ISSN 1592-7415 **eISSN 2282-8214**

Volume n. 33 – 2017

RATIO MATHEMATICA

Journal of Mathematics, Statistics, and Applications

Publisher
A.P.A.V.
Accademia Piceno – Aprutina dei Velati in Teramo
www.eiris.it – www.apav.it
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RATIO MATHEMATICA
Journal of Mathematics, Statistics, and Applications
ISSN 1592-7415 - eISSN 2282-8214

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Publisher: **APAV** - Accademia Piceno-Aprutina dei Velati in Teramo
Tax Code 92036140678, **Vat Id. Number** 02184450688
Registered Office Address: via del Concilio, 24 - 65121 Pescara
Operational Office: via Chiarini, 191 - 65126 Pescara

The Volume n. 33-2017 of Ratio Mathematica is published in honor of the Prof. Thomas Vougiouklis and his academic career. The Volume is edited with the kind participation of Prof. Haralambos Sakonidis and Prof. Achilles Dramalidis.

Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 5-20 **eISSN**: 2282-8214

On Some Applications of the Vougiouklis Hyperstructures to Probability Theory

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[3]**doi**: 10.23755/rm.v33i0.372

Abstract

Some important concepts about algebraic hyperstructures, especially from a geometric point of view, are recalled. Many applications of the H_v structures, introduced by Vougiouklis in 1990, to the de Finetti subjective probability theory are considered. We show how the wealth of probabilistic meanings of H_v-structures confirms the importance of the theoretical results obtained by Vougiouklis. Such results can be very meaningful also in many application fields, such as decision theory, highly dependent on subjective probability.
Keywords: algebraic hyperstructures; subjective probability; H_v structures, join spaces.
2010 AMS subject classification: 20N20; 60A05; 52A10.

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 Received: 15-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

The theory of the algebraic hyperstructures was born with the paper (Marty, 1934) at the VIII Congress of Scandinavian Mathematicians and it was developed in the last 40 years.

In the book "Prolegomena of hypergroup theory" (Corsini, 1993) all the fundamental results on the algebraic hyperstructures, until 1992, have been presented. A complete bibliography is given in the appendix. A review of the results until 2003 is in (Corsini, Leoreanu, 2003).

Perhaps the most important motivation for the study of algebraic hyperstructures comes from the basic text "Join Geometries" by Prenowitz and Jantosciak (1979), which in addition to giving an original and general approach to the study of Geometry, introduces an interdisciplinary vision of Geometry and Algebra, showing how the Euclidean Spaces can be drawn as Join Spaces, i.e. commutative hypergroups that satisfy an axiom called "incidence property". Moreover, various other geometries, such as the Projective Geometries (Beutelspacher, Rosembaum, 1998), are also Join Spaces.

Considering, for example, the Affine Geometries, it is seen that associative property is not satisfied in many important geometric spaces. This and other important geometric and algebraic issues have led to the study of weak associative hyperstructures. The theory of such hyperstructures, called H_v-structures, was carried out by Thomas Vougiouklis, who introduced the concept of H_v-structures in the work "The fundamental relation in hyperrings. The general hyperfield" (1991),**.** presented at the 4th AHA Conference, Xanthi, Greece, 1990. Subsequently Vougiouklis found many fundamental results on the H_v-structures in numerous works (e.g. Vougiouklis, 1991, 1992, 1994a, 1994b; Spartalis, Vougiouklis, 1994). A collection of all the results on the subject until 1994 is in the important book "Hyperstructures and their representations" (Vougiouklis, 1994c).

Subsequent insights into H_v-structures were made by Vougiouklis in many subsequent works (1996a, 1996b, 1996c, 1997, 1999a, 1999b, 2003a, 2003b, 2008, 2014), also in collaboration with other authors (Dramalidis, Vougiouklis, 2009, 2012; Vougiouklis et al., 1997; Nikolaidou, Vougiouklis, 2012).

From the H_v-structures of Vougiouklis, the idea in the Chieti-Pescara research group was conceived to interpret some important structures of subjective probability as algebraic structures. Some paper on this topic are (Doria, Maturo, 1995, 1996; Maturo, 1997a, 1997b, 1997c, 2000a, 2000b, 2001a, 2001b, 2003c, 2008, 2010).

The study of applications of hyperstructures to the treatment of uncertainty and decision-making problems in Architecture and Social Sciences begins with a series of lectures held at the Faculty of Architecture in Pescara by Giuseppe Tallini in 1993, on hyperstructures seen from a geometric point of view, and was developed at various AHA conferences (Algebraic Hyperstructures and Applications) as well as various seminars and conferences with Piergiulio Corsini from 1994 to 2014.

For example, in December 1994 and October 1995, two conferences on "Hyperstructures and their Applications in Cryptography, Geometry and Uncertainty Treatment" were organized by Corsini, Eugeni and Maturo, respectively in Chieti and Pescara, with which it was initiated the systematic study of the applications of hyperstructures to the treatment of uncertainty and Architecture.

In (Corsini, 1994), it is proved that the fuzzy sets are particular hypergroups. This fact leads us to examine properties of fuzzy partitions from a point of view of the theory of hypergroups. In particular, crisp and fuzzy partitions given by a clustering could be well represented by hypergroups. Some results on this topic and applications in Architecture are in the papers of Ferri and Maturo (1997, 1998, 1999a, 1999b, 2001a, 2001b). Applications of hyperstructures in Architecture are also in (Antampoufis et al., 2011; Maturo, Tofan, 2001). Moreover, the results on fuzzy regression by Fabrizio Maturo, Sarka Hoskova-Mayerova (2016) can be translate as results on hyperstructures.

A new research trend concerns the applications of hypergroupoid to Social Sciences. Vougiouklis, in some of his papers (e.g. 2009, 2011), propose hyperstructures as models in social sciences; Hoskova-Mayerova and Maturo analyze social relations and social group behaviors with fuzzy sets and H_v-structures (2013, 2014), and introduce some generalization of the Moreno indices.

2 Fundamental Definitions on Hyperstructures

Let us recall some of the main definitions on the hyperstructures that will be applied in this paper to represent concepts of Logic and Subjective Probability.

For further details on hyperstructure theory, see, for example, (Corsini, 1993; Corsini, Leoreanu, 2003; Vougiouklis, 1994c).

Definition 2.1 Let H be a non-empty set and let $\wp^*(H)$ be the family of non-empty subsets of H. A *hyperoperation* on H is a function $\sigma: H\times H \to \wp^*(H)$, such that to every ordered pair (a, b) of elements of H associates a non-empty subset of H, noted $a\sigma b$. The pair (H, σ) is called *hypergroupoid* with *support* H and *hyperoperation* σ.

If A and B are non-empty subsets of H, we put $A\sigma B = \cup\{a\sigma b: a\in A, b\in B\}$. Moreover, $\forall a, b\in H$, we put, $a\sigma B = \{a\}\sigma B$ and $A\sigma b = A\sigma\{b\}$.

Definition 2.2 A hypergroupoid (H, σ) is said to be:

- a *semihypergroup*, if $\forall a, y, c\in H$, $a\sigma(b\sigma c) = (a\sigma b)\sigma c$ (*associativity)*;
- a *quasihypergroup*, if $\forall a\in H$, $a\sigma H = H = H\sigma a$ (*riproducibility*);
- a *hypergroup* if it is both a *semihypergroup* and a *quasihypergroup*;
- *commutative*, if $\forall a, b\in H$, $a\sigma b = b\sigma a$;
- *idempotent*, if $\forall a\in H$, $a\sigma a=\{a\}$.
- *weak associative*, if $\forall a, b, c\in H$, $a\sigma(b\sigma c) \cap (a\sigma b)\sigma c \neq \emptyset$;
- *weak commutative*, if $\forall a, b\in H$, $a\sigma b \cap b\sigma a \neq\emptyset$.

The weak associative hypergroupoid, called also H_v-semigroup by Vougiouklis (1991), appear to be particularly significant in the Theory of Subjective Probability, and all results found by Vougiouklis in later papers (e.g.1992, 1994a, 1994b), should have important logic and probabilistic meanings. Vougiouklis (1991) introduced also the notation "H_v-group" for the weak associative quasihypergroups.

A H_v-semigroup is said to be *left directed* if $\forall a, b, c\in H$, $a\sigma(b\sigma c) \subseteq (a\sigma b)\sigma c$, and *right directed* if $a\sigma(b\sigma c) \supseteq (a\sigma b)\sigma c$.

Let (H, σ) a hypergroupoid. Using a geometric language, a singleton {a}, $a\in H$, is said to be a block of order 1 (briefly 1-block) generated by a. Every hyperproduct $a\sigma b$, $a, b\in H$, is a block of order 2 (2-block), called block generated by (a, b). For every $a_1, a_2, a_3 \in H$, the hyperproducts $a_1 \sigma (a_2 \sigma a_3)$ and $(a_1 \sigma a_2) \sigma a_3$ are the 3-blocks generated by (a_1, a_2, a_3). For recurrence, for every $a_1, a_2, \ldots, a_n\in H$, $n > 2$, the blocks generated by $(a_1, a_2, \ldots, a_n)$ are the hyperproducts $A\sigma B$, with A block of order $s < n$, generated by $(a_1, \ldots, a_s)$, and B block of order n-s generated by $(a_{s+1}, \ldots, a_n)$. In general, for every $n > 1$, a block of order n (or n-block) is a hyperproduct $A\sigma B$, with A block of order $s < n$, B block of order $n - s$.

For every $n\in N$, let Δ_n be the set of all the blocks of order n, and let $\Delta_0 = \cup\{\Delta_n, n\in N\}$. Then for every $n\in N_0$, a geometric space (H, Δ_n) is associated to the hypergroupoid (H, σ). A polygonal with length m of (H, Δ_n) is a n-tuple $(A_1, A_2, \ldots, A_m)$ of blocks of Δ_n such that $A_i \cap A_{i+1} \neq \emptyset$. Let Π_n be the set of all the polygonals of (H, Δ_n).

The relation β_n and β_n* are defined as:

$\forall a, b\in H$, $a \beta_n b \Leftrightarrow \exists A\in\Delta_n: \{a, b\}\subseteq A$,

$\forall a, b\in H$, $a \beta_n^* b \Leftrightarrow \exists P\in\Pi_n: \{a, b\}\subseteq P$.

β_n is reflexive and symmetric, β_n* is the transitive closure of β_n. For $n = 0$ we have the classical relations β and β* considered in many papers, e.g. (Freni,1991; Corsini, 1993; Gutan, 1997; Vougiouklis 1999b), A more restrictive condition than the weak associativity is the "strong weak associativity", called also feeble associativity.

Definition 2.3 A hypergroupoid (H, σ) is said to be *feeble associative* if, for every $a_1, a_2, \ldots, a_n \in H$, the intersection of all the blocks generated by $(a_1, a_2, \ldots, a_n)$ is not empty.

If (H, σ) is a commutative quasihypergroup, the σ-division is defined in H, as the hyperoperation $/_\sigma : H\times H \to \wp^*(H)$ that to every pair $(a, b) \in H\times H$ associates the nonempty set $\{x\in H: a\in b\sigma x\}$.

Definition 2.4 A commutative hypergroup (H, σ) is said to be a *join space* if the following "incidence property" hold:

$$\forall a, b, c, d \in H,\ a /_\sigma b \cap c /_\sigma d \neq \emptyset \Rightarrow a\sigma d \cap b\sigma c \neq \emptyset. \qquad (2.1)$$

A join space (H, σ) is:

- *open*, if, $\forall a, b \in H$, $a \neq b$, $a\, \sigma\, b \cap \{a, b\} = \emptyset$;
- *closed*, if, $\forall a, b \in H$, $\{a, b\} \subseteq a\, \sigma\, b$;
- *σ-idempotent*, if, $\forall a\in H$, $a\, \sigma\, a = \{a\}$;
- $/_\sigma$ *-idempotent*, if, $\forall a\in H$, $a /_\sigma a = \{a\}$.

A join space (H, σ) is said to be a *join geometry* if it is σ-idempotent and $/_\sigma$ -idempotent. We have the following theorem.

Theorem 2.1 Let $H = R^n$ and σ the hyperoperation that to every $(a, b)\in H\times H$ associates the open segment with extremes a and b if $a \neq b$, and $a\, \sigma\, a = \{a\}$. (H, σ) is a join geometry, called *Euclidean join geometry*.

Let (H, σ) be a join geometry. We can note that it is open. Using a notation like that of Euclidean join geometry, in this paper the elements of H are called points and a block aσb, with $a \neq b$, is called (open) "σ-segment" with extremes a and b or simply "segment" if only the hyperoperation σ is considered in the context.

The concept of join space leads to a unified vision of Algebra and Geometry, that can be very useful from the point of view of advanced didactics (Di Gennaro, Maturo, 2002). Also, as some of our papers show, join geometries have important applications in subjective probability. Moreover, we can introduce general uncertainty measures in join geometries such that in the Euclidean join geometries reduce to the de Finetti coherent probability (Maturo, 2003a, 2003b, 2006, 2008; Maturo et al., 2010).

3 Subjective Probability and Hyperstructures

Let us recall the concept of coherent probabilty and its geometric representation with the notation given in (Maturo, 2006).

The coherence of an assessment of probabilities $p = (p_1, p_2, \ldots, p_n)$ on a n-tuple $E = (E_1, E_2, \ldots, E_n)$ of events is defined by an hypothetical bet with a n-tuple of wins $S = (S_1, S_2, \ldots, S_n)$ (de Finetti, 1970; Coletti, Scozzafava, 2002; Maturo 2006).

For every $i \in \{1, 2, \ldots, n\}$ an individual A, called *the better*, pays the *stake* p_iS_i to an individual B, called *the bank*, and, if the event E_i occurs, A receives from B the win S_i. If $S_i < 0$ the verse of the bet on E_i is inverted, i. e. B pays the stake and A pays the win.

The total random gain G_A of A is given by the formula:

$$G_{A, p, S} = (|E_1| - p_1) S_1 + (|E_2| - p_2) S_2 + \ldots + (|E_n| - p_n) S_n. \quad (3.1)$$

where $|E_i| = 1$ if the event E_i is verified and $|E_i| = 0$ if the event E_i is not verified.

The atoms associated with the set of events $\mathbf{E} = \{E_1, E_2, \ldots, E_n\}$ are the intersections $A_1 \cap A_2 \cap \ldots \cap A_n$, where $A_i \in \{E_i, -E_i\}$, different from the impossible event $\varnothing$. Let At($\mathbf{E}$) be the set of the atoms. Then $G_A(p, S)$ can be interpret as the function:

$$G_{A, p, S}: a = A_1 \cap A_2 \cap \ldots \cap A_n \in At(\mathbf{E}) \rightarrow (|E_1| - p_1) S_1 + (|E_2| - p_2) S_2 + \ldots + (|E_n| - p_n) S_n. \quad (3.2)$$

Definition 3.1 The probability assessment $p = (p_1, p_2, \ldots, p_n)$ on the n-tuple $E = (E_1, E_2, \ldots, E_n)$ of events is said to be *coherent* if, for every $S = (S_1, S_2, \ldots, S_n) \in R^n$, there are $a, b \in At(\mathbf{E})$ such that $G_{A, p, S}(a) \geq 0$ and $G_{A, p, S}(b) \leq 0$.

We note that the previous definition implies a hyperoperation. Let Λ be an algebra of events containing the set $\mathbf{E}$. Then Λ also contains At($\mathbf{E}$) and we can define the hyperoperation α on Λ:

$$\alpha: (A, B) \in \Lambda \times \Lambda \rightarrow At(A, B). \quad (3.3)$$

The above considerations show that it may be important, in a probabilistic context, to know the properties of the algebraic hyperstructure (Λ, α), introduced in (Doria, Maturo, 2006), and called *hypergroupoid of atoms*.

The coatoms associated with $\mathbf{E}$ are the nonimpossible complementary events of the atoms. Let Co($\mathbf{E}$) be the set of coatoms, and k be the number of atoms. For k = 1, At($\mathbf{E}$) = $\{\Omega\}$, where Ω is the certain event and Co($\mathbf{E}$) is empty. For k = 2, At($\mathbf{E}$) = Co($\mathbf{E}$) and for k >2 the sets At($\mathbf{E}$) and Co($\mathbf{E}$) are disjoint and with the some number of elements.

For every $A, B, C \in \Lambda$, we have (we write X Y to denote $X \cap Y$):

$(A\alpha B)\alpha C = (\{X\ C, X\ (-C), X \in At(A, B)\} \cup \{Y\ C, Y\ (-C), Y \in Co(A, B)\}) - \{\varnothing\}$,

$A\alpha(B\alpha C)$ = ({A Z, (-A) Z, Z∈At(B, C)} ∪ {A T, (-A) T, T∈Co(B, C)}-{∅},

At{A, B, C} = {X C, X (-C), X∈At(A, B)}-{∅} = {A Z, (-A) Z, Z∈At(B, C)}-{∅}.

Then:

At{A, B, C} ⊆ $(A\alpha B)\alpha C \cap A\alpha(B\alpha C)$.

Therefore, the following theorem applies:

Theorem 3.1 Let Λ be an algebra of events, and α the hyperoperation defined by (3.3). Then (Λ, α) is a commutative H_v-semigroup.

The algebra associated with the set of events **E** = {E_1, E_2, …, E_n}, denoted with Alg(**E**) is the set containing the impossible event ∅ and all the unions of the elements of At(**E**), i.e. X∈Alg(**E**) iff ∃Y∈ ℘(At(**E**)) such that X is the union of the elements of Y. If |At(**E**)|=s, then |Alg(**E**)| = 2^s.

Let Λ be an algebra of events. We can introduce the following hyperoperation on Λ:

$$\beta: (A, B)\in \Lambda\times\Lambda\rightarrow Alg(A, B) \quad (3.4)$$

The hyperoperation β is commutative, and, since {A, B} ⊆ Alg(A, B), (Λ, β) is a quasihypergroup. Moreover At(A, B) ⊆ Alg(A, B) and so β is an extension of the operation α and we have:

At{A, B, C} ⊆ (A β B) β C ∩ A β (B β C).

Theorem 3.2 Let Λ be an algebra of events, and β the hyperoperation defined by (3.4). Then (Λ, β) is a commutative H_v-group.

Suppose A, B, C are logically independent events, then |At(A, B| = 4, |Alg(A, B| = 2^4 = 16, |At(A, B, C)| = 8, Alg(A, B, C)| = 2^8 = 256. Moreover Alg(A, B) contains ∅, Ω and other 7 elements with their complements. If X is one of these elements, then X β C contains ∅, Ω, C, -C and other 12 elements.

Then (A β B) β C has 7×12+4= 88 elements and 168 elements are in Alg(A, B, C) but not in (A β B) β C. So, in general, we can write:

At{A, B, C} ∪ Co{A, B, C} ⊂ (A β B) β C, A β (B β C) ⊂ Alg(A, B, C).

Let (H, σ) be a join geometry. From the associative and commutative properties, for every a_1, a_2, …, a_n∈H there is only a block $a_1\sigma\, a_2\sigma \ldots\sigma a_n$ generated by (a_1, a_2, …, a_n) and this block depend only by on the set {a_1, a_2, …, a_n} and not on the order of the elements. By the idempotence we can reduce to the case in which a_1, a_2, …, a_n are distinct.

Definition 3.2 For every A ⊆ H, A ≠ ∅, the *convex hull* of A, in (H, σ), is the set

$[A]_\sigma = \{x\in H: \exists n\in N, \exists\, a_1, a_2, \ldots, a_n\in A : x \in a_1\sigma\, a_2\sigma \ldots\sigma a_n\}$.

If A is finite then $[A]_\sigma$ is said to be the *polytope* generated by A.

Let $E = (E_1, E_2, \ldots, E_n)$ be a n-tuple of events set and let At(E) the set of atoms associated to E. For every $a = A_1 \cap A_2 \cap \ldots \cap A_n \in$ At(E), let $x_i(a) = 1$ if $A_i = E_i$ and $x_i(a) = 0$ if $A_i = - E_i$. The atom a is identified with the point $(x_1(a), x_2(a), \ldots, x_n(a)) \in R^n$. From definition 3.1, the following theorem applies (Maturo, 2006, 2008, 2009).

Theorem 3.3 Let (R^n, ε) the Euclidean join geometry. The probability assessment $p = (p_1, p_2, \ldots, p_n)$ on the n-tuple $E = (E_1, E_2, \ldots, E_n)$ of events is coherent iff $p \in [At(E)]_\varepsilon$.

The theorem 3.3 opens the way to introduce measures of uncertainty that are different from the probability and coherent with respect non-Euclidean join geometries. We can introduce many possible join geometries. The following is an example.

Example 3.1 Let $H = R^n$ and κ the hyperoperation that to every $(a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n)) \in H \times H$ associated the Cartesian product of the open segments I_r with extremes a_r and b_r belonging to (R, ε). We can prove that (H, κ) is a join geometry, called the Cartesian join geometry.

Some applications of the Cartesian join geometry to problems of Architecture and Town-Planning are in (Ferri, Maturo, 2001a, 2001b).

In a general join geometry with support R^n we can introduce the following definition:

Definition 3.3 Let (R^n, σ) be a join geometry. The measure assessment $m = (m_1, m_2, \ldots, m_n)$ on the n-tuple $E = (E_1, E_2, \ldots, E_n)$ of events is said to be coherent with respect to (R^n, σ) iff $m \in [At(E)]_\sigma$.

For example, σ can be the hyperoperation that to every $(a, b) \in H \times H$ associates a particular curve with extremes a and b, and the polytope $[At(E)]_\sigma$ is a deformation of the Euclidean polytope, obtained by replacing the segments with curves. It can have important meanings in appropriate contexts of Physics or Social Sciences.

In a generic join geometry (R^n, σ) can happen that some of the most intuitive properties of the Euclidean join geometry fall. To avoid this, you should restrict yourself to join geometries where some additional properties apply. Important is the following:

Ordering condition. If a, b, c, are distinct elements of R^n, at most one of the following formulas occurs: $a \in b\sigma c$, $b \in a\sigma c$, $c \in a\sigma b$.

A join geometry (R^n, σ) with the order condition is said to be an *ordered join geometry*.

4 Conditional Events, Conditional Probability and Hyperstructures

The "axiomatic probability" by Kolmogorov, usually considered as the "true probability" is based on the assessment of a universal set U, whose elements are called the atoms, a σ-algebra S of subsets of U, whose elements are called the events, and a finite measure p on S, called the probability, such that p(U) = 1.

Let S* = S-{∅}. In the Kolmogorov approach to probability no consideration is given to the logical concept of conditional event E|H, with E∈S and H∈S*, but only the conditional probability p(E|H) is defined, only in the case in which p(H) > 0, by the formula:

$$p(E|H) = p(E\ H)/p(H). \tag{4.1}$$

On the contrary, the "subjective probability" (de Finetti,1970; Dubins, 1975; Coletti, Scozzafava 2002; Maturo, 2003b, 2006, 2008b), don't consider the events as subsets of a given universal set U, but they are *logical propositions* that can assume only one of the truth values: *true* and *false*. A sharp separation is given among the concepts concerning the three areas of the *logic of the certain*, the *logic of the uncertain* and the *measure theory*.

The conditional event E|H is a concept belonging to the logic of the certain and it is a proposition that can assume three values: *true* if both E and H are verified, *false* if H is verified but E is not and *empty* (or *undetermined*) if H is not verified. The conditional event E|H reduces to the event E if H is the certain event Ω. In the appendix of his fundamental book (1970) de Finetti presents also some different interpretations of the logical concept of three valued proposition.

By the point of view of Reichenbach (1942) the value "empty" is replaced by the value "undetermined". In the following we assume the notation of Reichenbach and we write T for true, F for false and U for undetermined. The set V = {F, U, T} is also ordered by putting F < U < T.

A numerical representation of the ordered set V is given by associating 0 to F, 1 to T and the number 1/2 to U. An alternative, in a fuzzy contest, we can associate to U is the fuzzy number u with support and core the interval [0, 1], then the relation 0 < u <1 is a consequence of the usual order relation among the trapezoidal fuzzy numbers.

In the subjective probability, the conditional probability p(E|H) of the conditional event E|H is given by an expert and no condition is given about the belonging of the events E and H to a structured set, e.g. like an algebra. The only condition of H ≠∅ is required, because if H = ∅ we have the totally undetermined conditional event.

If C is a set of conditional events the assessment of a subjective conditional probability to the elements of C must satisfy some coherence conditions.

The coherence of an assessment of probabilities $p = (p_1, p_2, \dots, p_n)$ on a n-tuple $K = (E_1|H_1, E_2|H_2, \dots, E_n|H_n)$ of conditional events is defined by an hypothetical bet with a n-tuple of wins $S = (S_1, S_2, \dots, S_n)$ (de Finetti, 1970; Coletti, Scozzafava, 2002; Maturo, 2006).

For every $i \in \{1, 2, \dots, n\}$ an individual A, called *the better*, pays the *stake* p_iS_i to an individual B, called *the bank*, and,

- if the event E_iH_i occurs, A receives from B the win S_i;
- if the event $-H_i$ occurs, the amount paid p_iS_i is refunded to A;
- if the event $(-E_i)\, H_i$ occurs, no payment is made to A.

The total random gain G_A of A is given by the formula:

$$G_{A,\,p,\,S} = |H_1|\,(|E_1| - p_1)\, S_1 + \dots + |H_n|(|E_n| - p_n)\, S_n. \qquad (4.2)$$

where $|E_i| = 1$ if the event E_i is verified and $|E_i| = 0$ if the event E_i is not verified, and similarly to H.

The atoms associated with the set of conditional events $\mathbf{K} = \{E_1|H_1, E_2|H_2, \dots, E_n|H_n\}$ are the intersections $A_1 \cap A_2 \cap \dots \cap A_n$, where $A_i \in \{E_i\, H_i, -E_i\, H_i, -H_i\}$, different from the impossible event $\varnothing$. The complement of $H = \cup\{H_i, i \in \{1, 2, \dots, n\}\}$ is said to be the *inactive atom*.

Let Atc($\mathbf{E}$) be the set of the atoms associated to $\mathbf{K}$. Then $G_{A,\,p,\,S}$ can be interpret as the function:

$$G_{A,\,p,\,S}: a = A_1 \cap A_2 \cap \dots \cap A_n \in \mathrm{Atc}(\mathbf{E}) \rightarrow (|A_1| - p_1)\, S_1 + (|A_2| - p_2)\, S_2 + \dots + (|A_n| - p_n)\, S_n \qquad (4.3)$$

where $|A_i| = 1, 0, p_i$, if $A_i = E_i\, H_i, -E_i\, H_i, -H_i$, respectively.

Definition 4.1 The conditional probability assessment $p = (p_1, p_2, \dots, p_n)$ on the n-tuple $K = (E_1|H_1, E_2|H_2, \dots, E_n|H_n)$ of conditional events is said to be *quasi-coherent* if, for every $S = (S_1, S_2, \dots, S_n) \in R^n$, there are $a, b \in \mathrm{Atc}(\mathbf{E})$ such that $G_{A,\,p,\,S}(a) \geq 0$ and $G_{A,\,p,\,S}(b) \leq 0$. Moreover, $p = (p_1, p_2, \dots, p_n)$ is said to be *coherent* if, for any $s \leq n$ and for any $\{i1, i2, \dots, is\} \subseteq \{1, 2, \dots, n\}$, the conditional probability assessment $p_{i1,\, i2,\, \dots,\, is} = (p_{i1}, p_{i2}, \dots, p_{is})$ on $(E_{i1}|H_{i1}, E_{i2}|H_{i2}, \dots, E_{is}|H_{is})$ is quasi-coherent.

Let $K = (E_1|H_1, E_2|H_2, \dots, E_n|H_n)$ be a n-tuple of conditional events and let Atc($\mathbf{K}$) the set of atoms associated to $\mathbf{K} = \{E_1|H_1, E_2|H_2, \dots, E_n|H_n\}$. For every $a = A_1 \cap A_2 \cap \dots \cap A_n \in \mathrm{Atc}(E)$, let $x_i(a) = |A_i|$. The atom a is identified with the point $(x_1(a), x_2(a), \dots, x_n(a)) \in R^n$. From definition 4.1, the following theorems applies:

Theorem 4.1 Let (R^n, ε) the Euclidean join geometry. The probability assessment $p = (p_1, p_2, \dots, p_n)$ on the n-tuple $K = (E_1|H_1, E_2|H_2, \dots, E_n|H_n)$ of conditional events is quasi-coherent iff $p \in [\mathrm{Atc}(K)]_{\varepsilon}$.

Theorem 4.2 The probability assessment p = (p_1, p_2, …, p_n) on K = ($E_1|H_1$, $E_2|H_2$, …, $E_n|H_n$) is coherent iff for any s ≤ n and for any {i1, i2, …, is} ⊆ {1, 2, …, n}, the conditional probability assessment $p_{i1, i2, \ldots, is}$ = (p_{i1}, p_{i2}, …, p_{is}) on ($E_{i1}|H_{i1}$, $E_{i2}|H_{i2}$, …, $E_{is}|H_{is}$) belongs to [Atc($E_{i1}|H_{i1}$, $E_{i2}|H_{i2}$, …, $E_{is}|H_{is}$)]$_\varepsilon$ ⊆ R^s.

Let Λ be an algebra of events. An axiomatic formalization of the coherence conditions in the case in which **K** = {E|H, E∈Λ, H∈Λ - {∅}} is in Dubins (1975).

In terms of hyperstructures, conditional events can be defined by the following hyperstructure, introduced in (Doria, Maturo, 1996) and studied in (Maturo, 1997c).

Definition 4.2 Let Λ be an algebra of events. We define on Λ the hyperoperation:

γ: (E, H)∈ Λ×Λ → {E H, H}.

We have:

E γ H ∩ H γ E = {E H};

(E γ H) γ K = {E H K, H K, K}, E γ (H γ K) = {E H K, H K, E K, K};

E γ E = {E}.

Then we have the following theorem.

Theorem 4.3 The hyperstructure (Λ, γ), let us call the *hyperstructure of conditional events*, is a weak commutative and idempotent H_v-semigroup. Moreover (Λ, γ) is right directed, i.e. (E γ H) γ K ⊆ E γ (H γ K).

Any singleton {H} is the conditional event H|H and any set {E, H} with E ⊆ H is the conditional event E|H**,** *true* if E is verified, *false* if H is verified but not E, and it is not *undetermined* if H is not verified. Many other meanings, of the finite subsets of Λ, are considered in (Maturo, 1997c).

The coherence conditions of definition 4.1 and theorems 4.1 and 4.2 lead us to associate the n-tuple K = ($E_1|H_1$, $E_2|H_2$, …, $E_n|H_n$) of conditional events with the set all the conditional events A|B with A∈At{E_1, E_2, …, E_n} and B an union of elements of {H_1, H_2, …, H_n}. Then, if Λ is an algebra of events, and Θ ⊆ Λ is a set of nonempty events, closed with respect to the union, the following hyperoperation can be introduced:

δ: (E|H, F|K) ∈ (Λ×Θ)×(Λ×Θ) → {A|B: A∈At{E, F}, B∈{H, K, H∪K}}.

We can prove the following thorem

Theorem 4.4 The hyperstructure (Λ×Θ, δ) is a commutative H_v-semigroup, called *hypergroupoid of conditional atoms* and, for Θ = {Ω}, is isomorphic to (Λ, α).

5 Conclusions and Perspectives of Research

We have shown that all logical operations related to subjective probability can reduce to Vougiouklis hyperstructures. (Λ, α) and $(\Lambda\times\Theta, \delta)$ are commutative H_v-semigroups, and (Λ, β) is a commutative H_v-group. The hyperoperation γ isweak commutative and idempotent and (Λ, γ) is a right directed H_v-semigroup.

To verify the coherence of a subjective probability assignment $p = (p_1, p_2, \ldots, p_n)$ on the n-tuple $E = (E_1, E_2, \ldots, E_n)$ of events, we represent the atoms as points of the space R^n, in which the i-th axis is associated with the event E_i. The assessment p is coherent iff p belongs to the polytope of the join geometry (R^n, ε) generate from the atoms.

More complex is the coherence check of a conditional probability assessment $p = (p_1, p_2, \ldots, p_n)$ on the n-tuple $K = (E_1|H_1, E_2|H_2, \ldots, E_n|H_n)$ of conditional events, as in this case we must consider polytopes in all the join geometries (R^s, ε), $s \leq n$ associated to subsets of $\mathbf{K} = \{E_1|H_1, E_2|H_2, \ldots, E_n|H_n\}$.

A research perspective is to investigate the properties of the considered Vougiouklis structures, highlighting their meanings from the point of view of logic and subjective probability.

A further research perspective is studying the measures that can be obtained by applying the geometric coherence conditions in ordered join geometries other than the Euclidean join geometry.

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 21-38 **eISSN**: 2282-8214

An Overview of Topological and Fuzzy Topological Hypergroupoids

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†**doi**:10.23755/rm.v33i0.389

It is my honour to dedicate this paper to Professor Thomas Vougiouklis lifetime work.

Abstract

On a hypergroup, one can define a topology such that the hyperoperation is pseudocontinuous or continuous. This concepts can be extend to the fuzzy case and a connection between the classical and fuzzy (pseudo)continuous hyperoperations can be given. This paper, that is his an overview of results received by S. Hoskova-Mayerova with coauthors I. Cristea, M. Tahere and B. Davaz, gives examples of topological hypergroupoids and show that there is no relation (in general) between pseudotopological and strongly pseudotopological hypergroupoids. In particular, it shows a topological hypergroupoid that does not depend on the pseudocontinuity nor on strongly pseudocontinuity of the hyperoperation.
Keywords: Hyperoperation, hypergroupoid, continuous, pseudocontinuous and strongly pseudocontinuous hyperoperation, topology, topological hypergroupoid, (fuzzy) pseudocontinuous hyperoperation, (fuzzy) continuous hyperoperation, fuzzy topological space.
2010 AMS Mathematics Subject Classification: 20N20, 22A22.

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†©Šarka Hošková-Mayerová. Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

As was mentioned e.g. in [32], in various branches of mathematics we encounter important examples of topologico-algebraical structures like topological groupoids, groups, rings, fields etc. Therefore, there was a natural interest to generalize the concept of topological groupoid to topological hypergroupoid. First results of this type can be found e.g. in [6, 32].

Hypergroups are generalizations of groups. Group is a set with a binary operation on it satisfying a number of conditions. If this binary operation is taken to be multivalued, then we arrive at a hypergroup. The motivation for generalization of the notion of group resulted naturally from various problems in non-commutative algebra, another motivation for such an investigation came from geometry.

Hypergroups theory, born in 1934 with Marty's paper [39] presented in the 8th Congress of Scandinavian Mathematicians where he had given this renowned definition. "*Marty, managed to do the greatest generalisation anybody would ever do, acting as a pure and clever researcher. He left space for future generalisations "between" his axioms and other hypergroups, as the regular hypergroups, join spaces etc. The reproduction axiom in the theory of groups is also presented as solutions of two equations, consequently, Marty got round that hitch, too.* " [53]. He was followed in 1938 by Dresher with Ore [23] as well as by Griffiths [27] and in 1940 by Eaton [24] is now studied from the theoretical point of view and for its applications to many subjects of pure and applied mathematics (see [9, 15, 16, 57]) like algebra, geometry, convexity, topology, cryptography and code theory, graphs and hypergraphs, lattice theory, Boolean algebras, logic, probability theory, binary relations, theory of fuzzy and rough sets [12, 20], automata theory, economy, etc. [10, 11, 15].

Hypergroupoids, [17] quasi-hypergroups, semihypergroups [41, 42], hypergroups [1, 2], hyperrings [40, 52], hyperfields, [60] hyper vector spaces, hyperlattices, up to all kinds of fuzzy hyperstructures [49], have been studied theoretically as well as from the perspective of particular applications, see e.g. [5, 18, 21, 30, 33, 56]. In 1990, Th. Vougiouklis introduced the class of Hv-structures which satisfy the weak axioms where the non-empty intersection replaces the equality [55].

Moreover, topological and algebraic structures in fuzzy sets are strategically located at the juncture of fuzzy sets, topology, [26] algebra [7], lattices, etc. They has these unique features: major studies in uniformities and convergence structures, fundamental examples in lattice-valued topology, modifications and extensions of sobriety, categorical aspects of lattice-valued subsets, logic and foundations of mathematics, t-norms and associated algebraic and ordered structures. In the last decade a number of interesting applications to social sciences appear, e.g. [3, 43, 44, 59, 61, 62].

In [6], Ameri presented the concept of topological (transposition) hypergroups. He introduced the concept of a (pseudo, strong pseudo) topological hypergroup and gave some related basic results. R. Ameri studied the relationships between pseudo, pseudo topological polygroups and topological polygroups. In [28], Heidari et al. studied the notion of topological hypergroups as a generalization of topological groups. They showed - by considering the quotient topology induced by the fundamental relation on a hypergroup - that if every open subset of a topological hypergroup is complete part, then it's fundamental group is a topological group. Moreover, in [29], Heidari et al. defined the notion of topological polygroups and they investigated the topological isomorphism theorems it. Later on, Salehi Shadkami et al. [47, 48] established various relations between its complete parts and open sets and they used these facts to obtain some new results in topological polygroups. For example, they investigated some properties of cp-resolvable topological polygroups. In [32], the author of this note introduced the concept of topological hypergroupoid and found necessary and sufficient conditions for having a τ_U-topological hypergroupoid, a τ_L-topological hypergroupoid and a $\tau_\aleph$-topological hypergroupoid by using the concepts of pseudocontinuity, strong pseudocontinuity and both respectively.

When in 1965 Zadeh [63] introduced the fuzzy sets, than the reconsideration of the concept of classical mathematics began. Since then the connections between fuzzy sets and hyperstructures was studied. Using the structure of a fuzzy topological space and that of a fuzzy group (introduced by Rosenfeld [46]), Foster [26] defined the concept of *fuzzy topological group*. Later, Ma and Yu [38] changed Foster's definition in order to make sure that an ordinary topological group is a special case of a fuzzy topological group. An interesting book concerning fuzzy topology was published in 1997 by Liu [36].

Inspired by the definition of the *topological groupoid* I. Cristea and S. Hoskova-Mayerova in [19] extended these notions on a *fuzzy topological space*.

This paper is an overview of results received by S. Hoskova-Mayerova in [32] as well as the results with coauthors I. Cristea [19], M. Tahere, B. Davaz [50]. Paper is organized as follows: Firstly, we review some basic definitions and results on hypergroups and topology and fuzzy topological spaces. Section 3 recall the results concerning topological hypergroupoids. In Section 4 we recall some basic results on the fuzzy topological spaces that we use in the following Section 6. In Section 5 we recall the definition of fuzzy (pseudo)continuous hyperoperations, we explain relations between fuzzy continuous and continuous hyperoperations, between fuzzy continuous and fuzzy pseudocontinuous hyperoperations, respectively. Moreover, we give the condition when a product hypergroupoid is a fuzzy pseudotopological hypergroupoid. Finally, in Section 6 we recall some results - published in [19] concerning fuzzy topological hypergroupoids.

2 Basic Definitions

In this section, we present some definitions related to hyperstructures and topology that are used throughout the paper. They can be found in e.g. [4, 19, 31, 22].

Definition 2.1. *Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a binary hyperoperation on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a hypergroupoid.*

If A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a:

- *semihypergroup if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$;*
- *quasihypergroup if for every $x \in H$, $x \circ = H = H \circ x$ (This condition is called the reproduction axiom);*
- *hypergroup if it is a semihypergroup and a quasihypergroup.*

Definition 2.2. *Let (X, τ) be a topological space. Then*

1. *(X, τ) is a T_0-space if for all $x \neq y \in X$, there exists $U \in \tau$ such that $x \in U$ and y is not in U or $y \in U$ and x is not in U.*
2. *(X, τ) is a T_1-space if for all $x \neq y \in X$, there exist $U, V \in \tau$ such that $x \in U$ and y is not in U and $y \in V$ and x is not in V.*
3. *(X, τ) is a T_2-space if for all $x \neq y \in X$, there exist $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

So, every T_2-topological space is a T_1-topological space and every T_1-topological space is a T_0-topological space.

Definition 2.3. *Let $(H_1, \circ_1)$, $(H_2, \circ_2)$ be two hypergroupoids and define the topologies τ, τ' on H_1, H_2 respectively. A mapping f from H_1 to H_2 is said to be good topological homomorphism if for all $x, y \in H_1$,*

1. *$f(a \circ_1 b) = f(a) \circ_2 f(b)$;*
2. *f is continuous;*

3. f is open.

A good topological homomorphism is a topological isomorphism if f is one to one and onto and we say that H_1 is topologically isomorphic to H_2.

Let $(H, \circ)$ be a hypergroupoid and A, B be non empty subsets of H. By $A \approx B$ we mean that $A \cap B \neq \emptyset$.

3 Topological Hypergroupoids

Š. Hošková-Mayerová in [32] introduced some new definitions inspired by the definition of topological groupoid. Her results are summarized in this section.

Definition 3.1. [32] *Let $(H, \cdot)$ be a hypergroupoid and (H, τ) be a topological space. The hyperoperation " $\cdot$ " is called:*

1. *pseudocontinuous (p-continuous) if for every $O \in \tau$, the set $O_\star = \{(x, y) \in H^2 : x \cdot y \subseteq O\}$ is open in $H \times H$.*
2. *strongly pseudocontinuous (sp-continuous) if for every $O \in \tau$, the set $O^\star = \{(x, y) \in H^2 : x \cdot y \approx O\}$ is open in $H \times H$.*

A simple way to prove that a hyperoperation " $\cdot$ " is p- continuous (sp- continuous) is to take any open set O in τ and $(x, y) \in H^2$ such that $x \cdot y \subseteq O$ ($x \cdot y \approx O$) and prove that there exist $U, V \in \tau$ such that $u.v \subseteq O$ ($u.v \approx O$) for all $(u, v) \in U \times V$.

Definition 3.2. [32] *Let $(H, \cdot)$ be a hypergroupoid, (H, τ) be a topological space and $\tau_\star$ be a topology on $\mathcal{P}^*(H)$.*

The triple $(H, \cdot, \tau)$ is called a pseudotopological or strongly pseudotopological hypergroupoid if the hyperoperation " $\cdot$ " is p-continuous or sp-continuous respectively.

The quadruple $(H, \cdot, \tau, \tau_\star)$ is called $\tau_\star$-topological hypergroupoid if the hyperoperation " $\cdot$ " is $\tau_\star$-continuous.

Let (H, τ) be a topological space, $V, U_1, \ldots, U_k \in \tau$. We define S_V, I_V and $\aleph(U_1, \ldots, U_k)$ as follows:

- $S_V = \{U \in \mathcal{P}^*(H) : U \subseteq V\} = \mathcal{P}^*(V)$.
- $I_V = \{U \in \mathcal{P}^*(H) : U \approx V\}$.
- $\aleph(U_1, \ldots, U_k) = \{B \in \mathcal{P}^*(H) : B \subseteq \bigcup_{i=1}^k U_i \text{ and } B \approx U_i \text{ for } i = 1, \ldots, k\}$.

$S_\emptyset = I_\emptyset = \emptyset$. For all $V \neq \emptyset$, we have

$$S_V = \mathcal{P}^*(V) \text{ and } I_V \supseteq \{H, \mathcal{P}^*(V)\}.$$

Lemma 3.1. *Let (H, τ) be a topological space.*

Then $\{S_V\}_{V\in\tau}$ forms a base for a topology (τ_U) on $\mathcal{P}^*(H)$. Moreover, τ_U is called the upper topology.

Then $\{I_V\}_{V\in\tau}$ forms a subbase for a topology (τ_L) on $\mathcal{P}^*(H)$. Moreover, τ_L is called the lower topology.

Let (H, τ) be a topological space. Then $\{\aleph(U_1, \ldots, U_k)\}_{U_i\in\tau}$ forms a base for a topology ($\tau_\aleph$) on $\mathcal{P}^*(H)$. Moreover, $\tau_\aleph$ is called the Vietoris topology [51].

Following results was proved by S. Hoskova-Mayerova in [32].

Theorem 3.1. *Let $(H, \cdot)$ be a hypergroupoid and (H, τ) be a topological space.*

Then the triple $(H, \cdot, \tau)$ is a pseudotopological hypergroupoid if and only if the quadruple $(H, \cdot, \tau, \tau_U)$ is a τ_U-topological hypergroupoid.

Then the triple $(H, \cdot, \tau)$ is a strongly pseudotopological hypergroupoid if and only if the quadruple $(H, \cdot, \tau, \tau_L)$ is a τ_L-topological hypergroupoid.

Then the triple $(H, \cdot, \tau)$ is a pseudotopological hypergroupoid and strongly pseudotopological hypergroupoid if and only if the quadruple $(H, \cdot, \tau, \tau_\aleph)$ is a $\tau_\aleph$-topological hypergroupoid.

4 Fuzzy Topological Spaces

In this section we recall some basic results on the fuzzy topological spaces that we use in the following.

Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \longrightarrow [0, 1]$. We denote by $FS(X)$ the set of all fuzzy sets on X.

In this paper we use the definition of a fuzzy topological space given by Chang [8].

Definition 4.1. [8] *A fuzzy topology on a set X is a collection $\mathcal{T}$ of fuzzy sets in X satisfying*

(i) $\underline{0} \in \mathcal{T}$ *and* $\underline{1} \in \mathcal{T}$ *(where* $\underline{0}, \underline{1} : X \longrightarrow [0, 1]$, $\underline{0}(x) = 0, \underline{1}(x) = 1$, *for any* $x \in X$*).*

(ii) *If* $A_1, A_2 \in \mathcal{T}$, *then* $A_1 \cap A_2 \in \mathcal{T}$.

(iii) *If* $A_i \in \mathcal{T}$ *for any* $i \in I$, *then* $\bigcup_{i\in I} A_i \in \mathcal{T}$, *where* $\mu_{A_1 \cap A_2}(x) = \mu_{A_1}(x) \wedge \mu_{A_2}(x)$ *and* $\mu_{\bigcup_{i \in I} A_i}(x) = \bigvee_{i \in I} \mu_{A_i}(x)$.

The pair $(X, \mathcal{T})$ *is called a fuzzy topological space.*

In the definition of a fuzzy topology of Lowen [37], the condition (i) is substituted by

(i') for all $c \in [0, 1]$, $k_c \in \mathcal{T}$, where $\mu_{k_c}(x) = c$, for any $x \in X$.

Example 4.1. *Now we present some examples of fuzzy topologies on a set* X. *For more details see [19].*

(i) *The family* $\mathcal{T} = \{\underline{0}, \underline{1}\}$ *is called the indiscrete fuzzy topology on* X.

(ii) *The family of all fuzzy sets in* X *is called the discrete fuzzy topology on* X.

(iii) *If* τ *is a topology on* X, *then the collection* $\mathcal{T} = \{A_O \mid O \in \tau\}$ *of fuzzy sets* X, *where* μ_{A_O} *is the characteristic function of the open set O, is a fuzzy topology on* X.

(iv) *The collection of all constant fuzzy sets in* X *is a fuzzy topology on* X, *where a constant fuzzy set* A *in* x *has the membership function* μ_A *defined as follows :* $\mu_a\colon \longrightarrow [0, 1]$, $\mu_A(x) = k$, *with* k *a fix constant in* $[0, 1]$.

Definition 4.2. [8] *Given two topological spaces* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$, *a function* $f : X \longrightarrow Y$ *is fuzzy continuous if, for any fuzzy set* $A \in \mathcal{U}$, *the inverse image* $f^{-1}[A]$ *belongs to* $\mathcal{T}$, *where* $\mu_{f^{-1}[A]}(x) = \mu_A(f(x))$, *for any* $x \in X$.

Proposition 4.1. [8] *A composition of fuzzy continuous functions is fuzzy continuous function.*

Definition 4.3. [36] *A base for a fuzzy topological space* $(X, \mathcal{T})$ *is a subcollection* $\mathcal{B}$ *of* $\mathcal{T}$ *such that each member* A *of* $\mathcal{T}$ *can be written as the union of members of* $\mathcal{B}$.

A natural question is: 'How to judge whether some fuzzy subsets just form a base of some fuzzy topological space?' We have the following rule:

Proposition 4.2. [36] *A family* $\mathcal{B}$ *of fuzzy sets in* X *is a base for a fuzzy topology* $\mathcal{T}$ *on* X *if and only if it satisfies the following conditions:*

(i) *For any* $A_1, A_2 \in \mathcal{B}$, *we have* $A_1 \cap A_2 \in \mathcal{B}$.

(ii) $\bigcup_{A \in \mathcal{B}} A = \underline{1}$.

If $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are fuzzy topological spaces, we can speak about the product fuzzy topological space $(X_1 \times X_2, \mathcal{T}_1 \times \mathcal{T}_2)$, where the product fuzzy topology is given by a base like in the following result, which can be generalized to a family of fuzzy topological spaces.

Proposition 4.3. [36] *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be fuzzy topological spaces. The product fuzzy topology $\mathcal{T}$ on the product space $X = X_1 \times X_2$ has as a base the set of product fuzzy sets of the form $A_1 \times A_2$, where $A_i \in \mathcal{T}_i$, $i = 1, 2$, and $\mu_{A_1 \times A_2}(x_1, x_2) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2)$.*

Proposition 4.4. [26] *Let $\{(X_i, \mathcal{T}_i)\}_{i\in I}, \{(Y_i, \mathcal{U}_i)\}_{i\in I}$ be two families of fuzzy topological spaces and $(X, \mathcal{T}), (Y, \mathcal{U})$ the respective product fuzzy topological spaces. For each $i \in I$, let $f_i : (X_i, \mathcal{T}_i) \longrightarrow (Y_i, \mathcal{U}_i)$. Then the product mapping $f = \times f_i : (x_i) \longrightarrow (f_i(x_i))$ of $(X, \mathcal{T})$ into $(Y, \mathcal{U})$ is fuzzy continuous if f_i is fuzzy continuous, for each $i \in I$.*

5 Some Results on Relation between Topological Spaces on a Set and Topological Spaces on its Powerset

In this section, we use the results presented in [32] to study topological hypergroupoids. First, we show that there is no relation (in general) between pseudotopological and strongly pseudotopological hypergroupoids.

Proposition 5.1. [50] *Let $H = \{a, b\}$, $\tau = \{\emptyset, \{a\}, H\}$ and define a hyperoperation " $\circ_1$ " on H as follows:*

$\circ_1$	a	b
a	b	H
b	H	H

Then $(H, \circ_1, \tau)$ is a pseudotopological hypergroupoid.

Thus, the quadruple $(H, \circ_1, \tau, \tau_U)$ is a τ_U- topological hypergroupoid.
Moreover, $(H, \circ_1, \tau)$ is not strongly pseudotopological hypergroupoid.

Proposition 5.2. [50] *Let $H = \{a, b\}$, $\tau = \{\emptyset, \{a\}, H\}$ and define a hyperoperation " $\circ_2$ " on H as follows:*

$\circ_2$	a	b
a	H	a
b	a	b

Then $(H, \circ_2, \tau)$ is a strongly pseudotopological hypergroupoid.

Now we have: The quadruple $(H, \circ_2, \tau, \tau_L)$ is a τ_L- topological hypergroupoid. $(H, \circ_2, \tau)$ is not pseudotopological hypergroupoid. Not every strongly pseudotopological hypergroupoid is a pseudotopological hypergroupoid.

Let $(H, \circ)$ be a hypergroupoid and τ a topology on H. Then $(H, \circ, \tau)$ may be neither a pseudotopological hypergroupoid nor a strongly pseudotopological hypergroupoid. We illustrate this fact by the following example.

Example 5.1. [50] *Let $H = \{a, b\}$, $\tau = \{\emptyset, \{a\}, H\}$ and define a hyperoperation "$\circ_3$" on H as follows:*

$\circ_3$	a	b
a	b	a
b	a	b

It is easy to check, by taking $O = \{a\}$ and $a \circ_3 b \in O$, that $(H, \circ_3, \tau)$ is neither a pseudotopological hypergroupoid nor a strongly pseudotopological hypergroupoid.

Proposition 5.3. *Let $(H, \circ, \tau, \tau_U)$ be a topological hypergroupoid. Then (H, τ) is the trivial topology if and only if $(\mathcal{P}^*(H), \tau_U)$ is the trivial topology.*

For the proof see [50].

Corolary 5.1. *Let $(H, \circ, \tau, \tau_\aleph)$ be a topological hypergroupoid. Then (H, τ) is the trivial topology if and only if $(\mathcal{P}^*(H), \tau_\aleph)$ is the trivial topology.*

Proposition 5.4. *Let $(H, \circ, \tau, \tau_L)$ be a topological hypergroupoid. Then (H, τ) is the trivial topology if and only if $(\mathcal{P}^*(H), \tau_L)$ is the trivial topology.*

The proof is similar to that of Proposition 5.3.

Proposition 5.5. *Let $(H, \circ, \tau, \tau_U)$ be a topological hypergroupoid, $|H| \geq 2$ and (H, τ) be the powerset topology.*

Then $(\mathcal{P}^*(H), \tau_U)$ is not the powerset topology on $\mathcal{P}^*(H)$.

Then $(\mathcal{P}^*(H), \tau_L)$ is not the powerset topology on $\mathcal{P}^*(H)$.

Proposition 5.6. *Let $(H_1, \circ_1, \tau)$ and $(H_2, \circ_2, \tau')$ be two topologically isomorphic hypergroupoids.*

If $(H_1, \circ_1, \tau, \tau_U)$ is a τ_U- topological hypergroupoid then $(H_2, \circ_2, \tau', \tau'_U)$ is a τ_U- topological hypergroupoid.

If $(H_1, \circ_1, \tau, \tau_L)$ is a τ_L- topological hypergroupoid then $(H_2, \circ_2, \tau', \tau'_L)$ is a τ_L- topological hypergroupoid.

Corolary 5.2. *Let* $(H_1, \circ_1, \tau)$ *and* $(H_2, \circ_2, \tau')$ *be two topologically isomorphic hypergroupoids. If* $(H_1, \circ_1, \tau, \tau_\aleph)$ *is a* $\tau_\aleph$*- topological hypergroupoid then* $(H_2, \circ_2, \tau', \tau'_\aleph)$ *is a* $\tau_\aleph$*- topological hypergroupoid.*

We present now some $\tau_\aleph$- topological hypergroupoids.

Proposition 5.7. *Let* $(H, \circ)$ *be the total hypergroup (i.e.,* $x \circ y = H$ *for all* $(x, y) \in H^2$*) and* τ *be any topology on* H*. Then* $(H, \circ, \tau)$ *is both: pseudotopological hypergroupoid and strongly pseudotopological hypergroupoid.*

Corolary 5.3. *Let* $(H, \circ)$ *be the total hypergroup and* τ *be any topology on* H*. Then the quadruple* $(H, \circ, \tau, \tau_\aleph)$ *is a* $\tau_\aleph$*- topological hypergroupoid.*

Proposition 5.8. *Let* $H = \mathbb{R}$*,* $(H, \circ)$ *be the hypergroupoid defined by:*

$$x \circ y = \begin{cases} \{a \in \mathbb{R} : x \leq a \leq y\}, & \text{if } x \leq y; \\ \{a \in \mathbb{R} : y \leq a \leq x\}, & \text{if } y \leq x. \end{cases}$$

and τ *be the topology on* H *defined by:*

$$\tau = \{]-\infty, a[: a \in \mathbb{R} \cup \{\pm\infty\}\}.$$

Then $(H, \circ, \tau, \tau_\aleph)$ *is a* $\tau_\aleph$*- topological hypergroupoid.*

Proposition 5.9. *Let* $H = \mathbb{R}$*,* $(H, \circ)$ *be the hypergroupoid defined by:*

$$x \circ y = \begin{cases} \{a \in \mathbb{R} : x \leq a \leq y\}, & \text{if } x \leq y; \\ \{a \in \mathbb{R} : y \leq a \leq x\}, & \text{if } y \leq x. \end{cases}$$

and τ *be the topology on* H *defined by:*

$$\tau = \{]a, \infty[: a \in \mathbb{R} \cup \{\pm\infty\}\}.$$

Then $(H, \circ, \tau', \tau'_\aleph)$ *is a* $\tau'_\aleph$*- topological hypergroupoid.*

Proposition 5.10. *Let* $(H, \star)$ *be the hypergroupoid defined by* $x \star y = \{x, y\}$ *and* τ *be any topology on* H*. Then* $(H, \star, \tau, \tau_\aleph)$ *is a* $\tau_\aleph$*- topological hypergroupoid.*

Example 5.2. [50] *Let* $H = \{a, b\}$*,* $\tau = \{\emptyset, \{a\}, H\}$ *and define a hyperoperation "* $\circ_4$ *" on* H *as follows:*

$\circ_4$	a	b
a	a	H
b	H	b

Then, by Proposition 5.10, $(H, \circ_4, \tau, \tau_\aleph)$ *is a* $\tau_\aleph$*- topological hypergroupoid. Moreover,* $\tau_\aleph = \{\emptyset, \{\{a\}\}, \mathcal{P}^*(H)\}$.

Proposition 5.11. *Let $(H, \cdot)$ be any hypergroupoid and τ be the power set topology on H. Then $(H, \cdot, \tau, \tau_{\aleph})$ is a $\tau_{\aleph}$- topological hypergroupoid.*

Proposition 5.12. *Let $(H, \cdot)$ be any hypergroupoid and τ be the trivial topology on H. Then $(H, \cdot, \tau, \tau_{\aleph})$ is a $\tau_{\aleph}$- topological hypergroupoid.*

Next, we present a topological hypergroupoid that does not depend on the pseudocontinuity nor on strongly pseudocontinuity of the hyperoperation.

Proposition 5.13. *Let $(H, \cdot)$ be any hypergroupoid, τ be any topology on H and $\tau_{\star}$ be the trivial topology on $\mathcal{P}^*(H)$. Then $(H, \cdot, \tau, \tau_{\star})$ is a topological hypergroupoid.*

Next, we present some results on T_0, T_1, T_2- topological spaces.

Proposition 5.14. *Let $(H, \cdot, \tau, \tau_U)$ be a τ_U-topological hypergroupoid. If $(\mathcal{P}^*(H), \tau_U)$ is a T_0- topological space then (H, τ) is a T_0- topological space.*

The converse of Proposition 5.14 is not always true. An illustrating example is presented in [50].

Proposition 5.15. *Let $|H| \geq 2$ and $(H, \cdot, \tau, \tau_U)$ be a τ_U-topological hypergroupoid. Then $(\mathcal{P}^*(H), \tau_U)$ is neither a T_1- topological space nor a T_2- topological space.*

Proposition 5.16. *Let $|H| \geq 2$ and $(H, \cdot, \tau, \tau_L)$ be a τ_L-topological hypergroupoid. Then $(\mathcal{P}^*(H), \tau_L)$ is neither a T_1- topological space nor a T_2- topological space.*

Proposition 5.17. *Let $(H, \cdot, \tau, \tau_L)$ be a τ_L-topological hypergroupoid. If $(\mathcal{P}^*(H), \tau_L)$ is a T_0- topological space then (H, τ) is a T_0- topological space.*

Proposition 5.18. *Let $(H, \cdot, \tau, \tau_L)$ be a τ_L-topological hypergroupoid. If (H, τ) is a T_0- topological space then $(\mathcal{P}^*(H), \tau_L)$ may not be a T_0- topological space.*

It can be proved that: Let $(H, \cdot, \tau, \tau_{\aleph})$ be a $\tau_{\aleph}$-topological hypergroupoid. If $(\mathcal{P}^*(H), \tau_{\aleph})$ is a T_0- topological space then (H, τ) is a T_0- topological space.

Let $(H, \cdot, \tau, \tau_{\aleph})$ be a $\tau_{\aleph}$-topological hypergroupoid. Then $(\mathcal{P}^*(H), \tau_{\aleph})$ is neither a T_2- topological space nor a T_1- topological space.

6 Fuzzy Topological Hypergroupoids

In this section we recall some results - published in [19] concerning fuzzy topological hypergroupoids.

Definition 6.1. *Let $(H, \circ)$ be a hypergroupoid, $\mathcal{T}$ and $\mathcal{U}$ be fuzzy topologies on H and $\mathcal{P}^*(H)$, respectively. The hyperoperation "$\circ$" is called $\mathcal{U}$-fuzzy continuous if the map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is fuzzy continuous with respect to the fuzzy topologies $\mathcal{T} \times \mathcal{T}$ and $\mathcal{U}$.*

For any topology τ on a set X, we denote by $\mathcal{T}_c$ the fuzzy topology formed with the characteristic functions of the open sets of τ. In the following result we give a relation between the continuity and fuzzy continuity of a hyperoperation.

Proposition 6.1. *Let $(H, \circ)$ be a hypergroupoid, τ and τ^* be topologies on H and $\mathcal{P}^*(H)$, respectively. Let $\mathcal{T}_c$ and $\mathcal{U}_c$ be the fuzzy topologies on H and $\mathcal{P}^*(H)$, respectively, generated by τ and τ^*, respectively. The hyperoperation "$\circ$" is τ^*-continuous if and only if it is $\mathcal{U}_c$-fuzzy continuous.*

Let $(H, \mathcal{T})$ be a fuzzy topological space. Then the family $\mathcal{B} = \{\tilde{A} \in FS$ $(\mathcal{P}^*(H)) \mid A \in \mathcal{T}\}$, where $\mu_{\tilde{A}}(X) = \bigwedge_{x \in X} \mu_A(x)$, is a base for a fuzzy topology $\mathcal{T}^*$ on $\mathcal{P}^*(H)$.

Definition 6.2. *Let $(H, \circ)$ be a hypergroupoid endowed with a fuzzy topology $\mathcal{T}$. The hyperoperation "$\circ$" is called fuzzy pseudocontinuous (or briefly fuzzy p-continuous) if, for any $A \in \mathcal{T}$, the fuzzy set A_* in $H \times H$ belongs to $\mathcal{T} \times \mathcal{T}$, where $\mu_{A_*}(x, y) = \bigwedge_{u \in x \circ y} \mu_A(u)$.*

The triple $(H, \circ, \mathcal{T})$ is called a fuzzy pseudotopological hypergroupoid if the hyperoperation "$\circ$" is fuzzy p-continuous.

Now we can characterize a fuzzy pseudotopological hypergroupoid $(H, \circ, \mathcal{T})$ using the $\mathcal{T}^*$-fuzzy continuity of the hyperoperation "$\circ$", where the fuzzy topology $\mathcal{T}^*$ is that one given in Proposition 6.1.

Let $(H, \circ)$ be a hypergroupoid and $\mathcal{T}$ be a fuzzy topology on H. Then the triple $(H, \circ, \mathcal{T})$ is a fuzzy pseudotopological hypergroupoid if and only if the hyperoperation "$\circ$" is $\mathcal{T}^*$-fuzzy continuous.

Proposition 6.2. *Let $(H_1, \mathcal{T}_1)$ and $(H_2, \mathcal{T}_2)$ be two fuzzy topological spaces. We denote $H = H_1 \times H_2$ and $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$. Then the mapping*

$$\alpha : (H, \mathcal{T}) \times (H, \mathcal{T}) \longrightarrow (H_1 \times H_1, \mathcal{T}_1 \times \mathcal{T}_1) \times (H_2 \times H_2, \mathcal{T}_2 \times \mathcal{T}_2),$$

defined by $\alpha((x_1, x_2), (y_1, y_2)) = ((x_1, y_1), (x_2, y_2))$ is fuzzy continuous.

Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be two hypergroupoids. The product hypergroupoid $(H_1 \times H_2, \otimes)$ has the hyperoperation defined by $(x_1, x_2) \otimes (y_1, y_2) = (x_1 \circ_1 y_1, x_2 \circ_2 y_2)$, for any $(x_1, x_2), (y_1, y_2) \in H_1 \times H_2$.

So, we get:

If $(H_1, \circ_1, \mathcal{T}_1)$ and $(H_2, \circ_2, \mathcal{T}_2)$ are fuzzy pseudotopological hypergroupoids, then the product hypergroupoid $(H_1 \times H_2, \otimes, \mathcal{T}_1 \times \mathcal{T}_2)$ is a fuzzy pseudotopological hypergroupoid.

7 Conclusions

On a hypergroup, a topology such that the hyperoperation is pseudocontinuous can be defined. This paper highlighted the topological hypergroupoids by proving some of theirs properties. It illustrated the results achieved on topological hypergroupoids in [32] by examples and remarks. Moreover, it was shown that there is no relation (in general) between pseudotopological and strongly pseudotopological hypergroupoids. In particular, we presented a topological hypergroupoid that does not depend on the pseudocontinuity nor on strongly pseudocontinuity of the hyperoperation.

For future work, the existence of topological hypergroupoids on $\mathcal{P}^*(H)$ that are neither τ_U nor τ_L nor $\tau_{\aleph}$ can be investigated or the existence of n-ary topological hypergroupoids can be studied.

This work could be also continued in order to introduce the notion of fuzzy topological hypergroup as a generalization of a fuzzy topological group in the sense of Foster [26] or in the sense of Ma and Yu [38].

The author would like to express a wish for this beautiful discipline of mathematics to be continue and to be developed. Since there are already numbers of excellent mathematicians around the world who are concerned with this issue, lets believe their interest will not go away, and that the School of Professor P. Corsini and Professor T. Vougiouklis will find many followers.

Acknowledgement

The work presented in this paper was supported within the project for "Development of basic and applied research developed in the long term by the departments of theoretical and applied bases FMT (Project code: DZRO K-217) supported by the Ministry of Defence the Czech Republic.

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 39-46 **eISSN**: 2282-8214

Contributions in Mathematics: Hyperstructures of Professor Thomas Vougiouklis

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†**doi**:10.23755/rm.v33i0.391

Abstract

After presenting some basic notions of hyperstructures and their applications, I shall point out on the contribution of Professor Thomas Vougiouklis to this field of research: algebraic hyperstructures.
Keywords: weak hyperstructure
2010 AMS subject classifications: 20N20.

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†©Violeta Leoreanu - Fotea. Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Hyperstructures and applications

Theory of hyperstructures is a field of algebra, around 80 years old and very rich in applications, for instance in geometry, fuzzy and rough sets, automata, cryptography, codes, probabilities, graphs and hypergraphs (see [2], [3]).

Some basic definitions:

A *hyperoperation* on a nonempty set H is a map $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of nonempty subsets of H.

For subsets A, B of H, set $A \circ B = \bigcup_{a \in A; b \in B} a \circ b$, and for $h \in H$ write $h \circ A$ and $A \circ h$ for $\{h\} \circ A$ and $A \circ \{h\}$.

The pair $(H, \circ)$ is a *hypergroup* if for all a, b, c of H we have

$$(a \circ b) \circ c = a \circ (b \circ c) \text{ and } a \circ H = H \circ a = H.$$

If only the associativity is satisfied then $(H, \circ)$ is a *semihypergroup*. The condition $a \circ H = H \circ a = H$ for all a of H is called the *reproductive law*.

A nonempty subset K of H is a *subhypergroup* if $K \circ K \subseteq K$ and for all $a \in K, K \circ a = K = a \circ K$.

A commutative hypergroup $(H, \circ)$ is a *join space* iff the following implication holds: for all a, b, c, d, x of H,

$$a \in b \circ x, c \in d \circ x \Rightarrow a \circ d \cap b \circ c \neq \emptyset.$$

A *semijoin space* is a commutative semihypergroup satisfying the join condition.

Hypergroups have been introduced by Marty [5] and join spaces by Prenowitz [6]. Join spaces are an important tool in the study of graphs and hypergraphs, binary relations, fuzzy and rough sets and in the reconstruction of several types of noneuclidean geometries, such as the descriptive, spherical and projective geometries [3], [6]. Several interesting books have been writen on hyperstructures [2], [3], [4], [6], [8].

2 Emeritus Professor Thomas Vougiouklis and his contribution to hyperstructures

Professor Thomas Vougiouklis is an author of more than 150 research papers and seven text books in mathematics. He have over 3000 references. He also wrote eight books on poetry, one CD music and lyrics.

He participated in Congresses (invited) about 60 congresses, over 20 countries. His monograph:

Hyperstructures and their representations, Hadronic Press monograph in Mathematics, USA (1994)

is an important book on the theory of algebraic hyperstructures.

Let us mention here some of his main contributions in Hyperstructures, especially H_v -Structures, Lie Algebras of infinite dimension, ring theory, Mathematical Models.

He first introduced and studied:

- The term hope=hyperoperation (2008)
- P-hypergroups, single-power cyclicity (1981).
- Fundamental relations in hyper-rings (γ^*-relation) and Representations of hypergroups by generalized permutations and hypermatrices (1985).
- Very Thin hyperstructures, S-construction (1988).
- Uniting elements procedure (1989), with P.Corsini.
- General hyperring, hyperfield (1990).
- The weak properties and the Hv-structures (1990).
- General Hypermodules, hypervector spaces(1990).
- Representation Theory by Hv-matrices (1990).
- Fundamental relations in hyper-modulus and hyper-vector spaces (ε^* - relation) (1994).
- The e-hyperstructures, Hv-Lie algebras (1996).
- The h/v-structures (1998).
- ∂ - operations (2005),
- The helix hyperoperations, with S. Vougiouklis,
- n-ary hypergroups (2006), with B.Davvaz.,
- Bar instead of scale, (2008), with P. Vougioukli, etc

Let us present here some of these notions.

H_V**- structures**

These notions were introduced in 1990 and they satisfy the weak axioms, where the non-empty intersection replaces the equality.

WASS means *weak associativity*:

$$\forall x, y, z \in H,\ (xy)z \cap x(yz) \neq \emptyset.$$

COW means *weak commutativity*:

$$\forall x, y \in H,\ xy \cap yx \neq \emptyset.$$

A hyperstructure $(H, \cdot)$ is called H_V*-semigroup* if it is WASS and it is called H_V-*group* if it is a reproductive H_V-semigroup, i.e.

$$xH = Hx = H,\ \forall x \in H.$$

Similarly, H_V-vector spaces, H_V-algebras and H_V-Lie algebras are defined and their applications are mentioned in the above books.

Fundamental relations

The fundamental relations β^*, γ^* and ϵ^* are defined in H_V-groups, H_V-rings and H_V-vector spaces being the smallest equivalences, such that the quotient structures are a group, a ring or a vector space respectively.

The following theorem holds:

Theorem. Let $(H, \cdot)$ be an H_V-group and denote by U the set of all finite products of elements of H. We define the relation β in H as follows:

$$x\beta y \Leftrightarrow \exists u \in U : \{x, y\} \subseteq u$$

Then β^* is the transitive closure of β.

In a similar way, relation γ^* is defined in an H_V-ring and relation ϵ^* is defined in an H_V-vector space.

An H_V-ring $(R, +, \cdot)$ is called an H_V-field if R/γ^* is a field.

If $(H,\cdot),\ (H,*)$ are H_V-semigroups defined on the same set H, then the hyperoperation $(\cdot)$ is *smaller* than $(*)$ (and $(*)$ is *greater* than $(\cdot)$ if there exists an $f\in Aut(H,*)$, such that

$$x\cdot y\subseteq f(x*y).$$

Theorem. Greater hopes than the ones which are WASS or COW are also WASS or COW, respectively.

This theorem leads to a partial order on H_V-structures and mainly to a correspondence between hyperstructures and posets.

The determination of all H_V-groups and H_V-rings is very interesting, but difficult. There are many results of R. Bayon and N. Lygeros in this direction.

In paper [1] one can see how many H_V-groups and H_V-rings there exist, up to isomorphism, for several chasses of hyperstructures of two, three or four elements.

∂- operations

The hyperoperations, called theta-operations, are motivated from the usual property, which the derivative has on the derivation of a product of functions.

If H is a set endowed with n operations (or hyperoperations) $\circ_1,\circ_2,...,\circ_n$ and with one map or multivalued map $f:H\to H$ (or $f:H\to\mathcal{P}(H)$ respectively), then n hyperoperations $\partial_1,\partial_2,...,\partial_n$ on H can be defined as follows:

$\forall x,y\in H,\forall i\in\{1,2,...,n\}$,

$$x\partial_i y=\{f(x)\circ_i y,\ x\circ_i f(y)\}$$

or in the case $\circ_i$ is a hyperoperation or f is a multivalued map, we have

$\forall x,y\in H,\forall i\in\{1,2,...,n\}$,

$$x\partial_i y=(f(x)\circ_i y)\cup(x\circ_i f(y)).$$

If $\circ_i$ is WASS, then ∂_i is WASS too.

n-ary hypergroups

A mapping $f:\underbrace{H\times\cdots\times H}_{n}\longrightarrow\mathcal{P}^*(H)$ is called an *n-ary hyperoperation*, where $\mathcal{P}^*(H)$ is the set of all the nonempty subsets of H. An algebraic system (H,f), where f is an n-ary hyperoperation defined on H, is called an *n-ary hypergroupoid*.

We shall use the following abbreviated notation:
The sequence $x_i, x_{i+1}, ..., x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty symbol. When $y_{i+1} = \cdots = y_j = y$ the last expression will be written in the form $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$.

For nonempty subsets $A_1, ..., A_n$ of H we define

$$f(A_1^n) = f(A_1, ..., A_n) =$$

$$\bigcup\{f(x_1^n) \mid x_i \in A_i, i = 1, ..., n\}.$$

An n-ary hyperoperation f is called *associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) =$$
$$f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

hold for every $1 \leq i < j \leq n$ and all $x_1, x_2, ..., x_{2n-1} \in H$. An n-ary hypergroupoid with the associative n-ary hyperoperation is called an
n-ary semihypergroup.

An n-ary hypergroupoid (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an
n-ary quasihypergroup.

Moreover, if (H, f) is an n-ary semihypergroup, (H, f) is called an
n-ary hypergroup.

An n-ary hypergroupoid (H, f) is
commutative if for all $\sigma \in S_n$ and for every $a_1^n \in H$ we have
$f(a_1, ..., a_n) = f(a_{\sigma(1)}, ..., a_{\sigma(n)})$.

Let (H, f) be an n-ary hypergroup and B be a non-empty subset of H. B is called an *n-ary subhypergroup* of (H, f), if $f(x_1^n) \subseteq B$ for $x_1^n \in B$, and the equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has a solution $x_i \in B$ for every $b_1^{i-1}, b_{i+1}^n, b \in B$ and $1 \leq i \leq n$.

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 47-60 **eISSN**: 2282-8214

A Brief Survey on the two Different Approaches of Fundamental Equivalence Relations on Hyperstructures

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[‡]**doi**:10.23755/rm.v33i0.388

This paper is dedicated to Prof. Thomas Vougiouklis lifetime work.

Abstract

Fundamental structures are the main tools in the study of hyperstructures. Fundamental equivalence relations link hyperstructure theory to the theory of corresponding classical structures. They also introduce new hyperstructure classes.The present paper is a brief reference to the two different approaches to the notion of the fundamental relation in hyperstructures. The first one belongs to Koskas, who introduced the β^* - relation in hyperstructures and the second approach to Vougiouklis, who gave the name fundamental to the resulting quotient sets. The two approaches, the necessary definitions and the theorems for the introduction of the fundamental equivalence relation in hyperstructures, are presented.

Keywords: Fundamental equivalence relations, strongly regular relation, hyperstructures, H_v - structures.
2010 AMS subject classifications: 20N20, 01A99.

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 Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

Dealing with classical algebraic structures often leads to the study of the behaviours of the elements of these sets with respect to the introduced operation(s). This study focuses, very often, on looking for elements with similar behaviour. Therefore, the use of the quotient set is intertwined with the search for regularity and symmetry between elements of algebraic structures and 'similar' algebraic structures too.

It is well known that "... the most powerful tool in order to obtain a stricter structure from a given one is the quotient out procedure. To use this method in ordinary algebraic domains, one needs special equivalence relations. If one suggests a method that can applied for every equivalence relation, has to use the hyperstructures" [6].

In the commutative algebra, many problems of algebraic structures are not always visible, resulting in a large number of questions and obstacles appearing in the non-commutative algebra. For example, in classical theory if G is a group and $H \subseteq G$ is a subgroup, then G/H quotient is a group only when H is a normal subgroup. This obstacle [21] is overcome by the definition of Fr. Marty (1934) [17], since

"If G is a group and $H \subseteq G$ is a subgroup of it, then the quotient G/H is a hypergroup."

The previous proposition is generalized by the definition [26] of the weak hyperstructures by Th. Vougiouklis (1990), as follows:

"If G is a group and S is any partition of G, then the quotient G/H is a H_v-group".

In these cases, the quotient set functioned as a process that led to 'looser' structures than classic algebraic ones, but increased complexity.

The utility of outmost importance of the quotient set in hyperstructures is its use as a bridge between classical structures and hyperstructures. In 1970, this connection was achieved by M. Koskas [16] using the β - relation and its transitive closure. Observing the similar behaviour of elements belonging to the same hyperproduct leads to the introduction of the β - relation which, clearly, is reflective, symmetric but not always transitive. The next step is to use the transitive closure of β to obtain equivalence relation and partition in equivalence classes. Using the usual definition of operations between classes, we return to classical algebraic structures. This relation studied mainly by Corsini [5], Vougiouklis [25], Davvaz [8], Leoreanou-Fotea [7], Freni [12], Migliorato [19] and many others.

The quotient set not only links the hyperstructures with the classical structures

as a bridge, but also enhances the view of hyperstructures as a generalization of the corresponding classical algebraic structures. In this way, as reported in [22], the algebraic structures are contained in the corresponding hyperstructures as subcases. It seems that Fr. Marty defined the hypergroup replacing the axiom of the existence of a unitary and inverse element with the axiom of reproduction because he had ”sensed” this connection and chose the widest possible generalization in order to create space for the introduction of new types of hypergroups.
In 1988 at a congress in Italy, Th. Vougiouklis presents a paper titled ”How a hypergroup hides a group” [22], [27] and finds out that
a) Vougiouklis, eighteen years after Koskas worked on the same subject without having knowledge of his work.
b) Much of the study was completed with the work of Koskas and especially P. Corsini and his school.
c) Vougiouklis approach was different from that of Koskas and the others.
In the following we will present the two different approaches.

2 Preliminaries

In a set $H \neq \emptyset$ equipped with a hyperoperation $(\cdot) : H \times H \to \wp^*(H)$ we abbreviate by

WASS the weak associativity: $x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset, \forall (x, y, z) \in H^3$.

COW the weak commutativity: $x \cdot y \cap y \cdot x \neq \emptyset, \forall (x, y) \in H^2$.

Definition 2.1. *The hyperstructure $(H, \cdot)$ is called H_v-semigroup if it is WASS and it is called H_v-group if it is reproductive H_v-semigroup, that is $xH = Hx = H, \forall x \in H$.*

Definition 2.2. *The hyperstructure $(H, \cdot)$ is called semihypergroup if $x \cdot (y \cdot z) = (x \cdot y) \cdot z, \forall (x, y, z) \in H^3$ and it is called hypergroup if it is reproductive semihypergroup.*

Definition 2.3. *A H_v-group is called H_b-group if its hyperoperation contains operation which define a group. We define analogously H_b-ring, H_b-vector space.*

Definition 2.4. *Let $(H, \cdot)$ be a H_v - structure. An element $e \in H$ is called identity if $x \in ex \cap xe, \forall x \in H$. We define analogously the left (right) identity.*

Definition 2.5. *Let $\phi : H \to H/\beta^*$ be the fundamental map of a H_v-group then, the kernel of ϕ is called core and it is denoted by ω_H.*

Definition 2.6. *A H_v-semigroup or a semihypergroup H is called cyclic if there exists $s \in H$, called generator, such that: $H = s^1 \cup s^2 \cup ... \cup s^n \cup ..., n \in N, n > 0$.*

For more definitions and applications on H_v-structures, see also the papers [4], [9], [10], [14], [15], [18], [20], [28].

3 The two approaches of fundamental relations

Searching for the quotient set, the definition of the relation between the elements of the hyperstructure plays an important role. The observation of a hyperoperation leads to the conclusion that elements belonging to the same hyperproduct act in a similar way with respect to the hyperoperation. This observation is the basis for defining the relation β. This definition is common to both approaches. Another common finding of the two approaches is the fact that the relation β is reflective, symmetric but not always transitive. The need for an equivalence relation that produces a quotient set such that it is a classical algebraic structure, makes it necessary to consider the β^* - relation that is the transitive closure of the β - relation and is, obviously, an equivalence relation. The last common point of the two approaches is the search for the smallest (with respect to the inclusion) equivalence relation having as quotient set the corresponding algebraic structure.

3.1 Koskas approach

Koskas, in his approach, introduces the equivalence relation that obtains as quotient set the corresponding algebraic structure by using the *strongly regular equivalence relation*. It is then shown that the transitive closure of the β - relation is the smallest strongly regular equivalence relation, i.e. β^* is the targeted relation. The proof is completed by the obvious finding that β^* is the desired equivalence relation such that the H/β^* is the corresponding algebraic structure. One can say that Koskas approach is a *deductive way* of defining fundamental relation on hyperstructures, since he starts considering a general definition and then specifying the β^*-relation as a subcase.

Taking into consideration the approach, the necessary definitions and theorems are mentioned, having as main sources the books [5], [7], [8].

Definition 3.1. *Let $(H, \circ)$ be a hypergroupoid, a, b elements of H and ρ be an equivalence relation on H. Then ρ is strongly regular on the left if the following implication holds:*

$$a\rho b \Rightarrow \forall u \in H, \forall x \in u \circ a, \forall y \in u \circ b : x\rho y.$$

Similarly, the strong regularity on the right can be defined. We call ρ strongly regular if it is strongly regular on both the left and the right.

Definition 3.2. *Let $(H, \cdot)$ be a semihypergroup and $n > 1$ be a natural number. We define the β_n relation as follows:*

$$x\beta_n y \quad \text{if there exist} \quad a_1, a_2, ..., a_n \quad \text{elements of} \quad H, \quad \text{so subsets} \quad \{x, y\} \subseteq \prod_{i=1}^{n} a_i.$$

and let

$$\beta = \bigcup_{n\geq 1} \beta_n, \quad \text{where} \quad \beta_1 = \{(x,x)/x \in H\} \quad \text{is the diagonal relation on } H.$$

Notice that relation β is reflexive and symmetric but, generally, not a transitive one.

Definition 3.3. *We denote β^* the transitive closure of relation β.*

Theorem 3.1. *β^* is the smallest strongly regular equivalence relation on H with respect to the inclusion.*

Theorem 3.2. *Let $(H,\cdot)$ be a semihypergroup (hypergroup), then the transitive closure of relation β is the smallest equivalence relation such that the quotient $H/\beta^*$is a semigroup (group).*

Definition 3.4. *β^* is called the fundamental relation on H and H/β^* is called the fundamental semigroup (group).*

Notice that [22] the term fundamental, given by Vougiouklis, is subsequent of Koskas definitions but totally used nowadays.

3.2 Vougiouklis approach

Unlike the previous ones, Vougiouklis approaches the issue in a straightforward way starting with the acquired question about the appropriate equivalence relations. He defines the relation that has as quotient set the corresponding algebraic structure. He then defines the relation β in a more general manner than previously defined. The approach has been completed by proving that the fundamental relation is no other than the transitive closure of the relation β. We can assume that Vougiouklis approach is an *inductive* way of defining the fundamental relation in hyperstructures because it starts with the partial and ends in a more general result.
It is important to note that Vougiouklis definitions were given for H_v-groups which are a wider class than the one of hypergroups. Also, the proof of theorem about β^* relation (see below) follows a remarkable strategy [3].

Taking into consideration the approach, the necessary definitions and theorems are mentioned, having as main source the book [25] and the papers [24], [26].

Definition 3.5. *Let $(H,\cdot)$ be a H_v-group. The relation β^* on H is called fundamental equivalence relation if it is the smallest equivalence relation on H such that the quotient set H/β^* is a group, called fundamental group of H.*

Notice that the proof of the fundamental groups existence for any H_v-group is an obvious result of the following theorem's proof.

Let us denote by U the set of all finite hyperproducts of elements of H.

Definition 3.6. *Let $(H, \cdot)$ be a H_v-group. We define the relation β on H as follows:*

$$x\beta y \quad \textit{iff} \quad \{x, y\} \subseteq u, u \in U.$$

Theorem 3.3. *The fundamental equivalence relation β^* is the transitive closure of the relation β on H.*

Definition 3.7. *Let $(R, +, \cdot)$ be a H_v-ring. The relation γ^* on R is called fundamental equivalence relation on R if it is the smallest equivalence relation on R such that the quotient set R/γ^* is a ring, called fundamental ring of R.*

Let us denote by U the set of all finite polynomials of elements of R, over N.

Definition 3.8. *Let $(R, +, \cdot)$ be a H_v-ring. We define the relation γ on H as follows:*

$$x\gamma y \quad \textit{iff} \quad \{x, y\} \subseteq u, u \in U.$$

Theorem 3.4. *The fundamental equivalence relation γ^* is the transitive closure of the relation γ on R.*

Definition 3.9. [26] *A H_v-ring is called H_v-field if its fundamental ring is a field.*

Definition 3.10. *Let V be a H_v-vector space over a H_v-field R. The relation ε^* on V is called fundamental equivalence relation if it is the smallest equivalence relation on V such that the quotient set V/ε^* is a vector space over the field R/γ^*, called fundamental vector space of V over R.*

4 Fundamental classes

Searching for the classes of fundamental equivalence relations is a central question in studying the fundamental structures derived from hyperstructures. This quest is intertwined with the exploration of the conditions that must be accomplished so that the β relation is transitive, that is, $\beta = \beta^*$. It is clear that the two different approaches to the fundamental equivalence relation in hyperstructures settle on two different ways of searching or constructing the fundamental equivalence classes in a hyperstructure. We could also talk, in a similar way with 3.1 and 3.2, about the *deductive* and *inductive* way of finding equivalence classes.

4.1 Complete Parts

According to the deductive way that Koskas used and Corsini's school continued, the equivalence classes, that occur when the equivalence relation is strongly regular, are used. The notion of complete part of a hyperstructure's subset plays a key role in finding the β^* class of each element. The complete closure $C(A)$ of the part A is connected with an increasing chain of subsets of the hyperstructure which, in turn, are related to hyperproducts containing A. It then turns out that the introduction, in a natural way, of the equivalence relation K is essentially a consideration of the equivalence β^*. In this way the complete closure coincides to the fundamental equivalence class.
In particular, the definition of the complete part is used in the case of the singleton $\{x\}$, for each element x of the hyperstructure, so that we find ourselves in the environment of the fundamental equivalence relation. The increasing chain of sets created by the set $\{x\}$ constructs the fundamental class of the arbitrary element x. It is evident that, as in the introduction of the β^* - relation [16], a notion is used as a mediator, which comes in between the questions "how is the class" and "what is the class". This notion is K relation.
We now present the necessary propositions in order to describe the step by step approach of the fundamental classes notion. The main references we use are the books [5], [7], [8] and the papers [13], [19].

Definition 4.1. *Let $(H, \cdot)$ be a semihypergroup and A be a nonempty subset of H. We say that A is a complete part of H if for any nonzero natural number n and for all $a_1, a_2, , a_n$ elements of H, the following implication holds:*

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \quad \Rightarrow \quad \prod_{i=1}^{n} a_i \subseteq A.$$

Notice that complete part A absorbs every hyperproduct containing one, at least, element of A.
According to theorem 3.1, $\beta^*(x)$ is a complete part of $H, \forall x \in H$. (Step 1)

Definition 4.2. *Let $(H, \cdot)$ be a semihypergroup and A be a nonempty subset of H. The intersection of the complete parts of H which contain A is called the complete closure of A in H; it will be denoted by $C(A)$.*

Denote $K_1(A) = A$ and for all $n > 0$

$$K_{n+1}(A) =$$

$$\left\{ x \in H | \quad \exists p \in N, \exists (h_1, h_2, ..., h_p) \in H^p : x \in \prod_{i=1}^{p}(h_i), \quad K_n(A) \cap \prod_{i=1}^{p}(h_i) \neq \emptyset \right\}.$$

Obviously, $K_n(A), n > 0$ is an increasing chain of subsets of H as we mentioned. If $x \in H$, we denote $K_n(\{x\}) = K_n(x)$. This implies that

$$K_n(A) = \bigcup_{a \in A} K_n(a)$$

In particular, if P is the set of all finite hyperproducts of elements of H, and $x \in H$ we have:

$$K_1(x) = \{x\}, \quad K_2(x) = \bigcup_{u \in P} u : x \in u, \quad K_3(x) = \bigcup_{u \in P} u : u \cap K_2(x) \neq \emptyset,$$

$$..., K_{n+1}(x) = \bigcup_{u \in P} u : u \cap K_n(x) \neq \emptyset \quad \text{and} \quad C(A) = \bigcup_{a \in A} C(a), A \subseteq H.$$

Notice that $K_2(x) = \{z \in H | z\beta x\} = \beta(x)$. (Step 2)

Theorem 4.1. *Let $(H, \cdot)$ be a semihypergroup and K a binary relation defined as follows:*

$$xKy \Leftrightarrow x \in C(y), (x, y) \in H^2.$$

Then,K is an equivalence relation that coincides with β^*. (Step 3)

Thus, the relation K and the chain of sets K_n behave as a mediator, which comes in between $C(x)$ and β^*, connecting the construction of the class with the class itself.
Now we present some propositions mainly about the transitivity of β-relation.

