\lambda_j} \in \mathcal{S}$ which contradicts the fact that $A \notin \mathcal{S}$. It follows easily that L is an upper bound $\{L_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{S}$. By Zorn's Lemma there exists a proper M_Γ-submodule B of A that is maximal in $\mathcal{S}$. It is a clear that B a maximal M_Γ-submodule of A such that contain K.

(2) By part (1), it suffices we put $K = \langle 0|$. □

Definition 3.3. *A left M_Γ-module A is unitary if there exists an element, say 1 in M and an element $\gamma_0 \in \Gamma$, such that, $1\gamma_0 a = a$ and $1\gamma_0 m = m = m\gamma_0 1$ for every $(a, m) \in A \times M$.*

Corolary 3.1. *If M is a unitary left (right) M_Γ-module, then M has a left (right) maximal ideal.*

Proof. It is evident by Proposition 3.2. □

Let A be a left M_Γ-module. let $X \subseteq A$ and let $B \leq A$. Then the set $(B : X) := \{m \in M : m\Gamma X \subseteq B\}$ is a left ideal of M. In particular, if $a \in A$, then $(0 : a) := ((0) : \{a\})$ is called the left annihilator of a and $(0 : A) := ((0) : A)$ is an ideal of M called the annihilating ideal of A. Furthermore A is said to be faithful if and only if $(0 : A) = (0)$.

Definition 3.4. *A left M_Γ-module A is called a multiplication left M_Γ-module if each M_Γ-submodule of A is of the form $I\Gamma A$, where I is an ideal of M.*

Proposition 3.3. *Let B be a M_Γ-submodule of multiplication left M_Γ-module A. Then $B = (B : A)\Gamma A$.*

Proof. It is a clear that $(B : A)\Gamma A \subseteq B$. Since A is a multiplication left M_Γ-module, we conclude that there exists ideal I of Γ-ring M such that $B = I\Gamma A$, it follows that $B = I\Gamma A \subseteq (B : A)\Gamma A \subseteq B$. Therefore $B = (B : A)\Gamma A$. □

Proposition 3.4. *Let A be a left M_Γ-module. A is multiplication if and only if for every $a \in A$, there exists ideal I in M such that $\langle a| = I\Gamma A$.*

Proof. In view of Definition 3.4, it is enough to show that if for every $a \in A$, there exists ideal I in M such that $\langle a| = I\Gamma A$, then A is multiplication. Let B be an M_Γ-submodule of A. Then for every $b \in B$, there exists ideal I_b in M such that $\langle b| = I_b\Gamma A$. By Proposition 3.1, $(\sum_{b\in B} I_b)\Gamma A = \sum_{b\in B}(I_b\Gamma A) = \sum_{b\in B}\langle b| = B$, it follows that A is multiplication. □

Proposition 3.5. *Let M be a Γ-ring which has a unique maximal ideal Q and A be a unitary multiplication left M_Γ-module. If every ideal I in M is contained in Q, then for every $a \in A \setminus Q\Gamma A$, $\langle a| = A$.*

Proof. Suppose that $a \in A \setminus Q\Gamma A$. Since A is multiplication left M_Γ-module, we conclude that there exists ideal I in M such that $\langle a| = I\Gamma A$. Clearly $I \nsubseteq Q$ and hence $I = M$, which implies $\langle a| = M\Gamma A = A$. □

Corolary 3.2. *Let Γ-ring M be a unitary left M_Γ-module which has a unique maximal ideal Q and A be a unitary multiplication left M_Γ-module. Then for every $a \in A \setminus Q\Gamma A$, $\langle a| = A$.*

Proof. By Propositions 3.2 and 3.5, it is evident. □

Proposition 3.6. *Let L be a left operator ring of the Γ-ring M and let A be a unitary left M_Γ-module. If we define a composition on $L \times A$ into A by $(\sum_l [x_i, \alpha_i])a = \sum_i x_i\alpha_i a$ for $a \in A$, $\sum_i [x_i, \alpha_i] \in L$, then A is a left L-module. Also, for every $B \subseteq A$, B is a M_Γ-submodule of A if and only if B is a L-submodule of A.*

Proof. Suppose that $1 \in M$ and $\gamma_0 \in \Gamma$ such that for every $(a, m) \in A \times M$, $1\gamma_0 a = a$ and $1\gamma_0 m = m = m\gamma_0 1$. Let $\sum_{i=1}^t [x_i, \alpha_i] = \sum_{j=1}^s [y_j, \beta_j] \in L$ and $a = b \in A$. By definition of left operator ring of the Γ-ring M, we conclude that $\sum_{i=1}^t x_i\alpha_i 1 = \sum_{j=1}^s y_j\beta_j 1$, it follows that

$$\begin{array}{rcl} (\sum_{i=1}^t [x_i, \alpha_i])a & = & \sum_{i=1}^t x_i\alpha_i a \\ & = & \sum_{i=1}^t (x_i\alpha_i(1\gamma_0 a)) \\ & = & \sum_{i=1}^t (x_i\alpha_i 1)\gamma_0 a \\ & = & (\sum_{i=1}^t x_i\alpha_i 1)\gamma_0 a \\ & = & (\sum_{j=1}^s y_j\beta_j 1)\gamma_0 b \\ & = & \sum_{j=1}^s y_j\beta_j b \\ & = & (\sum_{j=1}^s [y_j, \beta_j])b \end{array}$$

