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Some Kinds of Homomorphisms on Hypervector Spaces

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Abstract

In this paper, we introduce the concepts of homomorphism of type 1 , 2 and 3 and good homomorphism . Then we investigate some properties of them.
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1 Introduction and Preliminaries

The concept of hyperstructure was first introduced by Marty [13] in 1934 . He defined hypergroups and began to analysis their properties and applied them to groups and rational algebraic functions . Tallini introduced the notion of hypervector spaces [14] , [15] and studied basic properties of them . Homomorphisms of hypergroups are studied by several authers ([2] - [12]) . Since some kinds of homomorphisms on hypergroup were defined , we encourage to define them on hypervector spaces . In this paper , we introduce the concept of homomorphism of type 1 , 2 and 3. And give an example of a homomorphism that is not a homomorphism of type 1, 2 and 3. We show that if f be a homomorphism of type 1, 2 and 3, then f is a homomorphism and every homomorphism of type 2 or 3

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is a homomorphism of type 1. Also, we define a good homomorphism and obtain that every homomorphism of type 2 is a good homomorphism and every good homomorphism is a homomorphism. Finally, w e prove that every onto strong homomorphism is a good homomorphism.

Let us recall some definitions which are useful in our results .

Definition 1.1. *A hypervector space over a field K is a quadruplet $(V, +, \circ, K)$ such that $(V, +)$ is an abelian group and*

$$\circ : K \times V \to P_*(V)$$

is a mapping of $K \times V$ into the power set of V (deprived of the empty set) , such that

$$\begin{aligned}
(a+b)\circ x &\subseteq (a\circ x)+(b\circ x), \quad \forall a,b\in K,\ \forall x\in V, &(1)\\
a\circ(x+y) &\subseteq (a\circ x)+(a\circ y), \quad \forall a\in K,\ \forall x,y\in V, &(2)\\
a\circ(b\circ x) &= (ab)\circ x, \quad \forall a,b\in K,\ \forall\, x\in V, &(3)\\
x &\in 1\circ x, \quad \forall x\in V, &(4)\\
a\circ(-x) &= -a\circ x, \quad \forall a\in K,\ \forall x\in V. &(5)
\end{aligned}$$

Definition 1.2. *Let $(V, +, \circ, K)$ be a hypervector space . Then $H \subseteq V$ is a subspace of V , if*

1) *the zero vector , 0 , is in H ,*

2) *$U, V \in H$, then $U + V \in H$,*

3) *$U \in H, r \in K$, then $r \circ U \subseteq H$.*

Definition 1.3. *Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces . A mapping*

$$f : V \to W$$

is called

1) *a homomorphism , if $\forall r \in K,\ \forall x, y \in V$:*

$$\begin{aligned}
f(x+y) &= f(x)\oplus f(y), &(6)\\
f(r\circ x) &\subseteq r * f(x). &(7)
\end{aligned}$$

2) *a strong homomorphism, if $\forall r \in K,\ \forall x, y \in V$:*

$$\begin{aligned}
f(x+y) &= f(x)\oplus f(y), &(8)\\
f(r\circ x) &= r * f(x). &(9)
\end{aligned}$$

2 The main results

In this paper, the ground field of a hypervector space V is presented with K, This field is usually considered by $\mathbb{R}$ or $\mathbb{C}$. Let $(V, +, \circ)$ and $(W, \oplus, *)$ be two hypervector spaces and $f : V \to W$ be a mapping. We employ for simplicity of notation $x_f = f^{-1}(f(x))$ and for a subset A of V, $A_f = f^{-1}(f(A)) = \bigcup\{x_f : x \in A\}$.

Lemma 2.1. *Let $r \in K$ and $x \in V$. Then the following statements are valid:*

i) $r \circ x \subseteq (r \circ x)_f$,

ii) $r \circ x \subseteq r \circ x_f$,

iii) $(r \circ x)_f \subseteq (r \circ x_f)_f$,

iv) $r \circ x_f \subseteq (r \circ x_f)_f$.

