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Helix-Hopes on Finite Hyperfields

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Abstract

Hyperstructure theory can overcome restrictions which ordinary algebraic structures have. A hyperproduct on non-square ordinary matrices can be defined by using the so called helix-hyperoperations. We study the helix-hyperstructures on the representations using ordinary fields. The related theory can be faced by defining the hyperproduct on the set of non square matrices. The main tools of the Hyperstructure Theory are the fundamental relations which connect the largest class of hyperstructures, the H_v-structures, with the corresponding classical ones. We focus on finite dimensional helix-hyperstructures and on small H_v-fields, as well.

Keywords: hyperstructures, H_v-structures, h/v-structures, hope.

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1 Introduction

We deal with the largest class of hyperstructures called *H_v-structures* introduced in 1990 [10], [11], which satisfy the *weak axioms* where the non-empty intersection replaces the equality.

Definitions 1.1 In a set H equipped with a ***hyperoperation*** (which we abbreviate it by ***hope***)

$$\cdot : H\times H\rightarrow P(H)-\{\varnothing\}: (x,y)\rightarrow x\cdot y\subset H$$

we abbreviate by

WASS the *weak associativity*: $(xy)z\cap x(yz)\neq\varnothing,\ \forall x,y,z\in H$ and by

COW the *weak commutativity*: $xy\cap yx\neq\varnothing,\ \forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called H_v-semigroup if it is WASS and is called ***H_v-group*** if it is reproductive H_v-semigroup: $xH=Hx=H,\ \forall x\in H$.

$(R,+,\cdot)$ is called ***H_v-ring*** if (+) and ($\cdot$) are WASS, the reproduction axiom is valid for (+) and ($\cdot$) is weak distributive with respect to (+):

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\ \forall x,y,z\in R.$$

For more definitions, results and applications on H_v-structures, see books and the survey papers as [2], [3], [11], [1], [6], [15], [16], [20]. An extreme class is the following: An H_v-structure is *very thin* iff all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Thus, in a very thin H_v-structure in a set H there exists a hope ($\cdot$) and a pair $(a,b)\in H^2$ for which $ab=A$, with $cardA>1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

The fundamental relations β^* and γ^* are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [9], [10], [11], [12], [13]. The main theorem is the following:

Theorem 1.2 Let $(H,\cdot)$ be an H_v-group and let us denote by U the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then the fundamental relation β^* is the transitive closure of the relation β.

An element is called *single* if its fundamental class is a singleton.

Motivation for H_v-structures:
The quotient of a group with respect to an invariant subgroup is a group.
Marty states that, the quotient of a group by any subgroup is a hypergroup.

Now, the quotient of a group with respect to any partition is an H_v-group.

Definition 1.3 Let $(H,\cdot)$, $(H,\otimes)$ be H_v-semigroups defined on the same H. $(\cdot)$ is *smaller* than $(\otimes)$, and $(\otimes)$ *greater* than $(\cdot)$, iff there exists automorphism

$$f\in Aut(H,\otimes) \text{ such that } xy\subset f(x\otimes y),\ \forall x\in H.$$

Then $(H,\otimes)$ *contains* $(H,\cdot)$ and write $\cdot\leq\otimes$. If $(H,\cdot)$ is structure, then it is *basic* and $(H,\otimes)$ is an *H_b-structure*.

The Little Theorem [11]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

Fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ^*:

Definition 1.4 [10], [11]. The H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if the quotient R/γ^* is a field.

Let ω^* be the kernel of the canonical map from R to R/γ^*; then we call *reproductive H_v-field* any H_v-field $(R,+,\cdot)$ if

$$x(R-\omega^*)=(R-\omega^*)x=R-\omega^*,\forall x\in R-\omega^*.$$

From this definition, a new class is introduced [15]:

Definition 1.5 The H_v-semigroup $(H,\cdot)$ is ***h/v-group*** if the H/β^* is a group.

Similarly *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces*, are defined. The h/v-group is a generalization of the H_v-group since the reproductivity is not necessarily valid. Sometimes a kind of *reproductivity of classes* is valid, i.e. if H is partitioned into equivalence classes $\sigma(x)$, then the quotient is reproductive $x\sigma(y)=\sigma(xy)=\sigma(x)y$, $\forall x\in H$.

