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Multi-criteria media mix decision model for advertising multiple product with segment specific and mass media

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Abstract

Judicious media planning decisions are crucial for successful advertising of products. Media planners extensively use mathematical models supplemented with market research and expert opinion to devise the media plans. Media planning models discussed in the literature largely focus on single products with limited studies related to the multi-product media planning. In this paper we propose a media planning model to allocate limited advertising budget among multiple products advertised in a segmented market and determine the number of advertisements to be given in different media. The proposed model is formulated considering both segment specific and mass media vehicles to maximize the total advertising reach for each product. The model also incorporates the cross product effect of advertising of one product on the other. The proposed formulation is a multi-objective linear integer programming model and interactive linear integer goal programming is discussed to solve the model. A real life case study is presented to illustrate the application of the proposed model.

Key words: Multiple products, Mass advertising, Segment Specific advertising, Spectrum effect, Media Planning, Multi-objective decision making, Interactive Approach.

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1 Introduction

A firm's market share and profit are driven by consumer demand and spending. Advertising carried by the firms to promote their products play a crucial role in fuelling consumer demand. It is through media that consumers receive advertising messages. It acts as a link between the advertisers and the consumers. Media such as television, radio, newspapers, magazines, and the internet act as distributors of the advertising messages. Media planning is a challenging process and the media choices are made such that the advertising objectives are met. The goal of a media planner is to reach the target audience with the right message through the right media. Advertising reach and frequency are the critical elements in setting up a media plan [19]. This study proposes a mathematical programming media allocation model to maximize the advertising reach of a firm that markets multiple products advertised through different media in a segmented market.

There are two major aspects of media planning, viz. selection of the media and allocation of the advertising budget. A media planner focuses on reaching its target customers with a right message that can convert them into potential buyers. The target market of a product can be taken as uniform or it can be bifurcated in to various segments based on the customer profile characteristics. When the market is considered as uniform, the advertising is carried at the mass level through the media that could reach all the customers. Though, the customers in the target market possesses some common characteristics that identify them as the potential customers still there exist differences in how they respond to the products and the advertising messages. If the product is advertised only at the mass level with a uniform advertising strategy, due to the differential behaviour of the potential market customers it may not be effective in influencing the customers to buy the product. In the recent years firms have tried to reach its customers with advertising that is tailored with respect to their individual characteristics so that the advertising not only reaches them but also convert them into potential buyers. Segmentation is an important concept of marketing that helps the advertisers to develop a media plan with respect to the customer's characteristics. Given the importance of segment driven marketing, importance of mass marketing can't be undermined as it creates a wider spectrum of reach. Hence the marketers choose to adopt the advertising strategy such that the product is advertised using mass media and also with segment driven advertising media. The reach obtained in segments can thus be obtained both from segment specific advertising and mass advertising. The model proposed in this paper incorporates this idea and develops a media plan that allocates advertising budget for both mass and segment specific media.

Companies are increasingly extending their products into product lines that are related or fall into distant categories. Marketing product lines instead of single product gives a competitive edge to the firms. It helps in meeting the diversified demand of products that are related which customer tend to use together and also provides a variety to the customers. Firms have limited resources in terms of value that can be used for advertising. For the case of single product advertised in a segmented market, the segments compete for media budget allocation among themselves and with mass media allocation. If a firm markets several products the competition for advertising budget first exists between the products and then at the segment and mass level. At any instant of time if several products are marketed by a firm advertising reach of an individual product no longer remains independent of other products. Due to substitution or complimentary effect that one product may have on other the advertising reach is also affected. Very limited research has investigated media planning model for multiple products jointly [16] .

In this paper we propose a multi-objective linear integer media planning model to allocate advertising budget between several products marketed by a firm through various media in a segmented market. The model allocates media budget and also determines the number of advertisements for each product, in all chosen media both at segment and mass level. It also incorporates cross product effect of advertising among products and maximizes the total advertising reach taking all products together. Interactive goal programming technique is discussed to solve the formulated model.

The paper is organized as follows: literature review is carried in section 2. In section 3, the model formulation and solution procedure are discussed. A case study is presented in section 4 illustrating the solution methodology. Concluding remarks are made in section 5.

2 Literature Review

The researchers have worked on various aspects of media planning such as the models for media selection, models concerning the "timing" aspect, market segmentation studies, budget allocation models, media scheduling models, media effectiveness models.

Broadbent [3] presented a review of the simulation and optimization procedures for the media planning models. The author discussed a number of media planning models and classified them into two approaches: mathematical model approach and algorithmic approach. A linear programming media allocation model was proposed by Bass and Lonsdale [1] with an objective to maximize the media exposures for one product for a single time unit. Authors explored the influence of several types of constraints on the model solution. Little and Lodish [14] formulated a media planning model based on a heuristic search algorithm to select and schedule media maximizing the total market response in different segments over the several time periods. Zufryden [20] developed media planning models with an objective of maximizing sales and determine the optimal media schedule over time. They considered stochastic response behavior in the objective function and later developed heuristics for solving the model [21]. Dwyer and Evans [7] proposed an optimization model for to select the best set of mailing lists in the direct mail advertising maximizing the proportion of customers reachable with direct mail pieces. The formulated binary integer model is solved through the branch and bound algorithm.

Korhonen et al. [12] proposed an evolutionary approach to media selection model. The model constraints and objective have interchangeable role in this approach. The iterations are performed for different set of objectives and constraints, computing the decision maker's value function in each iteration. Then the value function most suited to the decision maker is chosen as the solution. The study was carried out for a software company in Finland. Doyle and Saunders [6] developed a model to determine the spending on the promotion of multiple products for a retail store. The model optimally allocates budget to the promotional campaigns where each campaign is for a specific product. They considered cross product effect of advertising campaigns that lags or leads a particular campaign for up to four periods. A logarithmic linear regression model was proposed by the authors. Danaher and Rust [5] developed a model with an objective of maximizing the return on investment considering the diminishing return on the advertising and calculated the optimal amount of expenditure on the media campaign.

A media planning model based on the analytic hierarchy process was developed by Kwak et al. [13]. The model is developed to allocate the budget in the media categories and determine the number of advertisements for different media categories for digital products. Three criterion customer, advertising and budget were considered to be fulfilled through the model. The solution methodology based on pre-emptive goal programming technique was used. Buratto et al. [4] analyzed the media selection problem to choose an advertising channel for the pre-launch campaign for a new product (as cited in [11]). Authors considered a segmented market with several advertising channels that have different diffusion spectra and efficiencies. The problem is analyzed in two steps. First, an optimal control problem is solved explicitly in order to determine the optimal advertising policy for each channel. Then a maximum profit channel is chosen. They discussed a simulation where the choice of a newspaper among six Italian newspapers is presented.

Grosset and Viscolani [8] proposed a dynamic profit maximizing advertising model comparing the model performance under two strategies viz. 1) single medium advertising for a segmented market, that reaches segments with the same message but with varying effectiveness and 2) advertising independently for each segment through a single segment specific medium. The profit is measured in terms of goodwill where the growth of goodwill depends on the advertising effort and the goodwill decays due to forgetting of the advertised brand. Viscolani [18] proposed a non-linear programming advertising model for a segmented market to select a set of advertising media with an objective of maximizing profit. Using the approach similar to the Grosset and Viscolani [8] they suggested to use multiple media.

Hsu et al. [9] gave a fuzzy model using genetic algorithm to determine the optimum advertising mix and the number of insertions in different promotional instruments based on linguistic preferences of the domain experts. Bhattacharya [2] proposed an integer programming model to determine the optimal number of insertions in different media with an objective of maximizing the reach to the target population for a single product. Jha et al. [10] extended the model for the multiple products and a segmented market. Saen [17] proposed a model for the selection of media through the approach of data envelopment analysis in presence of flexible factors and imprecise data. Royo et al. [16] proposed an advertising budget allocation model for multiple products considering cross elasticity of products. They optimised the investment on advertising in multiple media for multiple products. This model was further extended by Royo et al. [16] under stochastic environment. Jha et al. [11] proposed an integer linear programming model of media planning for a single product advertised with multiple media with mass and segment specific advertising strategies. The model is developed with reach maximization objective.

As discussed above an extensive literature has been developed on optimization of media planning decisions. Most of the researchers have focused on media planning models for single product. In the present age, firms market several products simultaneously to provide product variety to the customer. The advertisement budget is to be divided among the products judiciously. In case of multi-product offering it is also observed that the one product advertising affects the advertising of other product[16]. The effect can either be substitutive or complimentary. It is important to measure and take account of this effect in media planning decisions. This necessitates joint media planning for the range of products such that the advertising budget can be shared between the products judiciously at the same time accounting for the cross-product effect of advertising which is considered in this paper. The study carried also integrates concept of media planning for multiple products with segmentation aspect. Another distinguished feature of the study is that we consider two types of advertising strategies viz. mass and segment specific in the model development. This differentiation between advertising strategies has been recently carried in some recent studies [11]. Both strategies affect advertising message reach in the potential market in different manner. While the mass advertising spread reach over the entire market widely, segment specific advertising plays crucial role in targeting segments.

The model developed in this paper maximizes the total reach of all the products taking in to consideration budgetary restrictions and bounds on the decision variables. The reach function is formulated considering the cross product effect of advertising. The model is tested on a real life case study.

3 Model Development

3.1 Notation

i	index for segments ($i = 0, 1, ..., N$)
j	index for advertising media ($j = 1, 2, ..., M_i$)
k	index for media options ($k = 1, 2, ..., K_{ij}$)
l	index for slot in a media ($l = 1, 2, ..., L_{ij}$)
p	index for products ($p = 1, 2, ..., P$)
q	index for customer profile characteristics ($q = 1, 2, ..., Q$)
jkl	j^{th} media, k^{th} media option, l^{th} slot
a^p_{ijkl}	reach per advertisement for p^{th} product in i^{th} segment, jkl^{th} media driver
C_{ijkl}	average number of readers/viewers of jkl^{th} media driver in segment i
c_{ijkl}	cost of inserting one advertisement in jkl^{th} media driver in segment i
v^p_{ijkl}	lower bound on the number of advertisements in jkl^{th} media driver of segment i for p^{th} product
u^p_{ijkl}	upper bound on the number of advertisements in jkl^{th} media driver of segment i for p^{th} product
x^p_{ijkl}	decision variable denoting the number of advertisements to be given in jkl^{th} media driver of segment i for p^{th} product
e^p_{irjkl}	percentage of people who follow jkl^{th} media driver in segment i, and are p^{th} product's potential customers possessing r^{th} profile characteristic.
α^p_{ijk}	spectrum effect of k^{th} media option of j^{th} mass media vehicle on i^{th} segment for p^{th} product; ; $0 < \alpha^p_{ijk} < 1$
w_{rp}	relative importance of r^{th} customer profile characteristic for p^{th} product
r	minimum proportion of budget allocated for mass advertisement
G	total advertising budget
Z_p	total reach of p^{th} product
A_p	reach component solely due to advertisement of product p
θ_{pf}	constant of proportionality representing CPE of advertising of product p on reach of product f

3.2 Model Formulation

Assuming a firm markets P products in a segmented market and the segments index vary from 1 to N and index 0 represents the mass media.

The mathematical model to maximize the total reach of advertising for each product through the mass and segment specific media is formulated as follows

$$\text{Vector Maximize} Z = [Z_1, Z_2, ..., Z_P]^T \tag{1}$$

subject to

(P1)

$$\sum_{p=1}^{P}\sum_{i=0}^{N}\sum_{j=1}^{M_i}\sum_{k=1}^{K_{ij}}\sum_{l=1}^{L_{ij}} c_{ijkl}x_{ijkl}^{p} \leq G \tag{2}$$

$$\sum_{p=1}^{P}\sum_{j=1}^{M_0}\sum_{k=1}^{K_{0j}}\sum_{l=1}^{L_{0j}} c_{0jkl}x_{0jkl}^{p} \geq rG \tag{3}$$

$$x_{ijkl}^{p} \geq v_{ijkl}^{p}\ \forall p = 1,2,...P; i = 0,1,2,\ldots N; j = 1,2,\ldots M_i; k = 1,2,\ldots K_{ij}; l = 1,2,\ldots L_{ij} \tag{4}$$

$$x_{ijkl}^{p} \leq u_{ijkl}^{p}\ \forall p = 1,2,...P; i = 0,1,2,\ldots N; j = 1,2,\ldots M_i; k = 1,2,\ldots K_{ij}; l = 1,2,\ldots L_{ij} \tag{5}$$

$$x_{ijkl}^{p} \geq 0 \text{ and integers}$$
$$\forall p = 1,2,...P; i = 0,1,2,\ldots N; j = 1,2,\ldots M_i; k = 1,2,\ldots K_{ij}; l = 1,2,\ldots L_{ij} \tag{6}$$

where

$$Z_p = A_p + \sum_{\substack{f=1\\ f\neq p}}^{P} \theta_{pf}A_f \tag{7}$$

$$A_p = \sum_{i=1}^{N}\left(\sum_{j=1}^{M_i}\sum_{k=1}^{K_{ij}}\sum_{l=1}^{L_{ij}} a_{ijkl}^{p}x_{ijkl}^{p} + \sum_{j=1}^{M_0}\sum_{k=1}^{K_{0j}}\sum_{l=1}^{L_{0j}} \alpha_{ijkl}^{p}\left(a_{0jkl}^{p}x_{0jkl}^{p}\right)\right) \tag{8}$$

$$a_{ijkl}^{p} = \left\{\sum_{r=1}^{R} w_{rp}\, e_{irjkl}^{p}\right\} C_{ijkl} \tag{9}$$

Equation (1) represented by Z is a vector of objective functions with the components denoting the advertising reach of each product p. Component of Z denoted by Z_p (expressed mathematically as (7)) represents the combined reach from advertising for the product p and the cross product effect from advertising of other products. Where the reach expected to obtain from advertising for the product p is expressed as A_p (given by (8)). The individual advertising reach of each product as given by equation (8) is the sum of the reach from segment specific advertising and spectrum effect of the mass advertising in the segments. The per unit advertisement reach as given in equation (9) is the product of the readership/viewership of the media driver and the relative proportion of potential customers among them.

