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Quasi-Order Hypergroups determinated by $\mathscr{T}$-Hypergroups

Šárka Hošková-Mayerová
University of Defence,
Faculty of Military Technology,
Department of Mathematics and Physics,
Kounicova 65, 612 00 Brno, Czech Republic
sarka.mayerova@unob.cz

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Abstract

Quasi-order hypergroups were introduced by Jan Chvalina in 90s of the twentieth century. He proved that they form a subclass of the class of all hypergroups, i.e. structures with one associative hyperoperation fulfilling the reproduction axiom. In this paper a theorem which allows an easy description of all quasi-order hypergroups is presented. Moreover, some results concerning the relation of quasi-order and upper quasi-order hypergroups are given. Furthermore, the transformation hypergroups acting on tolerance spaces are defined and an example of them is mentioned.
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The applications of mathematics in other disciplines, for example, in informatics, play a key role and they represent, in the last decades, one of the purpose, of the study of the experts of hyperstructures theory all over the world. Hyperstructure theory was introduced in 1934 by the French mathematician Marty [16], at the 8th Congress of Scandinavian Mathematicians, where he defined hypergroups based on the notion of hyperoperation, began to analyze their properties, and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Several books have been written on hyperstructure theory, see [6, 10, 17]. A recent book on hyperstructures [9] points out on their applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [10] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: hyperstructures and transposition hypergroups.

Hypergroups in the sense of Marty [16] form the largest class of multivalued systems that satisfies group-like axioms. It should be noted that various problems in non-commutative algebra lead to the introduction of algebraic systems in which the operations are not single-valued. The motivation for generalization of the notion of group resulted naturally from various problems in non-commutative algebra, another motivation for such an investigation came from geometry. Hypergroups have been used in algebra, geometry, convexity, automata theory, combinatorial problems of coloring, lattice theory, Boolean algebras, logic etc., over the years. Over the following decades, new and interesting results again appeared, but it is above all that a more luxuriant flourishing of hyperstructures has been seen in the last 20 years. It is not surprising that hypergroups as well as hypergroupoids, quasi-hypergroups, semihypergroups, hyperfields, hyper vector spaces, hyperlattices etc. have been studied.

The most complete bibliography up to 2002 can be found in the monograph of Pierguilio Corsini and Violeta Leoreanu: Applications of Hyperstructure Theory [9]. Another comprehensive list of literature is in monograph [17] and updated information is included in web site: http://aha.eled.duth.gr.

In the paper [2] special types of hypergroups, so called *quasi-order hypergroups* ($\mathbb{QOHG}$) and *order hypergroups* ($\mathbb{OHG}$), were introduced (cf. also [6, 9, 14, 5]).

First of all recall some basic terms and definitions. A *hyperoperation* "$\circ$" on a nonempty set H is a mapping from $H \times H$ to $\mathscr{P}^*(H)$ (all nonempty subsets of

H). The *hypergroupoid* is a pair $(H, \circ)$. The *quasi-hypergroup* is a hypergroupoid if the reproduction axiom ($a \circ H = H = H \circ a$ for any $a \in H$) is fulfilled. The quasi-hypergroup $(H, \circ)$ is called a *hypergroup* if moreover the hyperoperation "$\circ$" is associative $\big((a \circ b) \circ c = a \circ (b \circ c)$ for any $a, b, c \in H\big)$. Here for nonempty $A, B \subseteq H$ we put $A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b$. We denote $a \circ B$ instead of $\{a\} \circ B$, $a \in H$. See, e.g. [5, 6, 9, 7, 11].

Let $(H, *)$ and $(H', \star)$ be hypergroupoids. Then a mapping $f\colon H \longrightarrow H'$ is called *inclusion homomorphism* if it satisfies the condition:

$$f(x * y) \subseteq f(x) \star f(y) \qquad \text{for all pairs } x, y \in H.$$

Let X be a set and τ be a tolerance relation (i.e., reflexive and symmetric binary relation)—see [1]. Then the pair (X, τ) is a *tolerance space*.

Definition 1. The hypergroup $(H, \circ)$ is called a *quasi-order hypergroup*—cf. [2, 4, 9]—if

(i) $a \in a^3 = a^2$ for any $a \in H$, (1)

(ii) $a \circ b = a^2 \cup b^2$ for any $a, b \in H$. (2)

The hypergroup $(H, \circ)$ is called an *order hypergroup* if moreover

(iii) $a^2 = b^2$ implies $a = b$ for any $a, b \in H$. (3)

Using the methods occurring in [2, 4] the following theorem characterizing all quasi-order hypergroups can be proved. For the prove see [13]. (By a^2 we mean $a \circ a$.)