Theorem 4.2. [12] *Let $(H, \cdot)$ be a hypergroup then, $\beta = \beta^*$.*

Theorem 4.3. *Let $(H, \cdot)$ be a hypergroupoid. Then,*

$$\beta = \beta^* \Leftrightarrow C(x) = K_2(x), \forall x \in H.$$

Theorem 4.4. [13] *Let $(H, \cdot)$ be a H_v-group having, at least, one identity. Then,*

$$\beta = \beta^*.$$

Theorem 4.5. [13] *Let $(H, \cdot)$ be a H_b-group then, $\beta = \beta^*$.*

4.2 Constructing Fundamental Classes

According to the inductive way that Vougiouklis introduced, the direct approach to the fundamental class is used. Vougiouklis, while studying the fundamental classes, follows the same philosophy and strategy as he does in his approach to the introduction of the fundamental relationship and the corresponding

structure. He does not attempt to create an environment of desirable definitions in which he will then place the fundamental class of each element, as Koskas has chosen to do.
On the contrary, he considers the fundamental class of each element as a set and tries to specify its elements through their common properties. He prefers, in short, a straightforward reference to the common behaviour of these elements in the generation of hyperproducts. Therefore, he develops propositions that relate each other the elements which behave in a similar way, thus achieving the accumulation of all the equivalent, with respect to β^*, elements which belong to the same fundamental class. It is clear that this straightforward dealing with the class study is an inductive type of answer to the question of the nature and form of the fundamental classes.
The fundamental class which is a singleton, whenever it exists, plays an essential role in the study of H_v-groups and $H_v$rings. The element of each such class is called single element. Its value lies in the fact that each hyperproduct having a single element as a factor, is an entire fundamental class. Thus, finding the classes is achieved by multiplying a single element with each element of the hyperstructure. In fact, it is not necessary to perform the hyperoperation with all the elements. Moreover, the existence of one, at least, single element is a sufficient condition such that $\beta = \beta^*$ holds in H_v-groups.
Finally, the above mentioned approach also includes the technique of "translation" of hyperproducts which allows us to find the fundamental structure of an H_v-group and its classes using isomorphism between the quotient sets. We now present the necessary propositions in order to describe the direct approach of fundamental classes. The main references we use are the book [25] and the papers [24], [26].

Theorem 4.6. *Let $(H, \cdot)$ be a $H_v$group, then*

$$x\beta^* y \quad \textit{iff there exist} \quad A, A^{'} \subseteq \beta^*(a), \quad B, B^{'} \subseteq \beta^*(b), \quad (a,b) \in H^2,$$

$$\textit{such that} \quad xA \cap B \neq \emptyset \quad \textit{and} \quad yA^{'} \cap B^{'} \neq \emptyset.$$

Theorem 4.7. *Let $(H, \cdot)$ be a H_v-group, then $u \in \omega_H$ iff there exist $A \subseteq \beta^*(a)$, for some $a \in H$, such that $uA \cap A \neq \emptyset$.*

Definition 4.3. *Let H be a H_v-structure. An element $s \in H$ is called single if $\beta^*(s) = \{s\}$.*

We denote by S_H the set of singles elements of H

Theorem 4.8. *Let $(H, \cdot)$ be a H_v-group and $s \in S_H$. Let $(a, v) \in H^2$ such that $s \in av$, then*

$$\beta^*(a) = \{h \in H : hv = s\} \quad \textit{and the core of H is} \quad \omega_H = \{z \in H : zs = s\}.$$

Theorem 4.9. *Let $(H, \cdot)$ be a H_v-group and $S_H \neq \emptyset$, then $\beta^* = \beta$.*

Definition 4.4. *(Translations)* [25] *Let $(H, \cdot)$ be a H_v-group and $x \in H$, then $H/\beta^* \cong (H/l_x)/\beta^*$, where l_x is the translation equivalence relation.*

4.3 Using the fundamental equivalence relations

The fundamental equivalence relations on hyperstructures, on the one hand, connect the theory of hyperstructures to that of the corresponding classical structures, and on the other hand, are a tool for the introduction of new hyperstructure classes. The two approaches to the concept of the fundamental equivalence relations were initially referred to semihypergroups and hypergroups. However, they form the driving lever to apply similar definitions to other hyperstructures as hyperrrings and hyperfields or to study specific behaviour of some hyperstructures using the quotient set.
Freni [11] and others (P. Corsini, B. Davazz) introduced and studied equivalence relations that refer to individual properties or characteristics of elements of a hyperstructure. As an example, we mention the equivalence relations Δ_{h^*} and α^* which are related to the cyclicity and commutativity of a hyperstructure, respectively. Our main reference is [8].
Relation α was introduced by Freni who used the letter γ instead of α. Thus, there was a confusion on symbolism since Vougiouklis had already used the letter γ for the fundamental equivalence relation on hyperrings.

Vougiouklis focused on the study of hyperstructures which have a desirable quotient.(h/v)-structures are a typical example of generalization using the fundamental structures. They are a larger class than that of H_v-structures. Also, the use of the fundamental classes of equivalence as hyperproducts, led to constructions of new hyperstructure classes. As additional examples of hyperstructures with desirable quotient we mention the $s1, s2, ..., sn$-hyperstructures [3], the complete (in the sense of Corsini) H_v-groups and the constructions S_1 and S_2 [25].

Definition 4.5. [23] *The H_v-semigroup $(H, \cdot)$ is called h/v-group if H/β^* is a group.*

Notice that the reproductivity is not necessarily valid. However, the reproductivity of classes is valid.

Definition 4.6. [1], [2]. *We shall say that a hyperstructure is an sn-hyperstructure if all its fundamental classes are singleton except for n of them, $n \in N, n > 0$.*

5 Conclusions - Findings

The aim of this paper was to present the two different approaches to the concept of the fundamental equivalence relation through a comparison of the choices and methods applied by Koskas and Vougiouklis, but also through the results they have achieved. Summarizing our conclusions and findings, we can note that:
a) The approach of Koskas is a deductive way based on the introduction of general notions willing to draw conclusions to more specific ones. On the contrary, Vougiouklis makes the reverse. Beginning with the specific notion leads to propositions generalizing his conclusions.
b) The directness of Vougiouklis approach allows for the "transfer" of definitions and corresponding theorems from the H_v-groups to H_v-rings, H_v-modules and the other weak hyperstructures. In essence, it is a widely applied model that imposed, among other things, on the terminology of the "fundamental" relations.
c) The nature of the step-by-step approach of Koskas has led to frequent use of the *mathematical induction method* in proving many basic theorems. On the other hand, due to the direct reference of Vougiouklis to the equivalence classes and their view as sets of elements, the most frequent proving method used is the *proof by contradiction*.
d) The Koskas approach raises the question: *What is the quotient set of the hyperstructure that we are studying?* That is why increasing chains of relations are the main study tool. Vougiouklis approach reverses the question with the following one: *Which hyperstructure produces a particular desired fundamental quotient?* In this inversion, there is a trigger for the introduction of weak and h/v-structures.
Finally, we consider that the mathematical value of fundamental equivalence relations in hyperstructures is important and their asynchronized approaches by Koskas and Vougiouklis is an interesting moment of the short history of algebraic hyperstructures from their beginning at 1934 until today.

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 61-76 **eISSN**: 2282-8214

Some Remarks on Hyperstructures their Connections with Fuzzy Sets and Extensions to Weak Structures

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†**doi**:10.23755/rm.v33i0.384

Abstract

A brief excursus on the last results on Hyperstructures and their connections with Fuzzy Sets. At the end a calculation of the Fuzzy Grade of Hv-structures of order two.
Keywords: hyperstructure
2010 AMS Mathematics Subject Classification: 20N20

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 Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

One knows that to every fuzzy set (H, μ_0) one hypergroup can be associated (which I proved [9], is a join space) in the following way:
$\forall (x, y) \in H^2$, one sets

$$x \circ_0 y = \{z \mid min\{\mu_0(x), \mu_0(y)\} \leq \mu_0(z) \leq max\{\mu_0(x), \mu_0(y)\}.$$

I proved also [18] that to every hypergroupoid $(H, \circ)$ a fuzzy set corresponds, defined as you can see below:
Set $\forall u \in H,\ Q(z) = \{(x, y) \mid u \in x \circ y\},\ q(u) = |Q(u)|.$

$$A(u) = \sum_{(x,y) \in Q(u)} 1/|x \circ y|, \quad \mu_1(u) = A(u)/q(u).$$

I proved that $H_0 = (H, \circ)$ is a join space.
So, to every hypergroupoid, a sequence of hypergroupoids and fuzzy sets is associated: $(H, \mu_0),\ (H, \mu_1), ...$
If $|H| < \aleph_0$, then the sequence is clearly finite.
We call *fuzzy grade* [20] of $(H, \circ)$ the minimum natural number of k, such that two consecutive join spaces are isomorphic.
For the H_v-structures, notion introduced by T. Vougiouklis, one can proceed in a similar way.
So, one defines the fuzzy grade of a H_v-hypergroupoid as

$$min\{k \mid H_k \simeq H_{k+1}\}.$$

Thomas Vougiouklis is author of many papers on Hyperstructures.
Just at the beginnng of his activity he invented and studied a structure, defining the following hyperoperation: given a hypergroupoid $(H; *)$ and a non empty subset P of H, he set $x \circ y = x * P * y$ and found several interesting results on this hyperoperation.
But the most important theory that he introduced is that one of the H_v-hyperstructures. He replaced the notion of associativity with that one of "weak associativity". That is instead of supposing
for every $x, y, z \in H,\ (x * y) * z = x * (y * z)$, one supposes

$$(x * y) * z \cap x * (y * z) \neq \emptyset.$$

One has considered also weak rings. It is enough to set
for every a, b, c in R, $a \circ (b + c) \cap (a \circ b + a \circ c) \neq \emptyset$.

The idea by Vougiouklis of considering weak Hyperstructures opened a new branch of Mathematics. Many significant results have been obtained in this field and probably many others will be found in the future.

A theme which deserves to be considered in this context is that one of HX structures. HX-groups were born in China, invented by Li Hongxing [81], and studied by him, Wang and others, see [79], [80], [87], [117], [118], [119]. In Italy, Corsini extended this notion to Hyperstructures. He and Cristea in Italy, Fotea in Romania, Kellil and Bouaziz in Saudi Arabia worked in this direction.

Given a group G and the set $\mathcal{P}^*(G)$ of all nonempty subsets of G, endowed with the operation

$$\forall(A, B) \in \mathcal{P}^*(G) \times \mathcal{P}^*(G),\ A \circ B = \{xy \mid x \in A, y \in B\}$$

a subgroup of $\mathcal{P}^*(G)$ is called an *HX-group.* One has calculated the fuzzy grade for $\mathbf{Z}/n\mathbf{Z}$ for $n \leq 16$ and also for other structures, for instance for the multiplicative group $\mathbf{Z}_2^{2,2}$ and the direct product of some $\mathbf{Z}/n\mathbf{Z}$, see [22], [23], [24].

It would very interesting to consider the same problems in the such general context of weak structures, that is to calculate the fuzzy grade of HX-hypergroup $\mathbf{Z}_n$.

Given an HX-group F, one considers the set F' of all nonempty subsets of F. Let us suppose that K is a subgroup of F.

We define the following hyperoperation

$$x \otimes y = \bigcup_{x \in A,\ y \in B,\ \{A,B\} \subseteq K} AB$$

in the set $\cup_{A \in K} A$.

The structure $(H, \otimes)$ is called an *HX-hypergroupoid.*

One can extend the notion of HX-hypergroup to weak hyperstructures.

Some open problems on weak structures:

- find conditions for an HX-hypergroupoid to be a hypergroup;
- the fuzzy grade of HX- weak hypergroups already considered in the classic case.

2 H_v-hypergroupoids of order 2

The H_v-hypergroupoids of order 2, which are not associative, are 10.

The following [12], [13], [15] have fuzzy grade 1.
The others [9], [10], [11], [14], [16], [17], [18] have fuzzy grade 2.

-

H_{12}	a	b
a	H	H
b	b	a

We have $q_1(a) = 3,\ A_1(a) = 2$, so $\mu_1(a) = 2/3$.

$q_1(b) = 3,\ A_1(b) = 2$, so $\mu_1(b) = 2/3$.

Then $\partial H_{12} = 1$.

-

H_{13}	a	b
a	H	b
b	H	a

We have $q_1(a) = 3,\ A_1(a) = 2$, so $\mu_1(a) = 2/3$.

$q_1(b) = 3,\ A_1(b) = 2$, so $\mu_1(b) = 2/3$.

Then $\partial H_{13} = 1$.

-

H_{15}	a	b
a	H	a
b	b	H

Indeed we find $q_1(a) = 3,\ A_1(a) = 2$, so $\mu_1(a) = 2/3$.

$q_1(b) = 3,\ A_1(b) = 2$, so $\mu_1(b) = 2/3$.

Hence $H_1 = T$, the total hypergroup. Therefore $\partial H_{15} = 1$.

H_9	a	b
a	H	b
b	b	a

  We have $q_1(a) = 2,\ A_1(a) = 3/2$, so $\mu_1(a) = 3/4 = 0.75$.

  $q_1(b) = 3,\ A_1(b) = 5/2$, so $\mu_1(b) = 5/6 = 0.8333$.

  So we obtain

H_9^1	a	b
a	a	H
b	H	b

  Therefore $\mu_2(a) = \mu_2(b)$, whence H_9^2 is the total hypergroup, whence $\partial H_9 = 2$.

H_{10}	a	b
a	a	H
b	b	a

  We find $q_1(a) = 2,\ A_1(a) = 5/2$, so $\mu_1(a) = 0.833$.

  $q_1(b) = 2,\ A_1(b) = 3/2$, so $\mu_1(b) = 3/4 = 0.75$.

  We obtain

H_{10}^1	a	b
a	a	H
b	H	b

  so $\mu_2(a) = \mu_2(b)$, whence H_{10}^2 is the total hypergroup, whence $\partial H_{10} = 2$.

H_{11}	a	b
a	b	H
b	a	b

  We find $q_1(a) = 2,\ A_1(a) = 3/2$, so $\mu_1(a) = 0.75$.

  $q_1(b) = 3,\ A_1(b) = 5/2$, so $\mu_1(b) = 0.833$.

It follows

H_{11}^1	a	b
a	a	H
b	H	b

so $\mu_2(a) = \mu_2(b)$, so $\partial H_{11} = 2$.

•

H_{14}	a	b
a	H	a
b	a	H

We find $q_1(a) = 4,\ A_1(a) = 3$, so $\mu_1(a) = 0.75$.
$q_1(b) = 2,\ A_1(b) = 1$, so $\mu_1(b) = 0.50$.
whence we hve

H_{14}^1	a	b
a	a	H
b	H	b

By consequence H_{14}^2 is the total hypergroup, whence $\partial H_{14} = 2$.

•

H_{16}	a	b
a	a	H
b	H	a

We find $q_1(a) = 4,\ A_1(a) = 3$, so $\mu_1(a) = 3/4 = 0.75$.
$q_1(b) = 2, A_1(b) = 1, \mu_1(b) = 0.50$.
It follows

H_{16}^1	a	b
a	a	H
b	H	b

so $\mu_2(a) = \mu_2(b)$, whence H_{16}^2 is the total hypergroup, whence $\partial H_{16} = 2$.

-

H_{17}	a	b
a	H	H
b	a	H

We find $q_1(a) = 4,\ A_1(a) = 5/2$, so $\mu_1(a) = 0.625$.

$q_1(b) = 3,\ A_1(b) = 3/2$, so $\mu_1(b) = 0.50$.

By cnsequence

H^1_{17}	a	b
a	a	H
b	H	b

whence H^2_{17} is the total hypergroup, therefore $\partial H_{17} = 2$.

-

H_{18}	a	b
a	H	H
b	b	H

We find $q_1(a) = 3,\ A_1(a) = 3/2$, so $\mu_1(a) = 0.50$.

$q_1(b) = 4,\ A_1(b) = 5/2$, so $\mu_1(b) = 0.625$.

We obtain

H^1_{18}	a	b
a	a	H
b	H	b

By consequence, H^2_{18} is the total hypergroup, whence $\partial H_{18} = 2$.

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 77-88 **eISSN**: 2282-8214

Vougiouklis Contributions in the Field of Algebraic Hyperstructures

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†**doi**:10.23755/rm.v33i0.380

Abstract

Thomas Vougiouklis was born in 1948, Greece. He has many contributions to algebraic hyperstructures. H_v-structures are some of his main contributions. In this article, we study some of Vougiouklis ideas in the field of algebraic hyperstructures as follows: (1) Semi-direct hyperproduct of two hypergroups; (2) Representation of hypergroups; (3) Fundamental relation in hyperrings; (4) Commutative rings obtained from hyperrings; (5) H_v-structures; (6) The uniting elements method; (7) The e-hyperstructures; (8) Helix-hyperoperations.
Keywords: hyperoperation, hypergroup, hyperring, H_v-group, H_v-ring, fundamental relation.
2010 AMS subject classifications: 20N20, 16Y99.

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†©Bijan Davvaz. Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Semi-direct hyperproduct of two hypergroups

For all natural numbers $n > 1$, define the relation β_n on a semihypergroup H, as follows: $a\beta_n b$ if and only if there exist $x_1, \ldots, x_n \in H$ such that $\{a, b\} \subseteq \prod_{i=1}^{n} x_i$, and take $\beta = \bigcup_{n\geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. Denote by β^* the transitive closure of β. The relation β^* is a strongly regular relation. This relation was introduced by Koskas [10] and studied mainly by Freni [9], proving the following basic result: If H is hypergroup, then $\beta = \beta^*$. Note that, in general, for a semihypergroup may be $\beta \neq \beta^*$. Moreover, the relation β^* is the smallest equivalence relation on a hypergroup H, such that the quotient H/β^* is a group. The *heart* ω_H of a hypergroup H is defined like the set of all elements x of H, for which the equivalence class $\beta^*(x)$ is the identity of the quotient group H/β^*. Vougiouklis in [13] studied the fundamental relation introduced by Koskas. He used the quotient set in order to define a semi-direct hyperproduct of two hypergroups. He obtained an extension of hypergroups by hypergroups. Let A, B be two hypergroups and consider the group $AutA$. Let $\widehat{\ } : B/\beta^* \to AutA$ be an arbitrary homomorphism, where we denote $\widehat{\beta^*(b)}$ by $\widehat{b}$. Then, in $A \times B$ a hyperproduct can be defined as follows: $(a, b) \odot (c, d) = \{(x, y) \mid x \in a\widehat{b}(c),\ y \in bd\}$ Then, $A \times B$ becomes a hypergroup called semi-direct hyperproduct of A and B corresponding to $\widehat{\ }$ and it is denoted by $A \widehat{\times} B$. Vougiouklis proved that $A\widehat{\times}B/\beta^*_{A\widehat{\times}B} \cong A/\beta^*_A \widehat{\times} B/\beta^*_B$ [13].

2 Representation of hypergroups

Vougiouklis in a sequence of papers studied the representations of hypergroups. For instance, in [15], a class of hypermatrices to represent hypergroups is introduced and application on class of P-hypergroups is given. Hypermatrices are matrices with entries of a semihyperring. The product of two hypermatrices (a_{ij}) and (b_{ij}) is the hyperoperation given in the usual manner $(a_{ij}) \cdot (b_{ij}) = \{(c_{ij}) \mid c_{ij} \in \sum_{k=1}^{n} a_{ik} b_{kj}\}$. Vougiouklis problem is the following one: For a given hypergroup H, find a semihyperring R such that to have a representation of H by hypermatrices with entries from R. Recall that if $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$, then a map $T : H \to M_R$ is called a representation if $T(x) \cdot T(y) = \{T(z) \mid z \in xy\} = T(xy)$, for all $x, y \in H$. He obtained an induced representation T^* for the hypergroup algebra of H, see [14].

3 Fundamental relation in hyperrings

Vougiouklis introduced the notion of fundamental relation in the context of general hyperrings [16, 17]. A multivalued system $(R, +, \cdot)$ is a (general) hyperring if (1) $(R, +)$ is a hypergroup; (2) $(R, \cdot)$ is a semihypergroup; (3) $(\cdot)$ is (strong) distributive with respect to $(+)$, i.e., for all x, y, z in R we have $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. In this paragraph, we use the term of a hyperring, instead of the term of a general hyperring, intending the above definition. A hyperring may be commutative with respect to $(+)$ or $(\cdot)$. If R is commutative with respect to both $(+)$ and $(\cdot)$, then it is a commutative hyperring. The above definition contains the class of multiplicative hyperrings and additive hyperrings as well. In the above hyperstructures, Vougiouklis introduced the equivalence relation γ^*, which is similar to the relation β^*, defined in every hypergroup. Let $(R, +, \cdot)$ be a hyperring. He defined the relation γ as follows: $a\gamma b$ if and only if $\{a, b\} \subseteq u$, where u is a finite sum of finite products of elements of R. Denote the transitive closure of γ by γ^*. The equivalence relation γ^* is called the fundamental equivalence relation in R. According to the distributive law, every set which is the value of a polynomial in elements of R is a subset of a sum of products in R. Let $\mathcal{U}$ be the set of all finite sums of products of elements of R. We can rewrite the definition of γ^* on R as follows: $a\gamma^* b$ if and only if there exist $z_1, ..., z_{n+1} \in R$ with $z_1{=}a$, $z_{n+1}{=}b$ and $u_1, ..., u_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i$ for $i \in \{1, ..., n\}$. Let $(R, +, \cdot)$ be a hyperring. Then the relation γ^* is the smallest equivalence relation in R such that the quotient R/γ^* is a ring [16]. The both $\oplus$ and $\odot$ on R/γ^* are defined as follows: $\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c)$, for all $c \in \gamma^*(a) + \gamma^*(b)$ and $\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d)$, for all $d \in \gamma^*(a) \cdot \gamma^*(b)$. If $u = \sum_{j\in J}(\prod_{i\in I_j} x_i) \in \mathcal{U}$, then for all $z \in u$, we have $\gamma^*(u) = \oplus \sum_{j\in J}(\odot \prod_{i\in I_j} \gamma^*(x_i)) = \gamma^*(z)$, where $\oplus \sum$ and $\odot \prod$ denote the sum and the product of classes.

4 Commutative rings obtained from hyperrings

The commutativity in addition in rings can be related with the existence of the unit in multiplication. If e is the unit in a ring then for all elements a, b we have $(a + b)(e + e) = (a + b)e + (a + b)e = a + b + a + b$ and $(a + b)(e + e) = a(e + e) + b(e + e) = a + a + b + b$. So $a + b + a + b = a + a + b + b$ gives $b + a = a + b$. Therefore, when we say $(R, +, \cdot)$ is a hyperring, $(+)$ is not commutative and there is not unit in the multiplication. So the commutativity, as well as the existence of the unit, it is not assumed in the fundamental ring. Of course, we know there exist many rings (+ is commutative) while don't have unit. Davvaz and Vougiouklis were interested in the fundamental ring to be commutative with respect to both sum and product, that is, the fundamental ring be an ordinary commutative ring. Therefore they introduced the following definition. Let R be a hyperring. Define the relation α as follows: $x\alpha y$ if and only if $\exists n \in \mathbb{N}$, $\exists(k_1,\ldots,k_n) \in \mathbb{N}^n$, $\exists\sigma \in S_n$ and $[\exists(x_{i1},\ldots,x_{ik_i}) \in R^{k_i}$, $\exists\sigma_i \in S_{k_i}$, $(i = 1,\ldots,n)]$ such that $x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in \sum_{i=1}^{n} A_{\sigma(i)}$, where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$. The relation α is reflexive and symmetric. Let α^* be the transitive closure of α, then α^* is a strongly regular relation both on $(R,+)$ and $(R,\cdot)$ [4]. The quotient R/α^* is a commutative ring [4]. Notice that they used the Greek letter α for the relation because of the 'A'belian. The relation α^* is the smallest equivalence relation such that the quotient R/α^* is a commutative ring [4].

5 H_v-structures

During the 4^{th} Congress of Algebraic Hyperstructures and Applications (Xanthi, 1990), Vougiouklis introduced the concept of the weak hyperstructures which are now named H_v-structures. Over the last 27 years this class of hyperstructure, which is the largest, has been studied from several aspects as well as in connection with many other topics of mathematics. The hyperstructure $(H,\cdot)$ is called an H_v-group if (1) $x\cdot(y\cdot z)\cap(x\cdot y)\cdot z \neq \emptyset$, for all $x,y,z \in H$; (2) $a\cdot H = H\cdot a = H$, for all $a \in H$. A motivation to obtain the above structures is the following. Let $(G,\cdot)$ be a group and R an equivalence relation on G. In G/R consider the hyperoperation $\odot$ such that $\overline{x}\odot\overline{y} = \{\overline{z}|\ z \in \overline{x}\cdot\overline{y}\}$, where $\overline{x}$ denotes the class of the element x. Then $(G,\odot)$ is an H_v-group which is not always a hypergroup [20]. Let $(H_1,\cdot)$, $(H_2,\star)$ be two H_v-groups. A map $f : H_1 \to H_2$ is called an H_v-homomorphism or weak homomorphism if $f(x\cdot y)\cap f(x)\star f(y) \neq \emptyset$, for all $x,y \in H_1$. The map f is called an inclusion homomorphism if $f(x\cdot y) \subseteq f(x)\star f(y)$, for all $x,y \in H_1$. Finally, f is called a strong homomorphism if $f(x\cdot y) = f(x)*f(y)$, for all $x,y \in H_1$. If f is onto, one to one and strong homomorphism, then it is called isomorphism, if moreover f is defined on the same H_v-group then it is called automorphism. It is an easy verification that the set of all automorphisms in H, written $AutH$, is a group. On a set H several H_v-structures can be defined. A partial order on those hyperstructures is introduced as follows. Let $(H,\cdot)$, $(H,\star)$ be two H_v-groups defined on the same set H. We call $\cdot$ less than or equal to $\star$, and write $\cdot \leq \star$, if there is $f \in Aut(H,*)$ such that $x\cdot y \subseteq f(x\star y)$, for all $x,y \in H$ [20]. A quasi-hypergroup is called a hypergroupoid $(H,\cdot)$ if the reproduction axiom is valid. In [20], it is proved that all the quasi-hypergroups with two elements are H_v-groups. It is also proved that up to the isomorphism there are exactly 18 different H_v-groups. If a hyperoperation is weak associative then every greater hyperoperation, defined on the same set is also weak associative. In [21], using this property, the set of all H_v-groups with a scalar unit defined on a set with three elements is determined, also, see [22]. Let $(H, \cdot)$ be an H_v-group. The relation β^* is the smallest equivalence relation on H such that the quotient H/β^*, the set of all equivalence classes, is a group. β^* is called the fundamental equivalence relation on H. According to [19] if $\mathcal{U}$ denotes the set of all the finite products of elements of H, then a relation β can be defined on H whose transitive closure is the fundamental relation β^*. The relation β is as follows: for x and y in H we write $x\beta y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \mathcal{U}$. We can rewrite the definition of β^* on H as follows: $a\beta^* b$ if and only if there exist $z_1, \ldots, z_{n+1} \in H$ with $z_1 = a,\ z_{n+1} = b$ and $u_1, \ldots, u_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i \ \ (i = 1, \ldots, n)$. The product $\odot$ on H/β^* is defined as follows: $\beta^*(a) \odot \beta^*(b) = \{\beta^*(c) |\ c \in \beta^*(a) \cdot \beta^*(b)\}$, for all $a, b \in H$. It is proved in [19] that $\beta^*(a) \odot \beta^*(b)$ is the singleton $\{\beta^*(c)\}$ for all $c \in \beta^*(a) \cdot \beta^*(b)$. In this way H/β^* becomes a hypergroup. If we put $\beta^*(a) \odot \beta^*(b) = \beta^*(c)$, then H/β^* becomes a group. A multi-valued system $(R, +, \cdot)$ is an H_v-ring if (1) $(R, +)$ is an H_v-group; (2) $(R, \cdot)$ is an H_v-semigroup; (3) $(\cdot)$ is weak distributive with respect to $(+)$, i.e., for all x, y, z in R we have $(x \cdot (y + z)) \cap (x \cdot y + x \cdot z) \neq \emptyset$ and $((x + y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \emptyset$. Let $(R, +, \cdot)$ be an H_v-ring. Define γ^* as the smallest equivalence relation such that the quotient R/γ^* is a ring. Let us denote the set of all finite polynomials of elements of R over $\mathbb{N}$ by $\mathcal{U}$. Define the relation γ as follows: $x\gamma y$ if and only if $\{x, y\} \subseteq u$, where $u \in \mathcal{U}$. The fundamental equivalence relation γ^* is the transitive closure of the relation γ [12]. Vougiouklis also introduced H_v-vector spaces in [18].

6 The uniting elements method

In 1989, Corsini and Vougiouklis [1], introduced a method, the uniting elements method, to obtain stricter algebraic structures, from given ones, through hyperstructure theory. This method was introduced before the introduction of the H_v-structures, but in fact the H_v-structures appeared in the procedure. This method is the following. Let G be a structure and d be a property, which is not valid, and suppose that d is described by a set of equations. Consider the partition in G for which it is put together, in the same class, every pair of elements that causes the non-validity of d. The quotient G/d is an H_v-structure. Then quotient of G/d by the fundamental relation β^*, is a stricter structure $(G/d)/\beta^*$ for which d is valid. An application of the uniting elements is when more than one properties are desired. The reason for this is some of the properties lead straighter to the classes than others. The commutativity and the reproductivity are easily applicable. One can do this because the following statement is valid. Let $(G, \cdot)$ be a groupoid, and $F = \{f_1, ..., f_m, f_{m+1}, ..., f_{m+n}\}$ a system of equations on G consisting of two subsystems $F_m = \{f_1, ..., f_m\}$ and $F_n = f_{m+1}, ..., f_{m+n}\}$. Let

σ and σ_m be the equivalence relations defined by the uniting elements using the F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m = (G_m/\sigma_n)/\beta^*$. Then, we have $(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*$[19].

7 The e-hyperstructures

In 1996, Santilli and Vougiouklis point out that in physics the most interesting hyperstructures are the one called e-hyperstructures. The e-hyperstructures are a special kind of hyperstructures and, they can be interpreted as a generalization of two important concepts for physics: Isotopies and Genotopies. On the other hand, biological systems such as cells or organisms at large are open and irreversible because they grow. The representation of more complex systems, such as neural networks, requires more advances methods, such as hyperstructures. In this manner, e-hyperstructures can play a significant role for the representation of complex systems in physics and biology, such as nuclear fusion, the reproduction of cells or neural systems. They are the most important tools in Lie-Santilli theory too [2, 11]. A hypergroupoid $(H, \cdot)$ is called an e-hypergroupoid if H contains a scalar identity (also called unit) e, which means that for all $x \in H,\ x{\cdot}e = e{\cdot}x = x$. In an e-hypergroupoid, an element x' is called inverse of a given element $x \in H$ if $e \in x \cdot x' \cap x' \cdot x$. Clearly, if a hypergroupoid contains a scalar unit, then it is unique, while the inverses are not necessarily unique. In what follows, we use some examples which are obtained as follows: Take a set where an operation "$\cdot$" is defined, then we "enlarge" the operation putting more elements in the products of some pairs. Thus a hyperoperation "$\circ$" can be obtained, for which we have $x \cdot y \in x \circ y,\ \forall x, y \in H$. Recall that the hyperstructures obtained in this way are H_b-structures. Consider the usual multiplication on the subset $\{1,\ -1,\ i,\ -i\}$ of complex numbers. Then, we can consider the hyperoperation $\circ$ defined in the following table:

$\circ$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	$i, -i$
i	i	$-i$	-1	1
$-i$	$-i$	i	$1, i$	$-1, i$

We enlarged the products $(-1){\cdot}(-i)$, $(-i){\cdot}i$ and $(-i){\cdot}(-i)$ by setting

$$(-1)\circ(-i)=\{i,-i\},\ (-i)\circ i=\{1,i\} \text{ and } (-i)\circ(-i)=\{-1,i\}.$$

We obtain an e-hypergroupoid, with the scalar unit 1. The inverses of the elements $-1, i, -i$ are $-1, -i, i$ respectively. Moreover, the above structure is an

H_v-abelian group, which means that the hyperoperation $\circ$ is weak associative, weak commutative and the reproductive axiom holds. The weak associativity is valid for all H_b-structures with associative basic operations [19]. We are interested now in another kind of an e-hyperstructure, which is the e-hyperfield. A set F, endowed with an operation "+", which we call addition and a hyperoperation, called multiplication "$\cdot$", is said to be an e-hyperfield if the following axioms are valid: (1) $(F,+)$ is an abelian group where 0 is the additive unit; (2) the multiplication $\cdot$ is weak associative; (3) the multiplication $\cdot$ is weak distributive with respect to +,i.e., for all $x,y,z \in F$, $x(y+z)\cap(xy+xz) \neq \emptyset$, $(x+y)z\cap(xz+yz) \neq \emptyset$; (4) 0 is an absorbing element, i.e., for all $x \in F,\ 0{\cdot}x = x{\cdot}0 = 0$; (5) there exists a multiplicative scalar unit 1, i.e., for all $x \in F,\ 1\cdot x = x\cdot 1 = x$; (6) for every element $x \in F$ there exists an inverse x^{-1}, such that $1 \in x{\cdot}x^{-1}\cap x^{-1}{\cdot}x$. The elements of an e-hyperfield $(F,+,\cdot)$ are called e-hypernumbers. We can define the product of two e-matrices in an usual manner: the elements of product of two e-matrices $(a_{ij}),(b_{ij})$ are $c_{ij} = \sum a_{ik} \circ b_{kj}$, where the sum of products is the usual sum of sets. Let $(F,+,\cdot)$ be an e-hyperfield. An ordered set $a = (a_1, a_2, \ldots, a_n)$ of n e-hypernumbers of F is called an e-hypervector and the e-hypernumbers $a_i,\ i \in \{1,2,\cdot,n\}$ are called components of the e-hypervector a. Two e-hypervectors are equals if they have equal corresponding components. The hypersums of two e-hypervectors a,b is defined as follows: $a+b = \{(c_1,c_2,\ldots,c_n) \mid c_i \in a_i+b_i,\ i \in \{1,2,\cdot,n\}\}$. The scalar hypermultiplication of an e-hypervector a by an e-hypernumber λ is defined in a usual manner: $\lambda \circ a = \{(c_1,c_2,\ldots,c_n) \mid c_i \in \lambda \cdot a_i,\ i \in \{1,2,\ldots,n\}\}$. The set F^n of all e-hypervectors with elements of F, endowed with the hypersum and the scalar hypermultiplication is called n-dimensional e-hypervector space. The set of $m \times n$ hypermatrices is an mn-dimensional e-hypervector space. We refer the readers to [5, 6, 7, 8] for more details.

8 Helix-hyperoperations

Algebraic hyperstructures are a generalization of the classical algebraic structures which, among others, are appropriate in two directions: (a) to represent a lot of application in an algebraic model, (b) to overcome restrictions ordinary structures usually have. Concerning the second direction the restrictions of the ordinary matrix algebra can be overcome by the helix-operations. More precisely, the helix addition and the helix-multiplication can be defined on every type of matrices [3, 23, 24]. Let $A = (a_{ij}) \in M_{m\times n}$ be a matrix and $s,t \in \mathbb{N}$ be two natural numbers such that $1 \leq s \leq m$ and $1 \leq t \leq n$. Then we define the characteristic-like map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{cst} = (a_{ij})$, where $1 \leq i \leq s$ and $1 \leq j \leq t$. We call this map cut-projection of type $\underline{st}$. In other words, $A\underline{cst}$ is a matrix obtained from A by cutting the lines and columns greater than s and t respectively. Let $A = (a_{ij}) \in M_{m\times n}$ be a matrix and $s, t \in \mathbb{N}$ be two natural numbers such that $1 \leq s \leq m$ and $1 \leq t \leq n$. Then we define the mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st} = (A_{ij})$ which has as entries the sets $A_{ij} = \{a_{i+ks,j+\lambda t} \mid k, \lambda \in \mathbb{N},\ i + ks \leq m,\ j + \lambda t \leq n\}$, for $1 \leq i \leq s$ and $1 \leq j \leq t$. We call this multivalued map helix-projection of type $\underline{st}$. Therefore, $A\underline{st}$ is a set of $s \times t$-matrices $X = (x_{ij})$ such that $x_{ij} \in A_{ij}$ for all i, j. Obviously, $A\underline{mn} = A$. Let us consider the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 & 2 \\ 3 & 2 & 0 & 1 & 2 \\ 2 & 4 & 5 & 1 & -1 \\ 1 & -1 & 0 & 0 & 8 \end{bmatrix}.$$

Suppose that $s = 3$ and $t = 2$. Then

$$A\underline{c32} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix}$$

and $A\underline{32} = (A_{ij})$, where

$$\begin{aligned}
A_{11} &= \{a_{11}, a_{13}, a_{15}, a_{41}, a_{43}, a_{45}\} = \{2, 3, 2, 1, 0, 8\}, \\
A_{12} &= \{a_{12}, a_{14}, a_{42}, a_{44}\} = \{1, 4, -1, 0\}, \\
A_{21} &= \{a_{21}, a_{23}, a_{25}\} = \{3, 0, 2\}, \\
A_{22} &= \{a_{22}, a_{24}\} = \{2, 1\}, \\
A_{31} &= \{a_{31}, a_{33}, a_{35}\} = \{2, 5, -1\}, \\
A_{32} &= \{a_{32}, a_{34}\} = \{4, 1\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
A\underline{32} = (A_{ij}) &= \begin{bmatrix} \{2,3,1,0,8\} & \{1,4,-1,0\} \\ \{3,0,2\} & \{2,1\} \\ \{2,5,-1\} & \{4,1\} \end{bmatrix} \\
&= \{(x_{ij}) \mid x_{11} \in \{0,1,2,3,8\}, x_{12} \in \{-1,0,1,4\}, x_{21} \in \{0,2,3\}, \\
&\qquad x_{22} \in \{1,2\}, x_{31} \in \{-1,2,5\}, x_{32} \in \{1,4\}\}.
\end{aligned}$$

Therefore $|A\underline{32}| = 720$.

Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (a_{ij}) \in M_{u\times v}$ be two matrices and $s = \min(m, u)$, $t = \min(n, u)$. We define an addition, which we call cut-addition, as follows:

$$\begin{aligned}
\oplus_c :\ & M_{m\times n} \times M_{u\times v} \longrightarrow M_{s\times t} \\
& (A, B) \mapsto A \oplus_c B = A\underline{cst} + B\underline{cst}.
\end{aligned}$$

Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (a_{ij}) \in M_{u\times v}$ be two matrices and $s = \min(n,u)$. Then we define a multiplication, which we call cut-multiplication, as follows:

$$\begin{array}{rl} \otimes_c: & M_{m\times n} \times M_{u\times v} \longrightarrow M_{m\times v} \\ & (A,B) \mapsto A \otimes_c B = A\underline{cms} \cdot B\underline{csv}. \end{array}$$

The cut-addition is associative and commutative.

Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (a_{ij}) \in M_{u\times v}$ be two matrices and $s = \min(m,u)$, $t = \min(n,v)$. We define a hyper-addition, which we call helix-addition or helix-sum, as follows:

$$\begin{array}{rl} \oplus: & M_{m\times n} \times M_{u\times v} \longrightarrow \mathcal{P}(M_{s\times t}) \\ & (A,B) \mapsto A \oplus B = A\underline{st} +_h B\underline{st}, \end{array}$$

where $A\underline{st} +_h B\underline{st} = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in A_{ij}, b_{ij} \in B_{ij}\}$. For illustration, suppose that

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Then

$$A\underline{22} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B\underline{22} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$\begin{array}{ll} A_{11} = \{a_{11}, a_{31}\} = \{2\}, & B_{11} = \{b_{11}, b_{13}\} = \{1,0\}, \\ A_{12} = \{a_{12}, a_{32}\} = \{1,3\}, & B_{12} = \{b_{12}\} = \{4\}, \\ A_{21} = \{a_{21}\} = \{0\}, & B_{21} = \{b_{21}, b_{23}\} = \{2,1\}, \\ A_{22} = \{a_{22}\} = \{1\}, & B_{22} = \{b_{22}\} = \{0\}. \end{array}$$

So

$$A\underline{22} = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \right\},$$

and

$$B\underline{22} = \left\{ \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \right\}.$$

Therefore, we have

$$\begin{aligned} A\underline{22} +_h B\underline{22} = & \left\{ \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 7 \\ 2 & 1 \end{bmatrix}, \right. \\ & \left. \begin{bmatrix} 3 & 7 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 7 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 7 \\ 1 & 1 \end{bmatrix} \right\}. \end{aligned}$$

The helix-addition is commutative. Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (a_{ij}) \in M_{u\times v}$ be two matrices and $s = \min(n,u)$. Then we define a hyper-multiplication, which we call helix hyperoperation, as follows:

$$\begin{array}{ll} \otimes: & M_{m\times n}\times M_{u\times v}\longrightarrow \mathcal{P}(M_{m\times v}) \\ & (A,B)\mapsto A\otimes B = A\underline{ms}\cdot_h B\underline{sv}, \end{array}$$

where $A\underline{ms}\cdot_h B\underline{sv} = \{(c_{ij}) = (\sum a_{it}b_{tj}) \mid a_{ij}\in A_{ij}, b_{ij}\in B_{ij}\}$. We consider the matrices A and B as follows:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 1 & 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then

$$A\underline{22} = \begin{bmatrix} \{1,2\} & 0 \\ 3 & \{1,2\} \end{bmatrix}.$$

Therefore,

$$A\otimes B = \begin{bmatrix} \{-1,-2\} & \{1,2\} \\ -3 & \{5,7\} \end{bmatrix}$$

The cut-multiplication $\otimes_c$ is associative, and the helix-multiplication $\otimes$ is weak associative [23]. Note that the helix-multiplication is not distributive (not even weak) with respect to the helix-addition [23]. But if all matrices which are used in the distributivity are of the same type $M_{m\times n}$, then we have $A\otimes(B\oplus C) = A\otimes(B+C)$ and $(A\otimes B)\oplus(A\otimes C) = (A\otimes B)+(A\otimes C)$. Therefore, the weak distributivity is valid and more precisely the inclusion distributivity is valid.

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 89-102 **eISSN**: 2282-8214

Special Classes of H_b-Matrices

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†**doi**:10.23755/rm.v33i0.382

Abstract

In the present paper we deal with constructions of 2×2 diagonal or upper-triangular or lower-triangular H_b-matrices with entries either of an H_b-field on $\mathbb{Z}_2$ or on $\mathbb{Z}_3$. We study the kind of the hyperstructures that arise, their unit and inverse elements. Also, we focus our study on the cyclicity of these hyperstructures, their generators and the respective periods.
Keywords: hope; H_v-structure; H_b-structure; H_v-matrix
2010 AMS subject classifications: 20N20.

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 Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

F. Marty, in 1934 [13], introduced the hypergroup as a set H equipped with a hyperoperation $\cdot : H \times H \to \mathcal{P}(H) - \{\emptyset\}$ which satisfies the associative law: (xy)z = x(yz), for all $x, y, z \in H$ and the reproduction axiom: xH = Hx = H, for all $x \in H$. In that case, the reproduction axiom is not valid, the $(H, \cdot)$ is called semihypergroup.
In 1990, T. Vougiouklis [19] in the Fourth AHA Congress, introduced the H_v-structures, a larger class than the known hyperstructures, which satisfy the weak axioms where the non-empty intersection replaces the equality.

Definition 1.1. *[21], The $(\cdot)$ in H is called weak associative, we write WASS, if*

$$(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H.$$

The $(\cdot)$ is called weak commutative, we write COW, if

$$xy \cap yx \neq \emptyset, \forall x, y \in H.$$

The hyperstructure $(H, \cdot)$ is called H_v-semigroup if $(\cdot)$ is WASS. It is called H_v-group if it is H_v-semigroup and the reproduction axiom is valid.
Further more, it is called H_v-commutative group if it is an H_v-group and a COW.
If the commutativity is valid, then H is called commutative H_v-group.
Analogous definitions for other H_v-structures, as H_v-rings, H_v-module, H_v-vector spaces and so on can be given.

For more definitions and applications on hyperstructures one can see books [3], [4], [5], [6], [21] and papers as [2], [7], [9], [10], [12], [14], [20], [22], [23], [24], [26], [27].
An element $e \in H$ is called *left unit* if $x \in ex, \forall x \in H$ and it is called *right unit* if $x \in xe, \forall x \in H$. It is called *unit* if $x \in ex \cap xe, \forall x \in H$. The set of left units is denoted by E^ℓ [8]. The set of right units is denoted by E^r and by $E = E^\ell \cap E^r$ the set of units [8].
The element $a\prime \in H$ is called *left inverse* of the element $a \in H$ if $e \in a\prime a$, where e unit element (left or right) and it is called *right inverse* if $e \in aa\prime$. If $e \in a\prime a \cap aa\prime$ then it is called *inverse* element of $a \in H$. The set of the left inverses is denoted by $I^\ell(a, e)$ and the set of the right inverses is denoted by $I^r(a, e)$[8]. By $I(a, e) = I^\ell(a, e) \cap I^r(a, e)$, the set of inverses of the element $a \in H$, is denoted.
In an H_v-semigroup the *powers* are defined by: $h^1 = \{h\}, h^2 = h \cdot h, \cdots, h^n = h \circ h \circ \cdots \circ h$, where $(\circ)$ is the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H, \cdot)$ is *cyclic of period s*, if there is an h, called *generator* and a natural s, the minimum: $H = h^1 \cup h^2 \cup \cdots \cup h^s$. Analogously the cyclicity for the infinite period

is defined [17],[21]. If there is an h and s, the minimum: $H = h^s$, then $(H, \cdot)$, is called *single-power cyclic of period s*.