An H_v-group is called *cyclic* [11], if there is element, called *generator*, which the powers have union the underline set, the minimal power with this property is the *period* of the generator. If there exists an element and a special power, the minimum, is the underline set, then the H_v-group is called *single-power cyclic*.

Definitions 1.6 [11], [14]. Let $(R,+,\cdot)$ be an H_v-ring, $(M,+)$ be COW H_v-group and there exists an external hope $\cdot: R\times M\rightarrow P(M):(a,x)\rightarrow ax$, such that, $\forall a,b\in R$ and $\forall x,y\in M$ we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\quad (a+b)x\cap(ax+bx)\neq\varnothing,\quad (ab)x\cap a(bx)\neq\varnothing,$$

then M is called an ***H_v-module*** over R. In the case of an H_v-field F instead of H_v-ring R, then the ***H_v-vector space*** is defined.

Definition 1.7 [17]. Let $(L,+)$ be H_v-vector space on $(F,+,\cdot)$, $\varphi:F\rightarrow F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map φ': $L\to L/\varepsilon^*$ and denote again 0 the zero of L/ε^*. Consider the *bracket* (commutator) hope:

$$[\ ,\ ] : L\times L\to P(L): (x,y)\to[x,y]$$

then L is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear:

$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$

$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing$, $\forall x,x_1,x_2,y,y_1,y_2\in L$ and $\lambda_1,\lambda_2\in F$

(L2) $[x,x]\cap\omega_L\neq\varnothing$, $\forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing$, $\forall x,y\in L$

Two well known and large classes of hopes are given as follows [11], [16]:

Definitions 1.8 Let $(G,\cdot)$ be a groupoid, then for every subset $P\subset G$, $P\neq\varnothing$, we define the following hopes, called ***P-hopes***: $\forall x,y\in G$

$$\underline{P}: x\underline{P}y = (xP)y\cup x(Py),$$

$$\underline{P}_r: x\underline{P}_ry= (xy)P\cup x(yP), \quad \underline{P}_l: x\underline{P}_ly= (Px)y\cup P(xy).$$

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called ***P-hyperstructures***.

The usual case is for semigroup $(G,\cdot)$, then

$$x\underline{P}y=(xP)y\cup x(Py)=xPy$$

and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes: Let $(G,\cdot)$ be abelian group and P a subset of G with more than one elements. We define the hope $\times_P$ as follows:

$$x\times_P y = \begin{cases} x\cdot P\cdot y = \{x\cdot h\cdot y \mid h\in P\} & \text{if } x\neq e \text{ and } y\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this hope, *P_e-hope*. The hyperstructure $(G,\times_P)$ is an abelian H_v-group.

Definition 1.9 Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\to G$ be a map. We define a hope (∂), called *theta-hope*, we write *∂-hope*, on G as follows

$$x\partial y = \{f(x)\cdot y, x\cdot f(y)\} \text{ (resp. } x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)) \text{), } \forall x,y\in G.$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is *COW*, then ∂ is *COW*.

If $(G,\cdot)$ is groupoid (or hypergroupoid) and $f:G\rightarrow P(G)-\{\varnothing\}$ multivalued map. We define the ∂-hope on G as follows: $x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)),\ \forall x,y\in G$.

Motivation for the ∂-hope is the map *derivative* where only the product of functions can be used.

Basic property: if $(G,\cdot)$ is semigroup then $\forall f$, the ∂-hope is *WASS*.

2 Some Applications of H_v-Structures

Last decades H_v-structures have applications in other branches of mathematics and in other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too [2], [3], [7], [18].

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed correspond into the hyperstructures introduced by Santilli & Vougiouklis in 1999 [7] and they are called *e-hyperfields*. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology.