Equation (2) represents the budgetary constraint. Knowing the importance of mass advertising it is likely that media planner specify a lower bound on the budget to be spent on mass advertising as otherwise very little budget may be allocated to the mass media. Equation (3) represents the lower bound constraint on the mass media budget allocation. Constraint (4) and (5) are the lower and upper bounds specified by the media planner on the number of advertisements in different media for different products to ensure the diversity in advertising budget allocations rather than allocating the entire budget to some specific set of media. Constraint (6) imposes the decision variable to take integral values.

In the literature authors have suggested to formulate evolutionary model [12] wherein constraints and objectives roles can interchange. This allows flexibility to the decision maker, tradeoff the model variables and ensures that an efficient solution is obtained. In this direction in order to obtain an efficient solution and ensure some minimum reach for every product first we solve the model (P1) for each reach objective one by one to obtain the advertising reach aspirations for all products. These aspirations are set as lower bound constraints on reach objective and the resulting model is formulated as follows

$$\begin{aligned}&\text{Vector Maximize} Z = [Z_1, Z_2, ..., Z_P]^T\\&\text{subject to constraints (2)-(6) and}\\&Z_p \geq Z_p^* \ \forall p = 1, 2, ...P\end{aligned} \tag{P2}$$

Weighted sum multi-objective model using scalar weights μ_p; $\sum \mu_p = 1$; $(p = 1, 2, ...P)$ according to the relative importance of the products [15] is formulated using (P2) to obtain the media planning model as given in (P3)

$$\begin{aligned}&\text{Maximize} \sum_{p=1}^{P} \mu_p Z_p\\&\text{subject to constraints (2)-(6) and}\\&Z_p \geq Z_p^* \ \forall p = 1, 2, ...P\end{aligned} \tag{P3}$$

The weights in the model (P3) can be given by the decision maker or computed through the interactive approach (discussed in detail in [11]). The linear integer optimization model (P3) is solved by coding on LINGO optimization modelling software. The solution to model (P3) may result in infeasibility due to high aspirations on reach objective for products. Further a goal linear integer model is formulated for model (P2) to obtain a compromised solution and trade off the reach aspirations and budget.

Solution Methodology: Goal Programming

In goal programming, the solution is obtained such that the deviations from the goals are minimized. Deviations may be either positive or negative. Problem (P2) is solved in two stages. In Stage 1 the model is solved to minimize the deviations of the rigid constraints and in the second stage goal deviations are minimized incorporating the solution of first stage. The formulations of the two stages of goal programming are given as follows

Stage 1

$$\text{Minimize } \rho_1 + \eta_2 + \sum_{p=1}^{P}\sum_{i=0}^{N}\sum_{j=1}^{M_i}\sum_{k=1}^{K_{ij}}\sum_{l=1}^{L_{ij}} \eta^p_{ijkl} + \sum_{p=1}^{P}\sum_{i=0}^{N}\sum_{j=1}^{M_i}\sum_{k_j=1}^{K_{ij}}\sum_{l_j=1}^{L_{ij}} \rho'^p_{ijkl} \quad \text{(P4)}$$

subject to constraints

$$\sum_{p=1}^{P}\sum_{i=0}^{N}\sum_{j=1}^{M_i}\sum_{k=1}^{K_{ij}}\sum_{l=1}^{L_{ij}} c_{ijkl} x^p_{ijkl} + \eta_1 - \rho_1 = A \tag{10}$$

$$\sum_{p=1}^{P}\sum_{j=1}^{M_0}\sum_{k=1}^{K_{0j}}\sum_{l=1}^{L_{0j}} c_{0jkl} x^p_{0jkl} + \eta_2 - \rho_2 = rA \tag{11}$$

$$x^p_{ijkl} + \eta^p_{ijkl} - \rho^p_{ijkl} = v^p_{ijkl}$$
$$\forall p = 1,2,...P; i = 0,1,2,\ldots N; j = 1,2,\ldots M_i; k = 1,2,\ldots K_{ij}; l = 1,2,\ldots L_{ij} \tag{12}$$

$$x^p_{ijkl} + \eta^p_{ijkl} - \rho^p_{ijkl} = u^p_{ijkl}$$
$$\forall p = 1,2,...P; i = 0,1,2,\ldots N; j = 1,2,\ldots M_i; k = 1,2,\ldots K_{ij}; l = 1,2,\ldots L_{ij} \quad (13)$$

$$x^p_{ijkl} \geq 0 \text{ and integers}$$
$$\forall p = 1,2,...P; i = 0,1,2,\ldots N; j = 1,2,\ldots M_i; k = 1,2,\ldots K_{ij}; l = 1,2,\ldots L_{ij} \quad (14)$$

$$\eta^p_{ijkl}, \rho^p_{ijkl}, \eta'^p_{ijkl}, \rho'^p_{ijkl} \geq 0$$
$$\forall p = 1,2,...P; i = 0,1,2,\ldots N; j = 1,2,\ldots M_i; k = 1,2,\ldots K_{ij}; l = 1,2,\ldots L_{ij} \quad (15)$$

$$\eta_i, \rho_i \geq 0 \forall i = 1,2 \quad (16)$$

Stage 2

$$\begin{aligned} &\text{Minimize } g(\eta,\rho,X) = \sum_{p=1}^{P} \lambda_{p+2}\eta_{p+2} \\ &\text{subject to constraints (10)-(15) and} \\ &Z_p + \eta_{p+2} - \rho_{p+2} = Z^*_p \quad \forall p = 1,2,...P \\ &\eta_i, \rho_i \geq 0 \quad \forall i = 1,2,...,(P+2) \end{aligned} \quad \text{(P5)}$$

where $g(\eta,\rho,X)$ is objective function of (P5) and η_{p+2}, ρ_{p+2}, are negative and positive deviational variables of goals for p^{th} product objective function.

4 Case Study

To illustrate the application of the proposed model a case study is presented in this section demonstrating the media planning decision of a firm marketing five products (P1-P5) in the market. The name of the firm has not been disclosed due to the commercial confidentiality. The firm has to devise an advertising plan for its products for a period of one quarter. On the basis of geographic segmentation, the market for all the products is divided into fourteen segments (say S1-S14). The company wants to promote all products at the mass level as well as at the segment level. The firm's potential market is described on the basis of demographic characteristics: gender and income level, that is the potential market to which these products are targeted to, are females belonging to middle class group.

For segment level advertising in each segment, up to four newspapers (RNP1-RNP4), and two television channels (RCH1, RCH2) are selected. For the mass advertising four newspapers (NNP1-NNP4), and two television channels (NCH1, NCH2) are selected. Each of these media is chosen based on the potential market preferences, expert opinion and the market research. Further in each media there are different slots, such as in case of newspapers we can advertise on front page (FP) and/or other pages (OP). Similarly in case of television, slots can be classified as prime time (PT) and other time (OT). The total budget given by the firm for the media planning is Rs. 800 millions. The minimum proportion of the budget allocated to mass media is set as 30 %.

The data given by the firm is confidential and used with appropriate rescaling (given in Tables 3-12 in the appendix). The potential customer profile matrix corresponding to each media is computed for all segments by conducting a survey of on a sample. The percentage profile matrix computed for product 1 is given in Table 3. Similarly profile matrices are computed for all products. The weights defining relative importance of the potential customer profile characteristics gender and income level is given in Table 4. The values of the relative importance are inferred from the primary and secondary data with expert opinion. The cross product effect coefficient matrix is shown in Table 5. Table 6 gives the spectrum effect coefficient of the mass media on the various segments of the potential market.

The cost of advertisement in newspapers is measured in per square cm and an advertising space of 4cm x 6cm is considered. In case of television advertisement rates are given per 10 second slot and 30 second advertisement duration is preferred by the media planner. The advertising costs used in the study are given in Table 7. The media planner has also provided the lower and upper bounds on number of advertisements to be given for different products in different media as given in Table 8-12. These bounds are set to ensure the minimum visibility of ads in every media and distribute the advertising resources judiciously such that all chosen media can be used for advertising.

The optimization model (P1) is coded on LINGO optimization modelling software. In order to compute the target goals on the reach objectives for each product, first model (P1) is solved for each of the five products as a single objective model taking reach objective of one product at a time. The branch and bound method is used in the software to solve the model. Using these aspirations as the lower bounds on reach objectives for all products, the media planning model (P3) is coded. As the scalar weights of relative importance of product are not known, so we use interactive technique (for details of the method reader may refer [11]) to determine these weights. For the first iteration of interactive technique, 125 ($=V$) dispersed weighing vectors are generated randomly such that the components of each vector lie in the range $[0, 1]$ and the sum of all the components of each vector is equal to one. Taking a suitable value of d (computed using mathematical expression given in the algorithm) and through forward filtering approach 10 non-dominated distinct vectors (W) are filtered with L_2 metric distances between each set of vector. The problem (P3) is solved for all these 10 filtered weighing vectors. The model shows infeasibility with these filtered weighting vectors. Thus we form an interactive weighted sum goal programming model for (P1) to obtain a compromised solution using the reach targets as goals on the reach objectives.

The goal programming model is solved in two stages. In stage 1, the deviations corresponding to the rigid constraints are minimized and in stage 2 the deviations from the reach goals are minimized. First, the model (P4) is coded and solved in LINGO. In the next stage of goal programming, model (P5) is coded incorporating the solution obtained in stage 1. The weights given to the reach deviations are determined using interactive approach. Using the ten non-dominated distinct vectors generated earlier, the problem (P5) is solved 10 times. The solution and the objective function values are tabulated for all the runs and 5 ($=P$) best criterion vectors are filtered from 10 runs which are presented to the decision maker. On discussion with the decision maker, most preferred solution is selected. Using the weighing vector corresponding to the selected solution, the reduction factor is calculated and a new interval is formed between which new generation of weighting vectors is generated and the iteration is repeated. Five iterations of the interactive approach is carried based on the termination criteria ($t \lesssim k$) of the algorithm. Note that the parameters of the interactive algorithms are defined in Jha et al. [11] and same notations are used in this paper. All the calculations are carried out on a computing device with Intel Core Duo 1.40 GHz processor and 4 GB RAM. The average time taken to solve each problem is 2-4 seconds.

It can be seen from solution in Table 1 that as we move from iteration 1 to iteration 5, the total reach obtained from all the five products together improves. But the percentage change in the total objective function value decreases in successive iterations (except one iteration). As per the termination criteria of the interactive algorithm should converge in five iterations and we can see that the solutions of iteration 4 and 5 are very close to each other (% change=.09%), so the algorithm is terminated in five iterations.

The budget is fully utilized with 24.27 % of the total budget allocated to newspaper and the rest of 75.73% to TV. With these budget allocations among media it is expected to obtain approximately 20% of the reach from newspaper advertising and rest 80% from TV advertising. The distribution of budget among mass and segment level media is 31% and 69% (approx.) respectively. The product wise percentage allocation of the total budget and expected reach is given in Table 2. The optimal number of advertisements for different media for all the products is given in Table 13-17 in the appendix.