Definition 1.2. *The fundamental relations β^*, γ^* and ϵ^*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively [18],[19],[21],[22], (see also [1],[3],[4]).*

More general structures can be defined by using the fundamental structures. An application in this direction is the general hyperfield. There was no general definition of a hyperfield, but from 1990 [19] there is the following [20], [21]:

Definition 1.3. *An H_v-ring $(R, +, \cdot)$ is called H_v-field if R/γ^* is a field.*

H_v-matrix is a matrix with entries of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is considered to be the n-ary circle hope on the hyperaddition. The hyperproduct of H_v-matrices is not necessarily WASS. H_v-matrices is a very useful tool in Representation Theory of H_v-groups [15],[16], [25],[28] (see also [11], [29]).

2 Constructions of 2×2 H_b-matrices with entries of an H_v-field on $\mathbb{Z}_2$

Consider the field $(\mathbb{Z}_2, +, \cdot)$. On the set $\mathbb{Z}_2$ also consider the hyperoperation $(\odot)$ defined by setting:

$$1 \odot 1 = \{0, 1\} \text{ and } x \odot y = x \cdot y \text{ for all } (x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_2 - \{(0, 1)\}.$$

Then $(\mathbb{Z}_2, +, \odot)$ becomes an H_b-field.
All the 2×2 H_b-matrices with entries of the H_b-field $(\mathbb{Z}_2, +, \odot)$, are $2^4 = 16$. Let us denote them by:

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, a_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$a_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, a_5 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, a_6 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, a_7 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$a_8 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, a_9 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, a_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, a_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, a_{13} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, a_{14} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, a_{15} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

By taking a_i^2, $i = 1, \cdots, 15$ there exist 15 closed sets, let us say H_i , $i = 1, \cdots, 15$. Two of them are singletons, $H_2 = H_3 = \{\mathbf{0}\}$. Also, $H_7 = H_8$ and $H_{11} = H_{14} = H_{15}$.
So, we shall study, according to the hyperproduct $(\cdot)$ of two H_b-matrices, the following sets:

$$H_1 = \{\mathbf{0}, a_1\}, H_4 = \{\mathbf{0}, a_4\}, H_5 = \{\mathbf{0}, a_1, a_2, a_5\}, H_6 = \{\mathbf{0}, a_1, a_3, a_6\},$$

$$H_7 = \{\mathbf{0}, a_1, a_4, a_7\}, H_9 = \{\mathbf{0}, a_2, a_4, a_9\}, H_{10} = \{\mathbf{0}, a_3, a_4, a_{10}\},$$

$$H_{12} = \{\mathbf{0}, a_1, a_2, a_4, a_5, a_7, a_9, a_{12}\}, H_{13} = \{\mathbf{0}, a_1, a_3, a_4, a_6, a_7, a_{10}, a_{13}\},$$

$$H_{15} = \{\mathbf{0}, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}a_{11}, a_{12}, a_{13}, a_{14}, a_{15}\}.$$

2.1 The case of diagonal 2×2 H_b-matrices

Every set of H_1, H_4, H_7 consists of diagonal 2×2 H_b-matrices. Then, the multiplicative tables of the hyperproduct, are the following:

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$
$\mathbf{0}$	0	0
$\mathbf{a_1}$	0	H_1

,

$\cdot$	$\mathbf{0}$	$\mathbf{a_4}$
$\mathbf{0}$	0	0
$\mathbf{a_4}$	0	H_4

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_4}$	$\mathbf{a_7}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	0	$0, a_1$
$\mathbf{a_4}$	0	0	$0, a_4$	$0, a_4$
$\mathbf{a_7}$	0	$0, a_1$	$0, a_4$	H_7

In all cases:

$$x \cdot y = y \cdot x, \ \forall x, y \in H_i, \ i = 1, 4, 7$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \ \forall x, y, z \in H_i, \ i = 1, 4, 7$$

So, we get the next propositions:

Proposition 2.1. *Every set H, consisting of diagonal 2×2 H_b-matrices with entries of the H_b-field $(\mathbb{Z}_2, +, \odot)$, equipped with the usual hyperproduct $(\cdot)$ of matrices, is a commutative semihypergroup.*

Notice that $H_1, H_4 \subset H_7$ and since $H_1 \cdot H_1 \subseteq H_1$, $H_4 \cdot H_4 \subseteq H_4$ then H_1, H_4 are sub-semihypergroups of $(H_7, \cdot)$.

Proposition 2.2. *For all commutative semihypergroups $(H, \cdot)$, consisting of diagonal 2×2 H_b-matrices with entries of the H_b-field $(\mathbb{Z}_2, +, \odot)$:*

$$E = \{a_i\},\ I(a_i, a_i) = \{a_i\},\ where\ a_i^2 = H.$$

Remark 2.1. *According to the above construction, the commutative semihypergroups $(H_1, \cdot), (H_4, \cdot)$ and $(H_7, \cdot)$, are single-power cyclic commutative semihypergroups with generators the elements a_1, a_4 and a_7, respectively, with single-power period 2.*

2.2 The case of upper- and lower- triangular 2×2 H_b-matrices

Every set of H_5, H_9, H_{12} consists of upper-triangular 2×2 H_b-matrices and every set of H_6, H_{10}, H_{13} consists of lower-triangular 2×2 H_b-matrices. Then, the multiplicative tables of the hyperproduct, are the following:

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_5}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	$0, a_2$	H_5
$\mathbf{a_2}$	0	0	0	0
$\mathbf{a_5}$	0	$0, a_1$	$0, a_2$	H_5

,

$\cdot$	$\mathbf{0}$	$\mathbf{a_2}$	$\mathbf{a_4}$	$\mathbf{a_9}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_2}$	0	0	$0, a_2$	$0, a_2$
$\mathbf{a_4}$	0	0	$0, a_4$	$0, a_4$
$\mathbf{a_9}$	0	0	H_9	H_9

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_4}$	$\mathbf{a_5}$	$\mathbf{a_7}$	$\mathbf{a_9}$	$\mathbf{a_{12}}$
$\mathbf{0}$	0	0	0	0	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	$0, a_2$	0	$0, a_1, a_2, a_5$	$0, a_1$	$0, a_2$	$0, a_1, a_2, a_5$
$\mathbf{a_2}$	0	0	0	$0, a_2$	0	$0, a_2$	$0, a_2$	$0, a_2$
$\mathbf{a_4}$	0	0	0	$0, a_4$	0	$0, a_4$	$0, a_4$	$0, a_4$
$\mathbf{a_5}$	0	$0, a_1$	$0, a_2$	$0, a_2$	$0, a_1, a_2, a_5$	$0, a_1, a_2, a_5$	$0, a_2$	$0, a_1, a_2, a_5$
$\mathbf{a_7}$	0	$0, a_1$	$0, a_2$	$0, a_4$	$0, a_1, a_2, a_5$	$0, a_1, a_4, a_7$	$0, a_2, a_4, a_9$	H_{12}
$\mathbf{a_9}$	0	0	0	$0, a_2, a_4, a_9$	0	$0, a_2, a_4, a_9$	$0, a_2, a_4, a_9$	$0, a_2, a_4, a_9$
$\mathbf{a_{12}}$	0	$0, a_1$	$0, a_2$	$0, a_2, a_4, a_9$	$0, a_1, a_2, a_5$	H_{12}	$0, a_2, a_4, a_9$	H_{12}

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_3}$	$\mathbf{a_6}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	0	$0, a_1$
$\mathbf{a_3}$	0	$0, a_3$	0	$0, a_3$
$\mathbf{a_6}$	0	H_6	0	H_6

,

$\cdot$	$\mathbf{0}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{a_{10}}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_3}$	0	0	0	0
$\mathbf{a_4}$	0	$0, a_3$	$0, a_4$	H_{10}
$\mathbf{a_{10}}$	0	$0, a_3$	$0, a_4$	H_{10}

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{a_6}$	$\mathbf{a_7}$	$\mathbf{a_{10}}$	$\mathbf{a_{13}}$
$\mathbf{0}$	0	0	0	0	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	0	0	$0, a_1$	$0, a_1$	0	$0, a_1$
$\mathbf{a_3}$	0	$0, a_3$	0	0	$0, a_3$	$0, a_3$	0	$0, a_3$
$\mathbf{a_4}$	0	0	$0, a_3$	$0, a_4$	$0, a_3$	$0, a_4$	$0, a_3, a_4, a_{10}$	$0, a_3, a_4, a_{10}$
$\mathbf{a_6}$	0	$0, a_1, a_3, a_6$	0	0	$0, a_1, a_3, a_6$	$0, a_1, a_3, a_6$	0	$0, a_1, a_3, a_6$
$\mathbf{a_7}$	0	$0, a_1$	$0, a_3$	$0, a_4$	$0, a_1, a_3, a_6$	$0, a_1, a_4, a_7$	$0, a_3, a_4, a_{10}$	H_{13}
$\mathbf{a_{10}}$	0	$0, a_3$	$0, a_3$	$0, a_4$	$0, a_3$	$0, a_3, a_4, a_{10}$	$0, a_3, a_4, a_{10}$	$0, a_3, a_4, a_{10}$
$\mathbf{a_{13}}$	0	$0, a_1, a_3, a_6$	$0, a_3$	$0, a_4$	$0, a_1, a_3, a_6$	H_{13}	$0, a_3, a_4, a_{10}$	H_{13}

In all cases:

$$(x \cdot y) \cap (y \cdot x) \neq \emptyset, \ \forall x, y \in H_i, \ i = 5, 6, 9, 10, 12, 13$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \ \forall x, y, z \in H_i, \ i = 5, 6, 9, 10, 12, 13$$

So, we get the next proposition:

Proposition 2.3. *Every set H, consisting either of upper-triangular or lower-triangular 2×2 H_b-matrices with entries of the H_b-field $(\mathbb{Z}_2, +, \odot)$, equipped with the usual hyperproduct $(\cdot)$ of matrices, is a weak commutative semihypergroup.*

Notice that $H_5, H_9 \subset H_{12}$ and $H_6, H_{10} \subset H_{13}$. Since $H_5 \cdot H_5 \subseteq H_5$, $H_9 \cdot H_9 \subseteq H_9$, $H_6 \cdot H_6 \subseteq H_6$, $H_{10} \cdot H_{10} \subseteq H_{10}$, then H_5, H_9 are sub-semihypergroups of $(H_{12}, \cdot)$ and H_6, H_{10} are sub-semihypergroups of $(H_{13}, \cdot)$.

Proposition 2.4. *For all weak commutative semihypergroups $(H, \cdot)$, consisting either of upper-triangular or lower-triangular 2×2 H_b-matrices with entries of the H_b-field $(\mathbb{Z}_2, +, \odot)$, the following assertions hold*
i) If $a_i, a_j \in H : a_i \cdot a_j = H, \ a_i \in a_i^2, \ a_j^2 = H, \ a_i \in a_j \cdot a_i$, then

$$a) E^{\ell} = \{a_i, a_j\}, \ b) I(a_i, a_i) = I(a_j, a_i) = \{a_i, a_j\}$$

$$c)I(a_j, a_j) = I^r(a_i, a_j) = \{a_j\},\ d)I^\ell(a_i, a_j) = \emptyset$$

ii) If $a_i, a_j \in H : a_j \cdot a_i = H,\ a_i \in a_i^2,\ a_j^2 = H,\ a_i \in a_i \cdot a_j$, *then*

$$a)E^r = \{a_i, a_j\},\ b)I(a_i, a_i) = I(a_j, a_i) = \{a_i, a_j\}$$

$$c)I(a_j, a_j) = I^\ell(a_i, a_j) = \{a_j\},\ d)I^r(a_i, a_j) = \emptyset$$

iii) If $a_i, a_j \in H : a_i \cdot a_j = a_j \cdot a_i = H,\ a_i \in a_i^2,\ a_j^2 = H$, *then*

$$a)E = \{a_i, a_j\},\ b)I(a_i, a_i) = I(a_j, a_i) = I(a_j, a_j) = \{a_i, a_j\},\ c)I(a_i, a_j) = \{a_j\}$$

Remark 2.2. *According to the above construction, the weak commutative semihypergroups $(H_i, \cdot)$, i=5,6,9,10,12,13 are single-power cyclic weak commutative semihypergroups with generators the elements $a_5, a_6, a_9, a_{10}, a_{12}, a_{13}$ respectively, with single-power period 2.*

3 Constructions of 2×2 H_b-matrices with entries of an H_b-field on $\mathbb{Z}_3$

Consider the field $(\mathbb{Z}_3, +, \cdot)$. On the set $\mathbb{Z}_3$, we consider four cases for the hyperoperation $(\odot_i), i = 1, 2, 3, 4$ defined, each time, by setting:

1) $1 \odot_1 2 = \{1, 2\}$ and $x \odot_1 y = x \cdot y$ for all $(x, y) \in \mathbb{Z}_3 \times \mathbb{Z}_3 - \{(1, 2)\}$.

2) $2 \odot_2 1 = \{1, 2\}$ and $x \odot_2 y = x \cdot y$ for all $(x, y) \in \mathbb{Z}_3 \times \mathbb{Z}_3 - \{(1, 2)\}$.

3) $1 \odot_3 1 = \{1, 2\}$ and $x \odot_3 y = x \cdot y$ for all $(x, y) \in \mathbb{Z}_3 \times \mathbb{Z}_3 - \{(1, 2)\}$.

4) $2 \odot_4 2 = \{1, 2\}$ and $x \odot_4 y = x \cdot y$ for all $(x, y) \in \mathbb{Z}_3 \times \mathbb{Z}_3 - \{(1, 2)\}$.

Then, each time, $(\mathbb{Z}_3, +, \odot_i), i = 1, 2, 3, 4$ becomes an H_b-field.

Now, consider the set H of the diag(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_i)$. Let us denote them by:

$$a_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, a_{21} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, a_{22} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

So, H = $\{a_{11}, a_{12}, a_{21}, a_{22}\}$.

3.1 The case of $1 \odot_1 2 = \{1, 2\}$

The multiplicative table of the hyperproduct, is the following:

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	a_{11}	a_{11}, a_{12}	a_{11}, a_{21}	H
$\mathbf{a_{12}}$	a_{12}	a_{11}	a_{12}, a_{22}	a_{11}, a_{21}
$\mathbf{a_{21}}$	a_{21}	a_{21}, a_{22}	a_{11}	a_{11}, a_{12}
$\mathbf{a_{22}}$	a_{22}	a_{21}	a_{12}	a_{11}

Notice that in the above multiplicative table:
i) $x \cdot H = H \cdot x = H, \forall x \in H$
ii) $(x \cdot y) \cap (y \cdot x) \neq \emptyset, \forall x, y \in H$
iii) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset, \forall x, y, z \in H$

So, we get the next proposition:

Proposition 3.1. *The set H, consisting of the diag*(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_1)$*, equipped with the usual hyperproduct* $(\cdot)$ *of matrices, is an* H_v*-commutative group.*

Proposition 3.2. *For the* H_v*-commutative group (H,* $\cdot$*), consisting of the diag*(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_1)$:

$$i)\ E = \{a_{11}\}\ ii)\ I^r(x, a_{11}) = \{a_{22}\}, \forall x \in H\ iii)\ I^\ell(x, a_{11}) = \{a_{11}\}, \forall x \in H$$

Proposition 3.3. *The* H_v*-commutative group (H,* $\cdot$*), consisting of the diag*(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_1)$*, is a single-power cyclic* H_v*-commutative group with generator the element* a_{22}*, with single-power period 3.*

3.2 The case of $2 \odot_2 1 = \{1, 2\}$

The multiplicative table of the hyperproduct, is the following:

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	a_{11}	a_{12}	a_{21}	a_{22}
$\mathbf{a_{12}}$	a_{11}, a_{12}	a_{11}	a_{21}, a_{22}	a_{21}
$\mathbf{a_{21}}$	a_{11}, a_{21}	a_{12}, a_{22}	a_{11}	a_{12}
$\mathbf{a_{22}}$	H	a_{11}, a_{21}	a_{11}, a_{12}	a_{11}

As in the paragraph 3.1:

Proposition 3.4. *The set H, consisting of the diag*(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_2)$*, equipped with the usual hyperproduct* $(\cdot)$ *of matrices, is an* H_v*-commutative group.*

Now, take a map f onto and 1:1, $f : H \to H$, such that

$$f(a_{11}) = a_{22},\ f(a_{12}) = a_{21},\ f(a_{21}) = a_{12},\ f(a_{22}) = a_{11}$$

Then, the successive transformations of the above multiplicative table are:

$\cdot$	$\mathbf{a_{22}}$	$\mathbf{a_{21}}$	$\mathbf{a_{12}}$	$\mathbf{a_{11}}$
$\mathbf{a_{22}}$	a_{11}	a_{12}	a_{21}	a_{22}
$\mathbf{a_{21}}$	a_{11}, a_{12}	a_{11}	a_{21}, a_{22}	a_{21}
$\mathbf{a_{12}}$	a_{11}, a_{21}	a_{12}, a_{22}	a_{11}	a_{12}
$\mathbf{a_{11}}$	H	a_{11}, a_{21}	a_{11}, a_{12}	a_{11}

$\cdot$	$\mathbf{a_{22}}$	$\mathbf{a_{21}}$	$\mathbf{a_{12}}$	$\mathbf{a_{11}}$
$\mathbf{a_{11}}$	H	a_{11}, a_{21}	a_{11}, a_{12}	a_{11}
$\mathbf{a_{12}}$	a_{11}, a_{21}	a_{12}, a_{22}	a_{11}	a_{12}
$\mathbf{a_{21}}$	a_{11}, a_{12}	a_{11}	a_{21}, a_{22}	a_{21}
$\mathbf{a_{22}}$	a_{11}	a_{12}	a_{21}	a_{22}

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	a_{11}	a_{11}, a_{12}	a_{11}, a_{21}	H
$\mathbf{a_{12}}$	a_{12}	a_{11}	a_{12}, a_{22}	a_{11}, a_{21}
$\mathbf{a_{21}}$	a_{21}	a_{21}, a_{22}	a_{11}	a_{11}, a_{12}
$\mathbf{a_{22}}$	a_{22}	a_{21}	a_{12}	a_{11}

Then, the last multiplicative table is the table of the paragraph 3.1. So, we get:

Proposition 3.5. *The H_v-commutative group $(H, \cdot)$ consisting of the diag(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_2)$, is isomorphic to H_v-commutative group $(H, \cdot)$ consisting of the diag(b_{11}, b_{22}) , $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_1)$.*

3.3 The case of $1 \odot_3 1 = \{1, 2\}$

The multiplicative table of the hyperproduct, is the following:

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	H	a_{12}, a_{22}	a_{21}, a_{22}	a_{22}
$\mathbf{a_{12}}$	a_{12}, a_{22}	a_{11}, a_{21}	a_{22}	a_{21}
$\mathbf{a_{21}}$	a_{21}, a_{22}	a_{22}	a_{11}, a_{12}	a_{12}
$\mathbf{a_{22}}$	a_{22}	a_{21}	a_{12}	a_{11}

Notice that in the above multiplicative table:
i) $x \cdot H = H \cdot x = H,\ \forall x \in H$

ii) $x \cdot y = y \cdot x,\ \forall x, y \in H$
iii) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset,\ \forall x, y, z \in H$

So, we get the next proposition:

Proposition 3.6. *The set H, consisting of the diag*(b_{11}, b_{22})*,* $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_3)$*, equipped with the usual hyperproduct* $(\cdot)$ *of matrices, is a commutative* H_v*- group.*

Proposition 3.7. *For the commutative* H_v*-group* $(H, \cdot)$*, consisting of the diag*(b_{11}, b_{22})*,* $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_3)$:

$i)\ E = E^r = E^\ell = \{a_{11}\}\ ii)\ I(x, a_{11}) = I^r(x, a_{11}) = I^\ell(x, a_{11}) = \{x\}, \forall x \in H$

Proposition 3.8. *The commutative* H_v*-group* $(H, \cdot)$*, consisting of the diag*(b_{11}, b_{22})*,* $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_3)$:
i) is a single-power cyclic commutative H_v*-group with generator the element* a_{11}*, with single-power period 2.*
ii) is a single-power cyclic commutative H_v*-group with generator the element* a_{22}*, with single-power period 4.*
iii) is a cyclic commutative H_v*-group of period 3 to each of the generators* a_{12} *and* a_{21}*.*

3.4 The case of $2 \odot_4 2 = \{1, 2\}$

The multiplicative table of the hyperproduct, is the following:

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	a_{11}	a_{12}	a_{21}	a_{22}
$\mathbf{a_{12}}$	a_{12}	a_{11}, a_{12}	a_{22}	a_{21}, a_{22}
$\mathbf{a_{21}}$	a_{21}	a_{22}	a_{11}, a_{21}	a_{12}, a_{22}
$\mathbf{a_{22}}$	a_{22}	a_{21}, a_{22}	a_{12}, a_{22}	H

Notice that in the above multiplicative table:
i) $x \cdot H = H \cdot x = H,\ \forall x \in H$
ii) $x \cdot y = y \cdot x,\ \forall x, y \in H$
iii) $(x \cdot y) \cdot z = x \cdot (y \cdot z),\ \forall x, y, z \in H$

So, we get the next proposition:

Proposition 3.9. *The set H, consisting of the diag*(b_{11}, b_{22})*,* $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_4)$*, equipped with the usual hyperproduct* $(\cdot)$ *of matrices, is a commutative hypergroup.*

Proposition 3.10. *For the commutative hypergroup $(H,\cdot)$, consisting of the diag(b_{11},b_{22}), $b_{11},b_{22}\in\mathbb{Z}_3$ with $b_{11}b_{22}\neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3,+,\odot_4)$:*

$$i)\ E=\{a_{11}\}\ ii)\ I(x,a_{11})=\{x\},\forall x\in H$$

Proposition 3.11. *The commutative hypergroup $(H,\cdot)$, consisting of the diag(b_{11},b_{22}), $b_{11},b_{22}\in\mathbb{Z}_3$ with $b_{11}b_{22}\neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3,+,\odot_4)$ is a single-power cyclic commutative hypergroup with generator the element a_{22}, with single-power period 2.*

4 Construction of 2×2 upper-triangular H_b-matrices with entries of an H_b-field on $\mathbb{Z}_3$

On the set $\mathbb{Z}_3$, consider the hyperoperation $(\odot_1)$ defined, by setting:

$1\odot_1 2=\{1,2\}$ and $x\odot_1 y=x\cdot y$ for all $(x,y)\in\mathbb{Z}_3\times\mathbb{Z}_3-\{(1,2)\}$

Now, consider the set H of the 2×2 upper-triangular H_b-matrices with $b_{11},b_{22}\in\mathbb{Z}_3$ and $b_{11}b_{22}\neq 0$, with entries of the H_b-field $(\mathbb{Z}_3,+,\odot_1)$. Let us denote the elements of H by:

$$a_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},a_2=\begin{pmatrix}1&0\\0&2\end{pmatrix},a_3=\begin{pmatrix}1&1\\0&1\end{pmatrix},a_4=\begin{pmatrix}1&1\\0&2\end{pmatrix},$$

$$a_5=\begin{pmatrix}1&2\\0&1\end{pmatrix},a_6=\begin{pmatrix}1&2\\0&2\end{pmatrix},a_7=\begin{pmatrix}2&0\\0&1\end{pmatrix},a_8=\begin{pmatrix}2&0\\0&2\end{pmatrix},$$

$$a_9=\begin{pmatrix}2&1\\0&1\end{pmatrix},a_{10}=\begin{pmatrix}2&1\\0&2\end{pmatrix},a_{11}=\begin{pmatrix}2&2\\0&1\end{pmatrix},a_{12}=\begin{pmatrix}2&2\\0&2\end{pmatrix}$$

So, H = $\{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}\}$.
Since the multiplicative table is long enough, it is omitted. From this table we get:
i) $x\cdot H=H\cdot x=H,\ \forall x\in H$
ii) $(\cdot)$ is non-commutative
iii) $(x\cdot y)\cdot z\cap x\cdot(y\cdot z)\neq\emptyset,\ \forall x,y,z\in H$

So, we get the next proposition:

Proposition 4.1. *The set H, consisting of the 2×2 upper-triangular H_b-matrices with $b_{11},b_{22}\in\mathbb{Z}_3$ and $b_{11}b_{22}\neq 0$, with entries of the H_b-field $(\mathbb{Z}_3,+,\odot_1)$, equipped with the usual hyperproduct $(\cdot)$ of matrices, is a non-commtative H_v-group.*

Proposition 4.2. *For the non-commtative H_v-group $(H, \cdot)$, consisting of the 2×2 upper-triangular H_b-matrices with $b_{11}, b_{22} \in \mathbb{Z}_3$ and $b_{11}b_{22} \neq 0$, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_1) : E = E^{\ell} = E^r = \{a_1\}, \forall x \in H$.*

Proposition 4.3. *The non-commtative H_v-group $(H, \cdot)$, consisting of the 2×2 upper-triangular H_b-matrices with $b_{11}, b_{22} \in \mathbb{Z}_3$ and $b_{11}b_{22} \neq 0$, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_1)$:*
i) is a single-power cyclic non-commutative H_v-group with generator the element a_{12}, with single-power period 4.
ii) is a single-power cyclic non-commutative H_v-group with generator the element a_{10}, with single-power period 3.

Now, take any H_b-field $(\mathbb{Z}_p, +, \odot_1)$, $p = prime \neq 2$ and then consider a set H consisting of the 2×2 upper-triangular H_b-matrices with entries of this H_b-field, with $b_{11}b_{22} \neq 0$, $b_{11}, b_{22} \in \mathbb{Z}_p$.
Then, for any such a set $\mathbb{Z}_p$, take for example the elements $a_3, a_7 \in H$, then:

$$a_7 \cdot a_3 = a_{11} \; and \; a_3 \cdot a_7 = \{a_1, a_7\}$$

So, we get the next general proposition:

Proposition 4.4. *Any set H, consisting of the 2×2 upper-triangular H_b-matrices with $b_{11}b_{22} \neq 0$, $b_{11}, b_{22} \in \mathbb{Z}_p$, $p = prime \neq 2$, with entries of the H_b-field $(\mathbb{Z}_p, +, \odot_1)$, equipped with the usual hyperproduct $(\cdot)$ of matrices, is a non-commutative hyperstructure.*

Remark 4.1. *The above proposition means that, the* ***minimum non-commutative*** *H_v- group, equipped with the usual hyperproduct $(\cdot)$ of matrices and consisting of the 2×2 upper-triangular H_b-matrices with $b_{11}b_{22} \neq 0$, is that with entries of the H_b-field $(\mathbb{Z}_p, +, \odot_1)$, where $1 \odot_1 2 = \{1, 2\}$ and $x \odot_1 y = x \cdot y$ for all $(x, y) \in \mathbb{Z}_3 \times \mathbb{Z}_3 - \{(1, 2)\}$.*

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 103-114 **eISSN**: 2282-8214

On P-H_v-Structures in a Two-Dimensional Real Vector Space

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[‡]**doi**:10.23755/rm.v33i0.381

Abstract

In this paper we study P-H_v-structures in connection with H_v-structures, arising from a specific P-hope in a two-dimensional real vector space. The visualization of these P-H_v-structures is our priority, since visual thinking could be an alternative and powerful resource for people doing mathematics. Using position vectors into the plane, abstract algebraic properties of these P-H_v-structures are gradually transformed into geometrical shapes, which operate, not only as a translation of the algebraic concept, but also, as a teaching process.

Keywords: Hyperstructures; H_v-structures; hopes; P-hyperstructures.
2010 AMS subject classifications: 20N20.

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 Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

In a set $H \neq \varnothing$, a *hyperoperation* (abbr. *hyperoperation=hope*) $(\cdot)$ is defined:

$$\cdot : H \times H \to \mathbf{P}(H) - \{\varnothing\} : (x, y) \mapsto x \cdot y \subset H$$

and the $(H, \cdot)$ is called *hyperstructure*.
It is abbreviated by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$.

The largest class of hyperstructures is the one which satisfy the weak properties. These are called H_v-structures introduced by T. Vougiouklis in 1990 [13], [14] and they proved to have a lot of applications on several applied sciences such as linguistics, biology, chemistry, physics, and so on. The H_v-structures satisfy the weak axioms where the non-empty intersection replaces the equality. The H_v-structures can be used in models as an organized devise.

The hyperstructure $(H, \cdot)$ is called H_v*-group* if it is WASS and the reproduction axiom is valid, i.e., $xH = Hx = H, \forall x \in H$.
It is called *commutative* H_v*-group* if the commutativity is valid and it is called H_v*- commutative group* if it is COW.

The motivation for the H_v-structures [13] is that the quotient of a group with respect to any partition (or equivalently to any equivalence relation), is an H_v-group. The fundamental relation $\beta*$ is defined in H_v-groups as the smallest equivalence so that the quotient is a group [14].

In a similar way more complicated hyperstructures are defined [14].

One can see basic definitions, results, applications and generalizations on both hyperstructure and H_v-structure theory in the books and papers [1], [2], [3], [10], [12], [14], [18].

The element $e \in H$, is called *left unit element* if $x \in ex, \forall x \in H$, *right unit element* if $x \in xe, \forall x \in H$ and *unit element* if $x \in xe \cap x \in ex, \forall x \in h$.
An element $x' \in H$ is called *left inverse* of $x \in H$ if there exists a unit $e \in H$, such that $e \in x'x$, *right inverse* of $x \in H$ if $e \in xx'$ and inverse of $x \in H$ if $e \in x'x \cap xx'$.
By E_*^l is denoted the set of the left unit elements, by E_*^r the set of the right unit elements and by E_* the set of the unit elements with respect to hope (*) [7].
By $I_*^l(x, e)$ is denoted the set of the left inverses, by $I_*^r(x, e)$ the set of the right inverses and by $I_*(x, e)$ the set of the inverses of the element $x \in H$ associated with the unit $e \in H$ with respect to hope (*) [7].

The class of P-hyperstructures was appeared in 80's to represent hopes of constant length [16], [18]. Then many applications appeared [1], [2], [4], [5], [6], [8], [9], [15].
Vougiouklis introduced the following definition:

Definition 1.1. *Let $(G, \cdot)$ be a semigroup and $P \subset GP \neq \varnothing$. Then the following hyperoperations can be defined and they are called P-hyperoperations: $\forall x, y \in G$*

$$P^* : xP^*y = xPy,$$

$$P_r^* : xP_r^*y = (xy)P$$

$$P_l^* : xP_l^*y = P(xy).$$

The (G, P^*),(G, P_r^*), (G, P_l^*) are called P-hyperstructures.
One, combining the above definitions gets that the most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = xP^*y = xPy$ and $(G, \underline{P})$ is a semihypergroup, but we do not know about $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$. In some cases, depending on the choice of P, $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS. $(G, \underline{P})$, $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS.

In this paper we define in the $I\mathbb{R}^2$ a hope which is originated from geometry. This geometrically motivated hope in $I\mathbb{R}^2$ constructs H_v-structures and P-HV-structures in which the existence of units and inverses are studied. One using the above H_v-structures and P-H_v-structures into the plane can easily combine abstract algebraic properties with geometrical figures [11].

2 P-H_v-structures on $I\mathbb{R}^2$

Let us introduce a coordinate system into the $I\mathbb{R}^2$. We place a given vector $\overrightarrow{p}$ so that its initial point P determines an ordered pair (a_1, a_2). Conversely, a point P with coordinates (a_1, a_2) determines the vector $\overrightarrow{p} = \overrightarrow{OP}$, where O the origin of the coordinate system. We shall refer to the elements $x, y, z, \dots$ of the set $I\mathbb{R}^2$, as vectors whose initial point is the origin. These vectors are very well known as position vectors.
In [7] Dramalidis introduced and studied a number of hyperoperations originated from geometry. Among them he introduced in $I\mathbb{R}^2$ the hyperoperation $(\oplus)$ as follows:

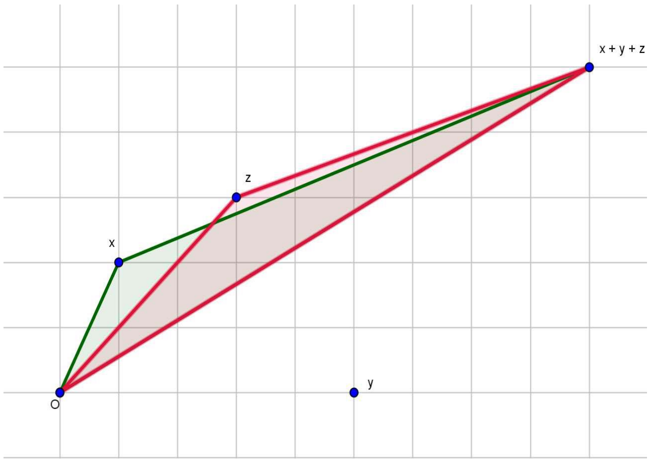
Definition 2.1. *For every $x, y \in I\mathbb{R}^2$*

$$\oplus : I\mathbb{R}^2 \times I\mathbb{R}^2 \to \mathbf{P}(I\mathbb{R}^2) - \{\varnothing\} : (x, y) \mapsto x \oplus y =$$

$$= [0, x + y] = \{\mu(x + y)/\mu \in [0, 1]\} \subset I\mathbb{R}^2$$

From geometrical point of view and for x, y linearly independent position vectors, the set $x \oplus y$ is the main diagonal of the parallelogram having vertices $0, x, x+y, y$.

Proposition 2.1. *The hyperstructure $(I\mathbb{R}^2, \oplus)$ is a commutative H_v-group.*



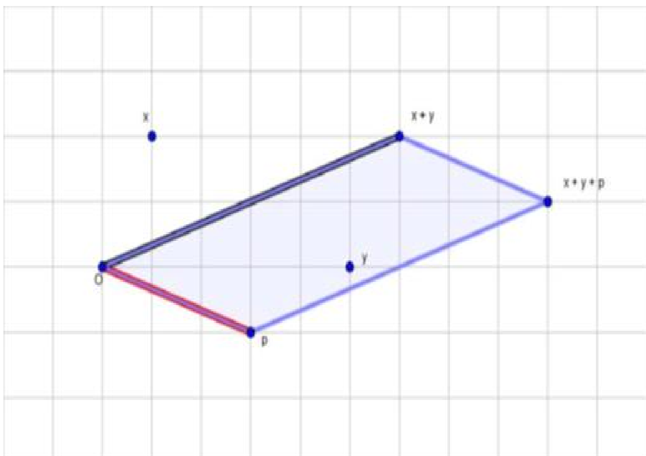
Now, let P be the set $P = [0, p] = \{\lambda p / \lambda \in [0, 1]\} \subset I\mathbb{R}^2$, where p is a fixed point of the plane. Geometrically, P is a line segment.

Consider the P-hyperoperation $(P^*_{r(\oplus)})$:

Definition 2.2. *For every $x, y \in I\mathbb{R}^2$*

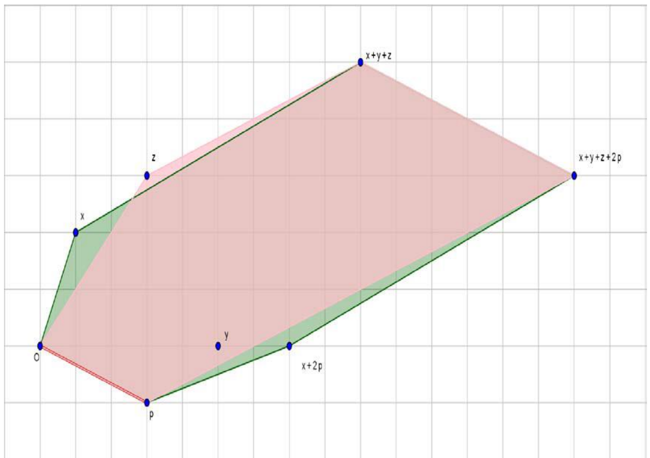
$$P^*_{r(\oplus)} : I\mathbb{R}^2 \times I\mathbb{R}^2 \to \mathbf{P}(I\mathbb{R}^2) - \{\varnothing\} : (x, y) \mapsto xP^*_{r(\oplus)}y = (x \oplus y) \oplus P \subset I\mathbb{R}^2$$

Obviously, $(P^*_{r(\oplus)})$ is commutative and geometrically, for x,y linearly independent position vectors, the set $xP^*_{r(\oplus)}y$ is the closed region of the parallelogram with vertices $0, x + y, x + y + p, p$.



Proposition 2.2. *The hyperstructure $(\mathbb{R}^2, P^*_{r(\oplus)})$ is a commutative P-H_v-group.*

Proof. Obviously, $xP^*_{r(\oplus)}\mathbb{R}^2 = \mathbb{R}^2 P^*_{r(\oplus)}x = \mathbb{R}^2, \forall x \in \mathbb{R}^2$.
For $x, y, z \in \mathbb{R}^2$
$(xP^*_{r(\oplus)}y)P^*_{r(\oplus)}z = \{[(x \oplus y)P] \oplus z\} \oplus P = [0, z, x+y+z, x+y+z+2p, p]$
$xP^*_{r(\oplus)}(yP^*_{r(\oplus)}z) = \{x \oplus [(y \oplus z) \oplus P]\} \oplus P =$
$= [0, x+y+z, x+y+z+2p, x+y+2p, x+2p, p]$

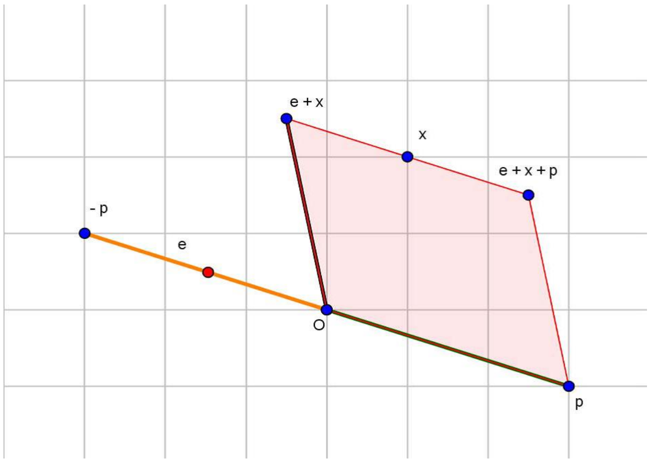


So,

$$(xP^*_{r(\oplus)}y)P^*_{r(\oplus)}z \cap xP^*_{r(\oplus)}(yP^*_{r(\oplus)}z) \neq \varnothing, \forall x, y, z \in \mathbb{R}^2. \square$$

Proposition 2.3. $E_{P^*_{r(\oplus)}} = [-p, 0] = \{-\lambda p/\lambda \in [0, 1]\}$

Proof. Let $e \in E^l_{P^*_{r(\oplus)}} \Leftrightarrow xeP^*_{r(\oplus)}x, \forall x \in \mathbb{R}^2 \Leftrightarrow x\{\mu\lambda e + \mu\lambda x + \mu\nu p/\mu, \nu, \lambda[0, 1]\}$.
That means that,
$\mu\lambda = 1$ and $\mu\lambda e + \mu\nu p = 0 \Leftrightarrow e + \mu\nu p = 0 \Leftrightarrow e = -\mu\nu p, -1 \leq -\mu\nu \leq 0$, then $e \in [-p, 0]$. So, $E^l_{P^*_{r(\oplus)}} = [-p, 0]$ and according to commutativity $E^r_{P^*_{r(\oplus)}} = [-p, 0] = E_{P^*_{r(\oplus)}} = [-p, 0]$.

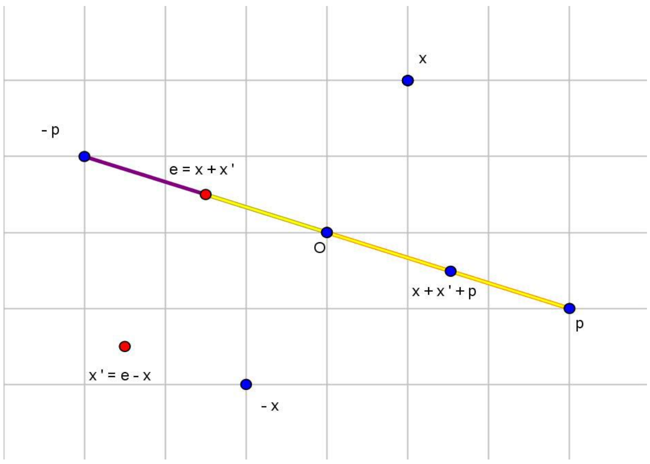


Proposition 2.4. $I_(P^*_{r(\oplus)})(x,e) = \{\frac{1}{\mu\lambda}e - x - \frac{\nu}{\lambda}p/\mu, \lambda \in (0,1], \nu \in [0,1]\}$, *where* $e \in E_{P^*_{r(\oplus)}}$.

Proof. Let $e \in E_{P^*_{r(\oplus)}}$ and $x' \in I^l_{P^*_{r(\oplus)}}(x,e) \Leftrightarrow e \in x'P^*_{r(\oplus)}x \Leftrightarrow e\{\mu\lambda x' + \mu\lambda x + \mu\nu p/\lambda, \mu, \nu[0,1]\}$.
That means there exist $\lambda_1, \mu_1, \nu_1[0,1]$:

$$e = \mu_1\lambda_1 x' + \mu_1\lambda_1 x + \mu_1\nu_1 p \Rightarrow x' = \frac{1}{\mu_1\lambda_1}e - x - \frac{\nu_1}{\lambda_1}p, \mu_1, \lambda_1 \neq 0.$$

So, $x' \in \{\frac{1}{\mu\lambda}e - x - \frac{\nu}{\lambda}p/\mu, \lambda \in (0,1], \nu \in [0,1]\}$.
Since $(P^*_{r(\oplus)})$ is commutative, we get $I_(P^*_{r(\oplus)})(x,e) = \{\frac{1}{\mu\lambda}e - x - \frac{\nu}{\lambda}p/\mu, \lambda \in (0,1], \nu \in [0,1]\}$.