Definition 2.1. *Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces and $f : V \to W$ be a map such that $f(x + y) = f(x) \oplus f(y)$, for all $a, b \in V$. Then, for any $r \in K$ and $x, y \in V$, f is called a homomorphism of*

i) type 1, if $f^{-1}(r * f(x)) = (r \circ x_f)_f$,

ii) type 2, if $f^{-1}(r * f(x)) = (r \circ x)_f$,

iii) type 3, if $f^{-1}(r * f(x)) = (r \circ x_f)$.

Theorem 2.1. *Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces, A be a non-empty subset of V and $f : V \to W$ be a map such that $f(a + b) = f(a) \oplus f(b)$, for all $a, b \in V$. Then, f is a homomorphism of*

i) type 1 implies $f^{-1}(r * f(A)) = (r \circ A_f)_f$,

ii) type 2 implies $f^{-1}(r * f(A)) = (r \circ A)_f$,

iii) type 3 implies $f^{-1}(r * f(A)) = (r \circ A_f)$.

Proof. Each part is established by a straightforward set theoretic argument. □

Example 2.1. *Let $(W, +, \cdot, K)$ be a classical vector space, P be a proper subspace of W, $W_1 = (W, +, \cdot, K)$ and $W_2 = (W, \oplus, \circ, K)$ that $r \circ a = r \cdot a + P$ for $r \in K$ and $a \in W$. Then W_1 and W_2 are hypervector spaces.*
Let $f : W_1 \to W_2$ be the function defined by $f(x) = k \cdot x$, where $k \in K$. We show

that f is a homomorphism, but not a homomorphism of type 1, 2 and 3.

For every $r \in K$ and $x \in W_1$ we have

$$f(r \cdot x) = rk \cdot x \subsetneq rk \cdot x + P = r \circ f(x).$$

Thus f is a homomorphism. Since f is one to one, we obtain $x_f = x$, for $x \in W$. It followes that

$$(r \cdot x_f)_f = (r \cdot x)_f = (r \cdot x_f) = (r \cdot x).$$

On the other hand,

$$f^{-1}(r \circ f(x)) = f^{-1}(kr \cdot x + P) = \{t \in W_1 : f(t) \in kr \cdot x + P\}$$

$$= \{t \in W_1 : k \cdot t \in kr \cdot x + P\} = \{t \in W_1 : k \cdot t - kr \cdot x \in P\}.$$

Hence,

$$f^{-1}(r \circ f(x)) \neq r \cdot x.$$

Therefore, f is not a homomorphism of type 1, 2 and 3.

Theorem 2.2. *Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces and $f : V \to W$ be a homomorphism of type n, for n=1,2,3. Then f is a homomorphism map.*

Proof. If f be a homomorphism of type 1. Then by using Lemma 2.1, we have

$$f(r \circ x) \subseteq f(r \circ x_f) \subseteq f((r \circ x_f)_f) = f(f^{-1}(r * f(x)) \subseteq r * f(x).$$

Suppose f is a homomorphism of type 2. Then

$$f(r \circ x) \subseteq f((r \circ x)_f) = f(f^{-1}(r * f(x)) \subseteq r * f(x).$$

Similarly, if f is a homomorphism of type 3, then

$$f(r \circ x) \subseteq f(r \circ x_f) = f(f^{-1}(r * f(x))) \subseteq r * f(x).$$

□

Lemma 2.2. *Let f be a homomorphism. Then*

$$(r \circ x_f)_f \subseteq f^{-1}(r * f(x)).$$

Proof. Since f is a homomorphism, for all $r \in K$ and $x \in V$, we have

$$f(r \circ x_f) \subseteq r * f(x_f).$$

Since $r * f(x_f) = r * f(f^{-1}(f(x)) \subseteq r * f(x)$, hence, $f(r \circ x_f) \subseteq r * f(x)$. Therefore,

$$(r \circ x_f)_f \subseteq f^{-1}(r * f(x)).$$

□

Proposition 2.1. *Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces and $f : V \to W$ be a homomorphism of type 2 or 3. Then f is a homomorphism of type 1.*