Hence composition on $L \times A$ into A is a well-defined. Let $r = \sum_{i=1}^t [x_i, \alpha_i]$ and $s = \sum_{j=1}^s [y_j, \beta_j]$. Then for every $a \in A$,

$$\begin{array}{rcl} (rs)a & = & (\sum_{i,j} [x_i\alpha_i y_j, \beta_j])a \\ & = & \sum_{i,j} (x_i\alpha_i y_j)\beta_j a \\ & = & \sum_{i,j} x_i\alpha_i(y_j\beta_j a) \\ & = & \sum_{i=1}^t x_i\alpha_i(\sum_{j=1}^s y_j\beta_j a) \\ & = & (\sum_{i=1}^t [x_i, \alpha_i])(\sum_{j=1}^s y_j\beta_j a) \\ & = & r((\sum_{j=1}^s [y_j, \beta_j])a) \\ & = & r(sa) \end{array}$$

The remainder of the proof is now easy. □

Proposition 3.7. *Let L be a left operator ring of the Γ-ring M. If A is a multiplication unitary left M_Γ-module, then A is a multiplication left L-module.*

Proof. Let B be a L-submodule of A. By Proposition 3.6, B is a M_Γ-submodule of A and there exists ideal I of Γ-ring M such that $B = I\Gamma A$. It well known that $[\Gamma, I]$ is an ideal of L. We show that $B = [I, \Gamma]A$. Suppose that $a_1, \ldots, a_t \in A$, and for every $1 \leq i \leq t$, $\sum_{j=1}^{k_i} [x_{i_j}, \alpha_{i_j}] \in [I, \Gamma]$. Then we have $\sum_{i=1}^{t}(\sum_{j=1}^{k_i}[x_{i_j}, \alpha_{i_j}])a_i = \sum_{i=1}^{t}\sum_{j=1}^{k_i} x_{i_j}\alpha_{i_j}a_i) \in B$ and it follows that $[I, \Gamma]A \subseteq B$. Also, if $b \in B$, then there exists $x_1, \ldots, x_t \in I$,$\gamma_1, \ldots, \gamma_t \in \Gamma$, and $a_1, \ldots, a_t \in A$ such that $b = \sum_{i=1}^{t} x_i\gamma_i a_i = \sum_{i=1}^{t}[x_i, \gamma_i]a_i \in [I, \Gamma]A$ and we conclude that $B = [I, \Gamma]A$. □

Proposition 3.8. *Let A be a unitary cyclic left M_Γ-module. If L is a left operator ring of the Γ-ring M and for every $l, l' \in L$, there exists $l'' \in L$ such that $ll' = l''l$, then A is a multiplication left L-module.*

Proof. Let B be a L-submodule of A and $I = \{l \in L : lA \subseteq B\}$, then $IA \subseteq B$. Since A is a unitary cyclic left M_Γ-module, we conclude that there exists $a \in A$ such that $A = M\Gamma a$. Let $b \in B$. Hence there exists $m_1, \ldots, m_t \in M$ and $\gamma_1, \ldots, \gamma_t \in \Gamma$ such that $b = \sum_{i=1}^{t} m_i\gamma_i a$. In view of operations of addition and multiplication in left L-module A, we have $b = \sum_{i=1}^{t}[m_i, \gamma_i]a = (\sum_{i=1}^{t}[m_i, \gamma_i])a$. We put $l = \sum_{i=1}^{t}[m_i, \gamma_i]$ and it follows that $b = la$. If $a' \in A$, then a similar argument shows that there exists $l' \in L$ such that $a' = l'a$. By hypothesis, there exists $l'' \in L$ such that $ll' = l''l$. Therefore $la' = ll'a = l''la = l''b \in B$ and it follows that $l \in I$, this is $b = la \in IA$. Hence $B = IA$ and the proof is now complete. □

Definition 3.5. *Let A be a unitary left M_Γ-module and B be a M_Γ-submodule in A and $P \in Max(M)$. A is called P-cyclic if there exist $p \in P$ and $b \in B$ such that $(1-p)\gamma_0 B \subseteq M\Gamma b$ and also, it is clear that $(1-p)\gamma_0 B = (1-p)\Gamma B$. Define $T_P B$ as the set of all $b \in B$ such that $(1-p)\gamma_0 b = 0$, for some $p \in P$.*

Lemma 3.1. *Let A be a unitary left M_Γ-module and B be a M_Γ-submodule in A and $P \in Max(M)$. If M is a commutative Γ-ring, then $T_P B$ is a M_Γ-submodule in A.*

Proof. Suppose $b_1, b_2 \in T_P B$. So there exist $p_1, p_2 \in P$ such that $b_1 = p_1\gamma_0 b_1$ and $b_2 = p_2\gamma_0 b_2$. Let $p_0 = p_1 + p_2 - p_1\gamma_0 p_2$. It is clear that $(1-p_0)\gamma_0(b_1 - b_2) = 0$. Hence $b_1 - b_2 \in T_P B$. Let $x \in M\Gamma(T_P B)$. So $x = \sum_{i=1}^{n} m_i\gamma_i b_i$, where $n \in \mathbb{N}$, $b_i \in T_P B$, $\gamma_i \in \Gamma$ and $m_i \in M$ $(1 \leq i \leq n)$. Suppose $i \in \{1, \cdots, n\}$. Since $b_i \in T_P N$, there exists $p_i \in P$ such that $(1 - p_i)\gamma_0 m_i\gamma_i b_i = 0$. Hence $m_i\gamma_i b_i \in T_P N$. Thus $x \in T_P B$. Hence $M\Gamma T_P B = T_P B$. □