The P-hyperoperation $P^*_{l(\oplus)} = P \oplus (x \oplus y)$ is identical to $(P^*_{r(\oplus)})$. But the P-hyperoperation $P^*_{(\oplus)} = x \oplus P \oplus y$ is different and even more $P^{*l}_{(\oplus)} = (x \oplus P) \oplus y \neq x \oplus (P \oplus y) = xP^{*r}_{(\oplus)}y$, since $(\oplus)$ is not associative.□

Definition 2.3. *For every $x, y \in I\mathbb{R}^2$*

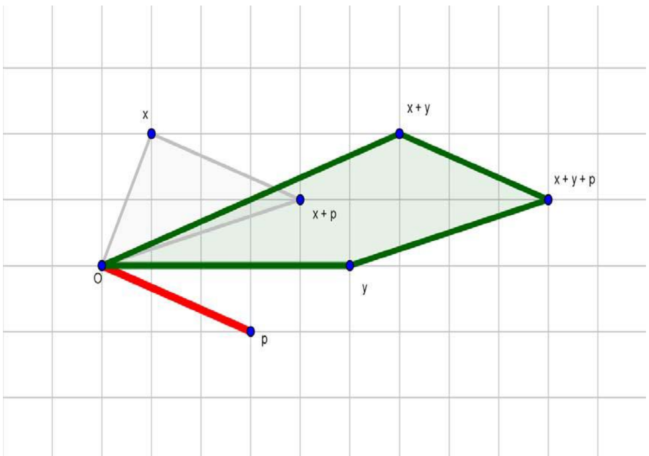
$$P^{*l}_{(\oplus)} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow (\mathbb{R}^2) : (x, y) \mapsto xP^{*l}_{(\oplus)}y = (x \oplus P) \oplus y$$

More specifically,

$$xP^{*l}_{(\oplus)}y = \{\lambda\kappa x + \lambda y + \lambda\kappa\mu p/\lambda, \kappa, \mu \in [0, 1]\}, \forall x, y \in \mathbb{R}^2.$$

Geometrically, for x,y linearly independent position vectors, the set $xP^{*l}_{(\oplus)}$ y is the closed region of the quadrilateral with vertices $0, x + y, x + y + p, y$. On the other hand the set $yP^{*l}_{(\oplus)}x$ is the closed region of the quadrilateral with vertices $0, x, x + y, x + y + p$. So,

$$(xP^{*l}_{(\oplus)}y) \cap (yP^{*l}_{(\oplus)}x) = [0, x + y, x + y + p] \neq \varnothing, \forall x, y \in \mathbb{R}^2.$$

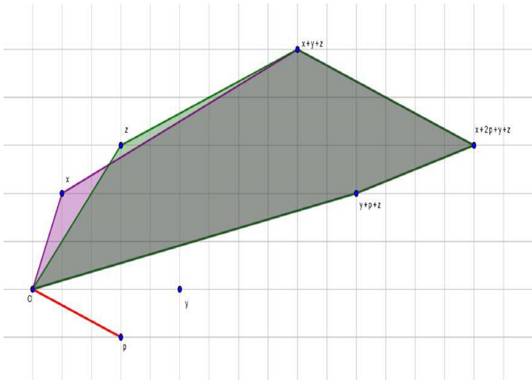


Proposition 2.5. *The hyperstructure $(\mathbb{R}^2, P^{*l}_{(\oplus)})$ is a P-H_v- commutative group.*

Proof. Obviously, $xP^{*l}_{(\oplus)}\mathbb{R}^2 = \mathbb{R}^2 P^{*l}_{(\oplus)}x = \mathbb{R}^2, \forall x \in \mathbb{R}^2$.
For $x, y, z\mathbb{R}^2$

$$(xP^{*l}_{(\oplus)}y)P^{*l}_{(\oplus)}z = \{[(x\oplus P)\oplus y]P\}\oplus z \equiv [O, z, x+y+z, x+2p+y+z, y+p+z]$$

$$xP^{*l}_{(\oplus)}(yP^{*l}_{(\oplus)}z) = (x\oplus P)\oplus[(y\oplus P)\oplus z] \equiv [O, x, x+y+z, x+2p+y+z, y+p+z]$$



So,
$$(xP^{*l}_{(\oplus)}y)P^{*l}_{(\oplus)}z \cap xP^{*l}_{(\oplus)}(yP^{*l}_{(\oplus)}z) \neq \varnothing, x, y, z \in \mathbb{R}^2.$$
□

Proposition 2.6. *i)* $E^l_{P^{*l}_{(\oplus)}} = \mathbb{R}^2$

ii) $E^r_{P^{*l}_{(\oplus)}} = [0, -p] = \{-\nu p/\nu \in [0,1]\} = E_{P^{*l}_{(\oplus)}}$

Proof.

i) Notice that $x \in eP^{*l}_{(\oplus)}x = [0, e+x, e+x+p, x], \forall x, e \in \mathbb{R}^2$. So, $E^l_{P^{*l}_{(\oplus)}} = \mathbb{R}^2$.

ii) Let $e \in E^r_{P^{*l}_{(\oplus)}} \Leftrightarrow x \in xP^{*l}_{(\oplus)}e, \forall x \in \mathbb{R}^2 \Leftrightarrow x \in \{\lambda\kappa x + \lambda e + \lambda\kappa\mu p/\lambda, \kappa, \mu \in [0,1]\}$. Then, there exist $\mu_1, \lambda_1, \kappa_1 \in [0,1] : x = \lambda_1\kappa_1 x + \lambda_1 e + \lambda_{11}\mu_1 p \Leftrightarrow e = \frac{1}{\lambda}\lambda_1[(1-\lambda_{11})x - \lambda_1\kappa_1\mu_1 p], \lambda_1 \neq 0$. The last one is valid $\forall x \in \mathbb{R}^2$, so by setting $x = 0$ we get $e = -\kappa_1\mu_1 p$. Since $\mu_1, \kappa_1 \in [0,1]$ there exists $\nu_1 \in [0,1] : \nu_1 = \kappa_1\mu_1 \Rightarrow e = -\nu_1 p \Rightarrow$
$$e \in \{-\nu p/\nu \in [0,1]\} = [0, -p].$$
Since $E^r_{P^{*l}_{(\oplus)}} \subset \mathbb{R}^2 = E^l_{P^{*l}_{(\oplus)}}$ we get $E^l_{P^{*l}_{(\oplus)}} \cap E^r_{P^{*l}_{(\oplus)}} = \{-\nu p/\nu[0,1]\} = E_{P^{*l}_{(\oplus)}}$.□

Proposition 2.7. $\alpha)$ $I^r_{P^{*l}_{(\oplus)}}(x, e) = \{-\kappa x - (\frac{\nu}{\lambda} + \kappa\mu)p/\kappa, \mu, \nu \in [0,1], \lambda \in (0,1]\}, e \in E^r_{P^{*l}_{(\oplus)}}$.

$\beta)$ $I^r_{P^{*l}_{(\oplus)}}(x, e) = \{\frac{e}{\lambda} - \kappa x - \kappa\mu p/\kappa, \mu \in [0,1], \lambda \in (0,1]\}, e \in E^l_{P^{*l}_{(\oplus)}}$

$\gamma)$ $I^l_{P^{*l}_{(\oplus)}}(x, e) = \{-\frac{x}{\kappa} - (\frac{\nu}{\lambda\kappa} + \mu)p/\kappa, \lambda \in (0,1], \mu \in (0,1]\}, e \in E^r_{P^{*l}_{(\oplus)}}$.

$\delta)$ $I^l_{P^{*l}_{(\oplus)}}(x,e) = \{\frac{1}{\kappa}(\frac{e}{\lambda} - x) - \mu p/\kappa, \lambda \in (0,1], \mu \in [0,1]\}, e \in E^l_{P^{*l}_{(\oplus)}}$

Proof.

$\alpha)$ Let $e \in E^r_{P^{*l}_{(\oplus)}} = [0, -p]$ and $x' \in I^r_{P^{*l}_{(\oplus)}}(x,e)$, then
$e \in xP^{*l}_{(\oplus)}x' \Rightarrow e \in \{\lambda\kappa x + \lambda x' + \lambda\kappa\mu p/\kappa, \lambda, \mu \in [0,1]\}$. That means there exist $\kappa_1, \lambda_1, \mu_1 \in [0,1]$:
$e = \lambda_1\kappa_1 x + \lambda_1 x' + \lambda_1\kappa_1\mu_1 p \Rightarrow x' = \frac{e}{\lambda_1} - \kappa_1 x - \kappa_1\mu_1 p, \lambda_1 \neq 0.$
But, $e \in \{-\nu p/\nu[0,1]\} \Rightarrow\ni \nu_1 \in [0,1] : e = -\nu_1 p.$
So, $x' = -\frac{\nu_1}{\lambda_1}p - \kappa_1 x - \kappa_1\mu_1 p, \lambda_1 \neq 0 \Rightarrow x' = -\kappa_1 x(\frac{\nu_1}{\lambda_1} + \kappa_1\mu_1)p, \lambda_1 \neq 0.$
Then we get $x' \in \{-\kappa x - (\frac{\nu}{\lambda} + \kappa\mu)p/\kappa, \mu, \nu \in [0,1], \lambda \in (0,1]\}$.

$\beta)$ Similarly as above.

$\gamma)$ Similarly as above.

$\delta)$ Similarly as above.

□

Definition 2.4. *For every* $x, y \in I\mathbb{R}^2$

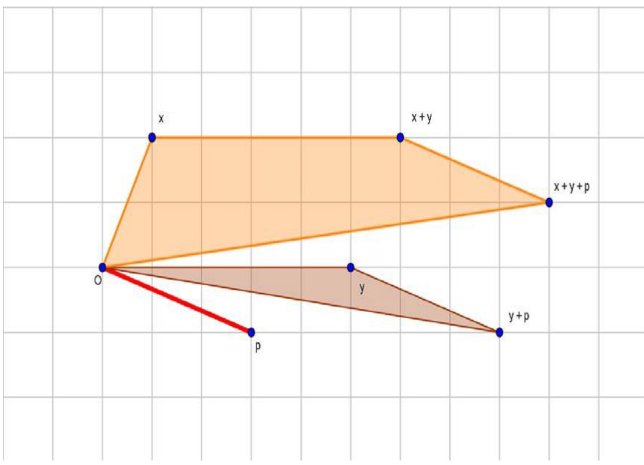
$$^{*r}_{(\oplus)} : \mathbb{R}^2 \times \mathbb{R}^2 \to (\mathbb{R}^2) : (x,y) \mapsto x^{*r}_{(\oplus)})y = x \oplus (P \oplus y)$$

More specifically,

$$x^{*r}_{(\oplus)}y = \{\lambda x + \lambda\kappa y + \lambda\kappa\mu p/\lambda, \kappa, \mu \in [0,1]\}, \forall x, y \in \mathbb{R}^2$$

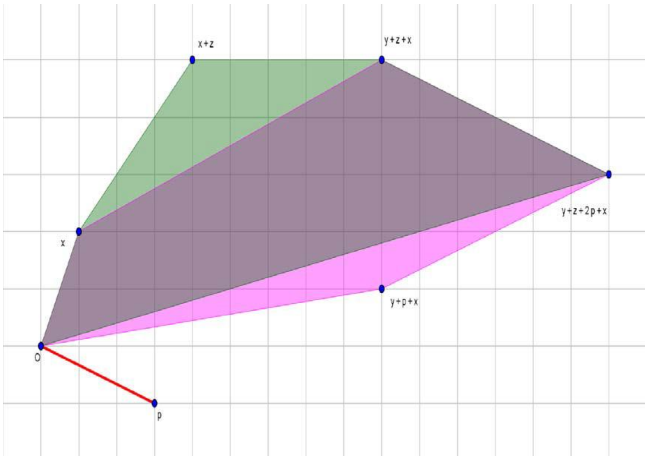
Geometrically, for x,y linearly independent position vectors, the set $xP^{*r}_{(\oplus)}y$ *is the closed region of the quadrilateral with vertices* $0, x, x+y, x+y+p$. *On the other hand the set* $yP^{*r}_{(\oplus)}x$ *is the closed region of the quadrilateral with vertices* $0, x+y, x+y+p, y$. *So,*

$$(xP^{*r}_{(\oplus)}y) \cap (yP^{*r}_{(\oplus)}x) = [0, x+y, x+y+p] \neq \varnothing, \forall x, y \in \mathbb{R}^2.$$



Proposition 2.8. *The hyperstructure* $(\mathbb{R}^2, P^{*r}_{(\oplus)})$ *is a P-*H_v*- commutative group.*

Proof. Obviously, $xP^{*r}_{(\oplus)}\mathbb{R}^2 = \mathbb{R}^2 P^{*r}_{(\oplus)}x = \mathbb{R}^2, \forall x \in \mathbb{R}^2$.
For $x, y, z \in R^2$
$(xP^{*r}_{(\oplus)}y)P^{*r}_{(\oplus)}z = [(x \oplus (P \oplus y)] \oplus (P \oplus z) \equiv [O, x, x+z, x+y+z, x+y+z+2p]$
$xP^{*r}_{(\oplus)}(yP^{*r}_{(\oplus)}z) = x \oplus \{P \oplus [y \oplus (P \oplus z)]\} \equiv [O, x, x+y+z, x+y+z+2p, y+p+x]$



So,

$$[(xP^{*r}_{(\oplus)}y)P^{*r}_{(\oplus)}z] \cap [xP^{*r}_{(\oplus)}(yP^{*r}_{(\oplus)}z)] \neq \varnothing, \forall x, y, z \in \mathbb{R}^2. \square$$

The following, are respective propositions of the Propositions 2.6. and 2.7. :

Proposition 2.9. *i)* $E^r_{P^{*r}_{(\oplus)}} = \mathbb{R}^2$

ii) $E^l_{P^{*r}_{(\oplus)}} = [0, -p] = \{-\nu p / \nu \in [0,1]\} = E_{P^{*r}_{(\oplus)}}$.

Proposition 2.10. $\alpha)$ $I^r_{P^{*r}_{(\oplus)}}(x, e) = \{\frac{1}{\kappa}(\frac{e}{\lambda} - x) - \mu p/\kappa, \lambda \in (0,1], \mu \in [0,1]\}, e \in E^r_{P^{*r}_{(\oplus)}}$

$\beta)$ $I^r_{P^{*r}_{(\oplus)}}(x, e) = \{-\frac{x}{\kappa} - (\frac{\nu}{\lambda\kappa} + \mu)p/\kappa, \lambda \in (0,1], \mu \in (0,1]\}, e \in E^l_{P^{*r}_{(\oplus)}}$.

$\gamma)$ $I^l_{P^{*r}_{(\oplus)}}(x, e) = \{\frac{e}{\lambda} - \kappa x - \kappa\mu p/\kappa, \mu \in [0,1], \lambda \in (0,1]\}, e \in E^r_{P^{*r}_{(\oplus)}}$

$\delta)$ $I^l_{P^{*r}_{(\oplus)}}(x, e) = \{-\kappa x - (\frac{\nu}{\lambda} + \kappa\mu)p/\kappa, \mu, \nu \in [0,1], \lambda \in (0,1]\}, e \in E^l_{P^{*r}_{(\oplus)}}$.

Remark 2.1. *Notice that,*

$$\alpha) x^{*l}_{(\oplus)}y = y^{*r}_{(\oplus)}x, \forall x, y \in \mathbb{R}^2$$
$$\beta) x^{*r}_{(\oplus)}y = y^{*l}_{(\oplus)}x, \forall x, y \in \mathbb{R}^2$$

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 115-126 **eISSN**: 2282-8214

Finite H_v-Fields with Strong-Inverses

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‡**doi**:110.23755/rm.v33i0.373

Abstract

The largest class of hyperstructures is the class of H_v-structures. This is the class of hyperstructures where the equality is replaced by the non-empty intersection. This extremely large class can used to define several objects that they are not possible to be defined in the classical hypergroup theory. It is convenient, in applications, to use more axioms and conditions to restrict the research in smaller classes. In this direction, in the present paper we continue our study on H_v-structures which have strong-inverse elements. More precisely we study the small finite cases.
Keywords: hyperstructure; H_v-structure; hope; strong-inverse elements.
2010 AMS subject classifications: 20N20, 16Y99.

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‡©Theodora Kaplani and Thomas Vougiouklis. Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

First we present some basic definitions on hyperstructures, mainly on the weak hyperstructures introduced in 1990 [7].

Definition 1.1. ***Hyperstructures*** *are called the algebraic structures equipped with, at least, one hyperoperation. Abbreviate:* ***hyperoperation=hope****. The* ***weak hyperstructures*** *are called* $\mathbf{H_v}$***-structures*** *and they are defined as follows:*
In a set H equipped with a hope $\cdot : H \times H \to \wp(H) - \{\varnothing\}$*, we abbreviate by*
WASS *the* ***weak associativity****:* $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ *and by*
COW *the* ***weak commutativity****:* $xy \cap yx \neq \varnothing, \forall x, y \in H$*.*
The hyperstructure $(H, \cdot)$ *is called an* $\mathbf{H_v}$**-semigroup** *if it is WASS, is called* $\mathbf{H_v}$**-group** *if it is reproductive* H_v*-semigroup:*

$$xH = Hx = H, \forall x \in H.$$

$(R, +, \cdot)$ *is called* $\mathbf{H_v}$***-ring*** *if the hopes* $(+)$ *and* $(\cdot)$ *are WASS, the reproduction axiom is valid for* $(+)$*, and* $(\cdot)$ *is* ***weak distributive*** *with respect to* $(+)$*:*

$$x(y+z) \cap (xy+xz) \neq \varnothing,\ (x+y)z \cap (xz+yz) \neq \varnothing,\ \forall x, y, z \in R.$$

An H_v-group is called *cyclic* [6], [8], if there is an element, called *generator*, which the powers have union the underline set. The minimal power with this property is called the period of the generator. If there is an element and a special power, the minimum one, is the underline set, then the H_v-group is called *single-power cyclic*.

For more definitions, results and applications on hyperstructures and mainly on the H_v-structures, see books as [1], [2], [8] and papers as [6], [9], [8], [10], [11], [12], to mention but a few of them. An extreme class of hyperstructures is the following: An H_v-structure is called **very thin** if and only if, all hopes are operations except one, with all hyperproducts to be singletons except only one, which is a subset with cardinality more than one.

The **fundamental relations** β^* and γ^* are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively. Normally to find the fundamental classes is very hard job. The basic theorems on the fundamental classes are analogous to the following:

Theorem 1.1. *[8] Let* $(H, \cdot)$ *be an* H_v*-group and let us denote by U the set of all finite products of elements of H. We define the relation* β *in H as follows:* $x\beta y$ *iff* $\{x, y\} \subset u$ *where* $u \in U$*. Then the fundamental relation* β^* *is the transitive closure of the relation* β*.*

Proof. See [7], [8].

An element of a hyperstructure is called **single** if its fundamental class is a singleton.

Motivation for H_v-structures:

The quotient of a group with respect to an invariant subgroup is a group.

F. Marty states that, the quotient of a group by any subgroup is a hypergroup.

Now, the quotient of a group with respect to any partition is an H_v-group.

We remark that in H_v-groups (or even in hypergroups in the sense of F. Marty) we do not have necessarily any 'unit' element, consequently neither 'inverses'. However, we may have more than one unit elements and for each element of an H_v-group we may have one inverse element or more than one inverse element. □

Definition 1.2. *Let $(H, \cdot)$ be an H_v-semigroup. An element e, is called left unit if $ex \ni x, \forall x \in H$, it is called right unit if $xe \ni x, \forall x \in H$ and it is called* ***unit*** *element if it is both left and right unit element. For given unit e, an element $x \in H$, has a left inverse with respect to e, any element x_{le} if $x_{le} \cdot x \ni e$, it has a right inverse element x_{re} if $x \cdot x_{re} \ni e$, and it has an inverse x_e with respect to e, if $e \in x_e \cdot x \cap x \cdot x_e$. Denote by E_l the set of all left unit elements, by E_r the set of all right unit elements, and by E the set of unit elements.*

Definition 1.3. *[16], [5] Let $(H, \cdot)$ be an H_v-semigroup. An element is called* ***strong-inverse*** *if it is an inverse to x with respect to all unit elements.*

Remark 1.1. *We remark that an element x_s is a strong-inverse to x, if $E \subset x_s \cdot x \cap x \cdot x_s$. Therefore the strong-inverse property it is not exists in the classical structures.*

Definition 1.4. *Let $(H, \cdot), (H, \otimes)$ be H_v-semigroups defined on the same H. $(\cdot)$ is* ***smaller*** *than $(\otimes)$, and $(\otimes)$* ***greater*** *than $(\cdot)$, if and only if, there exists an automorphism $f \in Aut(H, \otimes)$ such that $xy \subset f(x \otimes y)$, $\forall x, y \in H$. Then $(H, \otimes)$ contains $(H, \cdot)$ and write $\cdot \leq \otimes$. If $(H, \cdot)$ is a structure, then it is basic and $(H, \otimes)$ is an H_b-structure.*

The Little Theorem. In a set, greater hopes of the ones which are WASS or COW, are also WASS or COW, respectively.

The fundamental relations are used for general definitions, thus, for example, in order to define the general H_v-field one uses the fundamental relation γ^*:

Definition 1.5. *[7], [8], [9] The H_v-ring $(R, +, \cdot)$ is an* $\mathbf{H_v}$***-field*** *if the quotient R/γ^* is a field. The definition of the H_v-field introduced a new class of hyperstructures [12]: The H_v-semigroup $(H, \cdot)$ is h/v-group if the quotient H/β^* is a group.*

More complicated hyperstructures can be defined as well:

Definition 1.6. *Let [8]* $(R, +, \cdot)$ *be an* H_v*-ring,* $(\mathcal{M}, +)$ *be a COW* H_v*-group and there exists an external hope*

$$\cdot : \mathbf{R} \times \mathcal{M} \to \wp(\mathcal{M}) : (a, x) \to ax$$

such that $\forall a, b \in R$ *and* $\forall x, y \in \mathcal{M}$ *we have*

$$a(x+y) \cap (ax+ay) \neq \varnothing,\ (a+b)x \cap (ax+bx) \neq \varnothing,\ (ab)x \cap a(bx) \neq \varnothing,$$

then $\mathcal{M}$ *is called an* H_v*-module over F. In the case of an* H_v*-field F instead of an* H_v*-ring R, then the* $\mathbf{H_v}$***-vector space*** *is defined.*

In the above cases the fundamental relation ϵ^* is defined to be the smallest equivalence relation such that the quotient $\mathcal{M}/\epsilon^*$ is a module (resp. vector space) over the fundamental ring R/γ^* (resp. fundamental field F/γ^*).

The general definition of an H_v-Lie algebra was given in [14] as follows:

Definition 1.7. *Let* $(\mathbf{L}, +)$ *be an* H_v*-vector space over the* H_v*-field* $(\mathbf{F}, +, \cdot)$*,* $\phi : \mathbf{F} \to \mathbf{F}/\gamma^*$ *the canonical map and* $\omega_F = \{x \in \mathbf{F} : \phi(x) = 0\}$*, where 0 is the zero of the fundamental field* $\mathbf{F}/\gamma^*$*. Similarly, let* ω_L *be the core of the canonical map* $\phi' : \mathbf{L} \to \mathbf{L}/\epsilon^*$ *and denote by the same symbol 0 the zero of* L/ϵ^**. Consider the bracket (commutator) hope:*

$$[,] : \mathbf{L} \times \mathbf{L} \to \wp(L) : (x, y) \to [x, y]$$

*then **L** is an* H_v*-Lie algebra over F if the following axioms are satisfied:*

(L1) *The bracket hope is bilinear, i.e.*
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \varnothing,$
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in \mathbf{F}$

(L2) $[x, x] \cap \omega_L \neq \varnothing,\ \ \forall x \in \mathbf{L}$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing,\ \ \forall x, y \in \mathbf{L}$

Definition 1.8. *[10] Let* $(H, \cdot)$ *be a hypergroupoid. We say that we remove the element* $h \in H$*, if we simply consider the restriction of* $(\cdot)$ *on* $H - \{h\}$*. We say that the element* $\underline{h} \in H$ *absorbs the element* $h \in H$ *if we replace* h*, whenever it appears, by* $\underline{h}$*. We say that the element* $\underline{h} \in H$ *merges with the element* $h \in H$*, if we take as product of* $x \in H$ *by* $\underline{h}$*, the union of the results of* x *with both* h *and* $\underline{h}$*, and consider* h *and* $\underline{h}$ *as one class, with representative the element* $\underline{h}$*.*

2 Large classes of H_v-structures and applications

The large class of, so called, P-hyperstructures was appeared in 80's to represent hopes of constant length [6]. Since then several classes of P-hopes were introduced and studied [8], [4], [11].

Definition 2.1. *Let $(G, \cdot)$ be a groupoid, then for all P such that $\varnothing \neq P \subset G$, we define the following hopes called **P-hopes**: $\forall x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP),$$
$$\underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$,$(G, \underline{P}_r)$, $(G, \underline{P}_l)$ are called P-hyperstructures. The most usual case is when $(G, \cdot)$ is semigroup, then we have

$$x\underline{P}y = (xP)y \cup x(Py) = xPy$$

and $(G, \underline{P})$ is a semihypergroup.

It is immediate the following: Let $(G, \cdot)$ be a group, then for all subsets P such that $\varnothing \neq P \subset G$, the hyperstructure (G,$\underline{P}$), where the P-hope is x$\underline{P}$y = xPy, becomes a hypergoup in the sense of Marty, i.e. the strong associativity is valid. The P-hope is of constant length, i.e. we have $|x\underline{P}y| = |P|$. We call the hyperstructure (G,$\underline{P}$), *P-hypergroup.*

In [4], [15] a modified P-hope was introduced which is appropriate for the e-hyperstructures:

Construction 2.1. *Let $(G, \cdot)$ be abelian group and P any subset of G with more than one elements. We define the hyperoperation $\times_P$ as follows:*

$$x \times_p y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \textit{if } x \neq e \textit{ and } c \neq e \\ x \cdot y & \textit{if } x = e \textit{ or } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_p)$ is an abelian H_v-group.

Another large class is the one on which a new hope (∂) in a groupoid is defined.

Definition 2.2. *[13]. Let $(G, \cdot)$ be groupoid (resp. hypergroupoid) and $f : G \to G$ be a map. We define a hope (∂), called theta-hope or simply ∂-hope, on G as follows*

$x\partial y = \{f(x)\cdot y, x\cdot f(y)\}, \ \forall x, y \in G$. *(resp.* $x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)), \ \forall x, y \in G$*)*

If $(\cdot)$ *is commutative then* (∂) *is commutative. If* $(\cdot)$ *is COW, then* (∂) *is COW.*

Let $(G,\cdot)$ *be groupoid (resp. hypergroupoid) and* $f : G \to \mathbf{P}(G) - \{\varnothing\}$ *be multivalued map. We define the hope (∂), on G as follows*

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \ \ \forall x, y \in G$$

Properties. If $(G,\cdot)$ is a semigroup then:

(a) For every f, the hope (∂) is WASS.

(b) If f is homomorphism and projection, or idempotent: $f^2 = f$, then (∂) is associative.

Let $(G,\cdot)$ be a groupoid and $f_i : G \to G, i \in I$, be a set of maps on G. We consider the map $f_\cup : G \to \mathbf{P}(G)$ such that $f_\cup(x) = \{f_i(x)|i \in I,\}$ called the *the union* of the $f_i(x)$. We define the *union ∂-hope*, on G if we consider as map the $f_\cup(x)$. A special case for a given map f, is to take the union of this with the identity map. We consider the map $\underline{f} \equiv f \cup (id)$, so $\underline{f}(x) = \{x, f(x)\}, \forall x \in G$, which we call *b−∂−hope*. Then we have

$$x\partial y = \{xy, f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G$$

Motivation for the definition of the ∂-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for given functions s(x), t(x), we have $s\partial t = \{s't, st'\}$ where $(')$ denotes the derivative.

Proposition 2.1. *Let* $(G,\cdot)$ *be group and* $f(x) = a$*, a constant map. Then* $(G,\partial)/\beta^*$ *is a singleton.*

Proof. For all x in G we can take the hyperproduct of the elements, a^{-1} and $a^{-1}x$

$$a^{-1}\partial(a^{-1} \cdot x) = \{f(a^{-1}) \cdot a^{-1} \cdot x, a^{-1} \cdot f(a^{-1} \cdot x)\} = \{x, a\}.$$

thus $x\beta a, \forall x \in G$, so $\beta^*(x) = \beta^*(a)$ and $(G,\partial)/\beta^*$ is singleton. □

Special case if $(G,\cdot)$ be a group and $f(x) = e$, then $x\partial y = \{x, y\}$, is the incidence hope.

Taking the application on the derivative, consider all polynomials of the first degree $g_i(x) = a_i x + b_i$. We have $g_1\partial g_2 = \{a_1a_2x + a_1b_2, a_1a_2x + b_1a_2\}$, so it is a hope on the set of first degree polynomials. Moreover all polynomials $x + c$, where c be a constant, are units.

The Lie-Santilli *isotopies* born to solve Hadronic Mechanics problems. Santilli [4], [15], proposed a 'lifting' of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed correspond to H_v-structures called *e-hyperfields* which are used in physics or biology. Therefore, in this theory the units and the inverses are playing very important role. The Construction 2.1, is used, last years, in this theory.

Example 2.1. *Consider the 'small' ring $(\mathbf{Z}_4, +, \cdot)$, suppose that we want to construct non-degenerate H_v-field where 0 and 1 are scalars with respect for both addition and multiplication, and moreover every element of $\mathbf{Z}_4$ has a unique opposite and every non-zero element has a unique inverse. Then on the multiplication tables o these operations the lines and columns of the elements 0 and 1 remain the same. The sum 2+2=0 and the product $3 \cdot 3 = 1$, remain the same. In the results of the sums 2+3=1, 3+2=1 and 3+3=2 one can put respectively, the elements 3, 3 and 0. In the results of the products $2 \cdot 2 = 0$, $2 \cdot 3 = 2$ and $3 \cdot 2 = 2$ one can put respectively, the elements 2, 0 and 0. Then in all those enlargements, even only one enlargement is used, we obtain*

$$(\mathbf{Z}_4, +, \cdot)/\beta^* \cong \mathbf{Z}_2$$

Therefore this construction gives 49 H_v-fields.

3 Strong-inverse elements.

We present now some hyperstructures, results and examples of hyperstructures with strong-inverse elements.

Properties 3.1. *Let $(G, \cdot)$ be a group, take P such that $\varnothing \neq P \subset G$ and the P-hypergroup $(G, \underline{P})$, where $x\underline{P}y = xPy$. We have the following*

Units: In order an element u to be right unit of the P-hypergroup $(G, \underline{P})$, we must have $x\underline{P}u = xPu \ni x, \forall x \in G$. In fact the set Pu must contain the unit element e of the group $(G, \cdot)$. Thus, all the elements of the set P^{-1}, are right units. The same is valid for the left units, therefore, the set of all units is the P^{-1}.

Inverses: Let u be a unit in $(G, \underline{P})$, then, for given x in order to have an inverse element x' with respect to u, we must have $x\underline{P}x' = xPx' \ni u$, so taking $xpx' = u$, we obtain that all the elements of the form $x' = p^{-1}x^{-1}u$ are inverses to x with respect to the unit u.

Theorem 3.1. *[16] Let $(G, \cdot)$ be a group, then for all normal subgroups P of G, the hyperstructure $(G, \underline{P})$, where $x\underline{P}y = xPy, \forall x, y \in G$, is a hypergoup with strong inverses. Moreover, for any inverse x' of $x \in G$, with respect to any unit, we have $x\underline{P}x' = P$.*

Proof. Let $x \in G$, take an inverse $x' = p^{-1}x^{-1}u$ with respect to the unit $u = p_k^{-1}$, for any p. Then we have $x\underline{P}x' = xPx'$. But, since P is normal subgroup, we have

$$x\underline{P}x' = xp^{-1}x^{-1}p_k^{-1}P = xp^{-1}x^{-1}P = xp^{-1}Px^{-1} = xPx^{-1} = P$$

Remark that in this case, $P^{-1} = P$, is the set of all units, thus all inverses are strong. □

Properties 3.2. *Let $(G, \cdot)$ be groupoid and $f : G \to G$ be a map and (G, ∂) the corresponding ∂-structure, then we have the following:*
Units: In order an element u to be right unit, we must have

$$x\partial u = \{f(x) \cdot u, x \cdot f(u)\} \ni x.$$

But, the unit must not depend on the f(x), so $f(u) = e$, where e be unit in $(G, \cdot)$ which must be a monoid. The same it is obtained for the left units. So the elements of $kernf = \{u : f(u) = e\}$, are the units of (G, ∂).
Inverses: Let u be a unit in (G, ∂) , then $(G, \cdot)$ is a monoid with unit e and $f(u) = e$. For given x in order to have an inverse element x' with respect to u, we must have

$$x\partial x' = \{f(x) \cdot x', x \cdot f(x')\} \ni u \text{ and } x'\partial x = \{f(x') \cdot x, x' \cdot f(x)\} \ni u.$$

So the only cases, which do not depend on the image $f(x')$, are

$$x' = (f(x))^{-1}u \text{ and } x' = u(f(x))^{-1}$$

the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

Remark [16]: Since the inverses are depending on the units, therefore they are not strong.

The following constructions, originated from the properties the strong-inverse elements have, gives a minimal hyperstructure which have strong-inverse elements. This is a necessary enlargement in order all the elements to be strong-inverses.

Construction 3.1. *Let $(G, \cdot)$ be a group with unit e. Consider a finite set $E = \{e_i | i \in I\}$. On the set $\underline{G} = (G - \{e\}) \cup E$ we define a hope $(\times)$ as follows:*

$$\begin{cases} e_i \times e_j = \{e_i, e_j\}, \ \forall e_i, e_j \in E \\ e_i \times x = x \times e_j = x, \ \forall e_i \in E, x \in G - \{e\} \\ x \times y = x \cdot y \text{ if } x \cdot y \in G - \{e\} \text{ and } x \times y = E \text{ if } x \cdot y = e \end{cases}$$

Then the hyperstructure $(\underline{G}, \times)$ is a hypergroup. The set of unit elements is E and all the elements are strong-inverse. Moreover we have $(\underline{G}, \times)/\beta^* \cong (G, \cdot)$.

Proof. For the associativity we have the cases
$(e_i \times e_j) \times e_k = e_i \times (e_j \times e_k) = \{e_i, e_j, e_k\}, \forall e_i, e_j, e_k \in E$
$(x \times y) \times z = x \times (y \times z) = x \cdot y \cdot z$ or E, $\forall x, y, z \in \underline{G}$ and not all of them belong to E.

In the second case there is no matter if the product of two inverse elements appears. The only difference is that the result is singleton and in some cases the result is equal to the set E.

Therefore the strong associativity is valid. Moreover the reproductivity is valid and the set E is the set of units in $(\underline{G}, \times)$.

Two elements of G̲ are β^* equivalent if they belong to any finite $\times$-product of elements of G̲. Thus all fundamental classes are singletons except the set of units E. That means that we have $(\underline{G}, \times)/\beta^* \cong (G, \cdot)$. □

Construction 3.2. *Let $(G, \cdot)$ be an H_v-group with only one unit element e and every element has a unique inverse. Consider a finite set $E = \{e_i | i \in I\}$. On the set $\underline{\mathbf{G}} = (\mathbf{G} - \{e\}) \cup E$ we define a hope $(\times)$ as follows:*

$$\begin{cases} e_i \times e_j = \{e_i, e_j\}, \ \forall e_i, e_j \in E \\ e_i \times x = x \times e_j = x, \ \forall e_i \in E, x \in G - \{e\} \\ x \times y = x \cdot y \text{ if } x \cdot y \in G - \{e\} \text{ and } x \times y = E \text{ if } x \cdot y = e \end{cases}$$

Then the hyperstructure $(\underline{G}, \times)$ is an H_v-group. The set of unit elements is E and all the elements are strong-inverse. Moreover we have

$$(\underline{G}, \times)/\beta^* \cong (G, \cdot)/\beta^*.$$

Proof. For the associativity we have the cases
$(e_i \times e_j) \times e_k = e_i \times (e_j \times e_k) = \{e_i, e_j, e_k\}, \forall e_i, e_j, e_k \in E$
$(x \times y) \times z = x \times (y \times z) = x \cdot y \cdot z$ or E, $\forall x, y, z \in \underline{G}$ and not all of them belong to E.

Therefore the WASS is valid. Moreover the reproductivity is valid and the set E is the set of units in $(\underline{G}, \times)$.

Two elements of G̲ are β^* equivalent if they belong to any finite $\times$-product of elements of G̲. So, all fundamental classes correspond to the fundamental classes of $(G, \cdot)$, with an enlargement of the class of e into E. Thus, we have $(\underline{G}, \times)/\beta^* \cong (G, \cdot)/\beta^*$.□

We remark that the above constructions give a great number of hyperstructures with strong-inverses because we can enlarge then in any result except if the result is E.

Now we present a result on strong-inverses on a general finite case.

Theorem 3.2. *The minimum non-degenerate, i.e. have non-degenerate fundamental field, h/v-fields with strong-inverses with respect to both sum-hope and product-hope, obtained by enlarging the ring* $(\mathbf{Z}_{2p}, +, \cdot)$*, where* $p > 2$ *is prime number, and which has fundamental field isomorphic to* $(\mathbf{Z}_p, +, \cdot)$*, is defined as follows:*

The sum-hope $(\oplus)$ *is enlarged from (+) by setting*

(1). $p(\oplus)\kappa = \kappa(\oplus)p = \kappa + E, \forall\kappa \in \mathbf{Z}_{2p}$*, where* $E = \{0, p\}$ *be the set of zeros*

(2). whenever the result is 0 and p we enlarge it by setting p and 0, respectively.

The product-hope $(\otimes)$ *is enlarged from* $(\cdot)$ *by setting*

(3). $(p + 1) \otimes \kappa = \kappa \otimes (p + 1) = \kappa U, \forall\kappa \in \mathbf{Z}_{2p}$*, where* $U = \{1, p + 1\}$ *be the set of units*

(4). whenever the result is 1 and p+1 we enlarge it by setting p+1 and 1, respectively.

The fundamental classes are of the form $\underline{\kappa} = \{\kappa, \kappa + p\}, \forall\kappa \in \mathbf{Z}_{2p}$

Proof. In order to have non degenerate case, since we have 2p elements, in both, sum-hope and product-hope, is to take the zero-set $E = \{0, p\}$ and unit-set $U = \{1, p + 1\}$. In order to have strong-opposites, we have to enlarge, according to Remark 1.1, as in (2). Moreover, in order to have strong-inverses, we have to enlarge, again according to Remark 1.1, as in (4).

From the above definition of the sum-hope it is to see that the fundamental classes are of the form $\underline{\kappa} = \{\kappa, \kappa + p\}, \forall\kappa \in \mathbf{Z}_{2p}$.

For the above classes for the product-hope mod(2p), we have $\forall\kappa, \lambda \in \mathbf{Z}_{2p}$,

$$\underline{\kappa} \otimes \underline{\lambda} = \{\kappa, \kappa + p\} \cdot \{\lambda, \lambda + p\} = \{\kappa\lambda, \kappa(\lambda + p), (\kappa + p)\lambda, (\kappa + p)(\lambda + p)\} =$$

$$= \{\kappa\lambda, \kappa\lambda + \kappa p, \kappa\lambda + p\lambda, \kappa\lambda + \kappa p + p\lambda + pp\} = \{\kappa\lambda, \kappa\lambda + p\}$$

Because, if κ or λ are odd numbers then $\kappa\lambda + \kappa p$ or $\kappa\lambda + p\lambda$, respectively, are equal mod2p to $\kappa\lambda + p$. Moreover, if both κ and λ are even numbers then we have, $\kappa\lambda + \kappa p + p\lambda + pp = \kappa\lambda + p$.

From the above we remark that the fundamental classes $\underline{\kappa} = \{\kappa, \kappa + p\}, \forall\kappa \in \mathbf{Z}_{2p}$, are formed from sum-hope and they are remain the same in the product-hope. Finally the fundamental field is isomorphic to $(\mathbf{Z}_p, +, \cdot)$. □

As example of the above Theorem we present the case for p=5.

Example 3.1. *In the case of the h/v-field* $(\mathbf{Z}_{10}, \oplus, \otimes)$*, i.e. p=5, we have the following multiplicative tables:*

$\oplus$	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***
0	*0*	*1*	*2*	*3*	*4*	*5,0*	*6*	*7*	*8*	*9*
1	*1*	*2*	*3*	*4*	*5,0*	*6,1*	*7*	*8*	*9*	*0,5*
2	*2*	*3*	*4*	*5,0*	*6*	*7,2*	*8*	*9*	*0,5*	*1*
3	*3*	*4*	*5,0*	*6*	*7*	*8,3*	*9*	*0,5*	*1*	*2*
4	*4*	*5,0*	*6*	*7*	*8*	*9,4*	*0,5*	*1*	*2*	*3*
5	*5,0*	*6,1*	*7,2*	*8,3*	*9,4*	*0,5*	*1,6*	*2,7*	*3,8*	*4,9*
6	*6*	*7*	*8*	*9*	*0,5*	*1,6*	*2*	*3*	*4*	*5,0*
7	*7*	*8*	*9*	*0,5*	*1*	*2,7*	*3*	*4*	*5,0*	*6*
8	*8*	*9*	*0,5*	*1*	*2*	*3,8*	*4*	*5,0*	*6*	*7*
9	*9*	*0,5*	*1*	*2*	*3*	*4,9*	*5,0*	*6*	*7*	*8*

and

$\otimes$	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*
1	*0*	*1,6*	*2*	*3*	*4*	*5*	*6,1*	*7*	*8*	*9*
2	*0*	*2*	*4*	*6,1*	*8*	*0*	*2*	*4*	*6,1*	*8*
3	*0*	*3*	*6,1*	*9*	*2*	*5*	*8,3*	*1,6*	*4*	*7*
4	*0*	*4*	*8*	*2*	*6,1*	*0*	*4*	*8*	*2*	*6,1*
5	*0*	*5*	*0*	*5*	*0*	*5*	*0,5*	*5*	*0*	*5*
6	*0*	*6,1*	*2*	*8,3*	*4*	*0,5*	*6,1*	*2,7*	*8*	*4,9*
7	*0*	*7*	*4*	*1,6*	*8*	*5*	*2,7*	*9*	*6,1*	*3*
8	*0*	*8*	*6,1*	*4*	*2*	*0*	*8*	*6,1*	*4*	*2*
9	*0*	*9*	*8*	*7*	*6,1*	*5*	*4,9*	*3*	*2*	*1,6*

Moreover, it is easy to see that the fundamental field is isomorphic to $(\mathbf{Z}_5, +, \cdot)$.

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 127-138 **eISSN**: 2282-8214

A Fuzzy Coding Approach to Data Processing Using the Bar

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†**doi**:10.23755/rm.v33i0.387

Abstract

The bar is an alternative to Likert-type scale as a response format option used in closed-form questionnaires. An important advantage of using the bar is that it provides a variety of data post-processing options (i.e., ways of partitioning the values of a continuous variable into discrete groups). In this context, continuous variables are usually divided into equal-length or equal-area intervals according to a user-specified distribution (e.g. the Gaussian). However, this transition from continuous into discrete can lead to a significant loss of information. In this work, we present a fuzzy coding of the original variables which exploits linear and invertible triangular membership functions. The proposed coding scheme retains all of the information in the data and can be naturally combined with an exploratory data analysis technique, Correspondence Analysis, in order to visually investigate both linear and non-linear variable associations. The proposed approach is illustrated with a real-world application to a student course evaluation dataset.
Keywords: Likert scale; Bar; Correspondence Analysis; fuzzy coding; triangular membership functions
2010 AMS subject classifications: 62P25.

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†©Angelos Markos. Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

The closed-form questionnaire is the most commonly used data collection tool or instrument in quantitative studies. Likert scales are commonly used to measure attitude, providing a range of responses to a given question or statement (see e.g., [5]). Typically, Likert scales have an odd number of response categories, 3, 5 or 7, representing the degree of agreement with the corresponding statements. For example, a five-point scale ranges from 1 = strongly disagree to 5 = strongly agree. Although the response categories have a rank order, the invervals between values are frequently presumed equal. This assumption is often convenient in that it permits the calculation of descriptive and inferential statistics suitable for continuous variables.

The processing of questionnaire data obtained via Likert scales has certain advantages, but there are also major shortcomings [4, 5, 13, 27]. Firstly, the desicion on the number of categories of a Likert-type scale may affect the outcome of statistical analysis. Too many or too few response categories, may cause respondent fatigue with a corresponding drop-off in response rate and reliability [3]. Second, there is evidence that participants would give different ratings when using different versions of the same Likert-type scale [12]. This indicates that the decision on the verbal labels that will be used to describe the numerical values of a Likert scale is not a trivial one. Such a problem also involves a number of social and psychologica factors. [27]. A third issue is related to the legitimacy of assuming a continuous or interval scale for Likert-type categories, instead of an ordinal level of measurement. In fact, many authors advocate against this practice, given that the appropriate descriptive and inferential statistics differ for ordinal and interval variables [13]. Therefore, if the wrong statistical technique is used, researchers increase the chance of coming to the wrong conclusions about their findings. Finally, the fixed number of response categories limits the options for data processing and does not allow the direct comparison with the results of similar studies, where the same questions but with a different number of response categories were used [27].

Kambaki-Vougioukli & Vougiouklis [14] introduced the “bar”, an alternative to the Likert scale as a measurement instrument of a characteristic or attitude that is believed to range across a continuum of values. The bar is a straight horizontal line of fixed length, usually 62mm. The ends are defined as the extreme limits of the characteristic to be measured, orientated from the left (0) to the right (62). The study participants are asked to mark the bar at any point that expresses their answer to the specific question. Although similar to other concepts in the field of psychology, such as the Visual Analogue Scale [6], the idea of the bar originates from hyperstructure theory, a branch of mathematics that has recently found a wide range of applications in the social sciences (see e.g., [2, 7, 8, 21]). Consequently, the bar marks a transition from discrete into continuous and from single valued into fuzzy or multivalued [27].

A series of studies, mostly in quantitative linguistics [15, 16, 17, 18, 19], have shown that the bar can be widely used to a broad range of populations and settings due to its simplicity and adaptability. A questionnaire developed using the bar instead of a Likert scale takes less time to complete and no training or special skill of the participants is required other than to possess an understanding of distance on a ruler. Moreover, minimal translation difficulties can easily lead to a cross-cultural adaptation of a questionnaire. Recently, [19] developed a software for using the bar in online questionnaires.

The most important merit of using the bar, however, is the flexibility it offers to practitioners with regard to data analysis, without having to re-administer the questionnaire. After data collection, the analyst can decide how to split each variable at appropriate intervals. Instead, in the case of Likert-type scales, such a decision has to be taken before data collection and does not give any room for testing alternative ways of data processing. The number of groups per variable is chosen according to the distribution of the variable at hand. In this context, continuous variables are usually divided into equal-length or equal-area intervals according to a desirable distribution (e.g., the Gaussian or the parabola). A detailed justification of such a discretization scheme is given in [27]. Hereafter, we will refer to this procedure by crisp coding.

Crisp coding of a continuous value to a category obviously loses a substantial part of the original information and, subsequently, the advantage of continuity provided using the bar. This is because the original values are usually not uniformly distributed in the newly created intervals. To alleviate this problem, we discuss an alternative, fuzzy coding of the original data, which exploits linear and invertible triangular membership functions. A side advantage of the proposed fuzzy coding scheme is that the resulting data matrix can be given as input to Correspondence Analysis, a multivariate technique that can visualize both linear and non-linear variable associations.