Definition 2.1 A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

Definition 2.2 A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called *e-hyperfield* if the following axioms are valid: $(F,+)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0\cdot x=x\cdot 0=0,\ \forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x,\ \forall x\in F$, and $\forall x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. If the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Definition 2.3 *The Main e-Construction.* Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:

$$x\otimes y = \{xy, g_1, g_2,\ldots\},\ \forall x,y\in G-\{e\},\ \text{where } g_1, g_2,\ldots\in G-\{e\}$$

g_1, g_2,… are not necessarily the same for each pair (x,y). (G,⊗) is an H_v-group, it is an H_b-group which contains the (G,·). (G,⊗) is an e-hypergroup. Moreover, if for each x,y such that xy=e, so we have x⊗y=xy, then (G,⊗) becomes a strong e-hypergroup.

The main e-construction gives an extremely large number of e-hopes.

Example 2.4 Consider the quaternion group ***Q***={1,-1, i,-i, j,-j, k,-k} with defining relations $i^2 = j^2 = -1$, ij = -ji = k. Denoting $\underline{i}$={i,-i}, $\underline{j}$={j,-j}, $\underline{k}$={k,-k} we may define a very large number (∗) hopes by enlarging only few products. For example, (-1)∗k=$\underline{k}$, k∗i=$\underline{j}$ and i∗j=$\underline{k}$. Then the hyperstructure (Q,∗) is a strong e-hypergroup.

Mathematicalisation of a problem could make its results recognizable and comparable. This is because representing a research object or a phenomenon with numbers, figures or graphs might be simplest and in a recognizable way of reading the results. In questionnaires Vougiouklis & Vougiouklis proposed the substitution of Likert scales with the *bar* [5], [18].This substitution makes things simpler and easier for both the subjects of an empirical research and the researcher, either at the stage of designing or that of results processing, because it is really flexible. Moreover, the application of *the bar* opens a window towards the use of fuzzy sets in the whole procedure of empirical research, activating in this way more recent findings from different sciences, as well. The bar is closelly related with hyperstructure and fuzzy theories, as well.

More specifically, the following was proposed:

In every question, substitute the Likert scale with the 'bar' whose poles are defined with '0' on the left and '1' on the right:

0 ———————————————— 1

The subjects/participants are asked, instead of deciding and checking a specific grade on the scale, to cut the bar at any point they feel best expresses their answer to the specific question.

The suggested length of the bar is approximately 6.18cm, or 6.2cm, following the golden ration on the well known length of 10cm.

3 Small H_v-Numbers. H_v-Matrix Representations

In representations important role are playing the small hypernumbers.

Construction 3.1 On the ring ($\boldsymbol{Z}_4$,+,·) we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are,

(a) very thin minimal,

(b) COW (non-commutative),
(c) they have the elements 0 and 1, scalars.

Then, we have only the following isomorphic cases $2\otimes3=\{0,2\}$ or $3\otimes2=\{0,2\}$. Fundamental classes: $[0]=\{0,2\}$, $[1]=\{1,3\}$ and we have $(\boldsymbol{Z}_4,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$.

Thus it is isomorphic to $(\boldsymbol{Z}_2\times\boldsymbol{Z}_2,+)$. In this H_v-group there is only one unit and every element has a unique double inverse. Only f has one more right inverse element, the d, since $f\otimes d=\{I,b\}$. Moreover, the $(\boldsymbol{X},\otimes)$ is not cyclic.

Construction 3.2 On $(\boldsymbol{Z}_6,+,\cdot)$ we define, up to isomorphism, all multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are:

(a) very thin minimal
(b) COW (non-commutative)
(c) they have the elements 0 and 1, scalars

Then we have the following cases, by giving the only one hyperproduct,

(i) $2\otimes3=\{0,3\}$ or $2\otimes4=\{2,5\}$ or $2\otimes5=\{1,4\}$

$3\otimes4=\{0,3\}$ or $3\otimes5=\{0,3\}$ or $4\otimes5=\{2,5\}$

In all 6 cases the fundamental classes are $[0]=\{0,3\}$, $[1]=\{1,4\}$, $[2]=\{2,5\}$ and we have $(\boldsymbol{Z}_6,+,\otimes)/\gamma^* \cong (\boldsymbol{Z}_3,+,\cdot)$.