Table 1: Iteration parameters and the solution obtained

		Iteration 1 ($h=0$)	Iteration 2 ($h=1$)	Iteration 3 ($h=2$)	Iteration 4	Iteration 5 ($h=4$)
Interval width	$[\underline{\lambda}_1^{h+1}, \overline{\lambda}_1^{h+1}]$	$[0, 1]$	$[0, .732]$	$[0, .536]$	$[0.056, .449]$	$[0.0715, .359]$
	$[\underline{\lambda}_2^{h+1}, \overline{\lambda}_2^{h+1}]$	$[0, 1]$	$[0, .732]$	$[0, .536]$	$[0, .392]$	$[0.101, .389]$
	$[\underline{\lambda}_3^{h+1}, \overline{\lambda}_3^{h+1}]$	$[0, 1]$	$[0, .732]$	$[0, .536]$	$[0, .392]$	$[0.013, .301]$
	$[\underline{\lambda}_4^{h+1}, \overline{\lambda}_4^{h+1}]$	$[0, 1]$	$[0, .732]$	$[0.015, .552]$	$[0, .392]$	$[0.0848, .373]$
	$[\underline{\lambda}_5^{h+1}, \overline{\lambda}_5^{h+1}]$	$[0, 1]$	$[0, .732]$	$[0, .536]$	$[0.075, .468]$	$[0.0104, .298]$
D		0.066	0.0545	0.0536	0.0445	0.026
Reduction Factor		0.732	0.536	0.392	0.2877	0.2108
Vector Selected		Vector 1 [0.1039 0.1854 0.2556 0.1966 0.2584]	Vector 5 [0.1624 0.1752 0.1178 0.2834 0.2611]	Vector 1 [0.2524 0.1577 0.1956 0.1230 0.2713]	Vector 1 [0.2154 0.2447 0.1569 0.2287 0.1543]	Vector 39 [0.2025 0.1600 0.2225 0.2485 0.1725]
Reach		1858245000	1882864000	1899541000	1916792000	1918657000
% increase in Reach		-	1.32%	0.88%	0.91%	0.09%

Table 2: Product wise allocations from iteration 5

Products	Reach Achieved	Reach aspired	% reach achieved from aspired	% budget utilized
P1	641926400	720325700	89%	0.33%
P2	313761900	469876000	67%	0.16%
P3	572235900	607432200	94%	0.27%
P4	192188100	266835600	72%	0.13%
P5	198544300	310087000	64%	0.11%

5 Conclusion

A media planning model is proposed in this study to allocate advertising budget jointly among multiple products advertised in a segmented market. Media vehicles are chosen with respect to two types of advertising strategies namely, segment driven and mass media advertising. Segment specific media targets the segment potential while the mass media reaches the wider market with spectrum effect on the segments. The model determines the number of advertisement to be given in each of the media within the bounds suggested by media planner. When several products are advertised by a firm to serve the diverse need of a market, advertising of one product shows the cross product effect on other products. The study considers this effect in the model. Model applicability and solution methodology based on interactive linear integer goal programming is discussed with a case study and LINGO is used for computational support. The proposed model incorporates the cross product effect of advertising of a firms own products. Effect of competitive products can also be included in the future studies. The scope of the model is limited to media planning for a single period. The model can be further extended for dynamic media planning incorporating the retention and diminishing effect of advertising.

Appendix

Table 3: Customer percentage profile matrix for newspapers and television for product 1

Segment	RNP1				RNP2				RNP3				RNP4				RCH1				RCH2			
	gender		income		gender		income		gender		income		gender		income		gender		income		gender		income	
	FP	OP	FP	OP	FP	OP	FP	OP	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT	PT	OT	PT	OT
S1	0.29	0.13	0.12	0.04	0.35	0.08	0.12	0.06	0.2	0.14	0.19	0.06	0.24	0.13	0.15	0.09	0.23	0.19	0.15	0.12	0.27	0.17	0.13	0.08
S2	0.3	0.15	0.14	0.09	0.2	0.1	0.1	0.08	0.15	0.12	0.19	0.1	0.27	0.1	0.09	0.12	0.25	0.12	0.14	0.08	0.22	0.1	0.08	0.03
S3	0.29	0.14	0.16	0.07	0.19	0.07	0.12	0.04	0.17	0.15	0.18	0.09	0.22	0.1	0.1	0.05	0.32	0.14	0.21	0.13	0.27	0.13	0.14	0.09
S4	0.15	0.08	0.15	0.08	0.15	0.06	0.08	0.04	0.25	0.12	0.18	0.07	–	–	–	–	0.4	0.23	0.21	0.11	0.23	0.07	0.15	0.06
S5	0.27	0.17	0.2	0.1	0.1	0.05	0.07	0.03	0.15	0.11	0.18	0.06	0.33	0.09	0.07	0.06	0.37	0.15	0.22	0.08	0.18	0.06	0.12	0.03
S6	0.22	0.11	0.13	0.06	0.14	0.06	0.07	0.03	0.21	0.17	0.14	0.04	0.26	0.12	0.18	0.06	0.39	0.2	0.19	0.08	0.24	0.1	0.12	0.09
S7	0.3	0.18	0.18	0.08	0.24	0.14	0.2	0.12	0.26	0.17	0.13	0.05	–	–	–	–	0.3	0.2	0.13	0.09	0.16	0.07	0.11	0.08
S8	0.31	0.17	0.19	0.08	0.12	0.07	0.12	0.06	0.28	0.14	0.16	0.09	0.27	0.08	0.17	0.11	0.31	0.14	0.15	0.09	0.19	0.11	0.13	0.11
S9	0.26	0.12	0.16	0.06	0.25	0.1	0.16	0.07	0.21	0.13	0.17	0.1	–	–	–	–	0.28	0.2	0.17	0.11	0.25	0.16	0.18	0.1
S10	0.29	0.13	0.17	0.07	0.2	0.13	0.12	0.06	0.22	0.14	0.18	0.07	–	–	–	–	0.33	0.18	0.19	0.11	0.27	0.19	0.16	0.1
S11	0.28	0.13	0.13	0.08	0.2	0.13	0.15	0.07	0.22	0.17	0.15	0.09	–	–	–	–	0.31	0.13	0.13	0.1	0.25	0.2	0.14	0.09
S12	0.23	0.13	0.15	0.08	0.15	0.1	0.1	0.05	0.2	0.16	0.2	0.08	0.22	0.12	0.07	0.07	0.46	0.15	0.19	0.07	0.32	0.12	0.23	0.13
S13	0.28	0.13	0.18	0.08	0.23	0.1	0.13	0.07	–	–	–	–	–	–	–	–	0.33	0.17	0.3	0.22	0.27	0.19	0.23	0.11
S14	0.25	0.11	0.17	0.1	0.21	0.08	0.12	0.05	–	–	–	–	–	–	–	–	0.25	0.17	0.13	0.08	0.13	0.08	0.08	0.06
	RNP1				RNP2				RNP3				RNP4				RCH1				RCH2			
Mass	gender		income		gender		income		gender		income		gender		income		gender		income		gender		income	
Media	0.25	0.1	0.12	0.06	0.15	0.07	0.11	0.03	0.16	0.06	0.12	0.06	0.15	0.07	0.11	0.06	0.3	0.2	0.21	0.11	0.22	0.16	0.19	0.09

Table 4: Weights

	CR1	CR2
P1	0.65	0.35
P2	0.6	0.4
P3	0.36	0.64
P4	0.3	0.7
P5	0.55	0.45

Table 5: Cross Product Effect Matrix

Product	P1	P2	P3	P4	P5
P1	0	0.0109	0.034	0.02345	0.0034
P2	0.0234	0	0.0234	0.009	0.0054
P3	0.0195	0.0134	0	0.0156	0.00493
P4	0.0214	0.0093	0.0041	0	0.0067
P5	0.0145	0.011	0.0013	0.0078	0

Table 6: Spectrum effect coefficient of national newspapers and TV channels on regions

Segments	NNP1	NNP2	NNP3	NNP4	NCH1	NCH2
S1	0.09	0.12	0.1	0.12	0.1	0.11
S2	0.08	0.09	0.13	0.1	0.09	0.06
S3	0.12	0.09	0.09	0.15	0.12	0.1
S4	0.06	0.04	0.05	0.03	0.1	0.11
S5	0.07	0.07	0.08	0.09	0.03	0.04
S6	0.13	0.09	0.1	0.06	0.1	0.08
S7	0.03	0.06	0.04	0.07	0.04	0.05
S8	0.07	0.07	0.07	0.07	0.07	0.05
S9	0.1	0.14	0.1	0	0.04	0.04
S10	0.04	0.05	0.04	0.06	0.05	0.05
S11	0.06	0.06	0.05	0.02	0.04	0.05
S12	0.08	0.08	0.12	0.15	0.1	0.12
S13	0.04	0.02	0.01	0.05	0.05	0.05
S14	0.03	0.02	0.02	0.03	0.07	0.1

Table 7: Ad cost in different media

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	3750	1944	2423	1385	1400	1000	1719	1665	65480	26968	27548	12988
S2	2751	917	2221	1610	1650	900	1300	650	40400	19800	26400	12000
S3	3940	2225	2138	950	2040	1060	1285	1045	43628	15376	30464	13980
S4	1750	500	900	400	790	380	–	–	33800	12220	20908	9964
S5	3310	1572	2500	1200	1767	1010	1375	1100	19384	9408	14000	9100
S6	3800	2000	2331	1665	2200	1340	1400	900	45928	16480	41948	21472
S7	1200	600	1150	670	1100	550	–	–	8924	5948	6600	3200
S8	1200	600	1160	600	1000	550	900	500	14400	9000	8924	3964
S9	3700	1800	3960	2100	2500	1450	–	–	30980	12500	16700	9700
S10	1700	1000	1650	900	1450	850	–	–	17848	8956	14956	6980
S11	2500	1200	1640	1040	1100	870	–	–	8948	3980	5948	2980
S12	2920	1530	2100	1400	1100	890	2047	1575	41448	18984	30464	17476
S13	1800	900	1000	500	–	–	–	–	12250	6700	9945	4350
S14	700	527	595	424	–	–	–	–	34080	18900	21000	12340
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	9800	5640	8690	4250	6900	3540	5500	2900	104390	61019	86814	46570

Table 8: Upper and lower bounds on advertisements in different media for P1

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	[1, 22]	[12, 85]	[1, 15]	[12, 65]	[1, 20]	[12, 70]	[1, 12]	[11, 68]	[8, 36]	[18, 92]	[6, 39]	[15, 85]
S2	[1, 14]	[9, 50]	[1, 12]	[8, 41]	[1, 12]	[7, 42]	[1, 13]	[7, 48]	[7, 34]	[16, 88]	[4, 33]	[13, 73]
S3	[1, 24]	[11, 90]	[1, 16]	[9, 69]	[1, 20]	[10, 75]	[1, 15]	[8, 62]	[7, 39]	[19, 94]	[5, 38]	[14, 83]
S4	[1, 14]	[4, 56]	[1, 10]	[3, 44]	[1, 12]	[4, 42]	–	–	[6, 33]	[14, 78]	[6, 37]	[16, 81]
S5	[1, 20]	[7, 81]	[1, 15]	[5, 72]	[1, 18]	[6, 76]	[1, 13]	[4, 70]	[4, 31]	[8, 65]	[3, 25]	[7, 57]
S6	[1, 18]	[8, 76]	[1, 13]	[7, 61]	[1, 15]	[6, 65]	[1, 14]	[7, 60]	[7, 38]	[17, 85]	[5, 36]	[13, 79]
S7	[1, 12]	[6, 65]	[1, 11]	[5, 75]	[1, 14]	[6, 72]	–	–	[6, 31]	[10, 68]	[3, 31]	[10, 68]
S8	[1, 12]	[9, 49]	[1, 10]	[7, 45]	[1, 8]	[5, 40]	[1, 8]	[4, 38]	[7, 32]	[12, 72]	[3, 33]	[10, 72]
S9	[1, 18]	[10, 76]	[1, 14]	[11, 58]	[1, 14]	[8, 60]	–	–	[5, 33]	[12, 64]	[3, 29]	[8, 64]
S10	[1, 13]	[6, 49]	[1, 10]	[5, 40]	[1, 10]	[4, 42]	–	–	[4, 33]	[11, 62]	[3, 26]	[9, 59]
S11	[1, 19]	[9, 82]	[1, 14]	[7, 70]	[1, 14]	[6, 75]	–	–	[5, 34]	[12, 66]	[4, 27]	[10, 62]
S12	[1, 16]	[8, 71]	[1, 12]	[7, 55]	[1, 15]	[6, 71]	[1, 14]	[8, 63]	[8, 36]	[16, 79]	[7, 40]	[16, 86]
S13	[1, 12]	[4, 48]	[1, 9]	[3, 40]	–	–	–	–	[3, 31]	[11, 64]	[3, 25]	[12, 57]
S14	[1, 14]	[5, 64]	[1, 11]	[4, 40]	–	–	–	–	[5, 32]	[14, 85]	[5, 29]	[14, 65]
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	[1, 18]	[12, 84]	[1, 12]	[12, 64]	[1, 15]	[12, 70]	[1, 12]	[12, 64]	[8, 39]	[20, 94]	[8, 25]	[17, 86]