Section 2 presents the rationale behind utilizing a fuzzy instead of a crisp coding scheme to data obtained from questionnaires using the bar. Section 3 offers a brief introduction to Correspondence Analysis applied on fuzzy coded data. The proposed approach is illustrated with a real-world application in Section 4. Section 5 concludes the paper.

2 Crisp versus fuzzy coding of continuous variables

Let $A, B, C, \ldots$ be a number of continuous variables whose values range from 0 to 62 and were collected for a number of survey participants or subjects using the bar. A common discretization scheme is to split each variable into five intervals of equal length, 1 to 5, as follows:

$$1 : [0{-}12.4], 2 : [12.4{-}24.8), 3 : [24.8{-}37.2), 4 : [37.2{-}49.6), 5 : (49.6{-}62].$$

Then, for each subject a binary vector can be formed to summarize any value of each variable. For example, the value 35.7 for variable A lies in the third interval and can be coded into [0 0 1 0 0]. This type of binary coding is commonly referred to as *crisp coding* (e.g. see [1]) and a row-wise concatenation of all binary vectors forms a table, $\mathbf{Z}_A$, for variable A. The row margins of $\mathbf{Z}_A$ are the same, equal to a column of ones. The so-called *indicator matrix*, denoted by $\mathbf{Z}$, is composed of a set of subtables, $\mathbf{Z}_A, \mathbf{Z}_B, \mathbf{Z}_C, \ldots$ stacked side by side, one for each variable. Table 1 shows an example of crisp coding for some subjects on variable A with five categories (on the left), and their coding into a dummy variable (on the right). The matrix on the right is the subtable $\mathbf{Z}_A$ and $\mathbf{Z} = [\mathbf{Z}_A; \mathbf{Z}_B; \mathbf{Z}_C; \ldots]$ denotes the full indicator matrix. This matrix can be subsequently analyzed with Correspondence Analysis, a well-established exploratory data analysis technique (see e.g., [9] and Section 3).

Table 1: An example of crisp coding of a categorical variable with five categories into a dummy or indicator variable

A
3
1
5
.
.
.

A1	A2	A3	A4	A5
0	0	1	0	0
1	0	0	0	0
0	0	0	0	1
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

In the case of crisp coding, it is assumed that the original continuous values are uniformly distributed within each interval. However, this is a strong assumption to make and the discrete assignment of continuous values to categories obviously loses a substantial part of the original information. This problem can be alleviated by using a fuzzy instead of crisp coding scheme. Fuzzy coding (*codage flou* in French) has been successfully used in a variety of data analysis techniques and settings (see e.g., [1, 10, 11, 26].

The idea is to convert a continuous variable into a pseudo-categorical (i.e., fuzzy) variable using appropriate membership functions [11]. This is called "fuzzification" of the data. For example, 35.7 can be fuzzy coded into [0 0 0.75 0.25 0], instead of [0 0 1 0 0]. An important decision to make is the choice of membership functions that will be used for fuzzification. Following [1], we adopt the system of the so-called "three-point triangular membership functions", also known as piecewise linear functions, or second order B-splines [26]. Triangular membership functions have two nice properties that will be further illustrated below: they are linear and invertible.

A simple example of triangular membership functions is shown in Figure 1, defining a fuzzy variable with five categories. On the horizontal axis is the scale of the original variable and five hinge points or knots, chosen as the minimum, 1st quartile, median, 3rd quartile and maximum values of the variable. This choice of hinge points is a simple one and corresponds to the quantiles of the distribution; it has been argued that such a choice ensures robustness [24]. The five functions shown in Figure 1 are used for the recoding, and 35.7 is graphically shown to be recoded as 0 for category 1, 0 for category 2, 0.75 for category 3, 0.25 for category 4 and 0 for category 5. This coding scheme is linear and invertible, as shown below:

$$35.7 = 0.0 \times 0 + 0.0 \times 21 + 0.75 \times 31 + 0.25 \times 50 + 0.0 \times 62. \tag{1}$$

Given the fuzzy observation [0 0 0.75 0.25 0], the value of the original variable is unique and equals to 35.7.

An algebraic description of the proposed scheme is given below. Using triangular membership functions, the fuzzy values $z_1, z_2, \ldots, z_5$ for a five-category fuzzy coding, where x is the original value on the continuous scale and the hinge points are $m_1, m_2, \ldots, m_5$ are given by:

$$z_1(x) = \begin{cases} \frac{m_2 - x}{m_2 - m_1}, & \text{for } x \leq m_2 \\ 0, & \text{otherwise} \end{cases} \qquad z_2(x) = \begin{cases} \frac{x - m_1}{m_2 - m_1}, & \text{for } x \leq m_2 \\ \frac{m_3 - x}{m_3 - m_2}, & \text{for } m_2 \leq x \leq m_3 \\ 0 & \text{otherwise} \end{cases}$$

$$z_3(x) = \begin{cases} \frac{x - m_2}{m_3 - m_2}, & \text{for } m_2 \leq x \leq m_3 \\ \frac{m_4 - x}{m_4 - m_3}, & \text{for } m_3 \leq x \leq m_4 \\ 0 & \text{otherwise} \end{cases} \quad z_4(x) = \begin{cases} \frac{x - m_3}{m_4 - m_3}, & \text{for } m_3 \leq x \leq m_4 \\ \frac{m_5 - x}{m_5 - m_4}, & \text{for } x > m_4 \\ 0 & \text{otherwise} \end{cases}$$

$$z_5(x) = \begin{cases} \frac{x - m_4}{m_5 - m_4}, & \text{for } x > m_4 \\ 0 & \text{otherwise} \end{cases}$$

Table 2 shows the corresponding subtable $\mathbf{Z}_A$ in the case of fuzzy coding of some values of variable A. Let $\mathbf{Z}^*$ denote the full *fuzzy indicator matrix*, which is

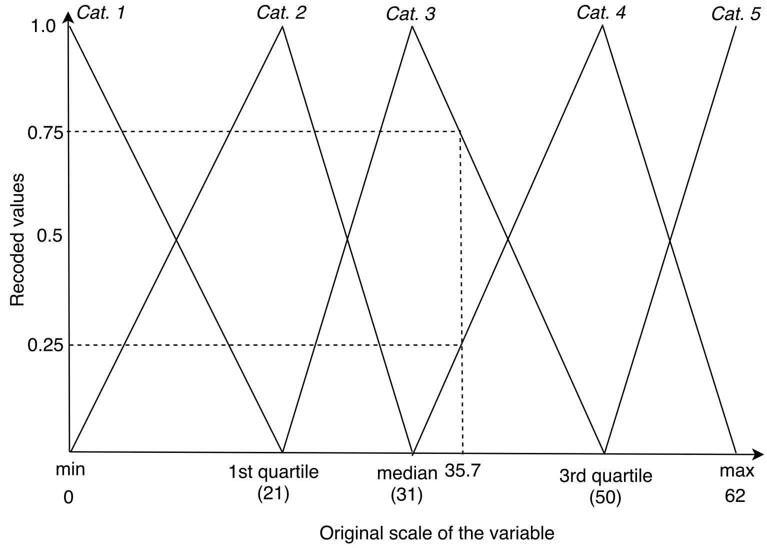

Figure 1: Triangular membership functions to code a continuous variable (horizontal axis) into five fuzzy categorical variables. An example is shown of a value on the orginal scale (35.7) being fuzzy coded as [0 0 0.75 0.25 0].

composed of a set of subtables stacked side by side, one for each fuzzy indicator variable. As it is obvious from Eq. 1, fuzzy coding transforms continuous variables into fuzzy categories with no loss of information, since a fuzzy-coded variable can be back-transformed to its original value. This is an improvement over crisp coding, where the information about the value of the variable within each interval is lost. Alternatives to triangular membership functions can be, for example, trapezoidal, Gaussian and generalized Bell membership functions [1, 25]. A thorough investigation of their properties in the context of questionnaire data obtained using the bar is beyond the scope of this work.

Table 2: An example of fuzzy coding of a continuous variable into a fuzzy indicator variable with five categories

A
35.7
43.1
25.0
.
.
.

A1	A2	A3	A4	A5
0	0	0.75	0.25	0
0	0	0.36	0.64	0
0	0.60	0.40	0	0
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

3 Correspondence Analysis on fuzzy-coded data

The fuzzy coding scheme described in Section 2, can be combined with Correspondence Analysis (CA), a well-established method of Geometric Data Analysis [23] for visualizing the rows and columns of a matrix of nonnegative data as points in a spatial representation. For a detailed treatment of CA we refer the reader to [9], for example. Aşan and Greenacre [1] showed that CA on the fuzzy indicator matrix $\mathbf{Z}^*$ (see Table 2) can visualize nonlinear relationships between variables and that this property holds for all forms of membership functions. The core of the CA algorithm is the Singular Value Decomposition (SVD) of a suitably transformed matrix. Next, we briefly present the algorithmic steps of CA on the fuzzy indicator matrix $\mathbf{Z}^*$ [20].

Step 1. Given a data table with continuous variables, apply the fuzzy coding scheme of Section 2 to obtain the fuzzy indicator matrix, $\mathbf{Z}^*$.
Step 2. Compute the matrix $\mathbf{P}$ as $\mathbf{Z}^*$ divided by its grand total, with row and column sums of $\mathbf{P}$ defined as $\mathbf{r} = \mathbf{P1}$, $\mathbf{c}^\mathbf{T} = \mathbf{1}^\mathbf{T}\mathbf{P}$, where $\mathbf{1}$ denotes a column vector of 1's of appropriate order and $^\mathbf{T}$ denotes vector and matrix transpose. The elements of $\mathbf{r}$ and $\mathbf{c}$ are called row and column *m*asses in CA terminology.
Step 3. Compute the matrix of standardized residuals $\mathbf{S}$:

$$\mathbf{S} = \mathbf{D}_r^{-1/2}(\mathbf{P} - \mathbf{rc})^\mathbf{T}\mathbf{D}_c^{-1/2}$$

where $\mathbf{D}_r$ and $\mathbf{D}_c$ denote diagonal matrices of the respective masses.
Step 4. Compute the SVD of $\mathbf{S}$:

$$\mathbf{S} = \mathbf{U}\mathbf{D}_\alpha\mathbf{V}^\mathbf{T}$$

where the singular vectors in $\mathbf{U}$ and $\mathbf{V}$ are normalized as $\mathbf{U}^\mathbf{T}\mathbf{U} = \mathbf{V}^\mathbf{T}\mathbf{V} = \mathbf{I}$, and $\mathbf{D}_\alpha$ is the diagonal matrix of the singular values, which are positive and in descending order, $\alpha_1 \geq \alpha_2 \geq \ldots > 0$.
Step 5. Compute the coordinates of the row and column points to obtain the so-called "symmetric" CA map:
rows: $\mathbf{F} = \mathbf{D}_r^{-1/2}\mathbf{U}\mathbf{D}_\alpha$, columns: $\mathbf{\Gamma} = \mathbf{D}_c^{-1/2}\mathbf{V}\mathbf{D}_\alpha$.

4 Application to real data

The real data set considered here consists of 159 pre-service teachers' evaluation ratings of an introductory statistics university course. The focus of the analysis is on the following 5 statements, A to E, that were used to evaluate the quality of the teaching-learning process.

How much has each of the following contributed to your understanding of the main ideas covered in this course?

A: The tutor's description of the aim, syllabus content and course objectives.
B: The tutor's encouragement of students to ask questions.
C: The connection of the course material with everyday life examples.
D: Your own effort and engagement in the course.
E: Your own consistency in attending classes.

The original five-point Likert-type scale was substituted by the bar (0 to 62mm). After data collection, each one of the five statements was coded into five fuzzy categories using triangular membership functions, as described in Section 2. Data analysis was perfomed using the `R` package `ca` [22] and `R` code written by the author.

The Correspondence Analysis symmetric map for these data (first and second dimension) is shown in Figure 2. This map explains a total of 27.3% of the variance (or inertia) in the data. Triangle points correspond to the fuzzy categories of each variable (A1 to A5, B1 to B5, etc). Variable category points close to each other indicate similar response profiles to the corresponding statements. The origin of the map corresponds to the average response profile.

The main interpretation of the CA map is carried out by evaluating the positions of the category points to each axis. On the left part of the first dimension (horizontal axis) lies a group of students who attribute their understanding of the course content to the tutor's quality of teaching and practices (strong agreement with statements A, B and C) but not to their own efforts and consistency in attending classes (strong disagreement with statements D and E). On the right part of the first dimension, there is a group of students that contrasts the one on the left. These students express strong agreement with statements D and E but strong disagreement with statements A, B and C. The second dimension, when projected on the vertical axis, separates extreme values on top from moderate responses below the cross of the axes. The resulting parabolic shape or "horse shoe" is a typical structure in CA that has a unidimensional structure and confirms that the items are articulated around a hierarchical scale (for more details on the horseshoe effect, see [23]). To sum up, CA on the fuzzy-coded data obtained using the bar, reveals an interesting negative association between statements $\{$A, B, C$\}$ and $\{$D, E$\}$.

5 Conclusions

The bar of Kambaki-Vougioukli and Vougiouklis is a suitable and useful continuous scale, similar to a rule, that serves to collect survey data. After data collection the analyst can decide how to split each variable at appropriate intervals.

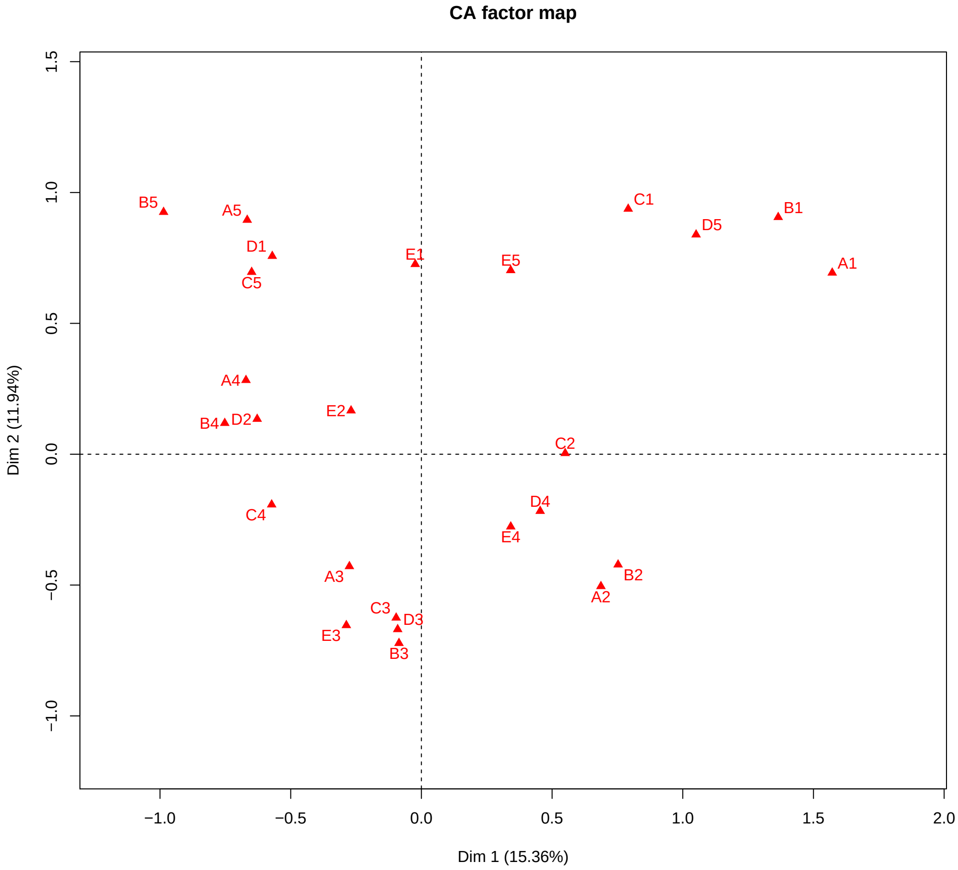

Figure 2: Correspondence Analysis symmetric map (1st and 2nd dimension).

This type of discrete or crisp coding, however, can lead to a significant loss of information and negate the important advantages of using the bar. Fuzzy instead of crisp coding preserves the original information lying in the original data. The original values are mapped, via triangular membership functions, to a 5-category recoding, using the minimum, quartiles and maximum as the hinge points, with the first and last functions not being “shouldered”. The proposed scheme is linear and invertible and can be paired with a well-established exploratory data analysis method, Correspondence Analysis, for the visual investigation of both linear and non-linear relationships among variables. A side advantage of fuzzy coding is that it transforms continuous data to a form that is comparable to categorical data, and so enables analysis of mixed measurement scales. Exploring this possibility is an important step for future work.

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 139-150 **eISSN**: 2282-8214

Multiple Ways of Processing in Questionnaires

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†**doi**:10.23755/rm.v33i0.376

Abstract

In social sciences when questionnaires are used, there is a new tool, the bar instead of Likert scale. The bar has been suggested by Vougiouklis & Vougiouklis in 2008, who have proposed the replacement of Likert scales, usually used in questionnaires, with bar. This new tool, gives the opportunity to researchers to elaborate the questionnaires in different ways, depending on the filled questionnaires and of course on the problem. Moreover, we improve the procedure of the filling the questionnaires, using the bar instead of Likert scale, on computers where we write down automatically the results, so they are ready for research. This new kind of elaboration is being applied on data obtained by a survey, studying the new results. The hyperstructure theory is being related with questionnaires and we study the obtained hyperstructures, which are used as an organized device of the problem and we focus on special problems.
Keywords: hyperstructures; questionnaires; bar;
2010 AMS subject classifications: 20N20, 16Y99.

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 Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Basic definitions

The main object of this paper is the class of hyperstructures called H_v-structures introduced in 1990 [17], which satisfy the weak axioms where the non-empty intersection replaces the equality. Some basic definitions are the following:

In a set H equipped with a hyperoperation (abbreviation *hyperoperation = hope)* $\cdot : H \times H \to P(H) - \{\varnothing\}$, we abbreviate by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H.$

The hyperstructure $(H, \cdot)$ is called an H_v*-semigroup* if it is *WASS*, it is called H_v*-group* if it is reproductive H_v-semigroup, i.e., $xH = Hx = H, \forall x \in H.$

Motivation. In the classical theory the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-group. This is the motivation to introduce the H_v-structures [17], [18].

$(R, +, \cdot)$ is called an H_v*-ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy + xz) \neq \varnothing,\ (x+y)z \cap (xz + yz) \neq \varnothing,\ \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be an H_v-ring, $(M, +)$ be a COW H_v-group and there exists an external hope

$$\cdot : R \times M \to P(M) : (a, x) \to ax$$

such that $\forall a, b \in R$ and $\forall x, y \in M$ we have

$$a(x+y) \cap (ax + ay) \neq \varnothing,\ (a+b)x \cap (ax + bx) \neq \varnothing,\ (ab)x \cap a(bx) \neq \varnothing,$$

then M is called an H_v*-module* over F. In the case of an H_v-field F, which is defined later, instead of an H_v-ring R, then the H_v*-vector space* is defined.

For more definitions and applications on H_v-structures one can see [2], [3], [4], [5], [6], [10], [14], [16], [18].

The main tool to study hyperstructures is the fundamental relation. In 1970 M. Koscas defined in hypergroups the relation β and its transitive closure $\beta*$. This relation connects the hyperstructures with the corresponding classical structures and is defined in H_v-groups as well. T. Vougiouklis introduced the $\gamma*$ and $\epsilon*$ relations, which are defined, in H_v-rings and H_v-vector spaces, respectively [17]. He also named all these relations $\beta*$, $\gamma*$ and $\epsilon*$, fundamental relations because they play very important role to the study of hyperstructures especially in the representation theory of them. For similar relations see [18], [22], [4].

Definition 1.1. *The fundamental relations β*, γ* and ϵ*, are defined, in H_v-groups, H_v-rings and H_v-vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively.*

Specifying the above motivation we remark the following: Let $(G, \cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R, \cdot)$ is an H_v-group, therefore we have the quotient $(G/R, \cdot)/\beta*$ which is a group, the fundamental one. Remark that the classes of the fundamental group $(G/R, \cdot)/\beta*$ are a union of some of the R-classes. Otherwise, the $(G/R, \cdot)/\beta*$ has elements classes of G where they form a partition which classes are larger than the classes of the original partition R.

The way to find the fundamental classes is given by the following [17], [20], [21], [22]:

Theorem 1.1. *Let $(\boldsymbol{H}, \cdot)$ be an H_v-group and denote by **U** the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x, y\} \subset$ **u** where **u** $\in$ **U**. Then β* is the transitive closure of β.*

A well known and large class of hopes is given as follows [15], [18], [12]:

Let $(G, \cdot)$ be a groupoid then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-*hopes*: for all $x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$,$(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P-*hyperstructures*. The most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup. We do not know what hyperstructures are $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$. In some cases, depending on the choice of P, the $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS. If more operations are defined in G, then for each operation several P-hopes can be defined.

2 The bar in questionnaires

Last decades hyperstructures seem to have a variety of application not only in mathematics, but also in many other sciences [1], [2], [9], [13], [19], [25], including the social ones.

An important application which can be used in social sciences is the combination of hyperstructure theory with fuzzy theory, by the replacement of the Likert Scale by the Bar. The suggestion is the following [9]:

Definition 2.1. *In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

0 ________________ *1*

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question.

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm, see [7], [8], [23]. Several advantages on the use of the bar instead of scale one can find in [9].

There are certain advantages concerning the use of the bar comparing to the Likert-scale during all stages of developing, filling and processing. The most important maybe advantage of the bar though is the fact that it provides the potential for different types of processing. Therefore, it gives the initiative to the researcher to explore if the given answers follow a special kind of distribution, as Gauss or parabola for example. In this case the researcher has the opportunity to correct any kind of tendency appeared, for more accurate results. A possibility of choosing among a number of alternatives is offered, by using fuzzy logic in the same way as it has already been done combining mathematical models with multivalued operation.

3 Evaluation

The following survey is based on the described theory that has been established in the department of Elementary Education of Democritus University of Thrace, in the frame of course evaluation, and especially of Algebra of first semester. The sample was 152 students, who were asked to answer questions related to the course, to the teacher and to the teaching of the course. The questionnaire used the bar, which was firstly divided into six equal-segments according to the first questionnaires which used a six-grade Likert scale.

The use of histograms helped in order to explore if the answers follow any kind of distribution or they present any kind of tendency. In this case, the bar is redivided into equal-area segments, for more accurate results.

The filling questionnaire procedure has been accomplished using computers, and especially a software developed for this purpose. Using this software the results can automatically be transferred for research elaboration. There are several advantages of the bar, the only disadvantage is to the data collection for further elaboration. The implemented program has been developed to overcome the problems raised during the data collection, inputting of data from questionnaires to processing. It eliminates the time of data collection, transferring data directly for any kind of elaboration [10].

3.1 Question category: Course

The first question category is about the course and consists of 9 questions. Gathering the answers on the bar, it is obvious that there is an upward trend, a fact that becomes even more obvious on the following histograms:

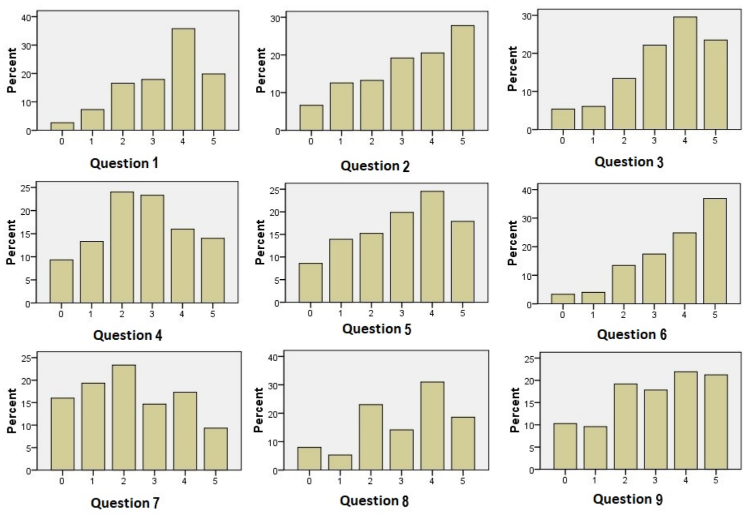

Figure 1: Question Category: Course

In the majority of the questions, on can notice a vast concentration in the last 2 or 3 grades and in some of the questions this is more obvious, as the concentration is the last grades is much higher.

More specifically, question number 1,2,3,5,6 and 9 present the biggest concentration rate in the last 2 grades, while in question 4, there is a remarkable concentration in the center of the bar.

Based mainly on this histograms and some other parameters that have been obtained by the correspondence analysis, the answers of questions 2,4 and 6 will be redistributed on the bar, which will be now divided in equal-area segments:
For question 2, the bar will be divided in 6 equal-ares segments according to the increasing-low parabola
For question 4, the be will be divided in 6 equal-ares segments according to the

Gauss distribution and,
for question 6, the bar will be divided in 6 equal-ares segments according to the increasing-upper parabola.

The new obtained histograms are the following:

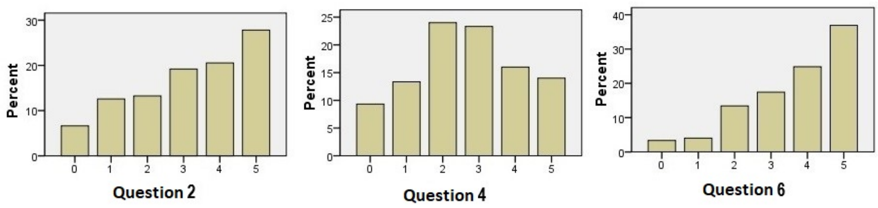

Figure 2: Equal segments

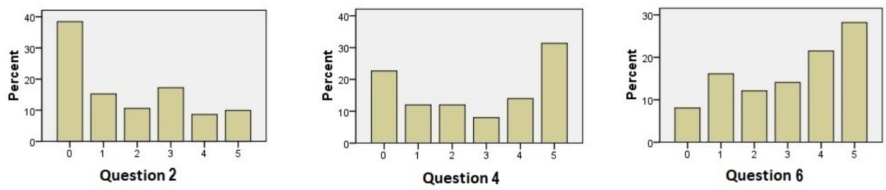

Figure 3: Equal-area segments

One can see that the use of the upper-low parabola on question 2, reveals that in question 2, more than 50% was concentrated at the last two grades of the scale, but with the new distribution there exists a tendency to the first grades. For questions number 4 and 6 the new histograms give no more information.

3.2 Question category:Teaching

The second question category consists of 6 questions relevant to the 'teaching of the lesson' and to the extend that some factors contributed to its comprehension. The related histograms are the following:

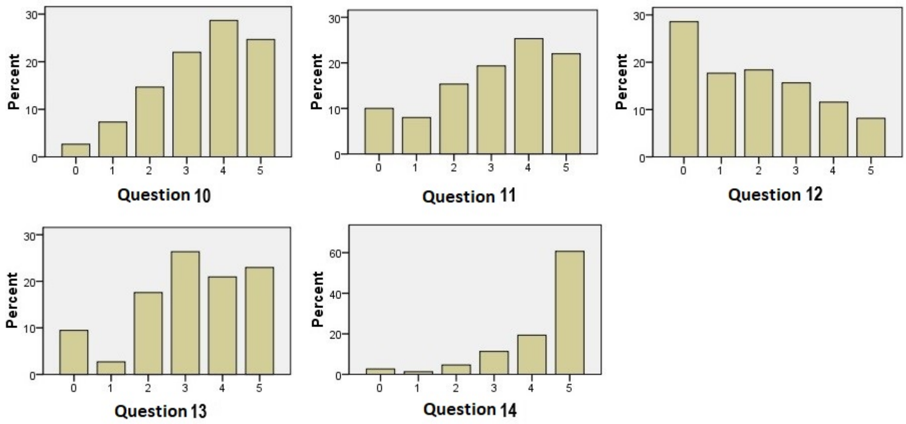

Figure 4: Question Category: Teaching

In this question category, there is also a general upward trend - with the exception of question number 12. More specifically, in questions 10, 11 and 13 the biggest concentration rate appears in the last grades, in opposition to question 12, in which the biggest rate appears in the first 2 grades. Question 14 present a vast rate in the last grade.

So, for question 12 the bar will be divided in equal-area segments according to decreasing-low parabola and for question 14, according to increasing upper parabola. The new obtained histograms are the following:

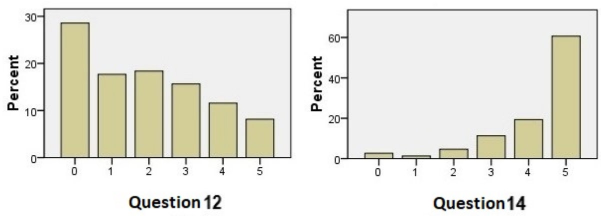

Figure 5: Equal segments

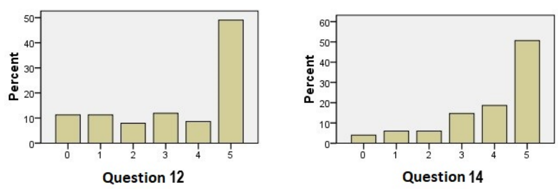

Figure 6: Equal-area segments

From the new distribution, the bar gives different results for question 12, as it reveals that the increasing-low trend not exists anymore. This fact is very important for the researcher as it gives him information he couldn't have only through the first subdivision of the bar. The second question leads to the same results.

3.3 Question category:Teacher

In the penultimate category there are 3 questions concerning the teacher.

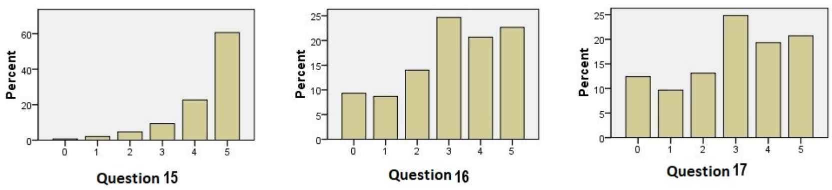

Figure 7: Question Category: Teacher

Once again, there is an obvious trend to the respondents according to the histograms, even more remarkable in the first question: there is a vast concentration rate in the last grade. Because of that, the bar will be divided into equal area segments following the increasing-upper parabola:

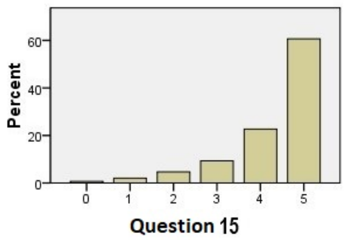

Figure 8: Equal segments

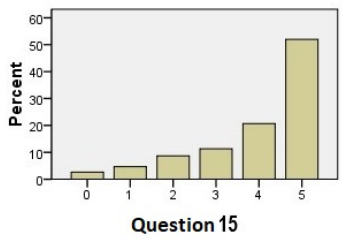

Figure 9: Equal-area segments

The new histogram is just confirming the first result.

4 Questionnares and Hyperstructures

In the research processing suppose that we want to use Likert scale through the bar dividing the continuum [0,62] into equal segments and into equal area division of Gauss distribution [9] or parabola distribution [24]. If we consider that the continuum [0,62] is divided into n segments, we can number the n segments starting with 0. We can define a hope on the segments as follows [11] :

Definition 4.1.

$$\textit{For all } i, j \in \{0, 1, ..., n-1\}, \textit{ if } e_n \textit{ the } n^{th} \textit{ segment ,then}$$

$$e_i \oplus e_j = \{e_k : x + y \in e_k, \forall x \in e_i, y \in e_j\}$$

Therefore, we can consider as an organized device the group $(Z_n, \oplus)$ where n the number of segments, as we have a modulo like hyperoperation. The multiplication tables obtained by this hyperoperation , referred in mm, are the following:

6 equal segments

0:$[0, 10.33]$, **1**:$(10.33, 20.66]$, **2**:$(20.66, 30.99]$, **3**:$(30.99, 41.32]$, **4**:$(41.32, 51.65]$, **5**:$(51.65, 62]$

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**
0	0,1	1,2	2,3	3,4	4,5	0,5
1	1,2	2,3	3,4	4,5	0,5	0,1
2	2,3	3,4	4,5	0,5	0,1	1,2
3	3,4	4,5	0,5	0,1	1,2	2,3
4	4,5	0,5	0,1	1,2	2,3	3,4
5	0,5	0,1	1,2	2,3	3,4	4,5

6 equal-area segments (Gauss distribution)

0:$[0, 22]$, **1**:$(22, 27]$, **2**:$(27, 31]$, **3**:$(31, 35]$, **4**:$(35, 40]$, **5**:$(40, 62]$,

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**
0	0,1,2,3,4,5	1,2,3,4,5	2,3,4,5	3,4,5	4,5	0,5
1	1,2,3,4,5	5	5	5	0,5	0,1
2	2,3,4,5	5	5	0,5	0	0,1,2
3	3,4,5	5	0,5	0	0	0,1,2,3
4	4,5	0,5	0	0	0	0,1,2,3,4
5	0,5	0,1	0,1,2	0,1,2,3	0,1,2,3,4	0,1,2,3,4,5

Increasing Low parabola $\mathbf{x = y^2}$

0:$[0, 34]$, **1**:$(34, 43]$, **2**:$(43, 49]$, **3**:$(49, 54]$, **4**:$(54, 58]$, **5**:$(58, 62]$

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**
0	0,1,2,3,4,5	0,1,2,3,4,5	0,2,3,4,5	0,3,4,5	0,4,5	0,5
1	0,1,2,3,4,5	0	0	0,1	0,1	0,1
2	0,2,3,4,5	0	0,1	0,1	1,2	1,2
3	0,3,4,5	0,1	0,1	1,2	1,2,3	2,3
4	0,4,5	0,1	1,2	1,2,3	2,3	3,4
5	0,5	0,1	1,2	2,3	3,4	4,5

Increasing Upper parabola $\mathbf{1 - y = (1 - x)^2}$

0:$[0, 22]$, **1**:$(22, 32]$, **2**:$(32, 40]$, **3**:$(40, 48]$, **4**:$(48, 55]$, **5**:$(55, 62]$

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**
0	0,1,2,3,	1,2,3,4	2,3,4,5	0,3,4,5	4,5,0	0,5
1	1,2,3,4	0,3,4,5	0,4,5	0	0,1	0,1
2	2,3,4,5	0,4,5	0	0,1	0,1,2	1,2
3	0,3,4,5	0	0,1	0,1,2	1,2,3	2,3
4	0,4,5	0,1	0,1,2	1,2,3	2,3	3,4
5	0,5	0,1	1,2	2,3	3,4	4,5

Decreasing low parabola $\mathbf{y = (1 - x)^2}$

0:$[0, 4]$, **1**:$(4, 8]$, **2**:$(8, 13]$, **3**:$(13, 19]$, **4**:$(19, 28]$, **5**:$(28, 62]$

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**
0	0,1	1,2	2,3	3,4	4,5	0,5
1	1,2	2,3	2,3,4	3,4	4,5	0,1,5
2	2,3	2,3,4	3,4	4,5	4,5	0,1,2,5
3	3,4	3,4	4,5	4,5	5	0,1,2,3,5
4	4,5	4,5	4,5	5	5	0,1,2,3,4,5
5	0,5	0,1,5	0,1,2,5	0,1,2,3,5	0,1,2,3,4,5	0,1,2,3,4,5

Decreasing upper parabola $\mathbf{1 - y = x^2}$

0:$[0, 7]$, **1**:$(7, 14]$, **2**:$(14, 22]$, **3**:$(22, 30]$, **4**:$(30, 40]$, **5**:$(40, 62]$

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**
0	0,1	1,2	2,3	3,4	4,5	0,5
1	1,2	2,3	2,3,4	3,4,5	4,5	0,1,5
2	2,3	2,3,4	3,4,5	4,5	5	0,1,2,5
3	3,4	3,4,5	4,5	5	0,1,5	0,1,2,3
4	4,5	4,5	5	0,1,5	0,1,2,5	1,2,3,4
5	0,5	0,1,5	0,1,2,5	0,1,2,3	1,2,3,4	2,3,4,5

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 151-165 **eISSN**: 2282-8214

Hyperstructures in Lie-Santilli Admissibility and Iso-Theories

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[‡]**doi**:10.23755/rm.v33i0.374

Abstract

In the quiver of hyperstructures Professor R. M. Santilli, in early 90'es, tried to find algebraic structures in order to express his pioneer Lie-Santilli's Theory. Santilli's theory on 'isotopies' and 'genotopies', born in 1960's, desperately needs 'units e' on left or right, which are nowhere singular, symmetric, real-valued, positive-defined for n-dimensional matrices based on the so called isofields.These elements can be found in hyperstructure theory, especially in H_v-structure theory introduced in 1990. This connection appeared first in 1996 and actually several H_v-fields, the e-hyperfields, can be used as isofields or genofields so as, in such way they should cover additional properties and satisfy more restrictions. Several large classes of hyperstructures as the P-hyperfields, can be used in Lie-Santilli's theory when multivalued problems appeared, either in finite or in infinite case. We review some of these topics and we present the Lie-Santilli admissibility in Hyperstructures.
Keywords: Lie-Santilli iso-theory, hyperstructures, hope, H_v-structures.
2010 AMS Mathematics Subject Classification: 20N20; 16Y99.

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 Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

In T. Vougiouklis, "The Santilli's theory 'invasion' in hyperstructures" [24], there is a first description on how Santilli's theories effect in hyperstructures and how new theories in Mathematics appeared by Santilli's pioneer research. We continue with new topics in this direction.

Last years hyperstructures have applications in mathematics and in other sciences as well. The applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons, in the Santilli's iso-theory, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, can be widely applicable in linguistic, in sociology, in industry and production, too. For all the above applications the largest class of the hyperstructures, the H_v-structures, is used, they satisfy the *weak axioms* where the non-empty intersection replaces the equality. The main tools of this theory are the *fundamental relations* which connect, by quotients, the H_v-structures with the corresponding classical ones. These relations are used to define hyperstructures as H_v-fields, H_v-vector spaces and so on. *Hypernumbers or H_v-numbers* are called the elements of H_v-fields and they are important for the representation theory.

The hyperstructures were introduced by F. Marty in 1934 who defined the hypergoup as a set equipped with an associative and reproductive hyperoperation. M. Koskas in 1970 was introduced the fundamental relation β^*, which it turns to be the main tool in the study of hyperstructures. T. Vougiouklis in 1990 was introduced the H_v-structures, by defining the weak axioms. The class of H_v-structures is the largest class of hyperstructures.

Motivation for $\mathbf{H_v}$-structures:
The quotient of a group with respect
to an invariant subgroup is a group.
The quotient of a group with respect to any subgroup is a hypergroup.
The quotient of a group with respect to any partition is an H_v-group.

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit.

According to Santilli's iso-theory [14], [8] on a field $F = (F, +, \cdot)$, *a general isofield* $\hat{\mathbf{F}} = \hat{\mathbf{F}}(\hat{a}, \hat{+}, \hat{\times})$ is defined to be a field with elements $\hat{a} = a \times \hat{1}$, called *isonumbers*, where $a \in F$, and $\hat{1}$ is a positive-defined element generally outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new product

$$\hat{a}\hat{\times}\hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}, \forall \hat{a}, \hat{b} \in \hat{\mathbf{F}}.$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of $\hat{\mathbf{F}}$,

$$\hat{1}\hat{\times}\hat{a} = \hat{a} \times \hat{1} = \hat{a}, \forall \hat{a} \in \hat{\mathbf{F}}$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli & Vougiouklis in 1996 [15], and called *e-hyperfields*. They point out that in physics the most interesting hyperstructures are the one called e-hyperstructures which contain a unique left ant right scalar unit.

2 Basic definitions on hyperstructures

In what follows we present the related hyperstructure theory, enriched with some new results. However one can see the books and related papers for more definitions and results on hyperstructures and related topics: [2], [4], [17], [18], [19], [20], [23], [31], [33].

In a set H is called **hyperoperation** (abbreviated: **hope**) or *multivalued operation*, any map from $H \times H$ to the power set of H. Therefore, in a hope

$$\cdot : H \times H \to \wp(H) : (x, y) \to x \cdot y \subset H$$

the result is subset of H, instead of element as we have in usually operations.

In a set H equipped with a hope $\cdot : H \times H \to \wp(H) - \{\varnothing\}$, we abbreviate by *WASS the weak associativity:* $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by
COW the weak commutativity: $xy \cap yx \neq \varnothing, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called H_v*-semigroup* if it is *WASS* and it is called $\mathbf{H_v}$***-group*** if it is reproductive H_v-semigroup, i.e. $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called H_v*-ring* if (+) and $(\cdot)$ are WASS, the reproduction axiom is valid for (+), and $(\cdot)$ is *weak distributive* to (+):

$$x(y+z) \cap (xy+xz) \neq \varnothing, (x+y)z \cap (xz+yz) \neq \varnothing, \forall x, y, z \in R.$$

An H_v-structure is ***very thin*** iff all hopes are operations except one, with all hyperproducts singletons except one, which is set of cardinality more than one.

The main tool to study all hyperstructures are the fundamental relations $\beta*$, $\gamma*$ and $\epsilon*$, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [17], [18].

A way to find fundamental classes is given by analogous to the following:

Theorem 2.1. *Let* $(H, \cdot)$ *be* H_v*-group and U all finite products of elements of H. Define the relation* β *by setting* $x\beta y$ *iff* $\{x, y\} \subset u, u \in U$*. Then* $\beta*$ *is the transitive closure of* β*.*

Let $(R, +, \cdot)$ be H_v-ring, U all finite polynomials of R. Define γ in R as follows: $x\gamma y$ iff $\{x, y\} \subset u$ where $u \in U$. Then γ^* is the transitive closure of γ.

An element is called *single* if its fundamental class is singleton.

The fundamental relations are used for general definitions. Thus, to define the H_v-field the γ^* is used [17], [18]: A H_v-ring $(R, +, \cdot)$ is called $\mathbf{H_v}$**-field** if R/γ^* is a field. In the sequence the H_v*-vector space* is defined.

Let $(F, +, \cdot)$ be H_v-field, $(\mathbf{V}, +)$ a COW H_v-group and there exists an external hope

$$\cdot : F \times \mathbf{V} \to \wp(\mathbf{V}) : (a, x) \to ax$$

such that, $\forall a, b \in F$ and $\forall x, y \in \mathbf{V}$, we have

$$a(x + y) \cap (ax + ay) \neq \varnothing, (a + b)x \cap (ax + bx) \neq \varnothing, (ab)x \cap a(bx) \neq \varnothing,$$

then V is called an $\mathbf{H_v}$**-vector space** over F. In the case of an H_v-ring instead of H_v-field then the $\mathbf{H_v}$**-modulo** is defined.

In the above cases the fundamental relation ϵ^* is the smallest equivalence such that the quotient $\mathbf{V}/\epsilon^*$ is a vector space over the fundamental field F/γ^*.

Let $(H, \cdot)$, $(H, *)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in Aut(H, *) \text{ such that } xy \subset f(x * y), \forall x, y \in H$$

Then we write $\cdot \leq *$ and we say that $(H, *)$ contains $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, *)$ is called H_b*-structure*.

The Little Theorem. Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.

The definition of H_v-field introduced a new class of hyperstructures:

The H_v-semigroup $(H, \cdot)$ is called **h/v-group** if the quotient H/β^* is a group.

In [20] the 'enlarged' hyperstructures were examined if an element, outside the underlying set, appears in one result. In enlargement or reduction, most useful in representations are H_v-structures with the same fundamental structure.

The Attach Construction. Let $((H, \cdot)$ be an H_v-semigroup and $v \notin H$. We extend $(\cdot)$ into $\underline{H} = H \cup \{v\}$ as follows: $x \cdot v = v \cdot x = v, \forall x \in H$, and $v \cdot v = H$.

Then $(\underline{H}, \cdot)$ is an h/v-group where $(\underline{H}, \cdot)/\beta^* \cong Z_2$ and v is single element.

We call the hyperstructure $(\underline{H}, \cdot)$ *attach h/v-group* of $(H, \cdot)$.

Definition 2.1. *Let $(H, \cdot)$ be a hypergroupoid. We say that remove $h \in H$, if simply consider the restriction of $(\cdot)$ on $H - \{h\}$. We say that $\underline{h} \in H$ absorbs $h \in H$ if we replace h, whenever it appears, by $\underline{h}$. We say that $\underline{h} \in H$ merges with $h \in H$, if we take as product of $x \in H$ by $\underline{h}$, the union of the results of x with both h and $\underline{h}$, and consider h and $\underline{h}$ as one class, with representative $\underline{h}$.*

The *uniting elements* method was introduced by Corsini & Vougiouklis [3]. With this method one puts in the same class more elements. This leads, through hyperstructures, to structures satisfying additional properties. The **uniting elements** method is the following: Let G be algebraic structure and d be a property, which is not valid and it is described by a set of equations; then, consider the partition in G for which it is put in the same partition class, all pairs that causes the non-validity of d. The quotient G/d is an H_v-structure. Then, quotient out the H_v-structure G/d by the fundamental relation β*, a stricter structure $(G/d)/\beta$* for which the property d is valid, is obtained.