Proposition 3.9. *Let M be a commutative Γ-ring and let A be a unitary left M_Γ-module. A is multiplication M_Γ-module if and only if for any ideal $P \in Max(M)$, either $A = T_P A$ or A is P-cyclic.*

Proof. Let A be a multiplication ideal and $P \in Max(M)$. First suppose that $A = P\Gamma A$. Since A is multiplication ideal, we conclude that for every $a \in A$, there exists an ideal I in M such that $< a >= I\Gamma A$. Hence $< a >= P\Gamma <$

$a >$. So there exists $p \in P$ such that $(1-p)\gamma_0 a = 0$, it follows that $a \in T_P B$ and then $A = T_P A$.

Now suppose that $A \neq P\Gamma A$ and $x \in A \setminus P\Gamma A$. Then there exists an ideal I in M such that $< x >= I\Gamma A$ and $P + I = M$. Obviously, if we assume that $p \in P$, then $(1-p)\gamma_0 A \subseteq M\Gamma x$. Therefore A is P-cyclic.

Conversely, suppose that B is a M_Γ-submodule in A. Define I as the set of all $m \in M$, where $m\gamma_0 a \in B$ for any $a \in A$. Clearly I is an ideal in M and $I\Gamma A \subseteq B$. Let $b \in B$. Define K as the set of all $m \in M$, where $m\gamma_0 b \in I\Gamma A$. We claim $K = M$. Assume that $K \subset M$. Hence by Zorns Lemma there exists $Q \in Max(M)$ such that $K \subseteq Q \subset M$. By hypothesis $A = T_Q A$ or A is Q-cyclic. If $A = T_Q A$, then there exists $s \in Q$ such that $(1-s)\gamma_0 b = 0$. Hence $(1-s) \in K \subseteq Q$, it follows that $1 \in Q$, a contradiction. If A is Q-cyclic, then there exist $t \in Q$ and $c \in A$ such that $(1-t)\gamma_0 A \subseteq M\Gamma c =< c >$. Define L as the set of all $m \in M$ such that $m\gamma_0 c \in (1-t)\gamma_0 B$. Clearly L is an ideal in M and $L\gamma_0 c \subseteq (1-t)\gamma_0 B \subseteq< c >$. Hence $(1-t)\gamma_0 B \subseteq L\gamma_0 c$. So $(1-t)\gamma_0 B = L\gamma_0 c$, it follows that $(1-t)\gamma_0 L\gamma_0 A \subseteq (1-t)\gamma_0 B \subseteq B$ and $(1-t)\gamma_0 L \subseteq I$. Therefore $(1-t)\gamma_0(1-t)\gamma_0 B \subseteq I\Gamma A$. Hence $(1-t)\gamma_0(1-t) \in K \subseteq Q$. Thus $1-t \in Q$, it follows that $1 \in Q$, a contradiction. Hence $K = M$ and $b \in I\Gamma A$. Thus A is a multiplication ideal. □

Let A be a left M_Γ-module. A is said to have the intersection property provided that for every non-empty collection of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of M,

$$\bigcap_{\lambda\in\Lambda} I_\lambda \Gamma A = (\bigcap_{\lambda\in\Lambda} I_\lambda)\Gamma A.$$

If left M_Γ-module of A has intersection property, then for every non-empty collection of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of M,

$$\bigcap_{\lambda\in\Lambda} I_\lambda \Gamma A = (\bigcap_{\lambda\in\Lambda}(I_\lambda + Ann(A)))\Gamma A.$$

Proposition 3.10. *Let M be a commutative Γ-ring and let A be a unitary left M_Γ-module.*

1. *If A has intersection property and for any M_Γ-submodule N in A any ideal I in M which $N \subset I\Gamma A$, there exists ideal J in M such that $J \subset I$ and $N \subseteq J\Gamma A$, then A is multiplication left M_Γ-module.*

2. *If A is faithful left multiplication M_Γ-module, then A has intersection property and for any M_Γ-submodule N in A any ideal I in M which $N \subset I\Gamma A$, there exists ideal J in M such that $J \subset I$ and $N \subseteq J\Gamma A$.*

Proof. (1) Let N be a M_Γ-submodule in A and

$$\mathcal{S} = \{I : \ I \text{ is an ideal of } M \text{ and } N \subseteq I\Gamma A\}.$$

Clearly $M \in \mathcal{S}$. Since A has intersection property, we conclude from Zorns Lemma that $\mathcal{S}$ has a minimal member I (say). Since $N \subseteq I\Gamma A$ and I is minimal element of $\mathcal{S}$, we can conclude that $N = I\Gamma A$. It follows that A is a multiplication ideal.