Table 9: Upper and lower bounds on advertisements in different media for P2

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	[1, 11]	[6, 42]	[1, 10]	[7, 36]	[1, 8]	[6, 30]	[1, 10]	[7, 32]	[7, 36]	[18, 77]	[5, 34]	[14, 65]
S2	[1, 8]	[4, 30]	[1, 6]	[3, 26]	[1, 7]	[4, 26]	[1, 6]	[4, 26]	[7, 38]	[16, 80]	[4, 32]	[13, 64]
S3	[1, 12]	[6, 49]	[1, 12]	[3, 31]	[1, 12]	[5, 35]	[1, 12]	[4, 32]	[8, 39]	[16, 82]	[6, 34]	[12, 66]
S4	[1, 8]	[4, 32]	[0, 7]	[2, 27]	[0, 7]	[2, 25]	–	–	[7, 29]	[13, 64]	[3, 35]	[12, 62]
S5	[1, 11]	[3, 49]	[1, 10]	[3, 32]	[1, 9]	[3, 36]	[1, 10]	[3, 31]	[5, 29]	[11, 65]	[3, 25]	[7, 59]
S6	[1, 13]	[5, 44]	[1, 12]	[4, 29]	[1, 12]	[4, 34]	[1, 12]	[3, 32]	[6, 33]	[15, 78]	[7, 33]	[10, 66]
S7	[1, 8]	[4, 29]	[0, 9]	[3, 21]	[1, 8]	[2, 25]	–	–	[7, 26]	[12, 63]	[3, 27]	[7, 52]
S8	[1, 9]	[6, 27]	[0, 8]	[4, 25]	[0, 6]	[3, 23]	[0, 6]	[3, 22]	[5, 28]	[12, 64]	[3, 31]	[8, 55]
S9	[1, 10]	[7, 45]	[1, 9]	[6, 42]	[1, 9]	[5, 35]	–	–	[3, 32]	[11, 72]	[3, 25]	[5, 61]
S10	[1, 8]	[4, 28]	[0, 6]	[3, 24]	[0, 7]	[2, 26]	–	–	[3, 27]	[10, 74]	[4, 29]	[8, 60]
S11	[1, 10]	[6, 50]	[0, 10]	[4, 48]	[0, 11]	[3, 36]	–	–	[2, 27]	[8, 71]	[3, 26]	[8, 49]
S12	[1, 9]	[4, 35]	[0, 8]	[2, 32]	[0, 6]	[1, 30]	[0, 8]	[2, 32]	[7, 36]	[14, 68]	[4, 33]	[12, 64]
S13	[1, 8]	[3, 28]	[0, 6]	[4, 26]	–	–	–	–	[5, 27]	[8, 65]	[5, 23]	[9, 59]
S14	[1, 8]	[4, 26]	[0, 4]	[3, 24]	–	–	–	–	[7, 33]	[14, 64]	[6, 33]	[16, 64]
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	[1, 12]	[10, 49]	[0, 10]	[9, 32]	[1, 12]	[8, 48]	[1, 10]	[9, 40]	[8, 39]	[18, 82]	[5, 33]	[16, 72]

Table 10: Upper and lower bounds on advertisements in different media for P3

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	[1, 9]	[8, 44]	[1, 7]	[6, 40]	[1, 8]	[7, 38]	[1, 7]	[6, 43]	[8, 39]	[16, 75]	[7, 33]	[13, 65]
S2	[1, 6]	[5, 40]	[0, 5]	[4, 30]	[1, 5]	[3, 35]	[1, 4]	[3, 28]	[7, 36]	[14, 68]	[5, 32]	[10, 57]
S3	[1, 10]	[4, 45]	[0, 9]	[2, 40]	[1, 9]	[4, 32]	[1, 8]	[3, 27]	[8, 38]	[17, 78]	[7, 34]	[12, 62]
S4	[1, 6]	[8, 40]	[1, 7]	[5, 30]	[1, 6]	[6, 32]	–	–	[8, 38]	[14, 73]	[6, 35]	[14, 64]
S5	[1, 7]	[4, 44]	[0, 6]	[3, 29]	[0, 7]	[3, 32]	[1, 6]	[3, 27]	[2, 31]	[5, 50]	[3, 22]	[7, 47]
S6	[1, 8]	[7, 43]	[1, 7]	[5, 32]	[1, 6]	[6, 30]	[1, 5]	[5, 23]	[7, 36]	[15, 70]	[6, 28]	[11, 59]
S7	[1, 6]	[5, 40]	[0, 4]	[4, 25]	[1, 5]	[4, 27]	–	–	[3, 33]	[10, 55]	[6, 25]	[8, 49]
S8	[1, 6]	[4, 40]	[1, 5]	[3, 27]	[1, 6]	[3, 32]	[1, 5]	[3, 27]	[6, 35]	[12, 63]	[5, 26]	[10, 52]
S9	[0, 9]	[7, 42]	[1, 8]	[7, 35]	[0, 7]	[5, 32]	–	–	[3, 31]	[7, 52]	[3, 23]	[7, 50]
S10	[1, 7]	[4, 40]	[1, 9]	[5, 25]	[1, 7]	[3, 23]	–	–	[4, 33]	[10, 61]	[6, 25]	[8, 51]
S11	[1, 8]	[7, 43]	[0, 7]	[5, 32]	[1, 7]	[5, 27]	–	–	[5, 34]	[8, 57]	[4, 26]	[11, 55]
S12	[1, 7]	[6, 43]	[1, 5]	[5, 34]	[1, 6]	[4, 29]	[1, 5]	[5, 26]	[8, 36]	[16, 73]	[7, 32]	[16, 66]
S13	[1, 5]	[5, 40]	[0, 4]	[4, 28]	–	–	–	–	[5, 34]	[9, 59]	[3, 21]	[10, 48]
S14	[1, 6]	[4, 40]	[0, 4]	[3, 31]	–	–	–	–	[7, 37]	[13, 65]	[6, 29]	[12, 61]
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	[1, 10]	[8, 47]	[1, 8]	[6, 38]	[1, 10]	[6, 45]	[1, 10]	[6, 40]	[8, 39]	[17, 78]	[7, 35]	[13, 66]

Table 11: Upper and lower bounds on advertisements in different media for P4

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	[1, 5]	[3, 20]	[1, 3]	[3, 20]	[1, 4]	[2, 16]	[0, 3]	[3, 19]	[4, 36]	[14, 65]	[4, 27]	[12, 57]
S2	[0, 4]	[2, 18]	[0, 2]	[1, 16]	[0, 3]	[1, 14]	[0, 2]	[1, 13]	[5, 35]	[14, 61]	[3, 27]	[8, 49]
S3	[0, 4]	[3, 17]	[0, 2]	[2, 13]	[0, 3]	[2, 14]	[0, 2]	[2, 14]	[5, 37]	[10, 70]	[4, 27]	[10, 55]
S4	[0, 3]	[2, 15]	[0, 1]	[1, 13]	[1, 2]	[2, 13]	–	–	[5, 37]	[13, 64]	[4, 26]	[11, 57]
S5	[1, 4]	[2, 16]	[1, 3]	[1, 12]	[0, 3]	[2, 14]	[0, 2]	[1, 12]	[2, 27]	[9, 51]	[3, 25]	[7, 41]
S6	[0, 5]	[3, 20]	[1, 3]	[2, 18]	[1, 2]	[3, 14]	[0, 3]	[2, 18]	[4, 32]	[12, 63]	[3, 25]	[9, 51]
S7	[0, 4]	[2, 18]	[1, 3]	[2, 16]	[1, 3]	[2, 14]	–	–	[3, 31]	[11, 55]	[3, 26]	[7, 43]
S8	[1, 3]	[2, 16]	[0, 2]	[3, 16]	[0, 2]	[2, 15]	[1, 1]	[3, 13]	[4, 31]	[13, 60]	[3, 27]	[7, 46]
S9	[1, 5]	[3, 20]	[0, 3]	[2, 18]	[0, 3]	[3, 16]	–	–	[2, 29]	[10, 52]	[3, 24]	[7, 42]
S10	[1, 4]	[1, 15]	[1, 3]	[2, 14]	[1, 4]	[1, 12]	–	–	[3, 33]	[13, 59]	[3, 27]	[7, 45]
S11	[0, 5]	[3, 20]	[1, 3]	[3, 18]	[0, 4]	[3, 19]	–	–	[3, 34]	[11, 56]	[3, 25]	[8, 49]
S12	[1, 5]	[3, 19]	[1, 4]	[3, 18]	[1, 3]	[2, 19]	[0, 3]	[3, 18]	[5, 33]	[14, 68]	[4, 24]	[12, 59]
S13	[0, 4]	[2, 17]	[0, 2]	[2, 15]	–	–	–	–	[3, 38]	[12, 57]	[3, 23]	[7, 47]
S14	[0, 4]	[2, 15]	[0, 2]	[1, 13]	–	–	–	–	[4, 33]	[12, 59]	[3, 26]	[10, 54]
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	[1, 5]	[3, 20]	[1, 3]	[3, 20]	[1, 4]	[3, 18]	[1, 4]	[3, 20]	[5, 37]	[14, 70]	[4, 27]	[12, 59]

Table 12: Upper and lower bounds on advertisements in different media for P5

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	[0, 5]	[3, 18]	[0, 4]	[2, 13]	[0, 5]	[2, 17]	[0, 4]	[2, 14]	[8, 38]	[12, 70]	[7, 29]	[11, 57]
S2	[0, 4]	[2, 15]	[0, 3]	[2, 13]	[0, 4]	[2, 9]	[0, 3]	[2, 6]	[7, 38]	[10, 68]	[6, 28]	[10, 51]
S3	[0, 5]	[2, 17]	[0, 3]	[2, 12]	[0, 4]	[2, 13]	[0, 2]	[2, 12]	[8, 37]	[13, 70]	[7, 29]	[10, 55]
S4	[0, 4]	[1, 13]	[0, 2]	[1, 10]	[0, 2]	[1, 10]	–	–	[7, 37]	[10, 64]	[6, 29]	[10, 55]
S5	[0, 3]	[3, 14]	[0, 3]	[2, 11]	[0, 3]	[2, 12]	[0, 3]	[2, 11]	[2, 31]	[5, 57]	[3, 25]	[5, 45]
S6	[0, 4]	[2, 16]	[0, 3]	[2, 12]	[0, 3]	[2, 13]	[0, 3]	[1, 14]	[7, 35]	[12, 62]	[7, 27]	[9, 51]
S7	[0, 3]	[2, 14]	[0, 2]	[1, 7]	[0, 2]	[2, 12]	–	–	[6, 33]	[7, 59]	[5, 25]	[7, 49]
S8	[0, 4]	[2, 15]	[0, 2]	[2, 8]	[0, 3]	[2, 10]	[0, 2]	[1, 8]	[6, 34]	[9, 62]	[6, 27]	[8, 51]
S9	[0, 5]	[3, 18]	[0, 4]	[3, 13]	[0, 3]	[2, 12]	–	–	[3, 31]	[6, 58]	[3, 24]	[7, 47]
S10	[0, 4]	[2, 15]	[0, 3]	[2, 12]	[0, 2]	[2, 11]	–	–	[4, 33]	[9, 60]	[5, 25]	[8, 51]
S11	[0, 5]	[3, 18]	[0, 4]	[2, 12]	[0, 3]	[2, 11]	–	–	[5, 34]	[7, 61]	[4, 24]	[9, 47]
S12	[0, 5]	[3, 17]	[0, 4]	[2, 16]	[0, 3]	[2, 15]	[0, 3]	[2, 11]	[7, 37]	[13, 66]	[6, 27]	[11, 55]
S13	[0, 3]	[1, 13]	[0, 2]	[1, 7]	–	–	–	–	[5, 34]	[8, 62]	[2, 24]	[8, 45]
S14	[0, 3]	[2, 13]	[0, 2]	[2, 8]	–	–	–	–	[7, 36]	[10, 62]	[7, 29]	[11, 52]
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	[0, 5]	[3, 18]	[0, 4]	[3, 13]	[0, 5]	[3, 15]	[0, 5]	[3, 18]	[8, 37]	[13, 70]	[7, 23]	[9, 57]

Table 13: Optimal number of advertisements in different media for P1

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	1	12	15	65	20	70	1	11	36	92	39	85
S2	14	50	1	8	1	42	13	48	34	88	33	13
S3	24	90	16	9	20	75	15	8	39	94	38	83
S4	14	56	1	3	12	42	–	–	33	78	37	81
S5	1	7	1	5	18	76	1	4	31	8	25	7
S6	18	76	1	7	1	6	14	7	38	85	17	13
S7	12	65	11	75	14	72	–	–	31	68	31	68
S8	12	49	10	45	8	40	8	38	32	72	33	72
S9	18	76	14	11	1	8	–	–	5	64	29	8
S10	1	6	1	5	1	4	–	–	33	62	26	59
S11	1	9	1	7	1	6	–	–	34	66	27	62
S12	1	71	1	7	1	6	14	8	36	79	40	86
S13	1	4	1	3	–	–	–	–	31	64	25	57
S14	1	5	1	4	–	–	–	–	32	85	29	14
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	18	84	12	12	15	70	12	64	39	94	25	86

Table 14: Optimal number of advertisements in different media for P2

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	1	6	10	7	8	30	1	7	7	18	5	14
S2	1	30	1	3	1	4	6	26	38	16	4	13
S3	1	6	1	3	12	35	12	4	39	82	6	12
S4	1	32	0	2	0	25	–	–	29	64	35	12
S5	1	3	1	3	1	36	1	3	5	11	3	7
S6	1	5	1	4	1	4	1	3	33	15	7	10
S7	1	4	0	3	8	25	–	–	26	63	27	52
S8	9	27	0	4	6	23	6	3	28	64	31	55
S9	1	7	1	6	1	5	–	–	3	11	3	5
S10	1	4	0	3	0	2	–	–	27	10	4	8
S11	1	6	0	4	0	3	–	–	27	71	26	49
S12	1	4	0	2	0	1	8	2	36	14	33	12
S13	1	3	0	4	–	–	–	–	27	65	23	9
S14	1	4	0	3	–	–	–	–	33	14	33	16
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	12	49	0	9	12	48	1	9	39	18	33	16