An application is when more than one properties are desired then:

Theorem 2.2. *[18] Let $(G, \cdot)$ be a groupoid, and $F = \{f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}\}$ be a system of equations on G consisting of two subsystems $F_m = \{f_1, \ldots, f_m\}$ and $F_n = \{f_{m+1}, \ldots, f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m resp., and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m = (G/\sigma_m)/\beta^*$. Then*

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*.$$

In a groupoid with a map on it, a hope is introduced [22]:

Definition 2.2. *Let $(G, \cdot)$ be groupoid (resp., hypergroupoid) and $f : G \to G$ be map. We define a hope (∂), called **theta** and we write ∂-**hope**, on G as follows*

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G.$$

$$(resp. x\partial y = (f(x) \cdot y) \cup (x \cdot f(y), \forall x, y \in G)$$

If $(\cdot)$ is commutative then (∂) is commutative. If $(\cdot)$ is COW, then (∂) is COW.

Motivation for a ∂-hope is the map derivative where only the product of functions is used. Thus for two functions s(x), t(x), we have $s\partial t = \{s't, st'\}$ where $(')$ is the derivative.

A large class of hyperstructures based on classical ones are defined by [18]:

Definition 2.3. *Let $(G, \cdot)$ be groupoid, then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-hopes: $\forall x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$,$(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P-hyperstructures. The usual case is for $(G, \cdot)$ semigroup, then

$$x\underline{P}y = (xP)y \cup x(Py) = xPy$$

and $(G, \underline{P})$ is a semihypergroup.

3 Representations. H_v-Lie algebras.

Representations of H_v-groups, can be faced either by H_v-matrices or by generalized permutations [18], [20], [31].

$\mathbf{H_v}$-matrix (or h/v-matrix) is called a matrix with entries elements of an H_v-ring or H_v-field (or h/v-field). The hyperproduct of H_v-matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ H_v-matrices, defined in a usual manner:

$$\mathbf{A} \cdot \mathbf{B} = (a_{ij}) \cdot (b_{ij}) = \{\mathbf{C} = (c_{ij}) | c_{ij} \in \oplus \sum a_{ik} \cdot b_{kj}\},$$

where $(\oplus)$ is the *n-ary circle hope* on the hypersum: the sum of products of elements is considered to be the union of the sets obtained with all possible parentheses. In the case of 2×2 H_v-matrices the 2-ary circle hope which coincides with the hypersum in the H_v-ring. Notice that the hyperproduct of H_v-matrices does not nessesarily satisfy WASS.

The representation problem by H_v-matrices is the following:

Definition 3.1. *Let $(H, \cdot)$ be H_v-group, $(R, +, \cdot)$ be H_v-ring and $\mathbf{M}_R = \{(a_{ij}) | a_{ij} \in R\}$, then any*

$$\mathbf{T} : H \to \mathbf{M}_R : h \to \mathbf{T}(h) \text{ with } \mathbf{T}(h_1 h_2) \cap \mathbf{T}(h_1)\mathbf{T}(h_2) \neq \varnothing, \forall h_1, h_2 \in H,$$

*is called **$\mathbf{H_v}$-matrix representation** If $\mathbf{T}(h_1 h_2) \subset \mathbf{T}(h_1)\mathbf{T}(h_2)$, then T is an inclusion representation, if $\mathbf{T}(h_1 h_2) = \mathbf{T}(h_1)\mathbf{T}(h_2)$, then T is a good representation. If T is one to one and good then it is a faithful representation.*

The main theorem of representations of H_v-structures is the following:

Theorem 3.1. *A necessary condition in order to have an inclusion representation T of an H_v-group $(H, \cdot)$ by $n \times n$ H_v-matrices over the H_v-ring $(R, +, \cdot)$ is the following:*
For all $\beta^*(x), x \in H$ there must exist elements $a_{ij} \in H, i, j \in \{1, \ldots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) | a'_i j \in \gamma^*(a_{ij}), i, j \in \{1, \ldots, n\}\}$$

Therefore, every inclusion representation $T : H \to M_R : a \mapsto T(a) = (a_{ij})$ induces an homomorphic representation T^* of H/β^* over R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \forall \beta^*(a) \in H/\beta^*$, where the element $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. Then T^* is called fundamental induced representation of T.

The **helix hopes** can be defined on any type of ordinary matrices [33], [34]:

Definition 3.2. *Let $A = (aij) \in \mathbf{M}_{m\times n}$ be matrix and $s, t \in N$, with $1 \le s \le m$, $1 \le t \le n$. The helix-projection is a map $\underline{st} : \mathbf{M}_{m\times n} \to \mathbf{M}_{s\times t} : A \to A\underline{st} = (\underline{a}_{ij})$, where $A\underline{st}$ has entries*

$$\underline{a}_i j = \{a_{i+\kappa s, j+\lambda t} | 1 \le i \le s, 1 \le j \le t \text{ and } \kappa, \lambda \in N, i + \kappa s \le m, j + \lambda t \le n\}$$

Let $A = (a_{ij}) \in \mathbf{M}_{m\times n}, B = (b_{ij}) \in \mathbf{M}_{u\times v}$ be matrices and $s = min(m, u)$, $t = min(n, v)$. We define a hyper-addition, called helix-sum, by

$$\oplus : \mathbf{M}_{m\times n} \times \mathbf{M}_{u\times v} \to \wp(\mathbf{M}_{s\times t}) : (A, B) \to A \oplus B =$$

$$= A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset \mathbf{M}_{s\times t}$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) | a_{ij} \in \underline{a}_{ij}$ and $b_{ij} \in \underline{b}_{ij})\}$.

Let $A = (a_{ij}) \in \mathbf{M}_{m\times n}, B = (b_{ij}) \in \mathbf{M}_{u\times v}$ and $s = min(n, u)$. Define the helix-product, by

$$\otimes : \mathbf{M}_{m\times n} \times \mathbf{M}_{u\times v} \to \wp(\mathbf{M}_{m\times v}) : (A, B) \to A \otimes B =$$

$$= A\underline{ms} \cdot B\underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset \mathbf{M}_{m\times v}$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it} b_{tj}) | a_{ij} \in \underline{a}_{ij}$ and $b_{ij} \in \underline{b}_{ij})\}$.

The helix-sum is commutative, WASS, not associative. The helix-product is WASS, not associative and not distributive to the helix-addition.

Using several classes of H_v-structures one can face several representations. Some of those classes are as follows [18], [19], [7]:

Definition 3.3. *Let $M = M_{m\times n}$, the set of $m \times n$ matrices on R and $P = \{P_i : i \in I\} \subseteq M$. We define, a kind of, a P-hope $\underline{P}$ on M as follows*

$$\underline{P} : M \times M \to \wp(M) : (A, B) A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq M$$

where P^t is the transpose of P. $\underline{P}$ is bilinear Rees' like operation where instead of one sandwich matrix a set is used. $\underline{P}$ is strong associative and inclusion distributive to sum:

$$A\underline{P}(B + C) \subseteq A\underline{P}B + A\underline{P}C, \forall A, B, C \in M.$$

So (M,+,$\underline{P}$) defines a multiplicative hyperring on non-square matrices.

Definition 3.4. *Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be module of $m \times n$ matrices on R and take the sets*

$$\mathbf{S} = \{s_k : k \in K\} \subseteq R, \mathbf{Q} = \{Q_j : j \in J\} \subseteq \mathbf{M}, \mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}.$$

Define three hopes as follows

$$\underline{S} : R \times \mathbf{M} \to \wp(\mathbf{M}) : (r, A) \to r\underline{S}A = \{(rs_k)A : k \in K\} \subseteq \mathbf{M}$$

$$\underline{Q}_+ : \mathbf{M} \times \mathbf{M} \to \wp(\mathbf{M}) : (A, B) \to A\underline{Q}_+ B = \{A + Q_j + B : j \in J\} \subseteq \mathbf{M}$$

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \wp(\mathbf{M}) : (A, B) \to A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

Then $(\mathbf{M}, \underline{S}, \underline{Q}_, \underline{P})$ *is a hyperalgebra on R called general matrix P-hyperalgebra.*

The general definition of an H_v-Lie algebra is the following [26], [31], [16]:

Definition 3.5. *Let* $(\mathbf{L}, +)$ *be* H_v*-vector space on* $(F, +, \cdot)$*,* $\phi : F \to F/\gamma^*$*, canonical map and* $\omega_F = \{x \in F : \phi(x) = 0\}$*, where 0 is the zero of the fundamental field* F/γ^**. Similarly, let* ω_L *be the core of the canonical map* $\phi' : \mathbf{L} \to \mathbf{L}/\epsilon^*$ *and denote by the same symbol 0 the zero of* $\mathbf{L}/\epsilon^*$*. Consider the bracket hope (commutator):*

$$[,] : \mathbf{L} \times \mathbf{L} \to \wp(L) : (x, y) \to [x, y]$$

*then **L** is an* $\mathbf{H_v}$***-Lie algebra*** *over F if the following axioms are satisfied:*

(L1) *The bracket hope is bilinear,*
i.e.$\forall x, x_1, x_2, y, y_1, y_2 \in L,$ *and* $\lambda_1, \lambda_2 \in F$
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \varnothing,$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \ \ \forall x \in \mathbf{L}$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \forall x, y, z \in \mathbf{L}$

4 The Santilli's: e-hyperstructures, iso-hyper theory.

The e-hyperstructures where introduced in [15], [25] and where investigates in several aspects depending from applications [5], [6], [16], [31].

Definition 4.1. *A hyperstructure* $(H, \cdot)$ *which contains a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse* x^{-1}*, i.e.* $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

Definition 4.2. *A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ a hope, is called e-hyperfield if the following axioms are valid: $(F,+)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0\cdot x = x\cdot 0 = 0, \forall x \in F$, there exists a multiplicative scalar unit 1, i.e. $1\cdot x = x\cdot 1 = x, \forall x \in F$, and $\forall x \in F$ there exists a unique inverse x^{-1}, such that $1 \in x\cdot x^{-1} \cap x^{-1}\cdot x$.*

The elements of an e-hyperfield are called e-hypernumbers. In the case that the relation: $1 = x\cdot x^{-1} = x^{-1}\cdot x$, is valid, then we say that we have a strong e-hyperfield.

Definition 4.3. *Main e-Construction. Given a group $(G,\cdot)$, where e is the unit, we define in G, an extremely large number of hopes $(\otimes)$ as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

$g_1, g_2,...$ are not necessarily the same for each pair (x,y). $(G,\otimes)$ is an H_v-group, it is an H_b-group which contains the $(G,\cdot)$. $(G,\otimes)$ is an e-hypergroup. Moreover, if for each x,y such that $xy = e$, so we have $x \otimes y = xy$, then $(G,\otimes)$ becomes a strong e-hypergroup

The proof is immediate since for both cases we enlarge the results of the group by putting elements from the set G and applying the Little Theorem. Moreover it is easy to see that the unit e is unique scalar element and for each x in G, there exists a unique inverse x^{-1}, such that $1 \in x\cdot x^{-1} \cap x^{-1}\cdot x$. Finally if the last condition is valid then we have $1 = x\cdot x^{-1} = x^{-1}\cdot x$, So the hyperstructure $(G,\otimes)$ is a strong e-hypergroup.

Example 4.1. *Consider the quaternion group $\mathbf{Q} = \{1,-1,i,-i,j,-j,k,-k\}$ with defining relations $i^2 = j^2 = -1, ij = -ji = k$. Denoting $\underline{i} = \{i,-i\}, \underline{j} = \{j,-j\}, \underline{k} = \{k,-k\}$ we may define a very large number $(*)$ hopes by enlarging only few products. For example, $(-1)*k = \underline{k}, k*i = \underline{j}$ and $i*j = \underline{k}$. Then the hyperstructure $(\mathbf{Q},*)$ is a strong e-hypergroup.*

Construction 4.1. *[31], [32]. On the ring $(\mathbf{Z}_4,+,\cdot)$ we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are,*

(a) very thin minimal,

(b) COW (non-commutative),

(c) they have 0 and 1, scalars.

We have the isomorphic cases: $2 \otimes 3 = \{0, 2\}$ *or* $3 \otimes 2 = \{0, 2\}$. *The fundamental classes are* $[0] = \{0, 2\}$, $[1] = \{1, 3\}$ *and we have* $(\mathbf{Z}_4, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

Thus it is isomorphic to $(\mathbf{Z}_2 \times \mathbf{Z}_2, +)$. *In this* H_v*-group there is only one unit and every element has a unique double inverse.*

We can also define the analogous cases for the rings $(\mathbf{Z}_6, +, \cdot)$, $(\mathbf{Z}_9, +, \cdot)$, *and* $(\mathbf{Z}_{10}, +, \cdot)$.

In order to transfer Santilli's iso-theory theory into the hyperstructure case we generalize only the new product $\hat{\times}$ by replacing it by a hope including the old one [15], [27], [29], [32] and [1], [5], [6], [13], [14], [21], [24]. We introduce two general constructions on this direction as follows:

Construction 4.2. *General enlargement. On a field* $\mathbf{F} = (\mathbf{F}, +, \cdot)$ *and on the isofield* $\hat{\mathbf{F}} = \hat{\mathbf{F}}(\hat{a}, \hat{+}, \hat{\times})$ *we replace in the results of the iso-product*

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}$$

of the element $\hat{T}$ *by a set of elements* $\hat{H}_{ab} = \{\hat{T}, \hat{x}_1, \hat{x}_2, \ldots\}$ *where* $\hat{x}_1, \hat{x}_2, \ldots \in \hat{\mathbf{F}}$, *containing* $\hat{T}$, *for all* $\hat{a} \hat{\times} \hat{b}$ *for which* $\hat{a}, \hat{b} \notin \{\hat{0}, \hat{1}\}$ *and* $\hat{x}_1, \hat{x}_2, \ldots \in \hat{\mathbf{F}} - \{\hat{0}, \hat{1}\}$. *If one of* $\hat{a}, \hat{b}$, *or both, is equal to* $\hat{0}$ *or* $\hat{1}$, *then* $\hat{H}_{ab} = \{\hat{T}\}$. *Therefore the new iso-hope is*

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{H}_{ab} \times \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}_1, \hat{x}_2, \ldots\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{\mathbf{F}}$$

$\hat{\mathbf{F}} = \hat{\mathbf{F}}(\hat{a}, \hat{+}, \hat{\times})$ *becomes iso*H_v*-field. The elements of* ***F*** *are called iso*H_v*-numbers or isonumbers.*

More important hopes, of the above construction, are the ones where only for few ordered pairs $(\hat{a}, \hat{b})$ the result is enlarged, even more, the extra elements $\hat{x}_i$, are only few, preferable one. Thus, this special case is if there exists only one pair $(\hat{a}, \hat{b})$ for which

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{F}$$

and the rest are ordinary results, then we have a very thin isoH_v-field.

The assumption $\hat{H}_{ab} = \{\hat{T}\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, with that $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the isoH_v-field has one scalar absorbing $\hat{0}$, one scalar $\hat{1}$, and $\forall \hat{a} \in \hat{\mathbf{F}}$ one inverse.

A **generalization of P-hopes**, used in Santilli's isotheory, is the following [5], [28], [31]: Let $(G, \cdot)$ be abelian group and P a subset of G with $\#P > 1$. We define the hope $(\times_p)$ as follows:

$$x \times_p y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_p)$ is abelian H_v-group.

Construction 4.3. *The P-hope. Consider an isofield* $\hat{\mathbf{F}} = \hat{\mathbf{F}}(\hat{a}, \hat{+}, \hat{\times})$ *with* $\hat{a} = a \times \hat{1}$*, the isonumbers, where* $a \in F$*, and* $\hat{1}$ *is positive-defined outside F, with two operations* $\hat{+}$ *and* $\hat{\times}$*, where* $\hat{+}$ *is the sum with the conventional unit 0, and* $\hat{\times}$ *is the iso-product*

$$\hat{a}\hat{\times}\hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}, \forall \hat{a}, \hat{b} \in \hat{\mathbf{F}}$$

Take a set $\hat{P} = \{\hat{T}, \hat{p}_1, ..., \hat{p}_s\}$*, with* $\hat{p}_1, \ldots, \hat{p}_s \in \hat{\mathbf{F}} - \{\hat{0}, \hat{1}\}$*, we define the* ***isoP-*** $\mathbf{H_v}$***-field***, $\hat{\mathbf{F}} = \hat{\mathbf{F}}(\hat{a}, \hat{+}, \hat{\times}_p)$ *where the hope* $\hat{\times}_P$ *as follows:*

$$\hat{a}\hat{\times}_P\hat{b} := \begin{cases} \hat{a} \times \hat{\hat{P}} \times \hat{b} = \{\hat{a} \times \hat{\hat{h}} \times \hat{b} | \hat{\hat{h}} \in \hat{\hat{P}}\} & \text{if } \hat{a} \neq \hat{1} \text{ and } \hat{b} \neq \hat{1} \\ \hat{a} \times \hat{\hat{T}} \times \hat{b} & \text{if } \hat{a} = \hat{1} \text{ or } \hat{b} = \hat{1} \end{cases}$$

The elements of $\hat{\mathbf{F}}$ *are called isoP-*$\mathbf{H_v}$*-numbers.*

Remark. If $\hat{P} = \{\hat{T}, \hat{p}\}$, that is that $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$. The inverses in isoP-H_v-fields, are not necessarily unique.

Example 4.2. *Non degenerate example on the above constructions:*

In order to define a generalized P-hope on $\hat{\mathbf{Z}}_7 = \hat{\mathbf{Z}}_7(\underline{\hat{a}}, \hat{+}, \hat{\times})$*, where we take* $\hat{P} = \{\underline{\hat{1}}, \underline{\hat{6}}\}$*, the weak associative multiplicative hope is described by the table:*

$\hat{\times}$	$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{\mathbf{6}}}$
$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$
$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{4}}$,$\underline{\hat{3}}$	$\underline{\hat{6}}$,$\underline{\hat{1}}$	$\underline{\hat{1}}$,$\underline{\hat{6}}$	$\underline{\hat{3}}$,$\underline{\hat{4}}$	$\underline{\hat{5}}$, $\underline{\hat{2}}$
$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\hat{6}}$,$\underline{\hat{1}}$	$\underline{\hat{2}}$,$\underline{\hat{5}}$	$\underline{\hat{5}}$,$\underline{\hat{2}}$	$\underline{\hat{1}}$,$\underline{\hat{6}}$	$\underline{\hat{4}}$,$\underline{\hat{3}}$
$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{1}}$,$\underline{\hat{6}}$	$\underline{\hat{5}}$,$\underline{\hat{2}}$	$\underline{\hat{2}}$,$\underline{\hat{5}}$	$\underline{\hat{6}}$,$\underline{\hat{1}}$	$\underline{\hat{3}}$,$\underline{\hat{4}}$
$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{0}}$	$\underline{\hat{5}}$	$\underline{\hat{3}}$,$\underline{\hat{4}}$	$\underline{\hat{1}}$,$\underline{\hat{6}}$	$\underline{\hat{6}}$,$\underline{\hat{1}}$	$\underline{\hat{4}}$,$\underline{\hat{3}}$	$\underline{\hat{2}}$,$\underline{\hat{5}}$
$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{5}}$,$\underline{\hat{2}}$	$\underline{\hat{4}}$,$\underline{\hat{3}}$	$\underline{\hat{3}}$,$\underline{\hat{4}}$	$\underline{\hat{2}}$,$\underline{\hat{5}}$	$\underline{\hat{1}}$,$\underline{\hat{6}}$

The hyperstructure $\hat{\mathbf{Z}}_7 = \hat{\mathbf{Z}}_7(\underline{\hat{a}}, \hat{+}, \hat{\times})$ *is commutative and associative on the product hope. Moreover the* $\beta*$ *classes on the iso-product* $\hat{\times}$ *are* $\{\hat{1}, \hat{6}\}, \{\hat{5}, \hat{2}\}, \{\hat{3}, \hat{4}\}$.

5 The Lie-Santilli's admissibility.

Another very important new field in hypermathematics comes straightforward from Santilli's Admissibility. We can transfer Santilli's theory in admissibility for representations in two ways: using either, the ordinary matrices and a hope on them, or using hypermatrices and ordinary operations on them [10], [11], [12], [14], [16] and [7], [9], [30], [31], [34].

Definition 5.1. *Let **L** be H_v-vector space over the H_v-field $(\mathbf{F}, +, \cdot)$, $\phi : \mathbf{F} \to \mathbf{F}/\gamma^*$, the canonical map and $\omega_F = \{x \in \mathbf{F} : \phi(x) = 0\}$, where 0 is the zero of the fundamental field $\mathbf{F}/\gamma^*$. Let ω_L be the core of the canonical map $\phi' : \mathbf{L} \to \mathbf{L}/\epsilon^*$ and denote by the same symbol 0 the zero of $\mathbf{L}/\epsilon^*$. Take two subsets $\mathbf{R}, \mathbf{S} \subseteq \mathbf{L}$ then a **Lie-Santilli admissible hyperalgebra** is obtained by taking the Lie bracket, which is a hope:*

$$[,]_{\mathbf{RS}} : \mathbf{L} \times \mathbf{L} \to \wp(\mathbf{L}) : [x, y]_{\mathbf{RS}} = x\mathbf{R}y - y\mathbf{S}x = \{xry - ysx | r \in \mathbf{R} \text{ and } s \in \mathbf{S}\}$$

Special cases, but not degenerate, are the 'small' and 'strict' ones:

*(a) When only **S** is considered, then* $[x, y]_{\mathbf{S}} = xy - y\mathbf{S}x = \{xy - ysx | s \in \mathbf{S}\}$

*(b) When only **R** is considered, then* $[x, y]_{\mathbf{R}} = x\mathbf{R}y - yx = \{xry - yx | r \in \mathbf{R}\}$

(c) When $\mathbf{R} = \{r_1, r_2\}$ *and* $\mathbf{S} = \{s_1, s_2\}$ *then*

$$[x, y]_{\mathbf{RS}} = x\mathbf{R}y - y\mathbf{S}x = \{xr_1y - ys_1x, xr_1y - ys_2x, xr_2y - ys_1x, xr_2y - ys_2x\}.$$

(d) We have one case which is 'like' P-hope for any subset $\mathbf{S} \subseteq \mathbf{L}$*:*

$$[x, y]_{\mathbf{S}} = \{xsy - ysx | s \in \mathbf{S}\}$$

On non square matrices we can define admissibility, as well:

Construction 5.1. *Let $\mathbf{L} = (\mathbf{M}_{m\times n}, +)$ be H_v-vector space of $m \times n$ hyper-matrices on the H_v-field $(\mathbf{F}, +, \cdot)$, $\phi : \mathbf{F} \to \mathbf{F}/\gamma^*$, canonical map and $\omega_F = \{x \in \mathbf{F} : \phi(x) = 0\}$, where 0 is the zero of the field $\mathbf{F}/\gamma^*$. Similarly, let ω_L be the core of $\phi' : \mathbf{L} \to \mathbf{L}/\epsilon^*$ and denote by the same symbol 0 the zero of $\mathbf{L}/\epsilon^*$. Take any two subsets $\mathbf{R}, \mathbf{S} \subseteq \mathbf{L}$ then a **Santilli's Lie-admissible hyperalgebra** is obtained by taking the Lie bracket, which is a hope:*

$$[,]_{\mathbf{RS}} : \mathbf{L} \times \mathbf{L} \to \wp(\mathbf{L}) : [x, y]_{\mathbf{RS}} = x\mathbf{R}^t y - y\mathbf{S}^t x.$$

Notice that $[x, y]_{\mathbf{RS}} = x\mathbf{R}^t y - y\mathbf{S}^t x = \{xr^t y - ys^t x | r \in \mathbf{R} \text{ and } s \in \mathbf{S}\}$ *Special cases, but not degenerate, is the 'small':*
$\mathbf{R} = \{r_1, r_2\}$ *and* $\mathbf{S} = \{s_1, s_2\}$ *then*

$$[x, y]_{\mathbf{RS}} = x\mathbf{R}^t y - y\mathbf{S}^t x =$$

$$= \{xr_1^t y - ys_1^t x, xr_1^t y - ys_2^t x, xr_2^t y - ys_1^t x, xr_2^t y - ys_2^t x\}$$

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 167-179 **eISSN**: 2282-8214

Helix-Hopes on S-Helix Matrices

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[‡]**doi**:10.23755/rm.v33i0.385

Abstract

A hyperproduct on non-square ordinary matrices can be defined by using the so called helix-hyperoperations. The main characteristic of the helix-hyperoperation is that all entries of the matrices are used. Such operations cannot be defined in the classical theory. Several classes of non-square matrices have results of the helix-product with small cardinality. We study the helix-hyperstructures on the representations and we extend our study up to H_v-Lie theory by using ordinary fields. We introduce and study the class of S-helix matrices.

Keywords: hyperstructures; H_v-structures; h/v-structures; hope; helix-hopes.
2010 AMS subject classifications: 20N20, 16Y99.

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[‡]©Souzana Vougioukli and Thomas Vougiouklis. Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

Our object is the largest class of hyperstructures, the H_v-structures, introduced in 1990 [10], satisfying the *weak axioms* where the non-empty intersection replaces the equality.

Definition 1.1. *In a set H equipped with a **hyperoperation** (abbreviate by **hope**)*

$$\cdot : H \times H \to P(H) - \{\emptyset\} : (x, y) \to x \cdot y \subset H$$

we abbreviate by
WASS the weak associativity: $(xy)z \cap x(yz) \neq \varnothing,\ \forall x, y, z \in H$ *and by*
COW the weak commutativity: $xy \cap yx \neq \varnothing,\ \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ *is called* H_v*-semigroup if it is WASS and is called* $\mathbf{H_v}$**−group** *if it is reproductive* H_v*-semigroup:* $xH = Hx = H, \forall x \in H$.

$(R, +, \cdot)$ *is called* $\mathbf{H_v}$**−ring** *if* $(+)$ *and* $(\cdot)$ *are WASS, the reproduction axiom is valid for* $(+)$ *and* $(\cdot)$ *is weak distributive with respect to* $(+)$*:*

$$x(y + z) \cap (xy + xz) \neq \varnothing,\ (x + y)z \cap (xz + yz) \neq \varnothing,\ \forall x, y, z \in R.$$

For more definitions, results and applications on H_v-structures, see [1], [2], [11], [12], [13], [17]. An interesting class is the following [8]: An H_v-structure is *very thin*, if and only if, all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in a set H there exists a hope $(\cdot)$ and a pair $(a, b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products, with respect to any other hopes, are singletons.

The fundamental relations $\beta*$ and $\gamma*$ are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [8], [9], [11], [12], [13], [17]. The main theorem is the following:

Theorem 1.1. *Let* $(H, \cdot)$ *be an* H_v*-group and let us denote by U the set of all finite products of elements of H. We define the relation* β *in H as follows:* $x\beta y$ *iff* $\{x, y\} \subset u$ *where* $u \in U$*. Then the fundamental relation* $\beta*$ *is the transitive closure of the relation* β*.*

An element is called *single* if its fundamental class is a *singleton*.

Motivation: The quotient of a group with respect to any partition is an H_v-group.

Definition 1.2. *Let* $(H, \cdot), (H, \otimes)$ *be* H_v*-semigroups defined on the same H.* $(\cdot)$ *is smaller than* $(\otimes)$*, and* $(\otimes)$ *greater than* $(\cdot)$*, iff there exists automorphism*

$$f \in Aut(H, \otimes) \text{ such that } xy \subset f(x \otimes y),\ \forall x, y \in H.$$

Then $(H, \otimes)$ *contains* $(H, \cdot)$ *and write* $\cdot \leq \otimes$. *If* $(H, \cdot)$ *is structure, then it is basic and* $(H, \otimes)$ *is an* H_b*-structure.*

The Little Theorem [11]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

Fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ^*:

Definition 1.3. *[10] The* H_v*-ring* $(R, +, \cdot)$ *is called* $\mathbf{H_v}$***-field*** *if the quotient* R/γ^* *is a field.*

This definition introduces a new class of which is the following [15]:

Definition 1.4. *The* H_v*-semigroup* $(H, \cdot)$ *is called* ***h/v-group*** *if* H/β^* *is a group.*

The class of h/v-groups is more general than the H_v-groups since in h/v-groups the reproductivity is not valid. The **h/v-fields** and the other related hyperstructures are defined in a similar way.

An H_v-group is called *cyclic* [8], if there is an element, called *generator*, which the powers have union the underline set, the minimal power with this property is the *period* of the generator.

Definition 1.5. *[11], [14], [18]. Let* $(R, +, \cdot)$ *be an* H_v*-ring,* $(M, +)$ *be COW* H_v*-group and there exists an external hope* $\cdot : R \times M \to P(M) : (a, x) \to ax$*, such that,* $\forall a, b \in R$ *and* $\forall x, y \in M$ *we have*

$$a(x + y) \cap (ax + ay) \neq \varnothing, \ (a + b)x \cap (ax + bx) \neq \varnothing, \ (ab)x \cap a(bx) \neq \varnothing,$$

then M is called an $\mathbf{H_v}$***-module*** *over R. In the case of an* H_v*-field F instead of an* H_v*-ring R, then the* $\mathbf{H_v}$***-vector space*** *is defined.*

Definition 1.6. *[16] Let* $(L, +)$ *be* H_v*-vector space on* $(F, +, \cdot)$*,* $\phi : F \to F/\gamma^*$*, the canonical map and* $\omega_F = \{x \in F : \phi(x) = 0\}$*, where 0 is the zero of the fundamental field* F/γ^**. Similarly, let* ω_L *be the core of the canonical map* $\phi' : L \to L/\epsilon^*$ *and denote again 0 the zero of* L/ϵ^**. Consider the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x, y) \to [x, y]$$

then L is an $\mathbf{H_v}$***-Lie algebra*** *over F if the following axioms are satisfied:*

(L1) The bracket hope is bilinear, i.e.
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \varnothing,$
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \ \ \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \ \ \forall x, y, z \in L$

A well known and large class of hopes is given as follows [8], [9], [11]:

Definition 1.7. *Let $(G, \cdot)$ be a groupoid, then for every subset $P \subset G, P \neq \varnothing$, we define the following hopes, called **P-hopes**: $\forall x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP),$$
$$\underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

*The $(G, \underline{P})$,$(G, \underline{P}_r)$, $(G, \underline{P}_l)$ are called **P-hyperstructures**.*

The usual case is for semigroup $(G, \cdot)$, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$, and $(G, \underline{P})$ is a semihypergroup.

A new important application of H_v-structures in Nuclear Physics is in the Santilli's isotheory. In this theory a generalization of P-hopes is used, [4], [5], [22], which is defined as follows: Let (G,) be an abelian group and P a subset of G with more than one elements. We define the hyperoperation $\times_P$ as follows:

$$x \times_p y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e-*hope*. The hyperstructure $(G, \times_p)$ is an abelian H_v-group.

2 Small hypernumbers and H_v-matrix representations

Several constructions of H_v-fields are uses in representation theory and applications in applied sciences. We present some of them in the finite small case [18].

Construction 2.1. *On the ring $(\mathbf{Z}_4, +, \cdot)$ we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are,*

(a) very thin minimal,

(b) COW (non-commutative),

(c) they have the elements 0 and 1, scalars.

Then, we have only the following isomorphic cases $2 \otimes 3 = \{0, 2\}$ *or* $3 \otimes 2 = \{0, 2\}$.

Fundamental classes: $[0] = \{0, 2\}, [1] = \{1, 3\}$ *and we have* $(\mathbf{Z}_4, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

Thus it is isomorphic to $(\mathbf{Z}_2 \times \mathbf{Z}_2, +)$. *In this* H_v*-group there is only one unit and every element has a unique double inverse.*

Construction 2.2. *On the ring* $(\mathbf{Z}_6, +, \cdot)$ *we define, up to isomorphism, all multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are:*

(a) very thin minimal, i.e. only one product has exactly two elements

(b) COW (non-commutative)

(c) they have the elements 0 and 1, scalars

Then we have the following cases, by giving the only one hyperproduct,

(I) $2 \otimes 3 = \{0, 3\}$ *or* $2 \otimes 4 = \{2, 5\}$ *or* $2 \otimes 5 = \{1, 4\}$
$3 \otimes 4 = \{0, 3\}$ *or* $3 \otimes 5 = \{0, 3\}$ *or* $4 \otimes 5 = \{2, 5\}$
In all 6 cases the fundamental classes are $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$ *and we have* $(\mathbf{Z}_6, +, \otimes)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot)$.

(II) $2 \otimes 3 = \{0, 2\}$ *or* $2 \otimes 3 = \{0, 4\}$ *or* $2 \otimes 4 = \{0, 2\}$ *or* $2 \otimes 4 = \{2, 4\}$ *or* $2 \otimes 5 = \{0, 4\}$ *or* $2 \otimes 5 = \{2, 4\}$ *or* $3 \otimes 4 = \{0, 2\}$ *or* $3 \otimes 4 = \{0, 4\}$ *or* $3 \otimes 5 = \{1, 3\}$ *or* $3 \otimes 5 = \{3, 5\}$ *or* $4 \otimes 5 = \{0, 2\}$ *or* $4 \otimes 5 = \{2, 4\}$
In all 12 cases the fundamental classes are $[0] = \{0, 2, 4\}$, $[1] = \{1, 3, 5\}$ *and we have* $(\mathbf{Z}_6, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

H_v-structures are used in Representation Theory of H_v-groups which can be achieved by generalized permutations or by H_v-matrices [11], [14], [18].

Definition 2.1. $\mathbf{H_v}$***-matrix*** *is a matrix with entries of an* H_v*-ring or* H_v*-field. The hyperproduct of two* H_v*-matrices* (a_{ij}) *and* (b_{ij})*, of type* $m \times n$ *and* $n \times r$ *respectively, is defined in the usual manner and it is a set of* $m \times r$ H_v*-matrices. The sum of products of elements of the* H_v*-ring is considered to be the n-ary circle hope on the hypersum. The hyperproduct of* H_v*-matrices is not necessarily WASS.*

The problem of the H_v-matrix representations is the following:

Definition 2.2. *Let* $(H, \cdot)$ *be* H_v*-group (or h/v-group). Find an* H_v*-ring* $(R, +, \cdot)$*, a set* $M_R = \{(a_{ij}) | a_{ij} \in R\}$ *and a map* $T : H \to M_R : h \mapsto T(h)$ *such that*

$$T(h_1 h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

T is H_v-matrix (or h/v-matrix) representation. If $T(h_1h_2) \subset T(h_1)(h_2)$ is called inclusion. If $T(h_1h_2) = T(h_1)(h_2) = \{T(h)|h \in h_1h_2\}$, $\forall h_1, h_2 \in H$, then T is good and then an induced representation T for the hypergroup algebra is obtained. If T is one to one and good then it is faithful.*

The main theorem on representations is [11]:

Theorem 2.1. *A necessary condition to have an inclusion representation T of an H_v-group $(H, \cdot)$ by $n \times n$, H_v-matrices over the H_v-ring $(R, +, \cdot)$ is the following:*

For all classes $\beta^*(x), x \in H$ must exist elements $a_{ij} \in H, i, j \in \{1, \ldots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij})|a_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \ldots, n\}\}$$

Inclusion $T : H \to M_R : a \mapsto T(a) = (a_{ij})$ induces homomorphic representation T^* of H/β^* on R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})]$, $\forall \beta^*(a) \in H/\beta^*$, where $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. T^* is called fundamental induced of T.

In representations, several new classes are used:

Definition 2.3. *Let $M = M_{m\times n}$ be the module of $m \times n$ matrices over R and $P = \{P_i : i \in I\} \subseteq M$. We define a P-hope $\underline{P}$ on M as follows*

$$\underline{P} : M \times M \to P(M) : (A, B) \to A\underline{P}B = \{AP_i^tB : i \in I\} \subseteq M$$

where P^t denotes the transpose of P.

The hope $\underline{P}$ is bilinear map, is strong associative and the inclusion distributive:

$$A\underline{P}(B + C) \subseteq A\underline{P}B + A\underline{P}C, \forall A, B, C \in M$$

Definition 2.4. *Let $M = M_{m\times n}$ the $m \times n$ matrices over R and let us take sets*

$$S = \{s_k : k \in K\} \subseteq R,\ \ Q = \{Q_j : j \in J\} \subseteq M,\ \ P = \{P_i : i \in I\} \subseteq M.$$

Define three hopes as follows

$$\underline{S} : R \times M \to P(M) : (r, A) \to r\underline{S}A = \{(rs_k)A : k \in K\} \subseteq M$$

$$\underline{Q}_+ : M \times M \to P(M) : (A, B) \to A\underline{Q}_+B = \{A + Q_j + B : j \in J\} \subseteq M$$

$$\underline{P} : M \times M \to P(M) : (A, B) \to A\underline{P}B = \{AP_i^tB : i \in I\} \subseteq M$$

Then $(M, \underline{S}, \underline{Q}_+, \underline{P})$ is hyperalgebra on R called general matrix P-hyperalgebra.

3 Helix-hopes

Recall some definitions from [3], [4], [6], [7], [19], [20], [21]:

Definition 3.1. *Let $A = (a_{ij}) \in M_{m\times n}$ be $m \times n$ matrix and $s, t \in N$ be naturals such that $1 \le s \le m$, $1 \le t \le n$. We define the map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to the matrix A, the matrix $A\underline{cst} = (a_{ij})$ where $1 \le i \le s$, $1 \le j \le t$. We call this map cut-projection of type $\underline{st}$. Thus $A\underline{cst} = (a_{ij})$ is matrix obtained from A by cutting the lines, with index greater than s, and columns, with index greater than t.*

We use cut-projections on all types of matrices to define sums and products.

Definition 3.2. *Let $A = (a_{ij}) \in M_{m\times n}$ be an $m \times n$ matrix and $s, t \in N$, such that $1 \le s \le m$, $1 \le t \le n$. We define the mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st} = (\underline{a}_{ij})$ which has as entries the sets*

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} | 1 \le i \le s, 1 \le j \le t \text{ and } \kappa, \lambda \in N, i + \kappa s \le m, j + \lambda t \le n\}.$$

Thus, we have the map

$$\underline{st} : M_{m\times n} \to M_{s\times t} : A \to A\underline{st} = (\underline{a}_{ij}).$$

We call this multivalued map helix-projection of type $\underline{st}$. $A\underline{st}$ is a set of $s \times t$-matrices $X = (x_{ij})$ such that $x_{ij} \in \underline{a}_{ij}, \forall i, j$. Obviously $A\underline{mn} = A$.

Let $A = (a_{ij}) \in M_{m\times n}$ be a matrix and $s, t \in N$ such that $1 \le s \le m$, $1 \le t \le n$. Then it is clear that we can apply the helix-projection first on the rows and then on the columns, the result is the same if we apply the helix-projection on both, rows and columns. Therefore we have

$$(A\underline{sn})\underline{st} = (A\underline{mt})\underline{st} = A\underline{st}.$$

Let $A = (a_{ij}) \in M_{m\times n}$ be matrix and $s, t \in N$ such that $1 \le s \le m$, $1 \le t \le n$. Then if A$\underline{st}$ is not a set but one single matrix then we call A *cut-helix matrix* of type $s \times t$. In other words the matrix A is a helix matrix of type $s \times t$, if A$\underline{cst}$= A$\underline{st}$.

Definition 3.3. **a.** *Let $A = (a_{ij}) \in M_{m\times n}, B = (b_{ij}) \in M_{u\times v}$, be matrices and s=min(m,u), t=min(n,u). We define a hope, called helix-addition or* ***helix-sum****, as follows:*

$$\oplus : M_{m\times n} \times M_{u\times v} \to P(M_{s\times t}) : (A, B) \to$$
$$A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s\times t},$$

where

$$(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij} = (a_{ij} + b_{ij}) | a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$$

b. *Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{u\times v}$, be matrices and s=min(m,u). We define a hope, called helix-multiplication or* ***helix-product****, as follows:*

$$\otimes : M_{m\times n} \times M_{u\times v} \to P(M_{m\times v}) : (A, B) \to$$

$$A \otimes B = A\underline{ms} \cdot B\underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset M_{m\times v},$$

where

$$(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij} = (\sum a_{it}b_{tj})|a_{ij} \in \underline{a}_{ij} \textit{ and } b_{ij} \in \underline{b}_{ij}\}$$

The helix-sum is an external hope and the commutativity is valid. For the helix-product we remark that we have $A \otimes B = A\underline{ms} \cdot B\underline{sv}$ so we have either $A\underline{ms} = A$ or $B\underline{sv} = B$, that means that the helix-projection was applied only in one matrix and only in the rows or in the columns. If the appropriate matrices in the helix-sum and in the helix-product are cut-helix, then the result is singleton.

Remark. In $M_{m\times n}$ the addition is ordinary operation, thus we are interested only in the 'product'. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the **helix-Lie Algebra** is defined [22], as well. This algebra is an H_v-Lie Algebra where the fundamental relation ϵ^* gives, by a quotient, a Lie algebra, from which a classification is obtained.

In the following we restrict ourselves on the matrices $M_{m\times n}$ where $m < n$. We have analogous results if $m > n$ and for $m = n$ we have the classical theory.

Notation. For given $\kappa \in \mathbb{N} - \{0\}$, we denote by $\underline{\kappa}$ the remainder resulting from its division by m if the remainder is non zero, and $\underline{\kappa} = m$ if the remainder is zero. Thus a matrix $A = (a_{\kappa\lambda}) \in M_{m\times n}, m < n$ is a cut-helix matrix if we have $a_{\kappa\lambda} = a_{\kappa\underline{\lambda}}, \forall \kappa\lambda \in \mathbb{N} - \{0\}$.

Moreover let us denote by $I_c = (c_{\kappa\lambda})$ the **cut-helix unit matrix** which the cut matrix is the unit matrix I_m. Therefore, since $I_m = (\delta_{\kappa\lambda})$, where $\delta_{\kappa\lambda}$ is the Kronecker's delta, we obtain that, $\forall \kappa, \lambda$, we have $c_{\kappa\lambda} = \delta_{\kappa\underline{\lambda}}$.

Proposition 3.1. *For $m < n$ in $(M_{m\times n}, \otimes)$ the cut-helix unit matrix $I_c = (c_{\kappa\lambda})$, where $c_{\kappa\lambda} = \delta_{\kappa\underline{\lambda}}$, is a left scalar unit and a right unit. It is the only one left scalar unit.*

Proof. Let $A, B \in M_{m\times n}$ then in the helix-multiplication, since $m < n$, we take helix projection of the matrix A, therefore, the result $A \otimes B$ is singleton if the matrix A is a cut-helix matrix of type $m \times m$. Moreover, in order to have $A \otimes B = A\underline{mm} \cdot B = B$, the matrix $A\underline{mm}$ must be the unit matrix. Consequently, $I_c = (c_\kappa\lambda)$, where $c_{\kappa\lambda} = \delta_{\kappa\underline{\lambda}}, \forall \kappa, \lambda \in \mathbb{N} - \{0\}$, is necessarily the left scalar unit.

Let $A = (a_{uv}) \in M_{m\times n}$ and consider the hyperproduct $A \otimes I_c$. In the entry $\kappa\lambda$ of this hyperproduct there are sets, for all $1 \leq \kappa \leq m$, $1 \leq \lambda \leq n$, of the form

$$\sum \underline{a}_{\kappa s} c_{s\lambda} = \sum \underline{a}_{\kappa s} \delta_{s\underline{\lambda}} = \underline{a}_{\kappa\underline{\lambda}} \ni a_{\kappa\lambda}.$$

Therefore $A \otimes I_c \ni A, \forall A \in M_{m\times n}$. $\square$

4 The S-helix matrices

Definition 4.1. *Let $A = (a_{ij}) \in M_{m\times n}$ be matrix and $s, t \in N$ such that $1 \le s \le m$, $1 \le t \le n$. Then if $A\underline{st}$ is a set of upper triangular matrices with the same diagonal, then we call A an* **S-helix matrix of type** $\mathrm{s} \times \mathrm{t}$*. Therefore, in an S-helix matrix A of type $s \times t$, the $A\underline{st}$ has on the diagonal entries which are not sets but elements.*

In the following, we restrict our study on the case of $A = (a_{ij}) \in M_{m\times n}$ with $m < n$.