(2) Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a nonempty collection of ideal in M and $I = \bigcap_{\lambda\in\Lambda} I_\lambda$. Clearly $I\Gamma A \subseteq \bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma A)$. Let $x \in \bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma A)$ and we put $L = \{m \in M : m\gamma_0 x \in I\Gamma A\}$. We claim $L = M$. Assume that $L \subset M$. By Proposition 3.2, there exists $P \in Max(M)$ such that $L \subseteq P$. It is clear that $x \notin T_P A$. Hence $T_P A \neq A$ and by Proposition 3.9, A is P-cyclic. Hence there exist $a \in A$ and $p \in P$ such that $(1-p)\gamma_0 A \subseteq M\Gamma a =< a >$. Thus $(1-p)\gamma_0 x \in \bigcap_{\lambda\in\Lambda}(I_\lambda\gamma_0 a)$ and so for any $\lambda \in \Lambda$, $(1-p)\gamma_0 x \in I_\lambda\gamma_0 a$. It is clear that $(1-p)\gamma_0(1-p) \in L \subseteq P$, in view of the fact that A is faithful. Hence $1 \in P$, a contradiction. Therefore $L = M$, it follows that $x = 1\gamma_0 x \in I\Gamma A$ and A has intersection property. Now suppose N be a M_Γ-submodule in A and I be an ideal in M which $N \subset I\Gamma A$. Since A is multiplication M_Γ-module, there exists an ideal J in M such that $N = J\Gamma A$. Let $K = I \cap J$. Clearly, $K \subset I$ and since A has intersection property, we conclude that $N \subseteq K\Gamma A$. The proof is now complete. □

Proposition 3.11. *Let A be a faithful multiplication M_Γ-module and I, J be two ideals in M. $I\Gamma A \subseteq J\Gamma A$ if and only if either $I \subseteq J$ or $A = [J : I]\Gamma A$.*

Proof. Let $I \not\subseteq J$. Note that $[J : I] = \bigcap_{i\in X}[J :< i >]$ where X is the set of all elements $i \in I$ with $i \notin J$. By Proposition 3.10,

$$[J : I]\Gamma A = \bigcap_{i\in X}([J :< i >]\Gamma A)$$

If for every $i \in X$, $A = [J :< i >]\Gamma A$, then $A = [J : I]\Gamma A$, which finishes the proof. Let $i \in X$ and $Q = [J :< i >]$. It is clear that $Q \neq M$. Let Ω denote the collection of all semi-prime ideals P in M containing Q. Suppose that there exists $P \in \Omega$ such that $A \neq P\Gamma A$ and $x \in A \setminus P\Gamma A$. Since A is a multiplication M_Γ-module, we conclude that there exists ideal D in M such that $< x >= D\Gamma A$ and $D \not\subseteq P$. Thus $c\Gamma A \subseteq < x >$ for some $c \in D \setminus P$. Now we have $c\Gamma a\Gamma A \subseteq J\Gamma < x >$. It is easily to show that for any $\gamma \in \Gamma$, there exists $\gamma_1 \in \Gamma$ and $b \in J$ such that $(c\gamma a - 1\gamma_1 b)\gamma_0 x = 0$, it follows that $(c\gamma a - 1\gamma_1 b)\Gamma c\Gamma A = 0$. Hence $c\gamma c \in [J :< i >] = Q$. Since P is a semi-prime ideal containing Q, we conclude that $c \in P$, a contradiction. Therefore for every $P \in \Omega$, $A = P\Gamma A$ and by Propositions 2.1 and 3.10,

$A = P(Q)\Gamma A$. Let $j \in A$. It is easily to show that $< j >= P(Q)\Gamma < j >$. Then there exists $s \in P(Q)$ such that for every $n \in \mathbb{N}$, $j = (s\gamma_0)^n j$. By Proposition 2.2, there exists $t \in \mathbb{N} \cup \{0\}$ such that $(s\gamma_0)^t s \in Q$, it follows that $j = (s\gamma_0)^t s\gamma_0 j \in Q\Gamma A$, i.e., $A \subseteq Q\Gamma A$. Hence $Q\Gamma A = A$. The converse is evident. □

4 Prime M_Γ-submodule

Through this section M and A will denote a commutative Γ-ring with unit and an unitary left M_Γ-module, respectively.

Definition 4.1. *A prime ideal P in M is called a minimal prime ideal of the ideal I if $I \subseteq P$ and there is no prime ideal P' such that $I \subseteq P' \subset P$. Let $Min(I)$ denote the set of minimal prime ideals of I in Γ-ring M, and every element of $Min((0))$ is called minimal prime ideal.*

Proposition 4.1. *If an ideal I of Γ-ring M is contained in a prime ideal P of M, then P contains a minimal prime ideal of I.*

Proof. Let

$$\mathcal{A} = \{Q : Q \text{ is prime ideal of } M \text{ and } I \subseteq Q \subseteq P\}.$$

By Zorn's Lemma, there is a prime ideal Q of R which is minimal member with respect to inclusion in $\mathcal{A}$. Therefore $Q \in Min(I)$ and $I \subseteq Q \subseteq P$. □

Lemma 4.1. *Let Γ be a finitely generated group. If I and J are finitely generated ideals of M, then $I\Gamma J$ is finitely generated ideal of M.*

Proof. Let $I = \langle a_1, \dots, a_n\rangle$, $J = \langle b_1, \dots, b_m\rangle$, and $\Gamma = \langle \gamma_1, \dots, \gamma_k\rangle$. It is clear that $I\Gamma J = \langle a_i\gamma_t b_j : 1 \leq i \leq n, 1 \leq t \leq k, 1 \leq j \leq m\rangle$. □

Proposition 4.2. *Let Γ be a finitely generated group. If I is a proper ideal of M and each minimal prime ideal of I is finitely generated, then $Min(I)$ is finite set.*