Table 15: Optimal number of advertisements in different media for P3

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	1	8	7	40	8	38	1	6	39	75	33	65
S2	6	40	5	4	1	35	4	28	36	68	32	57
S3	10	4	0	40	9	32	6	3	38	78	34	62
S4	6	40	7	30	6	32	–	–	38	73	35	64
S5	1	4	0	3	7	32	1	3	31	50	3	7
S6	8	43	7	5	6	6	5	5	36	70	6	11
S7	6	5	4	25	5	27	–	–	33	55	25	49
S8	6	40	5	27	6	32	5	27	35	63	26	52
S9	0	7	1	7	0	5	–	–	31	52	3	7
S10	1	4	1	5	1	3	–	–	33	61	25	51
S11	1	7	0	5	1	5	–	–	34	57	26	55
S12	1	43	1	5	1	4	5	5	36	73	32	66
S13	1	5	0	4	–	–	–	–	34	59	21	48
S14	1	4	0	3	–	–	–	–	37	65	29	61
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	10	47	8	38	10	45	10	40	39	78	35	66

Table 16: Optimal number of advertisements in different media for P4

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	1	3	3	3	4	16	0	3	4	14	4	12
S2	0	18	0	1	3	1	2	13	35	14	3	8
S3	0	3	0	2	3	14	2	2	37	70	4	10
S4	3	15	0	1	2	13	–	–	37	64	26	11
S5	1	2	1	1	3	2	0	1	2	9	3	7
S6	5	3	1	2	1	3	3	2	32	63	3	9
S7	0	2	3	16	3	14	–	–	31	55	26	43
S8	3	16	2	3	2	15	1	13	31	60	27	46
S9	1	3	0	2	3	3	–	–	2	10	3	7
S10	1	1	1	2	1	1	–	–	33	13	3	7
S11	0	3	1	3	0	3	–	–	34	56	25	49
S12	1	3	1	3	3	2	3	18	33	14	24	12
S13	0	2	0	2	–	–	–	–	38	57	23	47
S14	0	2	0	1	–	–	–	–	33	12	26	10
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	5	20	3	20	4	18	4	3	37	14	27	12

Table 17: Optimal number of advertisements in different media for P5

Segments	RNP1		RNP2		RNP3		RNP4		RCH1		RCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
S1	0	3	0	2	5	17	0	2	8	12	7	11
S2	0	2	0	2	0	2	3	6	38	10	6	10
S3	0	2	0	2	4	13	2	2	37	70	7	10
S4	0	1	0	1	2	10	–	–	37	64	29	10
S5	0	3	0	2	3	2	0	2	2	5	3	5
S6	0	2	0	2	0	2	0	1	35	12	7	9
S7	0	2	0	1	2	12	–	–	33	57	25	49
S8	4	15	0	2	3	10	2	8	34	9	27	51
S9	0	3	0	3	0	2	–	–	3	6	3	7
S10	0	2	0	2	0	2	–	–	33	9	5	8
S11	0	3	0	2	0	2	–	–	34	61	24	47
S12	0	3	0	2	0	2	3	2	37	13	27	11
S13	0	1	0	1	–	–	–	–	34	62	24	8
S14	0	2	0	2	–	–	–	–	7	10	7	11
Mass Media	NNP1		NNP2		NNP3		NNP4		NCH1		NCH2	
	FP	OP	FP	OP	FP	OP	FP	OP	PT	OT	PT	OT
	0	3	0	3	5	3	0	3	37	13	23	9

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Some properties of residual mapping and convexity in $\wedge$-hyperlattices

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Abstract

The aime of this paper is the study of residual mappings and convexity in hyperlattices. To get this point, we study principal down set in hyperlattices and we give some conditions for a mapping between two hyperlattices to be equivalent with a residual maping. Also, we investigate convex subsets in $\wedge$-hyperlattices.

Key words: residuated map, convex, down-set, hyperideal, hyperfilter.

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1 Introduction

Hyperalgebras (multialgebra) are generalization of classical algebras that are introduced by F. Marty in the eighth congress of Scandinavian in 1934 [11].

In [4], Ameri and M. M. Zahedi introduced and studied notion of hyperalgebraic systems. In [2], Ameri and Nozari Studied relationship between the categories of multialgebra and algebra. C. Pelea and I. Purdea have been proved that complete hyperalgebra can be obtained from a universal algebra and a appropriate congruence on it. Also, Pelea and others studied multialgebra, direct limit, and identities, for more details see [16, 17, 18, 19].

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Theory of hyperlattices introduced by Konstantinidou and J. Mittas in 1977[9]. In [10], G. A. Moghani and A. R. Ashrafi proved that in some cases the set of all subhypergroups G has a hyperlattice structure . In [24], X. L. Xin and X. G. Li studied hyperlattices and quotient hyperlattices. In [5], A. Asokkumar in 2007 proved that under certain conditions, the idempotent elements of a hyperring form a hyperlattice and the orthogonal idempotent elements form a quassi-distributive hyperboolean algebra. In [1], R. Ameri, M. Amiri Bideshki, and A. Borumand Said studied prime hyperfilters (hyperideals) in hyperlattices. Also, they gave some examples of $\wedge$-hyperlattices and dual distributive $\wedge$-hyperlattices.
In section 3, down set and residual maps in hyperlattices are studied and some properties of them are given. In section 4, convex subsets of a hyperlattice and some properties of them are given.

2 Preliminary

In this section we give some results of hyperlattices that we need to develop our paper.

Definition 2.1. [1] Let L be a nonempty set. L is called a $\wedge-$ hyperlattice if

(i) $a \in a \wedge a, a \vee a = a,$

(ii) $a \wedge b = b \wedge a, a \vee b = b \vee a,$

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c, a \vee (b \vee c) = (a \vee b) \vee c,$

(iv) $a \in (a \wedge (a \vee b)) \cap (a \vee (a \wedge b))$,

(v) $a \in a \wedge b \Longrightarrow a \vee b = b$,

for all $a, b, c \in L$.
Let $A, B \subseteq L$. Then:

$$A \wedge B = \cup\{a \wedge b | a \in A, b \in B\};$$
$$A \vee B = \{a \vee b | a \in A, b \in B\}.$$

Example 2.2. Let $(L, \vee, \wedge)$ be a lattice and define $a \oplus b = \{x \mid x \leq a \wedge b\}$. Then $(L, \vee, \oplus)$ is a $\wedge-$ hyperlattice.

Definition 2.3. [1] Let L be a $\wedge-$hyperlattice. We say that L is bounded If there exist $0, 1 \in L$, such that $0 \leq x \leq 1$, for all $x \in L$. We say that 0 is the least element of L and 1 is the greatest element of L.

Example 2.4. Let $L = \{0, a, 1\}$, and define $\wedge$-hyper operation and $\vee$-operation on L with tables 3. Then $(L, \wedge, \vee)$ is a bounded $\wedge$-hyperlattice.

$\wedge$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{a, 0\}$	$\{a, 0\}$
1	$\{0\}$	$\{a, 0\}$	L

(a)

$\vee$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

(b)

Table 1

Definition 2.5. [1] Let I and F are nonempty subsets of L. Then:

(i) I is called hyperideal if the following conditions hold.

  (a) If $x, y \in I$, then $x \vee y \in I$,

  (b) If $x \in I$ and $a \in L$, such that $a \leq x$, then $a \in I$.

(ii) F is called hyperfilter if the following conditions hold.

  (a) If $x, y \in F$, then $x \wedge y \subseteq F$,

  (b) If $x \in F$ and $a \in L$, such that $x \leq a$, then $a \in F$.

(iii) A hyperideal I is called prime if $x \wedge y \in I$, then $x \in I$ or $y \in I$, for all $x, y \in L$.

(iv) A hyperfilter F is called prime if $x \in F$ or $y \in F$, where $(x \wedge y) \cap F \neq \emptyset$, for all $x, y \in L$.

3 Resedual Mappings in $\wedge$-Hyperlattices

In this section, we are going to introduce down-set and resedual mapping in $\wedge$-hyperlattice. Let L be a $\wedge$-hyperlattice.

Definition 3.1. Let $\emptyset \neq A \subseteq L$. A is called a down-set, if $x \in A$ and $y \leq x$, then $y \in A$.

Example 3.2. every hyperideal of L is a down-set that is called principal down-set.

Example 3.3. Let $L = \{0, a, b, 1\}$. $\wedge$ and $\vee$ are given by Table 2 and 3.

$\wedge$	0	a	b	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0\}$	$\{0, a\}$
b	$\{0\}$	$\{0\}$	$\{0, b\}$	$\{0, b\}$
1	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{1\}$

Table 2:

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Table 3:

$I = \{0, a, b\}$ is a down-set, but it is not a hyperideal. We have $a, b \in I$ and $a \vee b = 1 \notin I$.

Let $x \in L$ and $x^{\downarrow} = \{y \in L | y \in x \wedge y\}$.

Proposition 3.4. *$\forall x \in L, x^{\downarrow}$ is a down set.*

$x^{\downarrow}$ is called a principal down-set.

Proposition 3.5. *Let L be a dual distributive $\wedge$-hyperlattice. Then every principal down-set is a hyperideal.*

Proof. $\square$

If $A \subseteq L$ and $a \vee b \subseteq A$, for all $a, b \in L$, then A is called join-closed.

Corollary 3.6. *Let $I \subseteq L$. Then I is an ideal if and only if I is a down-set and it is a join-closed set.*

Proposition 3.7. *Let L and K be hyperlattice. If $f : L \longrightarrow K$ is a isotone map and $A \subseteq L$ is a down-set, then $f(A)$ is a down-set.*

Proof. Since A is a down-set, there exists $x \in L$ such that $A = x^{\downarrow}$. It is sufficient set $f(A) = f(x)^{\downarrow}$. □

Let L and K be hyperlattices and $f : L \longrightarrow K$ is a mapping. We define two map $f^{\rightarrow}$ and $f^{\leftarrow}$ that $f^{\rightarrow}$ is called direct image map and $f^{\leftarrow}$ is called inverse image map. $f^{\rightarrow} : P(L) \longrightarrow P(K)$ is defined by $f^{\rightarrow}(X) = \{f(x)|x \in L\}$, for all $X \subseteq L$, and $f^{\leftarrow} : P(K) \longrightarrow P(L)$ is defined by $f^{\leftarrow}(Y) = \{x\{\in L|f(x) \in Y\}$ for all $Y \subseteq K$.

Definition 3.8. A mapping $F : L \longrightarrow K$ is called residuated if the inverse image under F of every principal down-set of K is a principal down-set of L.

Example 3.9. Let L be a $\wedge$-hyperlattice and $A \subseteq L$. We define $f^A : P(L) \longrightarrow P(L)$ by $f_A(B) = A \cap B$, for all $B \in P(L)$. Then f^A is a residuated and residual g is given by $g^A(C) = C \cup A'$, where that $A' = L \setminus A$.

Example 3.10. Let L be a $\wedge$-hyperlattice. Mapping $f : P(L) \longrightarrow P(L)$ that is defined by $f(A) = A$, for all $A \in P(L)$, is a residuated mapping.

Theorem 3.11. *Let L and K be two hyperlattices. A mapping $f : L \longrightarrow K$ is a residuated iff f is a is isotone and there exists an isotone mapping $g : K \longrightarrow L$ such that $gof \geq id_L$ and $fog \leq id_K$.*

Proof. For all $x \in L$, $x \in f^{\leftarrow}[f(x)^{\downarrow}]$. If $y \leq x$, then $y \in f^{\leftarrow}[f(x)^{\downarrow}]$. We have: $f(x)^{\downarrow} = \{y|y \leq f(x)\}$ and $f^{\leftarrow}[f(x)^{\downarrow}] = \{t \in L|f(t) \in f(x)^{\downarrow}\}$.
$y \in f^{\leftarrow}[f(x)^{\downarrow}]$, so $f(y) \leq f(x)$. Then f is isotone. By assumption we have $(\forall y \in K)(\exists x \in L)$ such that $f^{\leftarrow}(y^{\downarrow}) = x^{\downarrow}$. Now, for every given $y \in K$, this element x is clearly unique. So we can define a mapping $g : K \longrightarrow L$ by $g(y) = x$. Since $f^{\leftarrow}$ is isotone, it follow that so is g. For this mapping g, we have:

$$g(y) \in g(y)^{\downarrow} = x^{\downarrow} = f^{\leftarrow}(y^{\downarrow}).$$

So, $f[g(y)] \leq y$, for all $y \in K$ and therefore $fog \leq id_K$. Also, $x \in f^{\leftarrow}[f(x)^{\downarrow}] = g[f(x)]^{\downarrow}$, so that $x \leq g[f(x)]$, for all $x \in L$, and therefore $gof \geq id_L$.
Conversely, Since g is isotone, we have:

$$f(x) \leq y \Longrightarrow x \leq g[f(x)].$$

Also, we have:

$$x \leq g(y) \Longrightarrow f(x) \leq f[g(x)] \leq y.$$

It follows from these observations that $f(x) \leq y$ iff $x \leq g(y)$ and therefore $f^{\leftarrow}(y^{\downarrow}) = g(y)^{\downarrow}$. □

Proposition 3.12. *The residual of f is unique.*

Proof. Suppose that g and g' are residual of f. Then we have: $g = id_L og \leq (g'of)og = g'o(fog) \leq g'oid_K = g'$. Similarly, $g' \leq g$, then $g = g'$. □

We shall denote residual of f, by f^+.