(ii) $2\otimes3=\{0,2\}$ or $2\otimes3=\{0,4\}$ or $2\otimes4=\{0,2\}$ or $2\otimes4=\{2,4\}$ or

$2\otimes5=\{0,4\}$ or $2\otimes5=\{2,4\}$ or $3\otimes4=\{0,2\}$ or $3\otimes4=\{0,4\}$ or

$3\otimes5=\{1,3\}$ or $3\otimes5=\{3,5\}$ or $4\otimes5=\{0,2\}$ or $4\otimes5=\{2,4\}$.

In all 12 cases the fundamental classes are $[0]=\{0,2,4\}$, $[1]=\{1,3,5\}$ and we have $(\boldsymbol{Z}_6,+,\otimes)/\gamma^* \cong (\boldsymbol{Z}_2,+,\cdot)$.

Remark that if we need h/v-fields where the elements have at most one inverse element, then we must exclude the case of $2\otimes5=\{1,4\}$ from (i), and the case $3\otimes5=\{1,3\}$ from (ii).

H_v-structures are used in Representation Theory of H_v-groups which can be achieved by generalized permutations or by H_v-matrices [11], [12], [13], [14].

***H_v-matrix** (or **h/v-matrix**)* is a matrix with entries of an H_v-ring or H_v-field (or h/v-ring or h/v-field). The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m\times n$ and $n\times r$ respectively, is defined in the usual manner and it is a set of $m\times r$ H_v-matrices. The sum of products of elements of the H_v-ring is considered to be the n-ary circle hope on the hyperaddition. The hyperproduct of H_v-matrices is not necessarily WASS.

The problem of the H_v-matrix (or h/v-group) representations is the following:

Definition 3.3 Let $(H,\cdot)$ be H_v-group (or h/v-group). Find an H_v-ring (or h/v-ring) $(R,+,\cdot)$, a set $M_R=\{(a_{ij}) \mid a_{ij}\in R\}$ and a map $T:H\to M_R$: $h\mapsto T(h)$ such that

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \emptyset, \ \forall h_1,h_2 \in H.$$

T is *H_v-matrix (or h/v-matrix) representation*. If $T(h_1h_2) \subset T(h_1)T(h_2)$, $\forall h_1,h_2 \in H$, then T is called *inclusion*. If $T(h_1h_2)=T(h_1)T(h_2)= \{T(h) \mid h \in h_1h_2\}$, $\forall h_1,h_2 \in H$, then T is *good* and then an induced representation T* for the hypergroup algebra is obtained. If T is one to one and good then it is *faithful*.

The main theorem on representations is [13]:

Theorem 3.4 A necessary condition to have an inclusion representation T of an h/v-group $(H,\cdot)$ by $n\times n$, h/v-matrices over the h/v-ring $(R,+,\cdot)$ is the following:

For all classes $\beta^*(x)$, $x \in H$ must exist elements $a_{ij} \in H$, $i,j \in \{1,...,n\}$ such that

$$T(\beta^*(a)) \subset \{A=(a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}),\ i,j \in \{1,...,n\}\}$$

Inclusion $T:H \to M_R: a \mapsto T(a)=(a_{ij})$ induces homomorphic representation T* of H/β^* on R/γ^* by setting $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall \beta^*(a) \in H/\beta^*$, where $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. T* is called *fundamental induced* of T.

In representations, several new classes are used:

Definition 3.5 Let $M=M_{m\times n}$ be the module of $m\times n$ matrices over R and $P=\{P_i: i \in I\} \subseteq M$. We define a P-hope $\underline{P}$ on M as follows

$$\underline{P}: M\times M \to P(M): (A,B) \to A\underline{P}B=\{AP^t_iB:\ i \in I\} \subseteq M$$

where P^t denotes the transpose of P.

The hope $\underline{P}$ is bilinear map, is strong associative and inclusion distributive:

$$A\underline{P}(B+C) \subseteq A\underline{P}B+A\underline{P}C,\ \forall A,B,C \in M$$

Definition 3.6 Let $M=M_{m\times n}$ the $m\times n$ matrices over R and let us take sets

$$S=\{s_k: k \in K\} \subseteq R,\ \ Q=\{Q_i: j \in J\} \subseteq M,\ \ P=\{P_i: i \in I\} \subseteq M.$$

Define three hopes as follows

$$\underline{S}: R\times M \to P(M): (r,A) \to r\underline{S}A = \{(rs_k)A:\ k \in K\} \subseteq M$$

$$\underline{Q}_+: M\times M \to P(M): (A,B) \to A\underline{Q}_+B = \{A+Q_j+B:\ j \in J\} \subseteq M$$

$$\underline{P}: M\times M \to P(M): (A,B) \to A\underline{P}B = \{AP^t_iB:\ i \in I\} \subseteq M$$

Then $(M,\underline{S},\underline{Q}_+,\underline{P})$ is hyperalgebra on R called *general matrix P-hyperalgebra*.