Proof. Suppose that $r \in K$, $x \in V$ and $f : V \to W$ be a homomorphism of type 2, then by Lemma 2.2 we have

$$(r \circ x)_f \subseteq (r \circ x_f)_f \subseteq f^{-1}(r * f(x)) = (r \circ x)_f.$$

Similarly, if f is a homomorphism of type 3, then

$$r \circ x_f \subseteq (r \circ x_f)_f \subseteq f^{-1}(r * f(x)) = r \circ x_f.$$

□

Proposition 2.2. *Let $(V, +, \circ, K)$ and $(W, +\oplus, *, K)$ be two hypervector spaces and $f : V \to W$ be an onto mapping. Then, given $r \in K$ and $x \in V$, f is a homomorphism of*

i) type 1 if and only if $f(r \circ x_f) = r * f(x)$,

ii) type 2 if and only if $f(r \circ x) = r * f(x)$.

Proof. Since f is onto, we obtain

$$ff^{-1}(r * f(x)) = r * f(x).$$

Thus, (i) and (ii) are trivial. □

Corolary 2.1. *Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces, A and B be non-empty subsets of V and $f : V \to W$ be an onto mapping.Then, f is homomorphism of*

i) type 1 implies $f(r \circ A_f) = r * f(A)$,

ii) type 2 implies $f(r \circ A) = r * f(A)$.

Remark 2.1. *On onto homomorphisms between hypervector spaces, a homomorphism of type 2 is equivalent with a strong homomorphism.*

Theorem 2.3. *Let $(V_1, +_1, \circ_1, K)$, $(V_2, +_2, \circ_2, K)$ and $(V_3, +_3, \circ_3, K)$ be hypervector spaces. For $n = 1, 2, 3$, let f be a homomorphism of type n of V_1 onto V_2 and g be a homomorphism of type n of V_2 onto V_3. Then, gf is a homomorphism of type n of V_1 onto V_3.*

Proof. Let $x, y \in V$. We have $gf(x +_1 y) = g(f(x) +_2 f(y)) = gf(x) +_3 gf(y))$.
One can easily seen that $x_{gf} = f^{-1}(f(x)_g)$.
Let $n = 1$. By above relation, we obtain

$$gf(r \circ x_{gf}) = gf(r \circ f^{-1}(f(x)_g)).$$

Since f is onto, there exists a subset A of V such that $f(A) = f^{-1}(f(x)_g)$. By Corollary 2.1, we obtain

$$gf(r \circ_1 f^{-1}(f(x)_g)) = g(r \circ_2 f(x)_g).$$

Then, by Proposition 2.2, we have

$$g(r \circ_2 f(x)_g) = r \circ_3 gf(x).$$

Let $n = 2$. Similar to the previous case, but simpler.
Let $n = 3$. Since g is of type 3,

$$(gf)^{-1}(r \circ_3 (gf)(x)) = f^{-1}g^{-1}r \circ_3 (gf)(x)) = f^{-1}(r * f(x)_g).$$

Since f is onto, the item (iii) of Theorem 2.1 implies

$$f^{-1}(r \circ_2 f(x)_g) = r \circ_1 f^{-1}(f(x)_g) = r \circ_1 x_{gf}.$$

□

Definition 2.2. *Let $a \in V$ and $r \in K$. We define*

$$a/r = \{x \in V \ : \ a \in r \circ x\}.$$

Proposition 2.3. *Let $(V_1, +, \circ, K)$ and $(V_2, \oplus, *, K)$ be two hypervector spaces. If $f : V_1 \to V_2$ be an onto mapping. Then we have*

1) $f(a/r) = f(a)/r$, if f is a homomorphism of type 2.