Proof. Consider the set

$$\mathcal{S} = \{P_1\Gamma P_2 ... P_n; n \in \mathbb{N} \text{ and } P_i \in Min(I), \text{for each } 1 \leq i \leq n\}$$

and set

$$\Delta = \{K; K \text{ is an ideal of } M \text{ and } Q \nsubseteq K, \text{ for each } Q \in \mathcal{S}\}$$

which is the non-empty set, since $I \in \Delta$. $(\Delta, \subseteq)$ is the partial ordered set. Suppose $\{K_\lambda\}_{\lambda\in\Lambda}$ is the chain of Δ in which $\Lambda \neq \emptyset$ and set $K = \bigcup_{\lambda\in\Lambda} K_\lambda$. It is clear that K is an ideal of M. Also, if there exits $Q \in \mathcal{S}$ such that $Q \subseteq K$, then by Lemma 4.1, $Q = P_1\Gamma P_2 ... P_n$ is finitely generated ideal of M, i.e., $Q = \langle x_1, \ldots, x_n\rangle$. But $Q \subseteq K$ implies that $x_1, x_2, ..., x_n \in K$. Thus there exists $\lambda \in \Lambda$ such that $x_1, x_2, ..., x_n \in K_\lambda$ and so $Q \subseteq K_\lambda$, contradiction. Hence, for each $Q \in \mathcal{S}$, $Q \nsubseteq K$ and $K \in \Delta$ is the upper band of this chain.

By Zorhn's lemma Δ has maximal element such as Q. Now if $a \notin Q$ and $b \notin Q$ for $a, b \in M$, then $Q \subseteq \langle Q \cup \{a\}\rangle$ and $Q \subseteq \langle Q \cup \{b\}\rangle$. Maximality of Q implies that $\langle Q \cup \{a\}\rangle$, $\langle Q \cup \{b\}\rangle \notin \Delta$. So there exists Q_1 and Q_2 in $\mathcal{S}$ such that $Q_1 \subseteq \langle Q \cup \{a\}\rangle$ and $Q_2 \subseteq \langle Q \cup \{b\}\rangle$. It is clear that $Q_1\Gamma Q_2 \subseteq Q$ which is contradiction, since $Q_1\Gamma Q_2 \in \mathcal{S}$. Therefore $\langle a\rangle\Gamma\langle b\rangle \nsubseteq Q$ and Q is a prime ideal of M contained I. By Proposition 4.1, there exists a minimal prime ideal $P \subseteq Q$. But $P \in \mathcal{S}$, contradictory with $Q \in \Delta$. Above contradicts show that there exists $Q' = P_1\Gamma P_2 ... P_m \in \mathcal{S}$ such that $Q' \subseteq I$.

Now for each $P \in Min(I)$ we have $Q' \subseteq I \subseteq P$ and $P_1\Gamma P_2 ... P_m \subseteq P$. It is clear that $P_j \subseteq P$ for some $1 \leq j \leq m$. Thus $P_j = P$, since P is minimal. Hence $Min(I) = \{P_1, P_2, ..., P_m\}$ is finite. □

Proposition 4.3. *For proper M_Γ-submodule B of A, the following statements equivalent:*

1. *For every M_Γ-submodule C of A, if $B \subset C$, then $(B : A) = (B : C)$.*
2. *For every $(m, a) \in M \times A$, if $m\Gamma a \subseteq B$, then $a \in B$ or $m \in (B : A)$.*

Proof. (1) $\Rightarrow$ (2) Let $(m, a) \in M \times A$ such that $m\Gamma a \subseteq B$ and $a \notin B$. It is clear that $B \subset B + M\Gamma a$. Since $m\Gamma(B + M\Gamma a) \subseteq m\Gamma B + m\Gamma(M\Gamma a) = m\Gamma B + M\Gamma(m\Gamma a) \subseteq B$, we conclude from statement (1) that $m \in (B : B + M\Gamma a) = (B : A)$ and the proof is now complete.

(2) $\Rightarrow$ (1) Let C be a M_Γ-submodule of A such that $B \subset C$. It is clear that $(B : A) \subseteq (B : C)$. Now, suppose that $m \in (B : C)$. Since $B \subset C$, we infer that there exists $a \in C \setminus B$ such that $m\Gamma a \subseteq B$. By statement (2), $m \in (B : A)$ and the proof is now complete.

□

Definition 4.2. *A proper M_Γ-submodule B of A is said to be prime if $m\Gamma a \subseteq B$ for $(m, a) \in M \times A$ implies that either $a \in B$ or $m \in (B : A)$.*

Proposition 4.4. *If B is a prime M_Γ-submodule of A, then $(B : A)$ is a prime ideal of Γ-ring M.*

Proof. Let $x, y \in M$ such that $\langle x\rangle\Gamma\langle y\rangle \subseteq (B : A)$ and $x \notin (B : A)$. Then there exists $\gamma \in \Gamma$ and $a \in A$ such that $x\gamma a \notin B$, and also, $y\Gamma(x\gamma a) = (y\Gamma x)\gamma a = (x\Gamma y)\gamma a \subseteq B$. Since B is a prime M_Γ-submodule of A and $x\gamma a \notin B$, we conclude that $y\Gamma A \subseteq B$, i. e., $y \in (B : A)$. The proof is now complete. □

Proposition 4.5. *Let A be a multiplication left M_Γ-module, and B, B_1, ..., B_k be M_Γ-submodules of A. If B is a prime M_Γ-submodule of A, then the following statements are equivalent.*

1. *$B_j \subseteq B$ for some $1 \leq j \leq k$.*
2. *$\bigcap_{i=1}^k B_i \subseteq B$.*

Proof. (1) ⇒ (2) It is clear.