Theorem 1. *Let $(H, \circ)$ is a quasi-order hypergroup. Denote $K(a) = a^2$ for any $a \in H$. Then the system of sets $K(a)$ fulfills the following conditions:*

(i) $a \in K(a)$ *for any* $a \in H$, (4)

(ii) *if* $b \in K(a)$ *then* $K(b) \subseteq K(a)$. (5)

Conversely, if any system of subsets $K(a)$ of the set H, $a \in H$, fulfills (4) *and* (5)*, then there exists the only hyperoperation "$\circ$" on H such that $a \circ a = K(a)$ and $(H, \circ)$ is a quasi-order hypergroup.*

With respect to (3) the following corollary evidently holds:

Corollary 1. Under the assumptions of Theorem 1 the quasi-order hypergroup $(H, \circ)$ is an order hypergroup if and only if for $a \neq b$ there is $K(a) \neq K(b)$.

It is easy to show that if R is a quasi-ordering on a set H, then the pair $(H, \circ)$, where $a \circ b = R(a) \cup R(b)$, $a, b \in H$, is a quasi-order hypergroup. ($R(x)$ is an upper end of an element $x \in H$, i.e. the set $\{a \in H; a\,R\,x$ for each element $a \in H\}$). See e.g. [3, 11].

In [3] J. Chvalina introduced the concept of an upper quasi-order and upper order hypergroup.

Definition 2. A hypergroup $(H, \circ)$ is said an *upper quasi-order (upper order) hypergroup* if there exists a quasi-ordering (ordering) R such that $a \circ b = R(a) \cup R(b)$ for $a, b \in H$.

It can be shown that the classes of all quasi-order hypergroups and upper quasi-order hypergroups coincide. The same is true for the classes of all order hypergroups and upper order hypergroups. See [2, Theorem 1] or [9, Proposition 2 on p. 96]. These results can be easily proved using Theorem 1.

As we will need the above mentioned result of Prof. Jan Chvalina several times in this text we recall its formulation:

Proposition 1. [9, Proposition 2 on p. 96] A hypergroupoid $(H, \cdot)$ is a (quasi)-order hypergroup if and only if there exists a (quasi)-ordeg ρ on the set H, such that

$$\forall (a, b) \in H \times H, \qquad a \cdot b = \rho(a) \cup \rho(b),$$

where $\rho(a) = \{x \in H, a\,\rho\,x\}$.

Theorem 2. *Every quasi-order (order) hypergroup is an upper quasi-order (upper order) hypergroup.*

Proof. Let $(H, \circ)$ be a quasi-order hypergroup. Let us define a relation R on H as follows: $a\,R\,b$ iff $b \in a^2$ for each $a, b \in H$. Evidently $a\,R\,b$ iff $b^2 \subseteq a^2$. Then (4) and (5) imply that R is a quasi-ordering. Moreover, $R(a) = a^2$. Thus $a \circ b = a^2 \cup b^2 = R(a) \cup R(b)$.

If $(H, \circ)$ is even an order hypergroup, then by Corollary 1 there is $R(a) \neq R(b)$ for $a \neq b$, $a, b \in H$. Thus R is an ordering. □

In [12] a more general concept of *subquasi-order hypergroup* is introduced. It is an open question whether a similar representation result as in Theorem 1 can be found for this generalization.

Now let us recall the definition of a transformation hypergroup. It was introduced in [15].

Recall first that *tolerance relation* is a reflective and symmetric relation on a set. This relation yields the concept of singularity in abstract mathematical expressions. This relation namely in connection with other structures moves corresponding mathematical theories to useful applications. Many publications are devoted to systematic investigation to tolerances on algebraic structures compatible with all operations of corresponding algebras. A certain survey of important results including valuable investment can be found in [1]. Tolerance space is a set endowed with a tolerance relation.

Definition 3. Let X be a set, $(G, \bullet)$ be a hypergroup and $\pi\colon X \times G \to X$ a mapping such that $\pi(\pi(x,t),s) \in \pi(x, t \bullet s)$, where

$$\pi(x, t \bullet s) = \{\pi(x,u); u \in t \bullet s)\}$$

for each $x \in X$, $s,t \in G$.

Then the triple $\mathscr{T} = (X, G, \pi)$ is called a *discrete transformation hypergroup* or an action of the hypergroup G on the phase set X. The mapping π is also usually said to be simply an action.

More generally, it is possible to consider the situation, where the phase space X is endowed with some additional structure. The interesting case is given in the following definition.

Definition 4. Let (X, τ) be a tolerance space (so called phase tolerance space), $(G, \bullet)$ be a semihypergroup (so called phase semihypergroup) and $\pi\colon X \times G \to X$ a mapping such that

(i) $\pi(\pi(x,t),s) \in \pi(x, t \bullet s)$, where $\pi(x, t \bullet s) = \{\pi(x,u); u \in t \bullet s)\}$ for each $x \in X$, $s,t \in G$;

(ii) if $x, y \in X$ are such that $x \,\tau\, y$, then $\pi(x,g) \,\tau\, \pi(y,g)$ holds for any $g \in G$.