4 Helix-Hopes and Applications

Recall some definitions from [19], [8], [20], [4]:

Definition 4.1 Let $A=(a_{ij})\in M_{m\times n}$ be m×n matrix and $s,t\in N$ be naturals such that $1\leq s\leq m$, $1\leq t\leq n$. We define the map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to the matrix A, the matrix $A\underline{cst}=(a_{ij})$ where $1\leq i\leq s$, $1\leq j\leq t$. We call this map *cut-projection* of type $\underline{st}$. Thus $A\underline{cst}$ is matrix obtained from A by cutting the lines, with index greater than s, and columns, with index greater than t.

We use cut-projections on all types of matrices to define sums and products.

Definitions 4.2 Let $A=(a_{ij})\in M_{m\times n}$ be an m×n matrix and $s,t\in N$, $1\leq s\leq m$, $1\leq t\leq n$. We define the mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st}=(\underline{a}_{ij})$ which has as entries the sets

$$\underline{a}_{ij} = \{a_{i+\kappa s,j+\lambda t} \mid 1\leq i\leq s,\ 1\leq j\leq t. \text{ and } \kappa,\lambda\in N,\ i+\kappa s\leq m,\ \ j+\lambda t\leq n\}.$$

Thus we have the map

$$\underline{st}: M_{m\times n} \to M_{s\times t}: A \to A\underline{st} = (\underline{a}_{ij}).$$

We call this multivalued map *helix-projection* of type $\underline{st}$. $A\underline{st}$ is a set of s×t-matrices $X=(x_{ij})$ such that $x_{ij}\in\underline{a}_{ij}$, $\forall i,j$. Obviously $A\underline{mn}=A$.

Let $A=(a_{ij})\in M_{m\times n}$ be a matrix and $s,t\in N$ such that $1\leq s\leq m$, $1\leq t\leq n$. Then it is clear that we can apply the helix-projection first on the rows and then on the columns, the result is the same if we apply the helix-progection on both, rows and columns. Therefore we have

$$(A\underline{sn})\underline{st} = (A\underline{mt})\underline{st} = A\underline{st}.$$

Let $A=(a_{ij})\in M_{m\times n}$ be matrix and $s,t\in N$ such that $1\leq s\leq m$, $1\leq t\leq n$. Then if $A\underline{st}$ is not a set but one single matrix then we call A *cut-helix matrix* of type s×t. In other words the matrix A is a helix matrix of type s×t, if $A\underline{cst}= A\underline{st}$.

Definitions 4.3

(a) Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$ be matrices and s=min(m,u), t=min(n,u). We define a hope, called *helix-addition* or ***helix-sum***, as follows:

$$\oplus: M_{m\times n}\times M_{u\times v}\to P(M_{s\times t}): (A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset M_{s\times t},$$

where

$$(\underline{a}_{ij})+(\underline{b}_{ij})= \{(c_{ij})= (a_{ij}+b_{ij}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}.$$

(b) Let $A=(a_{ij})\in M_{m\times n}$ and $B=(b_{ij})\in M_{u\times v}$ be matrices and s=min(n,u). We define a hope, called *helix-multiplication* or ***helix-product***, as follows:

$$\otimes: M_{m\times n}\times M_{u\times v}\to P(M_{m\times v}):(A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v},$$

where

$$(\underline{a}_{ij})\cdot(\underline{b}_{ij})=\{(c_{ij})=(\sum a_{it}b_{tj})\mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}.$$

The helix-sum is external hope since it is defined on different sets and the result is also in different set. The commutativity is valid in the helix-sum. For the helix-product we remark that we have $A\otimes B=A\underline{ms}\cdot B\underline{sv}$ so we have either $A\underline{ms}=A$ or $B\underline{sv}=B$, that means that the helix-projection was applied only in one matrix and only in the rows or in the columns. If the appropriate matrices in the helix-sum and in the helix-product are cut-helix, then the result is singleton.