(2) ⇒ (1) We have $B_i = I_i\Gamma A$ for some ideals I_i, $(1 \leq i \leq k)$ of Γ-ring M. Then $(\bigcap_{i=1}^k I_i)\Gamma A \subseteq \bigcap_{i=1}^k (I_i\Gamma A) = \bigcap_{i=1}^k B_i \subseteq B$ and so $\bigcap_{i=1}^k I_i \subseteq (B : A)$. Since M is a commutative Γ-ring, we infer that for every permutations θ of $\{1, 2, \ldots, k\}$, $I_1\Gamma I_2 \cdots I_k = I_{\theta(1)}\Gamma I_{\theta(2)} \cdots I_{\theta(k)}$, it follows that $I_1\Gamma I_2 \cdots I_k \subseteq \bigcap_{i=1}^k I_i \subseteq (B : A)$. Since by Proposition 4.4, $(B : A)$ is prime ideal of Γ-ring M, we conclude that $I_j \subseteq (B : A)$ for some $1 \leq j \leq k$. Therefore, by Proposition 3.3, $B_j = I_j\Gamma A \subseteq B$ for some $1 \leq j \leq k$. □

Proposition 4.6. *If A is a multiplication left M_Γ-module, then for M_Γ-submodule B of A, the following statements are equivalent.*

1. *B is prime M_Γ-submodule of A.*
2. *$(B : A)$ is prime ideal of Γ-ring M.*
3. *There exists prime ideal P of Γ-ring M such that $B = P\Gamma A$ and for every ideal I of M, $I\Gamma A \subseteq B$ implies that $I \subseteq P$.*

Proof. (1) ⇒ (2) By Proposition 4.4, It is evident.

(2) ⇒ (3) We put

$$\mathcal{M} = \{P : B = P\Gamma A \text{ and } P \text{ is an ideal of } \Gamma\text{-ring } M\ \}$$

Since A is multiplication left M_Γ-module, we conclude that $(\mathcal{M}, \subseteq)$ is a non-empty partial order set. Let $\{P_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{M}$ be a chain. By Proposition 3.10, $\bigcap_{\lambda\in\Lambda} P_\lambda \in \mathcal{M}$ is an upper bound of $\{P_\lambda\}_{\lambda\in\Lambda}$. By Zorn's Lemma $\mathcal{M}$ has a maximal element. Thus, we can pick a P to be maximal element of $\mathcal{M}$. Let $x, y \in M$ and $\langle x\rangle\Gamma\langle y\rangle \subseteq P$. Hence $(\langle x\rangle\Gamma\langle y\rangle)\Gamma A \subseteq P\Gamma A = B$ and we infer that $\langle x\rangle\Gamma\langle y\rangle \subseteq (B : A)$. Now, by statement (2), $x \in (B : A)$ or $y \in (B : A)$. Since A is multiplication left M_Γ-module, we conclude from the Proposition

3.3 that $B = (B : A)\Gamma A$, it follows that $(B : A) \in \mathcal{M}$. On the other hand, clearly $P \subseteq (B : A)$ and so $P = (B : A)$, i.e., $x \in P$ or $y \in P$, Thus P is prime ideal of Γ-ring M.

(3) $\Rightarrow$ (1) Let prime ideal P of Γ-ring M such that $B = P\Gamma A$ and for every ideal I of Γ-ring M, $I\Gamma A \subseteq B$ implies that $I \subseteq P$. It is clear that $P = (B : A)$. Let $m \in M$ and $a \in A$ such that $m\Gamma a \subseteq B$. Since A is a multiplication S-act, we conclude that there exists an ideal I of Γ-ring M such that $\langle a\rangle = I\Gamma A$, it follows that $(m\Gamma I)\Gamma A = m\Gamma(I\Gamma A) = m\Gamma(M\Gamma a) = (m\Gamma M)\Gamma a = (M\Gamma m)\Gamma a = M\Gamma(m\Gamma a) \subseteq B$. Therefore $m\Gamma I \subseteq (B : A) = P$ and it is easy to see directly that $\langle m\rangle\Gamma I \subseteq (B : A)$. Then $m\Gamma A \subseteq B$ or $a \in I\Gamma A \subseteq B$ and the proof is now complete. □

Lemma 4.2. *Let A be a finitely generated left M_Γ-module. If I is an ideal of M such that $A = I\Gamma A$, then there exists $i \in I$ such that $(1 - i)\gamma_0 A = 0$.*

Proof. If $A =< a_1, \ldots, a_n >$, then for every $1 \le i \le n$, there exists $y_{i1}, \ldots, y_{in} \in I$ such that $a_i = \sum_{j=1}^{n} y_{ij}\gamma_0 a_j$, it follows that

$$-y_{i1}\gamma_0 a_1 - \cdots - y_{i(i-1)}\gamma_0 a_{i-1} + (1 - y_{ii})\gamma_0 a_i - y_{i(i+1)}\gamma_0 a_{i+1} - \cdots - y_{in}\gamma_0 a_n = 0.$$