Then $\mathscr{T} = (X, G, \pi)$ is a *transformation semihypergroup with phase tolerance space*. If, moreover, the pair $(G, \bullet)$ is a hypergroup (so called phase hypergroup), then the triple $\mathscr{T} = (X, G, \pi)$ is a *transformation hypergroup with phase tolerance space*.

In case the tolerance τ is trivial, i.e., $x \,\tau\, y$ if and only if $x = y$, the preceding definition coincides in fact with Definition 3.

Let us consider a discrete transformation hypergroup $\mathscr{T} = (X, G, \pi)$. It is possible to assign to each transformation hypergroup a commutative, extensive hypergroup with the support X (i.e., phase set of $\mathscr{T}$) as follows:

Let us define for arbitrary pair of elements $x, y \in X$ a binary hyperoperation $\odot\colon X \times X \to \mathscr{P}^*(X)$ in this way:

$$x \odot y = \pi(x, G) \cup \pi(y, G) \cup \{x, y\},$$

where $\pi(x, G) = \{\pi(x,u), u \in G\}$ and similarly for $\pi(y, G)$.

In the following we will need the next Lemma. The proof can be found in [4].

Lemma 1. *A hypergroupoid $(H, \cdot)$ such that $a \in a^3 \subset a^2$, $a \cdot b = a^2 \cup b^2$ for any $a, b \in H$ is a quasi-order hypergroup.*

Proposition 2. The pair $(X, \odot)$ is an extensive, commutative hypergroup.

The extensivity and commutativity of the hyperoperation is evident, so the pair $(X, \odot)$ is an extensive, commutative hypergroupoid. The conditions of Lemma 1 are satisfied too, so $(X, \odot)$ is an extensive, commutative hypergroup.

Remark 1. Even in case when $\mathscr{T}$ is a transformation semihypergroup we can assign a commutative, extensive hypergroup to this semihypergroup by the above described way.

The considered mapping is a functorial assignment which is described in the following way:

The above defined assignment determines a functor F from the category $\mathbb{DTH}$ of all discrete transformation hypergroups into the category $\mathbb{AH}$ of all commutative (abelian) hypergroups.

The functor $F = (F_O, F_m)$ (O-as objects, m-as morphisms) is defined as follows: $F_O(\mathscr{T}) = (X, \odot)$; $F_m(h_X, h_G) = h_X$. Consider $\mathscr{T}_i = (X_i, G_i, \pi_i) \in \mathbb{DTH}$ where $(X_i, \odot_i)$ are hypergroups, $i = 1, 2$ and the morhpisms $h_X \colon X_1 \to X_2$. Then

$$\begin{aligned} h_X(x \odot_1 y) = h\big(\pi(x, G) \cup \pi(y, G) \cup \{x, y\}\big) = & \big\{\pi(h_X(x), h_G(g), g \in G_1)\big\} \\ & \cup \big\{\pi(h_X(y), h_G(g), g \in G_1)\big\} \cup \big\{h_X(x), h_X(y)\big\} \subseteq \\ & \pi(h_X(x), G_2) \cup \pi(h_X(y), G_2) \cup \{h_X(x), h_X(y)\} \\ & = h_X(x) \odot_2 h_X(y) \end{aligned}$$

holds for all $x, y \in X_1$.

Theorem 3. *The pair $(X, \odot)$ is a quasi-order hypergroup determined by $\mathscr{T}$, shortly quasi-order $\mathscr{T}$-hypergroup.*

Proof. Let us define on $(X, \odot)$ a binary relation "ρ" as follows:

$$x \,\rho\, y \Leftrightarrow \exists\, u \in G \text{ such that, } \pi(x, u) = y \text{ or } x = y.$$

This relation is evidently reflexive. We will show, that it is transitive as well.
Let

1) $x = y$ and $y = z$, then $x = z$ and $x \,\rho\, z$,
2) $x = y$ and $\pi(y, v) = z$, then $\pi(x, v = z)$ so $x \,\rho\, z$,
3) $\pi(x, u) = y$ and $y = z$, then $\pi(x, u = z)$ so $x \,\rho\, z$,

4) $\pi(x,u) = y$ and $\pi(y,v) = z$, then $z = \pi(y,v) = \pi(\pi(x,u),v)$. From Definition 4 we have $\pi(\pi(x,u),v) \in \pi(x, u \odot v)$, thus there exists $w \in u \odot v$ such that $z = \pi(x,w)$. Hence we have $x\,\rho\,z$. So ρ is a quasi-order. It is well known that $\rho^2 = \rho \supset \operatorname{diag}(X)$, where $\operatorname{diag}(X) = \{(x,x); x \in X\}$.

Now for any pair of elements $x, y \in X$ we get $x \odot y = \rho(x) \cup \rho(y)$. So according Proposition 1 the pair $(X, \odot)$ is a quasi-order hypergroup determined by $\mathscr{T}$. Shortly it is a quasi-order $\mathscr{T}$-hypergroup. □

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