Proposition 3.13. *Mapping $f : L \Longrightarrow K$ is residuated iff for every $y \in K$, there exists $g(y) = max f^{\leftarrow}(y^{\downarrow}) = max\{x \in L | f(x) \leq y\}$. Moreover, $f^+ of \geq id_L$ and $fof^+ \leq id_K$.*

Definition 3.14. Let $f : L \longrightarrow K$ be a residuated mapping. Then f is called range closed if $Im(f)$ is a down-set of K.

Example 3.15. Let L be a $\wedge$-hyperlattice with a top element 1. Given $a \in L$, consider the mapping $f_a : L \longrightarrow L$ given by:
$f_a(x) = f_a$ is residuated. Clearly, $Im(f_a)$ is the down-set $a^{\downarrow}$ of L then f_a is a range closed.

Remark 3.16. In Example 3.15, L must have top element 1.

Example 3.17. Let N be the set of natural numbers. We define $\wedge$-hyperoperation and $\vee$ operation by:

$$a \wedge b = \{m \in N | m \leq min\{a, b\}\};$$

$$a \vee b = max\{a, b\}, forall a, b \in N.$$

Then $(L, \wedge, \vee)$ is a $\wedge$-hyperlattice. Consider $f : N \longrightarrow N$ by $f(x) = x$, for all $x \in N$. f is a residated mapping, but it is not range closed.

Theorem 3.18. *Let $f : L \longrightarrow K$ be a residuated mapping. Then $f = f^+$ iff $f^2 = id_L$.*

Proof. $\Longrightarrow$ It is obvious. $\Longleftarrow$ Since f is residuated, then $f^2 = id_L$. By $f^2 = id_L$, we have $fof \leq id_L$ and $fof \geq id_L$. So $f = f^+$. □

Theorem 3.19. *Let L and K be two $\wedge$-hyperlattices and Let L has a top element 1. If $f : L \longrightarrow K$ be a residuated mapping, then the following statements are equivalent.*

(i) *f is range closed.*

(ii) *$forally \in K$ $inf\{y, f(1)\}$ there exists and it equal to $ff^+(y)$.*

Proof. $(i \to ii)$:We have $f^+(y) \leq 1$, for all $y \in L$ and by isotonic f, $ff^+(y) \leq f(1)$. Also $ff^+(y) \leq y$, for all $y \in K$. So $ff^+(y)$ is a lower bound of $f(1)$ and y. We must show that $ff^+(y)$ is the greatest lower bound of $f(1)$ and y. Suppose that $x \in K$ is such that $x \leq y$ and $x \leq f(1)$. By (i), we have $x = f(z)$, for some $z \in L$ and $f(z) \leq y$; Since f^+ is isotone, $f^+f(x) \leq f^+(y)$. We have $z \leq f^+f(x)$, so $z \leq f^+(y)$. By isotonic f, $f(z) \leq ff^+(y)$, Then $x \leq ff^+(y)$. Thus $inf\{y, f(1)\} = ff^+(y)$.
$(ii \to i)$: We claim that $Im(f) = f(1)^{\downarrow}$. We have $x \leq 1$, for all $x \in L$, then $f(x) \leq f(1)$, for all $x \in L$. So $Im(f) \subseteq f(1)^{\downarrow}$. Let $y \in K$ be such that $y \leq f(1)$. Then by (ii), $ff^+(y) = inf\{y, f(1)\} = y$. We Know $ff^+(y) \in Im(f)$, so $y \in Im(f)$. Thus $f(1)^{\downarrow} \subseteq Im(f)$. Therefore $Im(f) = f(1)^{\downarrow}$. □

Proposition 3.20. *Let $f : L \longrightarrow K$ and $g : K \longrightarrow\longrightarrow M$ be residual map. Then gof so is, also $(gof)^+ = f^+og^+$.*

4 Convexity In $\wedge$-hyperlattice

In this section, we are going to introduce convex subsets in $\wedge$-hyperlattices and we are going to give some properties of convex subsets.

Proposition 4.1. *Let $F \subseteq L$. Then F is a hyperfilter of L, if and only if*

(i) *$a, b \in F$ implies that $a \wedge b \in F$.*

(ii) *$\forall a \in F$ and $\forall x \in L$, $a \vee x \in F$.*

Proof. Since F is a filter, $\forall a, b \in F$, $a \wedge b \in F$. We know $a \leq a \vee x$, then $a \vee x \in F$. So (i) and (ii) hold.
Conversely, Let $a \in F$ and $a \leq x$. So, $a \vee x = x$, by (ii) $a \vee x \in F$, then $x \in F$. □

Proposition 4.2. *Every hyperfilter of a $\wedge$-hyperlattice L is a $\wedge$-subhyperlattice.*

Remark 4.3. Converse of the above proposition does not hold. Consider hyperlattice in the Example 3.2. $A = \{0, a\}$ is a subhyperlattice. We have $a \leq 1$ and $1 \notin A$, then A is not a filter.

Remark 4.4. Every hyperideal of L is not a subhyperlattice. Also, every subhyperlattice is not an ideal.

Definition 4.5. Let $\emptyset \neq K \subseteq L$. We say K to be convex subset, if $a, b \in K$ and $c \in L$ such that $a \leq c \leq b$, then $c \in K$.

Example 4.6. Consider hyperlattice L in Example 3.2. Then $A = \{0, a\}$ is a convex subset, but $B = \{0, 1\}$ is not a convex subset. we have $0 \leq a \leq 1$ and $a \notin B$.

Proposition 4.7. *Every hyperideal (hyperfilter) of L is a convex subset of L.*

Remark 4.8. Every convex subset of L is not a hyperideal (a filter). Consider hyperlattice L in Example 3.2. Then $K = \{a, b, 1\}$ is a convex subset, but it is not a hyperideal ($0 \notin K$). Also, K is not a hyperfilter ($a \wedge b = \{0\}$ and $0 \notin K$).

Theorem 4.9. *Let L has a bottom element 0 and let K be a convex subset of L. If K is a chain and $0 \in K$, then K is a hyperideal of L.*

Remark 4.10. In Example 4.9, K must be a chain; also K must contain bottom element 0.

Example 4.11. Let L be hyperlattice in Example3.2

(i) $K_1 = \{a, b, 0\}$ is a convex subset, but it is a not chain(a, b are not comparable). Since $a \vee b \notin K_1$, K_1 is not a hyperideal.

(ii) $K_2 = \{a, b, 1\}$ is a convex subset, but it is not a hyperideal ($0 \notin K_2$).

Example 4.12. Consider hyperlattice L in Example 3.17. Then$K = \{2, 3, 4, ..., 10\}$ is a convex subset; Since K does not has bottom element 1, it is not a hyperideal.

Proposition 4.13. *Every principal down-set of L is a convex subset.*

Theorem 4.14. *Let I be a hyperideal and F be a hyperfiler of L, such that $I \cap F \neq \emptyset$, then $I \cap F$ is a convex sub-hyperlattice if and only if for all $a, b \in I \cap F$, $a \wedge b \subseteq I$.*

Proposition 4.15. *If K_i, $\forall i \in I$ is a convex sub-hyperlattice of L, then $\cap_{i \in I} K_i$ is so.*

Theorem 4.16. *Let K_1 and K_2 be convex sub-hyperlattices of L and let $0 \in K_1 \cap K_2$. Then $K_1 \cup K_2$ is a convex sub-hyperlattice if and only if $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$.*

Proof. Let $K_1 \cup K_2$ be a convex subhyperlattice, but $K_1 \nsubseteq K_2$ or $K_2 \nsubseteq K_1$. So, there exist $a, b \in L$, such that $a \in K_1 \setminus K_2$ and $K_1 \setminus K_2$. Since $a, b \in K_1 \cup K_2$ and $K_1 \cup K_2$ is a sub-hyperlattice, $a \vee b \in K_1 \cup K_2$; it implies that $a \vee b \in K_1$ or $a \vee b \in K_2$. If $a \vee b \in K_1$, $0 \leq a \leq a \vee b$, then $a \in K_2$, which is a contradiction; if $a \vee b \in K_2$, then we conclude that $b \in K_1$, which is a contradiction.
The converse is obvious. $\square$

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Multivalued linear transformations of hyperspaces

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Abstract

The purpose of this paper is the study of multivalued linear transformations of hypervector spaces (or hyperspaces) in the sense of Tallini. In this regards first we introduce and study various multivalued linear transformations of hyperspaces and then constitute the categories of hyperspaces with respect the different linear transformations of hyperspaces as the morphisms in these categories. Also, we construct some algebraic hyperoperations on $Hom_K(V, W)$, the set of all multivalued linear transformations from a hyperspace V into hyperspaces W, and obtaine their basic properties. Finally, we construct the fundamental functor F from $\mathcal{HV}_K$, category of hyperspaces over field K into $\mathcal{V}_K$, the category of clasical vector space over K.

Key words: hypervector space, multivalued linear transformation, category,fundamental relation

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1 Introduction

The theory of algebraic hyperstructures is a well-established branch of classical algebraic theory. Hyperstructure theory was first proposed in 1934 by Marty, who defined hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions [15]. It was later observed that the theory of hyperstructures has many applications in both pure and applied sciences; for example, semi-hypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. The theory of hyperstructures has been widely reviewed ([11], [12], [13],[14] and [20])(for more see [2, 3, 5, 6, 7, 8, 9]).

M.S. Tallini introduced the notion of hyperspaces(hypervector spaces) ([17], [18] and [19]) and studied basic properties of them. R. Ameri and O. Dehghan [2] introduced and studied dimension of hyperspaces and in [16] M. Motameni et. el. studied hypermatrix. R. Ameri in [1] introduced and studied categories of hypermodules. In this paper we introduce and study various types of multivalued linear transformations of hyperspaces. We will proceed by constructing various categories of hyperspaces based on various multilinear linear transformations of hyperspaces. Also, we construct some hyperalgebraic structures on $(Hom_K(V, W)$. Finally, we construct the fundumental functor from category of hyperspaces and multilinear transformations, as morphisms into the category of vectorspces.

2 Preliminaries

The concept of hyperspace, which is a generalization of the concept of ordinary vector space.

Definition 2.1. *Let H be a set. A map $. : H \times H \longrightarrow P_*(H)$ is called hyperoperation or join operation, where $P_*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A.B$ is given by*

$$A.B = \bigcup\{a.b : a \in A \textit{ and } b \in B\}.$$

the notations $a.A$ and $A.a$ are used for $\{a\}.A$ and $A.\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2.2. *[17] Let K be a field and $(V, +)$ be an abelian group. We define a hyperspace over K to be the quadrupled $(V, +, \circ, K)$, where $\circ$ is a* *mapping*

$$\circ : K \times V \longrightarrow P_*(V),$$

such that the following conditions hold:
(H_1) $\forall\ a \in K,\ \forall\ x, y \in V,\ a \circ (x + y) \subseteq a \circ x + a \circ y$, *right distributive law,*
(H_2) $\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x \subseteq a \circ x + b \circ x$, *left distributive law,*
(H_3) $\forall\ a, b \in K,\ \forall\ x \in V,\ a \circ (b \circ x) = (ab) \circ x$, *associative law,*
(H_4) $\forall\ a \in K,\ \forall\ x \in V,\ a \circ (-x) = (-a) \circ x = -(a \circ x)$,
(H_5) $\forall\ x \in V,\ x \in 1 \circ x$.

Remark 2.3. (i) *In the right hand side of* (H_1) *the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of* $a \circ x$ *with an element of* $a \circ y$. *Similarly we have in* (H_2).
(ii) *We say that* $(V, +, \circ, K)$ *is anti-left distributive, if*

$$\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and strongly left distributive, if

$$\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x = a \circ x + b \circ x,$$

In a similar way we define the anti-right distributive and strongly right distributive hyperspaces, respectvely. V *is called strongly distributive if it is both strongly left and strongly right distributive.*
(iii) *The left hand side of* (H_3) *means the set-theoretical union of all the sets* $a \circ y$, *where* y *runs over the set* $b \circ x$, *i.e.*

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) *Let* $\Omega_V = 0 \circ 0_V$, *where* 0_V *is the zero of* $(V, +)$, *In [17] it is shown if* V *is either strongly right or left distributive, then* Ω_V *is a subgroup of* $(V, +)$.

Let V be a hyperspace over a field K. $W \subseteq V$ is a subhyperspace of V, if

$$W \neq \emptyset,\ \ W - W \subseteq W,\ \ \forall a \in K,\ \ a \circ W \subseteq W.$$

Example 2.4. *[2] Consider abelian group* $(\mathbb{R}^2, +)$. *Define hyper-compositions*

$$\begin{cases} \circ : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2) \\ \quad a \circ (x, y) = ax \times \mathbb{R} \end{cases}$$

and

$$\begin{cases} \diamond : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2) \\ \quad a \diamond (x, y) = \mathbb{R} \times ay. \end{cases}$$

Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ *and* $(\mathbb{R}^2, +, \diamond, \mathbb{R})$ *are a strongly distributive hyperspaces.*

Example 2.5. *[2] Let $(V, +, ., K)$ be a classical vector space and P be a subspace of V. Define the hyper-composition*

$$\begin{cases} \circ : K \times V \longrightarrow P_*(V) \\ \quad a \circ x = a.x + P. \end{cases}$$

Then it is easy to verify that $(V, +, \circ, K)$ is a strongly distributive hyperspace.