If

$$B = \begin{bmatrix} 1 - y_{11} & -y_{12} & \cdots & -y_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ -y_{n1} & -y_{n2} & \cdots & 1 - y_{nn} \end{bmatrix},$$

then there exists $y \in I$ such that $det_\Gamma(B) = (1 + y)$, where

$$det_\Gamma(B) = \sum sign(\sigma) b_{1,\sigma(1)} \gamma_0 b_{2,\sigma(2)} \gamma_0 \cdots \gamma_0 b_{n,\sigma(n)}$$

and σ runs over all the permutation on $\{1, 2, \ldots, n\}$ (see [13]). Since for every $1 \le i \le n$, $det_\Gamma(B)\gamma_0 a_i = 0$, we conclude that $(1 + y)\gamma_0 A = 0$ and by setting $i = -y$ the proof will be completed. □

Proposition 4.7. *Let A be a finitely generated faitfull multiplication left M_Γ-module. For proper ideal of P in M, the following statements are equivalent.*

1. *P is a prime ideal of M.*
2. *$P\Gamma A$ is a prime M_Γ-submodule of A.*

Proof. (1) $\Rightarrow$ (2) Let I be an ideal of M such that $I\Gamma A \subseteq P\Gamma A$. Then by Proposition 3.11, either $I \subseteq P$ or $A = [P : I]\Gamma A$. If $A = [P : I]\Gamma A$, then by Lemma 4.2, there exists $i \in [P : I]$ such that $(1 - i)\gamma_0 A = 0$. Since A is a faitfull M_Γ-module, we conclude that $i = 1$ and $I \subseteq P$. Hence by Proposition 4.6, $P\Gamma A$ is a prime M_Γ-submodule of A.

(2) $\Rightarrow$ (1) Since A is a faitfull M_Γ-module and $[P\Gamma A : A]\Gamma A \subseteq P\Gamma A$, we conclude from the Proposition 3.11 and Lemma 4.2 that $[P\Gamma A : A] \subseteq P$. Hence $[P\Gamma A : A] = P$ and by Proposition 4.6, P is a prime ideal of M. □

Proposition 4.8. *Let A be a multiplication left M_Γ-module. Then*

1. *If M satisfies ACC (DCC) on prime ideals, then A satisfies ACC (DCC) on prime M_Γ-submodules.*

2. *If A is faitfull M_Γ-module and $(B : A)$ is a minimal prime ideal in M, then B is a minimal prime M_Γ-submodule of A.*

Proof. (1) Assume that $B_1 \subseteq B_2 \subseteq \ldots$ is a chain of prime M_Γ-submodule of A. By Proposition 4.4, $(B_1 : A) \subseteq (B_2 : A) \subseteq \ldots$ is a chain of prime ideal of Γ-ring M. By hypothesis there exists $k \in \mathbb{N}$ such that for every $i \geq k$, $(B_i : A) = (B_k : A)$. It follows from Proposition 3.3 that $B_i = (B_i : A)\Gamma A = (B_k : A)\Gamma A = B_k$. Thus A satisfies ACC on prime M_Γ-submodules.

(2) assume that B' is a prime M_Γ-submodule of A such that $B' \subseteq B$. By Proposition 4.6, $(B' : A) \subseteq (B : A)$ is a chain of prime ideal of Γ-ring M. By hypothesis $(B' : A) = (B : A)$, it follows from Proposition 3.3 that $B' = (B' : A)\Gamma A = (B : A)\Gamma A = B$. Thus B is a minimal prime M_Γ-submodule of A. □

Proposition 4.9. *Let A be a finitely generated faitfull multiplication left M_Γ-module. Then*

1. *If A satisfies ACC (DCC) on prime M_Γ-submodules, then Γ-ring M satisfies ACC (DCC) on prime ideals.*

2. *If B is a minimal prime M_Γ-submodule of A, then $(B : A)$ is a minimal prime ideal of Γ-ring M.*

Proof. (1) Assume that $P_1 \subseteq P_2 \subseteq \ldots$ is a chain of prime ideals of Γ-ring M. By Proposition 4.7, $P_1\Gamma A \subseteq P_2\Gamma A \subseteq \ldots$ is a chain of prime M_Γ-submodule of A. By hypothesis there exists $k \in \mathbb{N}$ such that for every $i \geq k$, $P_k\Gamma A = P_i\Gamma A$. Since A is a finitely generated faitfull multiplication M_Γ-module, we conclude from the Proposition 3.11 and Lemma 4.2 that $P_k = P_i$.

(2) By Proposition 4.6, $(B : A)$ is a prime ideal of Γ-ring M. Assume that P is a prime ideal of Γ-ring M such that $P \subseteq (B : A)$. Hence by Proposition 3.3, $P\Gamma A \subseteq (B : A)\Gamma A = B$. Since by Proposition 4.7, $P\Gamma A$ is a prime M_Γ-submodule of A, we conclude from our hypothesis that $P\Gamma A = (B : A)\Gamma A$.