Remark. In $M_{m\times n}$ the addition is ordinary operation, thus we are interested only in the 'product'. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the *helix-Lie Algebra* is defined [17], as well. This algebra is an H_v-Lie Algebra where the fundamental relation ε^* gives, by a quotient, a Lie algebra, from which a classification is obtained.

In the following we restrict ourselves on the matrices $M_{m\times n}$ where $m<n$. We have analogous results if $m>n$ and for $m=n$ we have the classical theory.

Notation. For given $\kappa\in\mathbb{N}-\{0\}$, we denote by $\underline{\kappa}$ the remainder resulting from its division by m if the remainder is non zero, and $\underline{\kappa}=m$ if the remainder is zero. Thus a matrix $A=(a_{\kappa\lambda})\in M_{m\times n}$, $m<n$ is a *cut-helix matrix* if we have $a_{\kappa\lambda}=a_{\kappa\underline{\lambda}}$, $\forall\kappa,\lambda\in\mathbb{N}-\{0\}$.

Moreover let us denote by $I_c=(c_{\kappa\lambda})$ the *cut-helix unit matrix* which the cut matrix is the unit matrix I_m. Therefore, since $I_m=(\delta_{\kappa\lambda})$, where $\delta_{\kappa\lambda}$ is the Kronecker's delta, we obtain that, $\forall\kappa,\lambda$, we have $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$.

Proposition 4.4 For $m<n$ in $(M_{m\times n},\otimes)$ the cut-helix unit matrix $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$, is a left scalar unit and a right unit. It is the only one left scalar unit.

Proof. Let $A,B\in M_{m\times n}$ then in the helix-multiplication, since $m<n$, we take helix projection of the matrix A, therefore, the result $A\otimes B$ is singleton if the matrix A is a cut-helix matrix of type $m\times m$. Moreover, in order to have $A\otimes B=A\underline{mm}\cdot B=B$, the matrix $A\underline{mm}$ must be the unit matrix. Consequently, $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$, $\forall\kappa,\lambda\in\mathbb{N}-\{0\}$, is necessarily the left scalar unit.

Let $A=(a_{uv})\in M_{m\times n}$ and consider the hyperproduct $A\otimes I_c$. In the entry $\kappa\lambda$ of this hyperproduct there are sets, for all $1\leq\kappa\leq m$, $1\leq\lambda\leq n$, of the form

$$\Sigma\underline{a}_{\kappa s}c_{s\lambda}=\Sigma\underline{a}_{\kappa s}\delta_{s\underline{\lambda}}=\underline{a}_{\kappa\underline{\lambda}}\ni a_{\kappa\lambda}.$$

Therefore $A\otimes I_c\ni A$, $\forall A\in M_{m\times n}$. ■

In the following examples of the helix-hope, we denote E_{ij} any type of matrices which have the ij-entry 1 and in all the other entries we have 0.

Example 4.5 Consider the 2×3 matrices of the forms,

$$A_{\kappa\lambda} = E_{11}+E_{13}+\kappa E_{21}+E_{22}+\lambda E_{23},\ \forall \kappa,\lambda\in\mathbb{Z}.$$

Then we obtain $A_{\kappa\lambda}\otimes A_{st} = \{A_{\kappa+s,\kappa+t}, A_{\kappa+s,\lambda+t}, A_{\lambda+s,\kappa+t}, A_{\lambda+s,\lambda+t}\}$.

Moreover $A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+s},A_{\kappa+s,\lambda+t},A_{\kappa+t,\lambda+s},A_{\kappa+t,\lambda+t}\}$, so

$$A_{\kappa\lambda}\otimes A_{st}\cap A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+t}\}, \text{ thus } (\otimes) \text{ is COW.}$$

The helix multiplication (⊗) is associative.