2) $f(a)/r \subseteq f(a_f)/r$, if f is a homomorphism of type 3.

Proof. 1) We know that an onto homomorphism of type 2 is a strong homomorphism. Suppose that $y \in f(a/r)$. Then, there exists $t \in a/r$ such that $f(t) = y$, so $a \in r \circ t$ and $f(a) \in r * f(t)$. It implies that $y = f(t) \in f(a)/r$. Therefore, $f(a/r) \subset f(a)/r$. Note that the inverse inclusion is always true. 2) If $y \in f(a)/r$, there is $t \in V_1$ such that $f(t) = y$. Since f is homomorphism of type 3, we have $a_f \in r \circ t_f$, which means that $t_f \in a_f/r$, therefore $y \in f(a_f)/r$. □

Definition 2.3. *Let* $(V, +, \circ, K)$ *and* $(W, *, \oplus, K)$ *be two hypervector spaces and* $f : V \to W$ *be a map such that* $f(a + b) = f(a) \oplus f(b)$*. Then* f *is called a good homomorphism if*

$$f(a/r) = f(a)/r,$$

for any $a,\ b \in V$ *and* $r \in K$*.*

Remark 2.2. *According to Proposition 2.3, if* f *is a homomorphism of type 2, then* f *is a good homomorphism.*

Theorem 2.4. *Let* $(V, +, \circ, K)$ *and* $(W, \oplus, *, K)$ *be two hypervector spaces. If* $f : V \to W$ *be a good homomorphism then,* f *is a homomorphism.*

Proof. Let $r \in K$ and $a \in V_1$. If $y \in f(r \circ a)$, then, there exists $t \in r \circ a$ such that $y = f(t)$. Hence, $f(a) \in f(t/r) = f(t)/r$. Abviously, $y = f(t) \in r * f(a)$. □

Theorem 2.5. *Let* $(V_1, +_1, \circ_1, K)$*,* $(V_2, +_2, \circ_2, K)$*, and* $(V_3, +_3, \circ_3, K)$ *be hypervector spaces. Let* f *be a good homomorphism of* V_1 *to* V_2 *and* g *be a good homomorphism of* V_2 *to* V_3*. Then,* gf *is a good homomorphism of* V_1 *to* V_3*.*

Proof. For every $r \in K$ and $a \in V_1$, we have

$$gf(a/r) = g(f(a)/r) = gf(a)/r.$$

□

Proposition 2.4. *Let* V *and* W *be two hypervector spaces over* K *and* $f : V \to W$ *be a good homomorphism. Then*

$$f(A/K) = f(A)/K,$$

where $A \subseteq V$ *and* $A/K = \bigcup\{a/r : a \in A, r \in K\}$*.*

Proof. Let $y \in f(A/K)$. There exist $r \in K$ and $a \in A$ such that $y \in f(a/r) = f(a)/r \subseteq f(A)/K$. Conversely, let $y \in f(A)/k$. Then, there exist $r \in k$ and $a \in V$ such that $y \in f(a)/r = f(a/r)$ and so $y \in f(A/K)$. □

Theorem 2.6. *Let* $(V, +, \circ, K)$ *and* $(W, \oplus, *, K)$ *be two hypervector spaces,* f *be onto strong homomorphism from* V *to* W*. Then* f *is a good homomorphism.*

Proof. Let $f(t) \in f(x/r)$. So $x \in r \circ t$. It followes that $f(t) \in f(x)/r$. Therefore $f(x/r) \subseteq f(x)/r$.
On the other hand, let $y \in f(x)/r$. Since f is an onto mapping, there exists a $t \in V$ such that $y = f(t)$. Hence, $f(x) \in r * f(t) = f(r \circ t)$. Thus $x \in r \circ t$ and then we have $t \in x/r$ and $y = f(t) \in f(x/r)$. Therefore $f(x)/r \subseteq f(x/r)$. This implies that $f(x/r) = f(x)/r$.

□

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