Since A is a finitely generated faitfull multiplication M_Γ-module, we conclude from the Proposition 3.11 and Lemma 4.2 that $P = (B : A)$. The proof is now complete. □

Proposition 4.10. *Let Γ be a finitely generated group. Let A be a finitely generated faitfull multiplication left M_Γ-module.*

1. *If every prime ideal of Γ-ring M is finitely generated, then A contains only a finitely many minimal prime M_Γ-submodule.*

2. *If every minimal prime M_Γ-submodule of A is finitely generated, then Γ-ring M contains only a finite number of minimal prime ideal.*

Proof. (1) Assume that $\{B_\lambda\}_{\lambda\in\Lambda}$ is the family of minimal prime M_Γ-submodules of A. Set $I_\lambda = (B_\lambda : A)$ for $\lambda \in \Lambda$. By Proposition 4.9, each I_λ is a minimal prime ideal of Γ-ring M. On the other hand, by Proposition 4.2, M contains only a finite number of minimal prime ideal as $\{I_1, I_2, \dots I_n\}$. Now suppose that $\lambda \in \Lambda$. So $I_\lambda = I_i$, for some $1 \leq i \leq n$ and by Proposition 3.3, $B_\lambda = I_\lambda \Gamma A = I_i \Gamma A$. Thus $\{I_1\Gamma A, I_2\Gamma A, \dots, I_n\Gamma A\}$ is the finite family of minimal prime M_Γ-submodule of A.

(2) Suppose that I and J are two distinct minimal prime ideal of Γ-ring M. By Proposition 3.11 and Lemma 4.2, $A \neq I\Gamma A \neq J\Gamma A$ and also, by Proposition 4.7, $I\Gamma A$ and $J\Gamma A$ are prime M_Γ-submodules of A. Assume that B_1 and B_2 are two prime M_Γ-submodules of A such that $B_1 \subseteq I\Gamma A$ and $B_2 \subseteq J\Gamma A$. By Proposition 3.3, $B_1 = (B_1 : A)\Gamma A$ and $B_2 = (B_2 : A)\Gamma A$. By Proposition 3.11 and Lemma 4.2, $(B_1 : A) \subseteq I$ and $(B_2 : A) \subseteq J$. Since I and J are two distinct minimal prime ideal of Γ-ring M, we conclude from the Proposition 4.4 that $(B_1 : A) = I$ and $(B_2 : A) = J$. This says that $I\Gamma A$ and $J\Gamma A$ are two distinct minimal prime M_Γ-submodules of A. Now if Γ-ring M contains infinite many minimal prime ideals, then A must have infinitely many minimal prime M_Γ-submodules which is contradiction. □

References

[1] J. Aliro and P. Penea, *A note on prime module*, Divulgaciones Matematicas **8** (2000), 31-42.

[2] D. D. Anderson, *A note on minimal prime ideals*, Proc. Amer. Math. Soc., **122** (1994), 13-14.

[3] A. Azizi, *Radical formula and prime submodules*, J. Algebra **307** (2007), No. 1, 454-460.

[4] A. Barnard, *Multiplication Modules*, J. Algebra **71** (1981), 174-178.

[5] W. E. Barnes, *On the* Γ*-ring of Nobusawa*, Pacific J. Math. **18** (1966), 411-422.

[6] M. Behboodi, O. A. Karamzadeh and H. Koohy, *Modules whose certain submodules are prime*, Vietnam Journal of Mathematics, **32** (2004), 303-317.

[7] M. Behboodi and H. Koohy, *On minimal prime submodules*, Far East J. Math. Sci., **6** (2002), 83-88.

[8] Sahin Ceran and Mustafa Asci, *Symmetric bi-*(σ, τ) *derivations of prime and semi prime gamma rings*, Bull. Korean Math. Soc. Vol. **43** (2006), No. 1, 9-16.

[9] J. Dauns, *Prime modules*, J. Reine Angew. Math., **298** (1978), 156-181.

[10] M. Dumitru, *Gamma-Rings: Some Interpretations Used In The Study Of Their Radicals*, U.P.B. Sci. Bull., Series A, Vol. **71** (2009), Iss. 3.

[11] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), 755-779.

[12] B. A. Ersoy, *Fuzzy semiprime ideals in Gamma-rings*, International Journal of Physical Sciences Vol. **5(4)** (2010), 308-312.

[13] A. A. Estaji, A. Saghafi Khorasani and S. Baghdari, *Multiplication ideals in* Γ*-rings*, Journal of Hyperstructures **2 (1)** (2013), 30-39.

[14] M. F. Hoque and A. C. Paul, *On Centralizers of Semiprime Gamma Rings*, International Mathematical Forum, Vol. **6** (2011), no. 13, 627 - 638.

[15] S. Kyuno, *On prime gamma ring*, Pacific J. Math. **75** (1978), 185-190.

[16] L. Luh, *On the theory of simple* Γ*-rings*, Michigan Math. J. **16** (1969), 65-75.

[17] N. Nobusawa, *On a generalization of the ring theory*, Osaka J. Math. **1** (1964), 81-89.

[18] M. A. Öztürk and H. Yazarl, *Modules over the generalized centroid of semi-prime Gamma rings*, Bull. Korean Math. Soc. **44** (2007), No. 2, 203-213.

[19] A.C. Paul and Md. Sabur Uddin, *Lie Structure in Simple Gamma Rings*, Int. J. Pure Appl. Sci. Technol., **4(2)** (2010), 63-70.

[20] C. Selvaraj and S. Petchimuthu, *Strongly Prime Gamma Rings and Morita Equivalence Of Rings*, Southeast Asian Bulletin of Mathematics **32** (2008), 1137-1147.