Example 2.6. *[?] In $(\mathbb{R}^2, +)$ define the hyper-composition $\circ$ as follows:*

$$\forall a \in \mathbb{R}, \forall x \in \mathbb{R}^2 \ : \ a \circ x = \begin{cases} line\ \bar{ox} & if\ x \neq 0_V \\ \{0_V\} & if\ x = 0_V, \end{cases}$$

where $0_V = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left, but not right distributive hyperspace.

Proposition 2.7. *[?] Every strongly right distributive hyperspace is strongly left distributive hyperspace. Let $(V, +)$ be an abelian group, Ω a subgroup of V and K a field such that $W = V/\Omega$ is a classical vector space over K. If $p : V \longrightarrow W$ is the canonical projection of $(V, +)$ onto $(W, +)$ and set:*

$$\begin{cases} \circ : K \times V \longrightarrow P_*(V) \\ \quad a \circ x = p^{-1}(a.p(x)). \end{cases}$$

Then $(V, +, \circ, K)$ is a strongly distributive hyperspace over K. Moreover every strongly distributive hyperspace can be obtained in such a way.

Proposition 2.8. *[?] If $(V, +, \circ, K)$ be a left distributive hyperspace, then for all $a \in K$ and $x \in V$*
1) *$0 \circ x$ is a subgroup of $(V, +)$;*
2) *Ω_V is a subgroup of $(V, +)$;*
3) *$a \circ 0_V = \Omega_V = a \circ \Omega_V$;*
4) *$\Omega_V \subseteq 0 \circ x$;*
5) *$x \in 0 \circ x \Longleftrightarrow 1 \circ x = 0 \circ x \Longleftrightarrow a \circ x = 0 \circ x,\ \forall a \in K$.*

Remark 2.9. *Let $(V, +, \circ, K)$ be a hyperspace and W be a subhyperspace of V. Consider the quotient abelian group $(V/W, +)$. Define the rule*

$$\begin{cases} * : K \times V/W \longrightarrow P_*(V/W) \\ \quad (a, x + W) \longmapsto a \circ x + W. \end{cases}$$

Then it is easy to verify that $(V/W, +, *, K)$ is a hyperspace over K and it is called the quotient hyperspace of V over W.

3 Multivalued linear transformations

Definition 3.1. *Let V and W be two hyperspaces over a field K. A multivalued linear transformation (MLT) $T : V \longrightarrow P_*(W)$ is a mapping such that :*
$\forall x, y \in V, \forall a \in K$
1) $T(x + y) \subseteq T(x) + T(y)$;
2) $T(a \circ x) \subseteq a \circ T(x)$;
3) $T(-a) = -T(a)$.

Remark 3.2. (i) *In Definition 3.1*(1) *and* (2)*, if the equality holds, then T is called a strong multivalued linear transformation $(SMLT)$.*
(ii) *In Definition 3.1, if we consider T as a mapping $T : V \longrightarrow W$, then is it is called a linear transformation. Here we consider only inclusion and equality cases.*
(iii) *If T is a MLT, then $0 \in T(x)$, since $T(x) \neq \emptyset$, so $\exists y \in T(x)$; $0 = y - y \in T(x) - T(x) = T(x) + T(-x) = T(x + (-x)) = T(x - x) = T(0)$.*

Definition 3.3. *[1] Let V and W be two hyperspaces over a field K and $T : V \longrightarrow P_*(W)$ be a $SMLT$. Then multivalued kernel and multivalued image of T, denoted by $\overline{Ker}T$ and $\overline{Im}T$, respectively, are defined as follows:*

$$\overline{Ker}T = \{x \in V \mid 0_W \in T(x)\};$$

and

$$\overline{Im}T = \{y \in W \mid y \in T(x) \textit{ for some } x \in V\}.$$

Remark 3.4. (i) *Note that $\overline{Ker}T \neq \emptyset$, by Remark 3.2(iii).*

(ii) *For hyperspaces V and W over a field K, by $Hom_K(V, W)$ and $Hom^s_K(V, W)$, we mean the set of all MLT and $SMLT$, respectively and sometimes we use morphism instead multivalued linear transformation, respectively. Also, by $hom_K(V, W)$ and $hom^s_K(V, W)$, we mean the set of all linear transformation LT and strong linear transformation SLT respectively and sometimes we use morphism instead multivalued , respectively.*

In the following we briefly introduced the categories of hyperspaces and study the relationship between monomorphism, epimorphism, isomorphism and*monic, epic* and iso objects in these category.

Definition 3.5. *The category of hyperspaces over a field K denoted by $\mathcal{HV}_K$ is defined as follows:*
1) *The objects of $\mathcal{HV}_K$ are all hyperspaces over K;*

2) *For the objects* V *and* W *of* $\mathcal{HV}_K$, *the set of all morphisms from* V *to* W *denoted by* $Hom_K(V, W)$, *is the set of all* MLT *from* V *to* W.
3) *The composition* $ST : V \longrightarrow P_*(W)$ *of morphisms* $T : V \longrightarrow P_*(L)$ *and* $S : L \longrightarrow P_*(W)$ *is defined as follows:*

$$ST(x) = \bigcup_{t \in T(x)} S(t).$$

4) *For any object* V, *the morphism* $1_V : V \longrightarrow P_*(V)$, $x \to \{x\}$ *is the identity.* 5) *The category of hyperspaces over a field* K *with (resp.* SLT*)*LT *is denoted by (resp.* $\langle\mathcal{V}_K$*)*$\langle\mathcal{V}_K$.

Remark 3.6. *If in Definition 3.5 part* (2) *we replace* $Hom_K(V, W)$ *by* Hom^s_K (V, W), *the set of all* $SMLT$, *then we will obtain a new category, which it denotes by* $\mathcal{HV}^s_K$. *In fact,* $\mathcal{HV}^s_K \preceq \mathcal{HV}_K$ *(by* $A \preceq B$ *we mean* A *is a subcategory of* B*). Also, denote the category of all vector spaces over a field* K *by* $\mathcal{V}_K$. *Clearly,* $\mathcal{V}_K \preceq \langle\mathcal{V}_K \preceq \langle\mathcal{V}^s_K \preceq \mathcal{HV}^s_K \preceq \mathcal{HV}_K$ *(for more details see [1]).*

Definition 3.7. *Let* $T : V \longrightarrow P_*(W)$ *be a* $SMLT$ *of hyperspaces. We say that* T *is weakly injective if*

$$\forall\ x, y \in V,\ T(x) \cap T(y) \neq \emptyset \Rightarrow x = y.$$

We say that T *is strongly injective if*

$$\forall\ x, y \in V,\ T(x) = T(y) \Rightarrow x = y.$$

Remark 3.8. *Clearly, every weakly injective morphism is also strongly injective. Note that* T *is strongly injective, means that* T *is injective as a function with values in* $P_*(W)$. *In the following example we show that a strongly injective morphism need not to be weakly injective.*

Example 3.9. *Consider the abelain group* $(\mathbb{R}, +)$. *Define a mapping*

$$\begin{cases} \circ : \mathbb{R} \times \mathbb{R} \longrightarrow P_*(\mathbb{R}) \\ \quad a \circ b = \{-ab, ab\}. \end{cases}$$

Then $(\mathbb{R}, +, \circ, \mathbb{R})$ *is a hyperspace. The mapping* $T : \mathbb{R} \longrightarrow P_*(\mathbb{R})$ *defined by* $T(a) = \{0, a\}$ *is a MLT, since* $T(a + b) = \{0, a + b\}$ *and* $T(a) + T(b) = \{0, a\} + \{0, b\} = \{0, a, b, a + b\}$, *so* $T(a + b) \subseteq T(a) + T(b)$. *Also,* $T(a \circ b) = \bigcup_{x \in a \circ b} T(x) = T(-ab) \cup T(ab) = \{0, -ab\} \cup \{0, ab\} = \{-ab, ab\}$ *and* $a \circ T(b) = a \circ \{0, b\} = \bigcup_{x \in \{0,b\}} a \circ x = a \circ 0 \cup a \circ b = \{0\} \cup \{-ab, ab\}$, *hence* $T(a \circ b) = a \circ T(b)$. *Then we have* $-T(a) = \{-x : x \in T(a)\} = T(-a)$. *Clearly,* T *is strongly injective, but it is not weakly injective, as desired.*

Proposition 3.10. *([4]) Let V and W be strongly left distributive hyperspaces such that $|1 \circ x| = 1$ for all $x \in V$. If $T : V \longrightarrow P_*(W)$ is monic in $\mathcal{HV}^s_K$, then T is strongly injective.*

4 Hyperoperations on $Hom_K(V, W)$

Next we proceed to constructs some algebraic hyperstructures on Hom_K (V, W)(resp. $Hom^s_K(V, W)$), the set of all MLT (resp. $SMLT$) as well as on $hom_K(V, W)$ and $hom^s_K(V, W)$), the set of all linear transformation LT (resp. strong linear transformation SLT) respectively and study some basic properties of them.

We start by $hom_K(V, W)$. Defin the operations $\oplus$ and $\odot$ on $hom_K(V, W)$ as follows:

$$T \oplus S(x) = T(x) + S(x); \quad \text{and} \quad (a \circ T)(x) = a \circ T(x).$$

Clearly, in general $T \odot S$ and $a \circ T$ are not members of $hom_K(V, W)$, but they are members of $Hom_K(V, W)$. These shows that the study of multivalued linear transformations are more useful than the linear transformations in a hyperspace. As by Remark4.6 we can consider morphisms in $hom_K(V, W)$ as morphisms of $Hom_K(V, W)$, we will prefer to work by multivalued linear transformations as a general case. Also, for $T \in Hom_K(V, W)$, $-T$ is defined by $-T(x) = -T(x)$. Then the following holds:

Lemma 4.1. *For $S, T \in Hom_K(V, W)$. The following statements are satisfies:*
(i) $(Hom(V, W), \oplus)$ *is a monoid;*
(ii) $0 \in -T \oplus T$;
(iii) $(Hom_K(V, W), \oplus, \odot K)$*, is a quasi vector space(that is a monoid $(M, +)$ by a function $. : K \times V \longrightarrow V$ that satisfies the all axioms of a K-vector space).*

Proof. The proof is strightforward. □

Now we define a hyperoperation $\oplus$, and operation $\odot$ on $Hom_K(V, W)$ as follows:

$$(T \oplus S)(x) = \{U| \ \ U(x) \subseteq T(x) + S(x)\}.$$

$$(a \odot T)(x) = a \circ T(x).$$

Theorem 4.2. *The following statements are satisfies for every $S, T \in Hom_K(V, W)$ and every $a, b \in K$:*

(i) $(Hom_K(V, W), \oplus)$ *is a commutative hypergroup, with the zero map as identity element ;*

(ii) $(Hom_K(V, W), \oplus)$ *is a commutative hypergroup;*

(iii) $(Hom_K(V, W), \oplus, \odot, K)$ *is a general hypervector space(that is a commutative hypergroup with an scalar identity, together with function* $\odot : K \times V \longrightarrow V$ *that satisfies the all axioms of a vector space over field* K*).*

Proof. The proof is routin and omitted. □

5 Fundamental relation of hyperspaces

Let $(V, +, \circ, K)$ be a hypervector space over K. The smallest equivalence relation ε^* on V, such that the quotient V/ε^* is a vector space over K is called the *fundamental relation* of V. T. Vougiouklis in [20] introduced and studied the fundamental relation of H_v-vector space (a general class of hypervector spaces). In the following we characterize the fundamental relation on hypervector spaces (in the sense of Tallini) and study the relationship between V and V/ε^*(for more detailes see [2]). In the following we consider the category $\mathcal{HV}_K$, the category of hyperspaces(with basis) and multivalued linear transformations to construct the fundamental functor from $\mathcal{HV}_K^s$ into $\mathcal{V}_K$, the category of vector spaces over K.

Let $\mathbf{U}$ be the set of all finite linear combinations of elements of V with coefficient in K, that is

$$\mathbf{U} = \left\{ \sum_{i=1}^{n} a_i \circ x_i \,:\, a_i \in K \text{ and } x_i \in V \;,\, n \in \;\mathbb{N} \right\}.$$

Define the relation ε over V by

$$x\varepsilon y \Longleftrightarrow \exists \mathbf{u} \in \mathbf{U} : \{x, y\} \subseteq \mathbf{u}, \quad \forall x, y \in V.$$

Then ε^* is the *transitive closure* of ε. Define addition operation and scalar multiplication on V/ε^* by

$$\begin{cases} \oplus : V/\varepsilon^* \times V/\varepsilon^* \longrightarrow V/\varepsilon^* \\ \varepsilon^*(x) \oplus \varepsilon^*(y) = \{\varepsilon^*(t) : t \in \varepsilon^*(x) + \varepsilon^*(y)\}, \end{cases}$$

and
$$\begin{cases} \odot : K \times V/\varepsilon^* \longrightarrow V/\varepsilon^* \\ a \odot \varepsilon^*(x) = \{\varepsilon^*(z) : z \in a \circ \varepsilon^*(x)\}, \end{cases}$$

Lemma 5.1. *([2]) The following statement are satisfied:*

(i) $\varepsilon^*(a \circ x) = \varepsilon^*(y)$ *for all* $y \in a \circ x$, $\forall a \in K$, $\forall x \in V$, *where* $\varepsilon^*(a \circ x) = \bigcup\limits_{b \in a \circ x} \varepsilon^*(b)$.