Example 4.6 Consider all *traceless* matrices $A=(a_{ij})\in M_{2\times 3}$, in the sense that we have $a_{11}+ a_{22}=0$. The cardinality of the helix-product of any two matrices is 1, or 2^3, or 2^6. These correspond to the cases: $a_{11}=a_{13}$ and $a_{21}=a_{23}$, or only $a_{11}=a_{13}$ either only $a_{21}=a_{23}$, or if there is no restriction, respectively.

Proposition. The Lie-bracket of two traceless matrices $A=(a_{ij})$, $B=(b_{ij})\in M_{m\times n}$, $m<n$, contain at least one traceless matrix.

Example 4.7 Let us denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Then take the following 2×2 upper triangular h/v-matrices on the above h/v-field $(\boldsymbol{Z}_4,+,\otimes)$, on the set $\boldsymbol{Z}_4=\{0,1,2,3\}$, of the case that only $2\otimes 3=\{0,2\}$ is a hyperproduct:

$$I=E_{11}+E_{22},\quad a=E_{11}+E_{12}+E_{22},\quad b=E_{11}+2E_{12}+E_{22},\quad c=E_{11}+3E_{12}+E_{22},$$

$$d=E_{11}+3E_{22},\quad e=E_{11}+E_{12}+3E_{22},\quad f=E_{11}+2E_{12}+3E_{22},\quad g=E_{11}+3E_{12}+3E_{22},$$

A hyper-matrix representation of four dimensional case with helix-hope:

Example 4.8 On the field of real or complex numbers we consider the four dimensional space of all 2×4 matrices of type, called helix-upper triangular matrices,

$$\boldsymbol{A} = \begin{pmatrix} a & b & a & c \\ 0 & d & 0 & d \end{pmatrix}$$

This set is closed under the helix-hope. That means that the helix-product of two such matrices is a 2×4 matrix, of the same type. In fact we have

$$\mathbf{A}\otimes\mathbf{A}' = \begin{pmatrix} a & b & a & c \\ 0 & d & 0 & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' & a' & c' \\ 0 & d' & 0 & d' \end{pmatrix} =$$

$$= \begin{pmatrix} a & \{b,c\} \\ 0 & d \end{pmatrix} \bullet \begin{pmatrix} a' & b' & a' & c' \\ 0 & d' & 0 & d' \end{pmatrix} =$$

$$= \begin{pmatrix} aa' & \{ab'+bd', ab'+cd'\} & aa' & \{ac'+bd', ac'+cd'\} \\ 0 & dd' & 0 & dd' \end{pmatrix}$$

Therefore the result is a set with 4 matrices.

Examples 4.9

(a) On the same type of matrices using the Construction 4.1, on $(\boldsymbol{Z}_4,+,\cdot)$ we take the small h/v-field $(\boldsymbol{Z}_4,+,\otimes)$, where only $2\otimes 3=\{0,2\}$, where we remind that the fundamental classes are $\{0,2\}$, $\{1,3\}$. We take from the set of all matrices

$$\boldsymbol{A} = \begin{pmatrix} a & b & a & c \\ 0 & d & 0 & d \end{pmatrix}$$

the matrix

$$\boldsymbol{X} = \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then the powers of this matrix are

$$\boldsymbol{X}^2 = \begin{pmatrix} 0 & \{1,3\} & 0 & \{1,3\} \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{X}^3 = \begin{pmatrix} 0 & \{1,3\} & 0 & \{1,3\} \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

We obtain that the generating set is the following

$$\left\{ \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right\} \cup \begin{pmatrix} 0 & \{1,3\} & 0 & \{1,3\} \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

The classes remain the same.

(b) If we take the matrix

$$\boldsymbol{Y} = \begin{pmatrix} 2 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then the powers of this matrix are

$$\boldsymbol{Y}^2 = \begin{pmatrix} 0 & \{0,3\} & 0 & \{1,2\} \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{Y}^3 = \begin{pmatrix} 0 & \boldsymbol{Z}_4 & 0 & \boldsymbol{Z}_4 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

We obtain that the generating set is the following

$$\left\{ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right\} \cup \begin{pmatrix} 0 & \boldsymbol{Z}_4 & 0 & \boldsymbol{Z}_4 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

We have only one class.

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