Remark. According to the cut-helix notation, we have,

$$a_{\kappa\lambda} = a_{\kappa\underline{\lambda}} = 0, \text{ for all } \kappa > \lambda \text{ and } a_{\kappa\lambda} = a_{\kappa\underline{\lambda}}, \text{ for } \kappa = \underline{\lambda}.$$

Proposition 4.1. *The set of S-helix matrices $A = (a_{ij}) \in M_{m\times n}$ with $m < n$, is closed under the helix product. Moreover, it has a unit the cut-helix unit matrix I_c, which is left scalar.*

Proof. It is clear that the helix product of two S-helix matrices, $X = (x_{ij}), Y = (a_{ij}) \in M_{m\times n}, X \otimes Y$, contain matrices $Z = (z_{ij})$, which are upper diagonals. Moreover, for every z_{ii}, the entry ii is singleton since it is product of only $z_{(i+km),(i+km)} = z_{ii}$, entries.

The unit is, from Proposition 3.1, the matrix $I_c = I_{m\times n}$, where we have $I_{m\times n} = I_{mm} = I_m$. $\square$

An example of hyper-matrix representation, seven dimensional, with helix-hope is the following:

Example 4.1. *Consider the special case of the matrices of the type 3×5 on the field of real or complex. Then we have*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{11} & x_{15} \\ 0 & x_{22} & x_{23} & 0 & x_{22} \\ 0 & 0 & x_{33} & 0 & 0 \end{pmatrix} \textit{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix}$$

$$X \otimes Y = \begin{pmatrix} x_{11} & \{x_{12}, x_{15}\} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} x_{11}y_{11} & x_{11}y_{12} + \{x_{12}, x_{15}\}y_{22} & x_{11}y_{13} + \{x_{12}, x_{15}\}y_{23} + x_{13}y_{33} & x_{11}y_{11} & x_{11}y_{15} + \{x_{12}, x_{15}\}y_{22} \\ 0 & x_{22}y_{22} & x_{22}y_{23} + x_{23}y_{33} & 0 & x_{22}y_{22} \\ 0 & 0 & x_{33}y_{33} & 0 & 0 \end{pmatrix}$$

Therefore the helix product is a set with cardinality up to 8.

The unit of this type is $I_c = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

Definition 4.2. *We call a matrix $A = (a_{ij}) \in M_{m\times n}$ an* $\mathbf{S_0}$***-helix matrix*** *if it is an S-helix matrix where the condition $a_{11}a_{22}\ldots a_{mm} \neq 0$, is valid. Therefore, an S_0-helix matrix has no zero elements on the diagonal and the set S_0 is a subset of the set S of all S-helix matrices. We notice that this set is closed under the helix product not in addition. Therefore it is interesting only when the product is used not the addition.*

Proposition 4.2. *The set of S_0-helix matrices $A = (a_{ij}) \in M_{m\times n}$ with $m < n$, is closed under the helix product, it has a unit the cut-helix unit matrix I_c, which is left scalar and S_0-helix matrices X have inverses X^{-1}, i.e. $I_c \in X \otimes X^{-1} \cap X^{-1} \otimes X$.*

Proof. First it is clear that on the helix product of two S_0-helix matrices, the diagonal has not any zero since there is no zero on each of them. Therefore, the helix product is closed. The entries in the diagonal are inverses in the H_v-field. In the rest entries we have to collect equations from those which correspond to each element of the entry-set. □

Example 4.2. *Consider the special case of the above Example 4.1, of the matrices of the type 3×5. Suppose we want to find the inverse matrix $Y = X^{-1}$, of the matrix X. Then we have $I_c \in X \otimes Y \cap Y \otimes X$. Therefore, we obtain*

$$x_{11}y_{11} = x_{22}y_{22} = x_{33}y_{33} = 1$$

$$x_{11}y_{12} + \{x_{12}, x_{15}\}y_{22} \ni 0, x_{11}y_{13} + \{x_{12}, x_{15}\}y_{23} + x_{13}y_{33} \ni 0,$$

$$x_{11}y_{15} + \{x_{12}, x_{15}\}y_{22} \ni 0, x_{23}y_{22} + x_{33}y_{23} \ni 0,$$

Therefore a solution is

$$y_{11} = \frac{1}{x_{11}}, y_{22} = \frac{1}{x_{22}}, y_{33} = \frac{1}{x_{33}}$$

$$y_{23} = \frac{-x_{23}}{x_{22}x_{33}}, y_{12} = \frac{-x_{12}}{x_{11}x_{22}}, y_{15} = \frac{-x_{15}}{x_{11}x_{22}}, \text{ and}$$

$$y_{13} = \frac{-x_{13}}{x_{11}x_{33}} + \frac{x_{23}x_{12}}{x_{11}x_{22}x_{33}} \text{ or } y_{13} = \frac{-x_{13}}{x_{11}x_{33}} + \frac{x_{23}x_{14}}{x_{11}x_{22}x_{33}}$$

Thus, a left and right inverse matrix of X is

$$X^{-1} = \begin{pmatrix} \frac{1}{x_{11}} & \frac{-x_{12}}{x_{11}x_{22}} & \frac{-x_{13}}{x_{11}x_{33}} + \frac{x_{23}x_{12}}{x_{11}x_{22}x_{33}} & \frac{1}{x_{11}} & \frac{-x_{15}}{x_{11}} \\ 0 & \frac{1}{x_{22}} & \frac{-x_{23}}{x_{22}x_{33}} & 0 & \frac{1}{x_{22}} \\ 0 & 0 & \frac{1}{x_{33}} & 0 & 0 \end{pmatrix}$$

An interesting research field is the finite case on small finite H_v-fields. Important cases appear taking the generating sets by any S_0-helix matrix.

Example 4.3. *On the type 3×5 of matrices using the Construction 2.1, on $(\mathbf{Z}_4, +, \cdot)$ we take the small H_v-field $(\mathbf{Z}_4, +, \otimes)$, where only $2 \otimes 3 = \{0, 2\}$ and fundamental classes $\{0, 2\}, \{1, 3\}$. We consider the set of all S_0-helix matrices and we take the S_0-helix matrix:*

$$X = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Then the powers of X are:

$$X^2 = \begin{pmatrix} 1 & \{0,2\} & \{0,2\} & 1 & \{0,2\} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 1 & \{0,2\} & \{0,2\} & 1 & \{0,2\} \\ 0 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \textit{ and so on}$$

We obtain that the generating set is the following

$$\begin{pmatrix} 1 & \{0,2\} & \{0,2\} & 1 & \{0,2\} \\ 0 & \{1,3\} & \{0,1\} & 0 & \{1,3\} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

where in the 22 and 25 entries appears simultaneously 1 or 3.

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Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 181-201 **eISSN**: 2282-8214

H_v-Fields, h/v-Fields

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†**doi**:10.23755/rm.v33i0.386

Abstract

In the last decades, the hyperstructures have had a lot of applications in mathematics and in other sciences. These applications range from biomathematics and hadronic physics to linguistic and sociology. For applications the largest class of the hyperstructures, the H_v-*structures*, is used, they satisfy the *weak axioms* where the non-empty intersection replaces the equality. The main tools in the theory of hyperstructures are the fundamental relations which connect, by quotients, the H_v-structures with the corresponding classical ones. These relations are used to define hyperstructures as H_v-fields, H_v-vector spaces and so on, as well. The extension of the reproduction axiom, from elements to fundamental classes, introduces the extension of H_v-structures to the class of h/v-structures. We focus our study mainly in the relation of these classes and we present some constructions on them.
Keywords: hope; H_v-structure; h/v-structure; H_v-field; h/v-field.
2010 AMS subject classifications: 20N20, 16Y99.

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†©Thomas Vougiouklis. Received: 31-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

The main object in this paper is the largest class of hyperstructures called H_v-*structures* introduced in 1990 [35], which satisfy the weak axioms where the non-empty intersection replaces the equality. Abbreviation: *hyperoperation*=**hope**.

Definition 1.1. *An algebraic hyperstructure is called a set H equipped with at least one* ***hope*** $\cdot : H \times H \to P(H) - \{\varnothing\}$. *We abbreviate by WASS the weak associativity:* $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ *and by COW the weak commutativity:* $xy \cap yx \neq \varnothing, \forall x, y \in H$. *The hyperstructure* $(H, \cdot)$ *is called an* H_v-*semigroup if it is WASS, it is called* $\mathbf{H_v}$-**group** *if it is reproductive* H_v-*semigroup, i.e.,* $xH = Hx = H, \forall x \in H$.

Motivation. The quotient of a group by an invariant subgroup, is a group. F. Marty (1934), *'Sur une generalization de la notion de groupe'.* 8^{eme} *Congres Math. Scandinaves, Stockholm, pp.45-49,* states: the quotient of a group by a subgroup is a hypergroup. The quotient of a group by a partition (or equivalently to any equivalence) is an H_v-group.

In an H_v-semigroup the *powers* are defined by: $h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h$, where $(\circ)$ is the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H, \cdot)$ is *cyclic of period s*, if there is an element h, called *generator*, and a natural number s, the minimum : $H = h^1 \cup h^2 ... \cup h^s$. Analogously the cyclicity for the infinite period is defined [30], [33], [39]. If there is an h and s, the minimum: $H = h^s$, then $(H, \cdot)$, is called *single-power cyclic of period s.*

Definition 1.2. *An* $(R, +, \cdot)$ *is called* $\mathbf{H_v}$*–**ring** if* $(+)$ *and* $(\cdot)$ *are WASS, the reproduction axiom is valid for* $(+)$ *and* $(\cdot)$ *is weak distributive with respect to* $(+)$*:*

$$x(y+z) \cap (xy+xz) \neq \varnothing,\ (x+y)z \cap (xz+yz) \neq \varnothing,\ \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ *be an* H_v*-ring,* $(M, +)$ *be a COW* H_v*-group and there exists an external hope*

$$\cdot : R \times M \to P(M) : (a, x) \to ax$$

such that $\forall a, b \in R$ *and* $\forall x, y \in M$ *we have*

$$a(x+y) \cap (ax+ay) \neq \varnothing,\ (a+b)x \cap (ax+bx) \neq \varnothing,\ (ab)x \cap a(bx) \neq \varnothing,$$

then M is called an H_v*-module over F. In the case of an* H_v*-field F, which is defined later, instead of an* H_v*-ring R, then the* $\mathbf{H_v}$*–**vector space** is defined.*

For more definitions and applications on hyperstructures one can see books [4], [5], [9], [10], [11], [39] and papers as [3], [7], [8], [15], [16], [20], [21], [27], [38], [40], [41], [43], [48], [55], [68].

Definition 1.3. *Let $(H, \cdot), (H, *)$ be H_v-semigroups on the same set H, the hope $(\cdot)$ is called smaller than the $(*)$, and $(*)$ greater than $(\cdot)$, iff there exists an*

$$f \in Aut(H, *) \text{ such that } xy \subset f(x * y),\ \forall x, y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H, *)$ contains $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called basic structure and $(H, *)$ is called $H_b - structure$.

The Little Theorem. Greater hopes than ones which are WASS or COW, are also WASS or COW, respectively.

This Theorem leads to a partial order on H_v-structures and to posets [39], [42], [43], [21].

Let $(H, \cdot)$ be hypergroupoid. We *remove* $h \in H$, if we take the restriction of $(\cdot)$ in the set $H - \{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear. $\underline{h} \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

The main tool in hyperstructures is the *fundamental relation*. M. Koscas 1970, [20], defined in hypergroups the relation β and its transitive closure $\beta*$. This relation is defined in H_v-groups, as well, and connect hyperstructures with the classical structures. T. Vougiouklis [34], [35], [39], [40], [41], [53], [54], [60], introduced the $\gamma*$ and $\epsilon*$ relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations, *fundamental*. (see also [4], [5], [1], [8], [10], [11]).

Definition 1.4. *The* ***fundamental relations*** *$\beta*$, $\gamma*$ and $\epsilon*$, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.*

Specifying the above motivation we remark that: Let $(G, \cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R, \cdot)$ is an H_v-group, therefore we have the quotient $(G/R, \cdot)/\beta^*$ which is a group, the *fundamental* one.

The main Theorem to find the fundamental classes is the following:

Theorem 1.1. *Let $(H, \cdot)$ be an H_v-group and denote by U the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then $\beta*$ is the transitive closure of β.*

Notation. We denote by [x] the fundamental class of the element $x \in H$. Therefore $\beta^*(x) = [x]$.

Analogous theorems are for H_v-rings, H_v-vector spaces and so on. For proof, see [34], [39]. An element is called **single** [39] if its fundamental class is singleton so, $[x] = \{x\}$.

More general structures can be defined by using the fundamental structures. An application in this direction is the general hyperfield. There was no general definition of a hyperfield, but from 1990 [35] there is the following [38], [39]:

Definition 1.5. *An H_v-ring $(R,+,\cdot)$ is called* $\mathbf{H_v}$***-field*** *if R/γ^* is a field.*

Since the algebras are defined on vector spaces, the analogous to Theorem 1.1, on H_v-vector spaces is the following: Let $(V,+)$ be an H_v-vector space over the H_v-field F. Denote by U the set of all expressions consisting of finite hopes either on F and V or the external hope applied on finite sets of elements of F and V. We define the relation ϵ, in V as follows: $x\epsilon y$ iff $\{x,y\}\in u$ where $u\in U$. Then the relation ϵ^* is the transitive closure of the relation ϵ.

Definition 1.6. *[53], [54], [57]. Let $(L,+)$ be an H_v-vector space over the H_v-field $(F,+,\cdot)$, $\phi: F\to F/\gamma^*$ the canonical map and $\omega_F=\{x\in F:\phi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Let ω_L be the core of the canonical map $\phi': L\to L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^*. Consider the bracket (commutator) hope:*

$$[,]: L\times L\to P(L):(x,y)\to[x,y]$$

then L is an $\mathbf{H_v}$***-Lie algebra*** *over F if the following axioms are satisfied:*

(L1) *The bracket hope is bilinear, i.e.*
$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$
$[x,\lambda_1y_1+\lambda_2y_2]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing,$
$\forall x,x_1,x_2,y,y_1,y_2\in L,\lambda_1,\lambda_2\in F$

(L2) $[x,x]\cap\omega_L\neq\varnothing,\ \ \forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing,\ \ \forall x,y\in L$

In the Definition 1.5, was introduced a new class of which is the following [45] (for a preliminary report see: T. Vougiouklis. A generalized hypergroup, Abstracts AMS, Vol. 19.3, Issue 113, 1998, p.489):

Definition 1.7. *The H_v-semigroup $(H,\cdot)$ is called* ***h/v-group*** *if H/β^* is a group.*

An important and well known class of hyperstructures defined on classical structures are defined as follows [30], [33], [36], [57], [60]:

Definition 1.8. *Let $(G,\cdot)$ be groupoid, then for every $P\subset G, P\neq\varnothing$, we define the following hopes called P-hopes: $\forall x,y\in G$*

$$\underline{P}: x\underline{P}y=(xP)y\cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP),$$
$$\underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

*The $(G, \underline{P})$,$(G, \underline{P}_r)$, $(G, \underline{P}_l)$ are called **P-hyperstructures**. The most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup.*

A **generalization of P-hopes**, used in Santilli's isotheory, is the following [12], [13], [14]: Let $(G, \cdot)$ be abelian group and P a subset of G with $\#P > 1$. We define the hope $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_P)$ is abelian H_v-group.

Definition 1.9. *[36]. An H_v-structure is called **very thin** if all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in H there exists a hope $(\cdot)$ and a pair $(a, b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products, are singletons.*

From the properties of the very thin hopes the *Attach Construction* is obtained [43], [54]: Let $(H, \cdot)$ be an H_v-semigroup and $v \notin H$. We extend the $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The $(\underline{H}, \cdot)$ is an H_v-group, where $(\underline{H}, \cdot)/\beta^* \cong Z_2$ and v is a single.

A class of H_v-structures is the following [47], [49], [57], [60]:

Definition 1.10. *Let $(G, \cdot)$ be groupoid (resp. hypergroupoid) and $f : G \to G$ be a map. We define a hope (∂), called theta-hope, we write ∂-hope, on G as follows*

$x\partial y = \{f(x)\cdot y, x\cdot f(y)\}, \ \forall x, y \in G.$ *(resp.* $x\partial y = (f(x)\cdot y) \cup (x\cdot f(y)), \ \forall x, y \in G)$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW.

If $(G, \cdot)$ is a groupoid (or hypergroupoid) and $f : G \to P(G) - \{\varnothing\}$ be any multivalued map. We define the ∂-hope on G as follows:

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \ \forall x, y \in G.$$

The ∂-hopes can be defined in H_v-vector spaces and H_v-Lie algebras:

Let $(\mathbf{A}, +, \cdot)$ be an algebra over the field F. Take any map $f : \mathbf{A} \to \mathbf{A}$, then the *$\partial$-hope on the Lie bracket* $[x, y] = xy - yx$, is defined as follows

$$x\partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

then $(\mathbf{A}, +, \partial)$ is an H_v-algebra over F, with respect to the ∂-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

Motivation for the theta-hope is the map *derivative* where only the multiplication of functions can be used. Basic property: if $(G, \cdot)$ is semigroup then $\forall f$, the ∂-hope is WASS.

Example.

(a) In integers $(\mathbf{Z}, +, \cdot)$ fix $n \neq 0$, a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x$, $\forall x \in \mathbf{Z} - \{0\}$. Then $(Z, \partial_+, \partial_.)$, where ∂_+ and $\partial_.$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(\mathbf{Z}, \partial_+, \partial_.)/\gamma^* \cong \mathbf{Z}_n.$$

(b) In $(\mathbf{Z}, +, \cdot)$ with $n \neq 0$, take f such that $f(n) = 0$ and $f(x) = x$, $\forall x \in \mathbf{Z} - \{n\}$. Then $(\mathbf{Z}, \partial_+, \partial_.)$ is an H_v-ring, moreover, $(\mathbf{Z}, \partial_+, \partial_.)/\gamma^* \cong \mathbf{Z}_n$.

Special case of the above is for $n = p$, prime, then $(\mathbf{Z}, \partial_+, \partial_.)$ is an H_v-field.

The uniting elements method was introduced by Corsini-Vougiouklis [6] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

The *uniting elements* method is the following: Let **G** be algebraic structure and d, a property which is not valid. Suppose that d is described by a set of equations; then, take the partition in **G** for which it is put together, in the same class, every pair of elements that causes the non-validity of the property d. The quotient by this partition $\mathbf{G}/d$ is an H_v-structure. Then, quotient out the H_v-structure $\mathbf{G}/d$ by the fundamental relation β^*, a stricter structure $(\mathbf{G}/d)/\beta^*$ for which the property d is valid, is obtained.

It is very important if more properties are desired, then we have the following [39]:

Theorem 1.2. *Let $(\mathbf{R}, +, \cdot)$ be a ring, and $F = \{f_1, ..., f_m, f_{m+1}, ..., f_{m+n}\}$ be a system of equations on $\mathbf{R}$ consisting of two subsystems $F_m = \{f_1, ..., f_m\}$ and $F_n = \{f_{m+1}, ..., f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the ring $\mathbf{R}_m = (\mathbf{R}/\sigma_m)/\gamma^*$. Then,*

$$(\mathbf{R}/\sigma)/\gamma^* \cong (\mathbf{R}_m/\sigma_n)\gamma^*.$$

Combining the uniting elements procedure with the enlarging theory or the ∂-theory, we can obtain analogous results [39], [51], [54], [60], [22].

Theorem 1.3. *In the ring $(\mathbf{Z}_n, +, \cdot)$, with n=ms we enlarge the multiplication only in the product of the special elements $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$ and the rest results remain the same. Then*

$$(\mathbf{Z}_n, +, \otimes)/\gamma^* \cong (\mathbf{Z}_m, +, \cdot).$$

Remark that we can enlarge other products as well, for example $2 \cdot m$ by setting $2 \otimes m = \{2, m+2\}$, then the result remains the same. In this case 0 and 1 remain scalars.

Corollary. In the ring $(\mathbf{Z}_n, +, \cdot)$, with n=ps where p is prime, we enlarge only the product $0 \cdot p$ by $0 \otimes p = \{0, p\}$ and the rest remain the same. Then $(\mathbf{Z}_n, +, \otimes)$ is very thin H_v-field.

2 Constructions of H_v-fields and h/v-fields

The class of h/v-groups is more general than the H_v-groups since in h/v-groups the reproductivity is not valid. The r*eproductivity of classes* is valid, i.e. if H is partitioned into equivalence classes, then

$$x[y] = [xy] = [x]y, \forall x, y \in H,$$

because the quotient is reproductive. In a similar way the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc are defined.

Remark 2.1. *From definition of the H_v-field, we remark that the reproduction axiom in the product, is not assumed, the same is also valid for the definition of the h/v-field. Therefore, an H_v-field is an h/v-field where the reproduction axiom for the sum is also valid.*

We know that the reproductivity in the classical group theory is equivalent to the axioms of the existence of the unit element and the existence of an inverse element for any given element. From the definition of the h/v-group, since a generalization of the reproductivity is valid, we have to extend the above two axioms on the equivalent classes.

Definition 2.1. *Let $(H, \cdot)$ be an H_v-semigroup, and denote $[x]$ the fundamental, or equivalent class, of the element $x \in H$. We call **unit class** the class $[e]$ if we have*

$$([e] \cdot [x]) \cap [x] \neq \varnothing \text{ and } ([x] \cdot [e]) \cap [x] \neq \varnothing, \forall x \in H,$$

*and for each element $x \in H$, we call **inverse class** of $[x]$, the class $[x']$, if we have*

$$([x] \cdot [x']) \cap [e] \neq \varnothing \text{ and } ([x'] \cdot [x]) \cap [e] \neq \varnothing.$$

The 'enlarged' hyperstructures were examined in the sense that a new element appears in one result. In enlargement or reduction, most useful are those H_v-structures or h/v-structures with the same fundamental structure [43], [53].

Construction 2.1. *(***a***) Let $(H,\cdot)$ be an H_v-semigroup and $v \notin H$. We extend the $(\cdot)$ into $\underline{H} = H \cup \{v\}$ as follows:*

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The $(\underline{H},\cdot)$ is an h/v-group, called ***attach****, where $(\underline{H},\cdot)/\beta^* \cong \mathbf{Z}_2$ and v is a single element.*

We have core $(\underline{H},\cdot) = H$. The scalars and units of $(H,\cdot)$ are scalars and units (resp.) in $(\underline{H},\cdot)$. If $(H,\cdot)$ is COW (resp. commutative) then $(\underline{H},\cdot)$ is also COW (resp. commutative).

*(***b***) Let $(H,\cdot)$ be an H_v-semigroup and $\{v_1,\ldots,v_n\} \cap H = \varnothing$, is an ordered set, where if $v_i < v_j$, when $i < j$. Extend $(\cdot)$ in $\underline{H}_n = H \cup \{v_1,\ldots,v_n\}$ as follows:*

$$x \cdot v_i = v_i \cdot x = v_i, v_i \cdot v_j = v_j \cdot v_i = v_j, \forall i < j \text{ and}$$

$$v_i \cdot v_i = H \cup \{v_1,\ldots,v_{i-1}\}, \forall x \in H, i \in \{1,\ldots,n\}.$$

Then $(\underline{H}_n,\cdot)$ is h/v-group, called ***attach elements****, where $(\underline{H}_n,\cdot)/\beta^* \cong Z_2$ and v_n is single.*

*(***c***) Let $(H,\cdot)$ be an H_v-semigroup, $v \notin H$, and $(\underline{H},\cdot)$ be its attached h/v-group. Take an element $0 \notin \underline{H}$ and define in $\underline{H}_o = H \cup \{v, 0\}$ two hopes:*
hypersum (+): $0+0 = x+v = v+x = 0$, $0+v = v+0 = x+y = v$, $0+x = x+0 = v+v = H$, $\forall x,y \in H$
hyperproduct $(\cdot)$: remains the same as in $\underline{H}$ moreover $0\cdot 0 = v\cdot x = x\cdot 0 = 0, \forall x \in \underline{H}$
Then $(\underline{H}_o,+,\cdot)$ is h/v-field with $(\underline{H}_o,+,\cdot)/\gamma^* \cong Z_3$. (+) is associative, $(\cdot)$ is WASS and weak distributive with respect to (+). 0 is zero absorbing and single but not scalar in (+). $(\underline{H}_o,+,\cdot)$ is called the ***attached h/v-field*** *of the H_v-semigroup $(H,\cdot)$.*

Let us denote by U the set of all finite products of elements of a hypergroupoid $(H,\cdot)$. Consider the relation defined as follows:

$$xLy \text{ iff there exists } u \in U \text{ such that } ux \cap uy \neq \varnothing.$$

Then the transitive closure L^* of L is called *left fundamental reproductivity relation.* Similarly, the *right fundamental reproductivity relation* R^* is defined.

Theorem 2.1. *If $(H, \cdot)$ is a commutative semihypergroup, i.e. the strong commutativity and the strong associativity is valid, then the strong expression of the above L relation: $ux = uy$, has the property: $L^* = L$.*

Proof. Suppose that two elements x and y of H are L^* equivalent. Therefore, there are $u_1, \ldots, u_{n+1}$ elements of U, and $z_1, \ldots, z_n$ elements of H, such that

$$u_1x = u_1z_1, u_2z_1 = u_2z_2, \ldots, u_nz_{n-1} = u_nz_n, u_{n+1}z_n = u_{n+1}y.$$

From these relations, using the strong commutativity, we obtain

$$u_{n+1}\ldots u_2u_1x = u_{n+1}\ldots u_2u_1z_1 = u_{n+1}\ldots u_1u_2z_1 =$$

$$= u_{n+1}\ldots u_2u_1z_2 = \cdots = u_{n+1}\ldots u_2u_1y$$

Therefore, setting $u = u_{n+1}\ldots u_2u_1 \in U$, we have $ux = uy$. □

Corollary. Let $(S, \cdot)$ be commutative semigroup which has an element $w \in S$ such that the set wS is finite. Consider the transitive closure L^* of the relation L defined by:

xLy iff there exists $z \in S$ such that $zx = zy$.

Then $< S/L^*,^\circ /\beta^*$ is a finite commutative group, where $(^\circ)$ is the induced operation on classes of S/L^*.

Open problem: Prove that L^*, is the smallest equivalence: H/L^*, is reproductive.

We present now the **small non-degenerate $\mathbf{H_v}$-fields** on $(\mathbf{Z}_n, +, \cdot)$ which satisfy the following conditions, appropriate in Santilli's iso-theory:

1. multiplicative very thin minimal,
2. COW (non-commutative),
3. they have the elements 0 and 1, scalars,
4. when an element has inverse element, then this is unique.

Remark that last condition means than we cannot enlarge the result if it is 1 and we cannot put 1 in enlargement. Moreover we study only the upper triangular cases, in the multiplicative table, since the corresponding under, are isomorphic since the commutativity is valid for the underline rings. From the fact that the reproduction axiom in addition is valid, we have always H_v-fields.

Theorem 2.2. *All multiplicative H_v-fields defined on $(\mathbf{Z}_4, +, \cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*

The only product which is set is $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$.

The fundamental classes are $[0] = \{0, 2\}, [1] = \{1, 3\}$ and we have $(\mathbf{Z}_4, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

Example. Let us denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Then take the following 2×2 upper triangular H_v-matrices on the above H_v-field $(\mathbf{Z}_4, +, \cdot)$ of the case that only $2 \otimes 3 = \{0, 2\}$ is a hyperproduct:

$$I = E_{11} + E_{22}, a = E_{11} + E_{12} + E_{22}, b = E_{11} + 2E_{12} + E_{22}, c = E_{11} + 3E_{12} + E_{22},$$

$$d = E_{11} + 3E_{22}, e = E_{11} + E_{12} + 3E_{22}, f = E_{11} + 2E_{12} + 3E_{22}, g = E_{11} + 3E_{12} + 3E_{22},$$

then, we obtain for $\mathbf{X} = \{I, a, b, c, d, e, f, g\}$, that $(\mathbf{X}, \otimes)$ is non-COW H_v-group and the fundamental classes are $\underline{a} = \{a, c\}, \underline{d} = \{d, f\}, \underline{e} = \{e, g\}$ and the fundamental group is isomorphic to $(\mathbf{Z}_2 \times \mathbf{Z}_2, +)$. In this H_v-group there is only one unit and every element has a unique double inverse.

Theorem 2.3. *All multiplicative H_v-fields defined on $(\mathbf{Z}_6, +, \cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*
We have the only one hyperproduct,

(I) $2 \otimes 3 = \{0, 3\}$ *or* $2 \otimes 4 = \{2, 5\}$ *or*
$3 \otimes 4 = \{0, 3\}$ *or* $3 \otimes 5 = \{0, 3\}$ *or* $4 \otimes 5 = \{2, 5\}$
Fundamental classes: $[0] = \{0, 3\}, [1] = \{1, 4\}, [2] = \{2, 5\},$ *and*
$(\mathbf{Z}_6, +, \cdot)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot)$.

(II) $2 \otimes 3 = \{0, 2\}$ *or* $2 \otimes 3 = \{0, 4\}$ *or* $2 \otimes 4 = \{0, 2\}$ *or* $2 \otimes 4 = \{2, 4\}$ *or*
$2 \otimes 5 = \{0, 4\}$ *or* $2 \otimes 5 = \{2, 4\}$ *or* $3 \otimes 4 = \{0, 2\}$ *or* $3 \otimes 4 = \{0, 4\}$ *or*
$3 \otimes 5 = \{3, 5\}$ *or* $4 \otimes 5 = \{0, 2\}$ *or* $4 \otimes 5 = \{2, 4\}$
Fundamental classes: $[0] = \{0, 2, 4\}, [1] = \{1, 3, 5\},$ *and*
$(\mathbf{Z}_6, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

Theorem 2.4. *All multiplicative H_v-fields defined on $(\mathbf{Z}_9, +, \cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*
We have the only one hyperproduct,
$2 \otimes 3 = \{0, 6\}$ *or* $\{3, 6\}$, $2 \otimes 4 = \{2, 8\}$ *or* $\{5, 8\}$, $2 \otimes 6 = \{0, 3\}$ *or* $\{3, 6\}$,
$2 \otimes 7 = \{2, 5\}$ *or* $\{5, 8\}$, $2 \otimes 8 = \{1, 7\}$ *or* $\{4, 7\}$, $3 \otimes 4 = \{0, 3\}$ *or* $\{3, 6\}$,
$3 \otimes 5 = \{0, 6\}$ *or* $\{3, 6\}$, $3 \otimes 6 = \{0, 3\}$ *or* $\{0, 6\}$, $3 \otimes 7 = \{0, 3\}$ *or* $\{3, 6\}$,
$3 \otimes 8 = \{0, 6\}$ *or* $\{3, 6\}$, $4 \otimes 5 = \{2, 5\}$ *or* $\{2, 8\}$, $4 \otimes 6 = \{0, 6\}$ *or* $\{3, 6\}$,
$4 \otimes 8 = \{2, 5\}$ *or* $\{5, 8\}$, $5 \otimes 6 = \{0, 3\}$ *or* $\{3, 6\}$, $5 \otimes 7 = \{2, 8\}$ *or* $\{5, 8\}$,
$5 \otimes 8 = \{1, 4\}$ *or* $\{4, 7\}$, $6 \otimes 7 = \{0, 6\}$ *or* $\{3, 6\}$, $6 \otimes 8 = \{0, 3\}$ *or* $\{3, 6\}$,
$7 \otimes 8 = \{2, 5\}$ *or* $\{2, 8\}$,
Fundamental classes: $[0] = \{0, 3, 6\}$, $[1] = \{1, 4, 7\}$, $[2] = \{2, 5, 8\}$, *and*
$(\mathbf{Z}_9, +, \otimes)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot)$.

Theorem 2.5. *All H_v-fields defined on $(\mathbf{Z}_{10}, +, \cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*

(I) We have the only one hyperproduct,
$2 \otimes 4 = \{3,8\}$, $2 \otimes 5 = \{2,5\}$, $2 \otimes 6 = \{2,7\}$, $2 \otimes 7 = \{4,9\}$, $2 \otimes 9 = \{3,8\}$, $3 \otimes 4 = \{2,7\}$, $3 \otimes 5 = \{0,5\}$, $3 \otimes 6 = \{3,8\}$, $3 \otimes 8 = \{4,9\}$, $3 \otimes 9 = \{2,7\}$, $4 \otimes 5 = \{0,5\}$, $4 \otimes 6 = \{4,9\}$, $4 \otimes 7 = \{3,8\}$, $4 \otimes 8 = \{2,7\}$, $5 \otimes 6 = \{0,5\}$, $5 \otimes 7 = \{0,5\}$, $5 \otimes 8 = \{0,5\}$, $5 \otimes 9 = \{0,5\}$, $6 \otimes 7 = \{2,7\}$, $6 \otimes 8 = \{3,8\}$, $6 \otimes 9 = \{4,9\}$, $7 \otimes 9 = \{3,8\}$, $8 \otimes 9 = \{2,7\}$.
Fundamental classes: $[0] = \{0,5\}$, $[1] = \{1,6\}$, $[2] = \{2,7\}$, $[3] = \{3,8\}$, $[4] = \{4,9\}$ *and* $(\mathbf{Z}_{10}, +, \otimes)/\gamma^* \cong (\mathbf{Z}_5, +, \cdot)$.

(II) The cases where we have two classes
$[0] = \{0,2,4,6,8\}$ *and* $[1] = \{1,3,5,7,9\}$, *thus we have fundamental field* $(\mathbf{Z}_{10}, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$, *can be described as follows:*
Taking in the multiplicative table only the results above the diagonal, we enlarge each of the products by putting one element of the same class of the results. We do not enlarge setting the element 1, and we cannot enlarge only the product $3 \otimes 7 = 1$. *The number of those H_v-fields is 103.*

Example 2.1. *In order to see how hard is to realize the reproductivity of classes and the unit class and inverse class, we consider the above H_v-field $(\mathbf{Z}_{10}, +, \otimes)$ where we have $2 \otimes 4 = \{3,8\}$. Then the Multiplicative Table of the hyperproduct is the following:*

$\otimes$	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*
1	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
2	*0*	*2*	*4*	*6*	***3,8***	*0*	*2*	*4*	*6*	*8*
3	*0*	*3*	*6*	*9*	*2*	*5*	*8*	*1*	*4*	*7*
4	*0*	*4*	*8*	*2*	*6*	*0*	*4*	*8*	*2*	*6*
5	*0*	*5*	*0*	*5*	*0*	*5*	*0*	*5*	*0*	*5*
6	*0*	*6*	*2*	*8*	*4*	*0*	*6*	*2*	*8*	*4*
7	*0*	*7*	*4*	*1*	*8*	*5*	*2*	*9*	*6*	*3*
8	*0*	*8*	*6*	*4*	*2*	*0*	*8*	*6*	*4*	*2*
9	*0*	*9*	*8*	*7*	*6*	*5*	*4*	*3*	*2*	*1*

On this table it is easy to see that the reproductivity is not valid but it is very hard to see that the reproductivity of classes is valid. We can see the reproductivity of classes easier if we reformulate the Multiplicative Table according to the fundamental classes, $[0] = \{0,5\}, [1] = \{1,6\}, [2] = \{2,7\}, [3] = \{3,8\}, [4] = \{4,9\}$. *Then we obtain:*

$\otimes$	***0***	***5***	***1***	***6***	***2***	***7***	***3***	***8***	***4***	***9***
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*
5	*0*	*5*	*5*	*0*	*0*	*5*	*5*	*0*	*0*	*5*
1	*0*	*5*	*1*	*6*	*2*	*7*	*3*	*8*	*4*	*9*
6	*0*	*0*	*6*	*6*	*2*	*2*	*8*	*8*	*4*	*4*
2	*0*	*0*	*2*	*2*	*4*	*4*	*6*	*6*	***3,8***	*8*
7	*0*	*5*	*7*	*2*	*4*	*9*	*1*	*6*	*8*	*3*
3	*0*	*5*	*3*	*8*	*6*	*1*	*9*	*4*	*2*	*7*
8	*0*	*0*	*8*	*8*	*6*	*6*	*4*	*4*	*2*	*2*
4	*0*	*0*	*4*	*4*	*8*	*8*	*2*	*2*	*6*	*6*
9	*0*	*5*	*9*	*4*	*8*	*3*	*7*	*2*	*6*	*1*

From this it is easy to see the unit class and the inverse class of each class.

3 The h/v-representations and applications

H_v-structures are used in Representation Theory of H_v-groups which can be achieved either by generalized permutations or by H_v-matrices [31], [32], [38], [39], [44], [46], [57], [58]. The representations by generalized permutations can be faced by translations [37]. Moreover in hyperstructure theory we can define hyperproduct on non-square ordinary matrices by using the so called helix hopes where we use all entries of them [65], [28], [29] and [13], [14], [66], [67]. Thus, we face the representations of the hyperstructures by non-square matrices as well.

$\mathbf{H_v}$**-matrix** (or **h/v-matrix**) is a matrix with entries of an H_v-ring or H_v-field (or h/v-ring or h/v-field). The hyperproduct of two H_v-matrices (a_{ij}) and $= (b_{ij})$, of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is considered to be the n-ary circle hope on the hyperaddition. The hyperproduct of H_v-matrices is not necessarily WASS.

The problem of the H_v-matrix (or h/v-group) representations is the following:

Definition 3.1. *Let $(H, \cdot)$ be an H_v-group (or h/v-group). Find an H_v-ring (or h/v-ring) $(R, +, \cdot)$, a set M_R=$\{(a_{ij})|a_{ij}{\in}R\}$ and a map $T : H \to M_R : h \mapsto T(h)$ such that*

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

T is an $\mathbf{H_v}$***-matrix (or h/v matrix) representation***.
If $T(h_1h_2) \subset T(h_1)T(h_2), \forall h_1, h_2 \in H$, then T is an inclusion representation.
If $T(h_1h_2) = T(h_1)T(h_2), \forall h_1, h_2 \in H$, then T is a good representation and an induced representation T of the hypergroup algebra is obtained. If T is one to one and the good condition is valid then it is called faithful representation.*

The main theorem of the theory of representations is the following [31], [32], [38]:

Theorem 3.1. *A necessary condition in order to have an inclusion representation T of an h/v-group $(H, \cdot)$ by $n \times n$, h/v-matrices over the h/v-ring $(R, +, \cdot)$ is the following:*
For all classes $\beta*(x)$, $x \in H$ there must exist elements $a_{ij} \in H, i, j \in \{1, ..., n\}$ such that

$$T(\beta*(a)) \subset \{A = (a'_{ij})|a'_{ij} \in \gamma*(a_{ij}), i, j \in \{1, ..., n\}\}$$

Thus, inclusion representation $T : H \to M_R : a \mapsto T(a) = (a_{ij})$ induces an homomorphic $T*$ of $H/\beta*$ over $R/\gamma*$ by setting
$T*(\beta*(a)) = [\gamma*(a_{ij})], \forall\beta*(a) \in H/\beta*$, where $\gamma*(a_{ij})R/\gamma*$ is the ij entry of $T*(\beta*(a))$. $T*$ is called fundamental induced representation of T.

Let T a representation of an h/v-group H by h/v-matrices and $tr_\phi(T(x)) = \gamma^*(Tx_{ii})$ be the fundamental trace, then is called *fundamental character*, the mapping

$$X_T : H \to R/\gamma* : x \mapsto X_T(x) = tr_\phi(T(x)) = trT^*(x)$$

In representations of H_v-groups there are two difficulties: First to find an H_v-ring or an H_v-field and second, an appropriate set of H_v-matrices. Notice that the more interesting cases are for the small H_v-fields, where the results have one or few elements.

Example 3.1. *In the case of the H_v-field $(\mathbf{Z}_6, +, \otimes)$ where the only one hyperproduct is $2 \otimes 4 = \{2, 5\}$ we consider the 2×2 h/v-matrices of type*

$$\underline{i} = E_{11} + iE_{12} + 4E_{22}, \text{ where } i = 0, 1, 2, 3, 4, 5,$$

then an h/v-group is obtained and the multiplicative table of the hyperproduct of those H_v-matrices is given by

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}$	$\underline{0},\underline{3}$	$\underline{1},\underline{4}$	$\underline{2},\underline{5}$	$\underline{0},\underline{3}$	$\underline{1},\underline{4}$
$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$

where the fundamental classes are $(0) = \{\underline{0}, \underline{3}\}, (1) = \{\underline{1}, \underline{4}\}, (2) = \{\underline{2}, \underline{5}\}$ and the fundamental group is isomorphic to $(\mathbf{Z}_3, +)$. Remark that $(\mathbf{Z}_6, \otimes)$ is an h/v-group which is cyclic where the elements $\underline{2}$ and $\underline{4}$ are generators of period 4. Notice that the hope $(\otimes)$ is a hyperproduct of h/v-matrices although (0) stands for the unit matrix, this is so because the symbolism follows the entry 12.

Example 3.2. *Let us denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Then take the following 2×2 upper triangular h/v-matrices on the above h/v-field $(\mathbf{Z}_4, +, \otimes)$ of the case that only $2 \otimes 3 = \{0, 2\}$ is a hyperproduct:*

$$I = E_{11} + E_{22}, a = E_{11} + E_{12} + E_{22}, b = E_{11} + 2E_{12} + E_{22}, c = E_{11} + 3E_{12} + E_{22},$$

$$d = E_{11} + 3E_{22}, e = E_{11} + E_{12} + 3E_{22}, f = E_{11} + 2E_{12} + 3E_{22}, g = E_{11} + 3E_{12} + 3E_{22},$$

then, we obtain the following multiplicative table for the set X={I,a,b,c,d,e,f,g}

$\otimes$	I	a	b	c	d	e	f	g
I	I	a	b	c	d	e	f	g
a	a	b	c	I	g	d	e	f
b	b	c	I	a	d,f	e,g	d,f	e,g
c	c	I	a	b	e	f	g	d
d	d	e	f	g	I	a	b	c
e	e	f	g	d	c	I	a	b
f	f	g	d	e	I,b	a,c	I,b	a,c
g	g	d	e	f	a	b	c	I

The $(\mathbf{X}, \otimes)$ is non-COW, H_v-group and we can see that the fundamental classes are the $\underline{a} = \{a, c\}$, $\underline{d} = \{d, f\}$, $\underline{e} = \{e, g\}$ and the fundamental group is isomorphic to $(\mathbf{Z}_2 \times \mathbf{Z}_2, +)$. In this H_v-group there is only one unit and every element has a unique double inverse. Only f has one more right inverse element, the d, since $f \otimes d = \{I, b\}$.

Remark that if we need h/v-fields where the elements have at most one inverse element, then we must exclude the case of $2 \otimes 5 = \{1, 4\}$ from (I), and the case $3 \otimes 5 = \{1, 3\}$ from (II).

Last decades H_v-structures have applications in other branches of mathematics and in other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructure theory is related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too [2], [5], [11], [12], [23], [25], [43], [47], [59].

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields needed, correspond into the hyperstructures were introduced by Santilli & Vougiouklis in 1996 [25] and they are called *e-hyperfields*, [12], [24], [52], [56], [61].

Definition 3.2. *A hyperstructure $(H, \cdot)$ which contains a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

Definition 3.3. *A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following axioms are valid: $(F, +)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$, there exists a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$, and $\forall x \in F$ there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield*.

Definition 3.4. *Main e-Construction. Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hopes ($\otimes$) as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

g_1, g_2,... are not necessarily the same for each pair (x,y). $(G, \otimes)$ is an H_v-group, in fact it is an H_b-group which contains the $(G, \cdot)$. $(G, \otimes)$ is an e-hypergroup. Moreover, if for each x,y such that $xy = e$, then $(G, \otimes)$ becomes a strong e-hypergroup

The main e-construction gives an extremely large number of e-hopes.

Example. Consider the quaternions $\mathbf{Q} = \{1, -1, i, -i, j, -j, k, -k\}$, with $i^2 = j^2 = -1, ij = -ji = k$, and denote $\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$. We define a lot of hopes $(*)$ by enlarging few products. For example, $(-1) * k = \underline{k}, k * i = \underline{j}$ and $i * j = \underline{k}$. Then the hyperstructure $(Q, *)$ is a strong e-hypergroup.

The Lie-Santilli admissibility on non-quare matrices [12], [14], [24], [26], [57], [61]:

Construction 3.1. *Let $L = (M_{m \times n}, +)$ be an H_v-vector space of $m \times n$ hypermatrices over the H_v-field $(F, +, \cdot), \phi : F \to F/\gamma^*$, the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ*. Similarly, let ω_L be the core of the canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ*. Take any two subsets $R, S \subseteq L$ then a Santilli's Lie-admissible hyperalgebra is obtained by taking the Lie bracket, which is a hope:*

$$[,]_{RS} : L \times L \to P(L) : [x, y]_{RS} = xR^t y - yS^t x.$$

Notice that $[x, y]_{RS} = xR^t y - yS^t x = \{xr^t y - ys^t x | r \in R \text{ and } s \in S\}$

An application, which combines the ∂-structures and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [19]:

Definition 3.5. *In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

0 __________________ *1*

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question

The use of the Vougiouklis & Vougiouklis bar instead of a Likert scale has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm, [17], [18], [50], [51], [62], [63], [64].

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Ratio Mathematica, 33, 2017

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Published by Accademia Piceno - Aprutina dei Velati in Teramo (A.P.A.V.)