(ii) $\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(x + y)$.

(iii) $\varepsilon^*(\underline{0})$ *is the identity element of* $(V/\varepsilon^*, \oplus)$.

(iv) $(V/\varepsilon^*, \oplus, \odot, K)$ *is a vector space over* K.

The vector space $(V/\varepsilon^*, \oplus, \odot, K)$ *is called the fundamental vector space of* V.

Theorem 5.2. *([2]) Let* $(V, +, \circ, K)$ *be a hypervector space and* $(V/\varepsilon^*, \oplus, \odot, K)$ *be the fundamental vector space of* V*. Then*
$$\dim V = \dim V/\varepsilon^*.$$

Lemma 5.3. *Let* V *and* W *be two hypervector spaces and* $T : V \longrightarrow W$ *be a SM. Then*

(i) $\forall x \in V$, $T(\varepsilon^*(x)) \subseteq \varepsilon^*(T(x))$;

(ii) The map
$$\begin{cases} T^* : V/\varepsilon^* \longrightarrow W/\varepsilon^* \\ T^*(\varepsilon^*(x)) = \varepsilon^*(T(x)) \end{cases}$$
is a linear transformation.

Proof. First note that since $T(x)$ is a nonempty subset of V for every $x \in V$. Then $\varepsilon^*(T(x)) = \bigcup_{y \in T(x)} \varepsilon^*(y) = \varepsilon^*(y), \forall y \in T(x)$. Now since T maps every linear combination of V to a linear combination of W. Then (i) follows. (ii) is strightforward. $\square$

Theorem 5.4. *The mapping* $F : HV^s_K \longrightarrow \mathcal{V}_K$ *is defined by* $F(V) = V/\varepsilon^*$ *is a functor. Moreover, the functor* F *preserves the dimension.*

Proof. The proof is similar to the proof of [2] by some manipulation. $\square$

Corollary 5.5. *Let* $T : V \longrightarrow W$ *be a morphism in* $\mathcal{HV}^g_K$*. Then the following diagram is commutative:*
$$\begin{array}{ccc} V & \xrightarrow{T} & P_*(W) \\ \varphi_V\downarrow & & \downarrow\varphi_W \\ V/\varepsilon^* & \xrightarrow{T^*} & W/\varepsilon^* \end{array}$$
where φ_V *and* φ_W *are the canonical projections of* V *and* W*, respectively.*

Proof. Let $x \in V$. Then

$$\begin{aligned}
\varphi_W(T(x)) &= \varepsilon^*(T(x)) \\
&= T^*(\varepsilon^*(x)) \\
&= T^*(\varphi_V(x)) \\
&= T^*\varphi_V(x).
\end{aligned}$$

□

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The views of primary education teachers on the verification of multiplication

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Abstract

Learning and using the four mathematical operations -addition, subtraction, multiplication and division- are very important in the primary school syllabus curriculum.
The verifications for the correctness of the operations are simple since they can be justified with the use of basic mathematical properties. However, this is not the case for one of the verifications of multiplication which seems to be preferred by most of the elementary school teachers in their practice. With this verification, the control of multiplication's correctness is only a necessary but not sufficient condition and it is based on the Numbers' Theory.
In this paper we present the findings of a study on the views of a group consisted of twenty four elementary school teachers using activities related to the operation of the multiplication and its verification.

Key words: multiplication and its verifications; elementary school teachers.

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1 Introduction

University education and continuous training of primary school teachers must include the essential scientific and technological knowledge which will enable them to contribute to education promotion taking into account the added value of their pedagogical role and practices. The scientific knowledge of the cognitive objects taught in elementary school and of the way that teaching is performed are essential prerequisites in order for the teachers to be successful in their work.

During their continuous education practicing teachers can face a variety of issues such as completion of their basic education, introduction of new methods of teaching or even reforms of the educational system.

In the framework of the continuous education in Mathematics of a team consisted of twenty four primary school teachers, we introduced a curriculum which included three hours of teaching multiplication and its verifications.

In the beginning we analyzed the conceptual field of multiplicative structures.

As it is well known the theory of conceptual fields according to Vergnaud [1], has two aims: "to describe and analyze the progressive complexity of competences that students develop in Mathematics inside and outside the school and to establish better connections between the operational form of knowledge, which consists in action in the physical and social world, and the predicative form of knowledge, which consists in the linguistic and symbolic expressions of this knowledge."

The conceptual field is "a set of problems and situations for the treatment of which concepts, procedures, and representations of different but narrowly interconnected types are necessary" [2]. The multiplication structures are "a conceptual field of multiplicative type, as a system of different but interrelated concepts, operations, and problems such as multiplication, division, fractions, ratios, similarity" [2] . "A single concept does not refer to only one type of situation and a single situation cannot be analyzed with only one concept" [3]. In addition, conceptual field is "a set of situations, the mastering of which requires mastery of several concepts of different natures" [3]. "Concepts-in-action serve to categorize and select information whereas theorems-in-action serve to infer appropriate goals and rules from the available and relevant information"[4].

Even the most complicated concepts, in order to be meaningful and functional should be placed in a framework and be explained via examples. Thus a concept is simultaneously a set of situations, a set of operational constants and a set of linguistic and symbolic representations. The use of the framework of conceptual fields is necessary for the analysis of the continuities and the discontinuities of development in Mathematics and for the invention of situations that will prompt and help students to move along the multifaceted complexity of conceptual field" [1] .

The multiplicative structures constitute a part of the field of the additive structures, if multiplication is considered as repeated addition. However, due to the fact that multiplication has its own internal structure and organization, it can also be considered as an independent operation.

The approach of multiplication as a repeated addition has a lot of limitations. Some of them are that "it does not easily generalize to rationals, it does not demonstrate commutativity and it emphasizes grouping over sharing approaches to division" [5]. The difficulties in learning multiplicative reasoning are often due to the different ways in which students think about multiplication problems and the models they use. The standard algorithm for teaching the multiplication of larger numbers requires memorization of the basic multiplication facts. However, a wide variety of efficient, alternative algorithms exists based on the history of Mathematics, such as finger multiplication, multiplication's area model, lattice multiplication, line, circle/radius, paper strip, egyptian, russian peasant, etc. [6].

Despite the fact that the development of procedural techniques for multiplication in the Greek school textbooks is performed mainly via the "grid method", the area model does address some of the limitations of repeated addition, even if it does not easily relate to rate. However, most applications of multiplicative reasoning include the rates, therefore, certain researchers propose the use of double number line [5].

The main types of multiplicative structures are:

1. Isomorphism of measures
2. Multiplication factor, or an area of measures
3. Product of measures or Cartesian product
4. Multiple proportion [2]

For the multiplication's problems most researchers identify four different categories of multiplicative structures. Two of them, the equal groups (repeated addition) and the multiplicative comparison, are the most prevalent in the elementary school. The two others, combinations (Cartesian products) and problems with product of measures (length on width equal acreage), are used less frequently [7].

The difference of multiplicative problems from the problems of addition or abstraction is due to the fact that their numbers represent different types of things. A number or a factor counts how many sets, groups, or parts of equal size are involved (multiplier) and the other tells the size of each set or part (multiplicand) while the third number is the whole or the total of all the parts [7].

Kindergarten and first-grade children can solve multiplication and divisions problems, even if division involves remainders. The strategies they follow are not reflection of multiplicative reasoning, but their involvement in all four operations improves their level of understanding and guides them early enough to the development of multiplicative strategies.

Strategies used for the algorithm of multiplication are more complex than these for addition and subtraction. In addition the ability to break numbers apart in flexible ways is even more important in multiplication [7].

2 Brief discussion on the verifications of the operations

During our group discussions a teacher proposed to also refer to the other operations and theirs verifications. In particular, the teachers suggested that in the set of natural numbers the verifications of the operations of addition, subtraction and division are based on their simple properties.

Thus, for the addition $\alpha+\beta=\gamma$ the verification is $\beta+\alpha=\gamma$, meaning the use of the commutative property. Certainly we can also check the correctness of the addition through subtraction, if from the sum we subtract one of the two addends, so we will find the other, that is to say $\alpha+\beta=\gamma \Leftrightarrow \gamma-\alpha=\beta$ or $\gamma-\beta=\alpha$. The second method, as it was pointed out by the teachers, can be used only if the students have already been taught subtraction.

For the subtraction $\alpha-\beta=\gamma$ the verification is $\beta+\gamma=\alpha$. Moreover the correctness of the subtraction can also be checked using the relation $\alpha/\gamma=\beta$. For the division $\Delta:\delta$ where $\frac{\Delta}{\delta}=\Pi+\frac{\nu}{\delta}$ the verification is $\Delta=\delta\cdot\Pi+\nu$ with $0\leq\nu<\delta$.

In each one of the previous cases verification is a necessary and sufficient cond for an operation to be correct.

3 The discussion on multiplication and its verification

In order to study the multiplication's verification the teachers were assigned the following activity: The numbers 4789 and 635 were given and they were asked to:

1. Find their product.

2. Perform the verification of multiplication and interpret it.

Before moving to algorithms of multiplication we mentioned various useful representations (material pieces of decimal base, area model, etc), which subsequently were excluded because the numbers used were large.

One teacher (Teacher 1) was then asked to perform the multiplication using the known traditional algorithm which is also considered as the most difficult (Figure 1). If the students fail to understand it, they place -as it was reported by the teachers- the numbers in error columns, they add the carries before they multiply and in this way they make a lot of errors.

$$\begin{array}{r} 4789 \\ \times\ \ 635 \\ \hline 23945 \\ 14367\ \\ +\ \ 28734\ \ \\ \hline 3041015 \end{array} \tag{1}$$

After extensive discussion the following ways of multiplication's performance were presented such as:
1. (Teacher 2) Analyze (635) to (600+30+5) and then multiply the multiplicand with 5, 30 and 600 meaning: perform three multiplications and then add their products (Figure 2).(Teacher 3) This method is correct and it is based on the distributive property of multiplication in regard to addition , which is an important concept for multiplication. (Teacher 1) In this method the final products are the partial products of the initial multiplication.

$$\begin{array}{r} 4789 \\ \times\ \ 5 \\ \hline 23945 \end{array} \qquad \begin{array}{r} 4789 \\ \times\ \ 30 \\ \hline 143670 \end{array} \qquad \begin{array}{r} 4789 \\ \times\ \ 600 \\ \hline 2873400 \end{array}$$

$$\begin{array}{r} 23945 \\ 143670 \\ +\ \ 2873400 \\ \hline 3041015 \end{array} \tag{2}$$

2. (Teacher 4) Another method is to change the position of the multiplicand by the multiplier, for example to perform the multiplication. This method is based on the use of the commutative property. The resulting partial products are different from the partial products of the multiplication(initial case).(Figure 3)

$$\begin{array}{r}
635 \\
\times \quad 4789 \\
\hline
5715 \\
5080 \\
4445 \\
+\ 2540 \\
\hline
3041015
\end{array} \tag{3}$$

3. (Teacher 5) We could also perform the multiplication without using carries, as shown below, but in this way we would have 12 products and a "great" addition afterwards. (Figure 4) In that case, emphasis should be given to the proper placement of the obtained products and to the following addition.

$$\begin{array}{r}
4789 \\
\times \quad 635 \\
\hline
45 \\
40 \\
35 \\
20 \\
27 \\
24 \\
21 \\
12 \\
54 \\
48 \\
42 \\
+\quad 24 \\
\hline
3041015
\end{array} \tag{4}$$

4. (Teacher 6) Because the multiplication can be performed as follows and has twelve partial products (Figure 5).

$$\begin{array}{rr} & 4789 \\ \times & 635 \\ \hline & 2400000 \\ & 420000 \\ & 48000 \\ & 5400 \\ & 120000 \\ & 21000 \\ & 270 \\ & 20000 \\ & 3500 \\ & 400 \\ + & 45 \\ \hline & 3041015 \end{array} \tag{5}$$

5. Other types of strategies for multiplication's performance were also mentioned but were later excluded by the same teachers because the numbers were large. In particular the teachers reported the following: 1) strategies without breaking numbers into parts (they usually use successive additions in different ways (Teacher 7), 2) partitioning strategies (breaking the numbers in a variety of ways and subsequently use the distributive property Teacher 7) and 3) compensation strategies (breaking the numbers in a variety of ways so that the calculations are easier and result to partial products which are then added, Teacher 8) [7].
The other issue that was discussed included whether the previous ways constitute verification of the standardized multiplication's algorithm. The view of most schoolteachers was that they represent a different way of finding the product by which the correctness of the result of standardized algorithm can be checked, without them constituting verification.
As it was found by the discussion that followed, the verification that teachers chose in their practice wasn't based on the usual properties but followed a special method, that one of the cross [8].
A detailed report of this method was presented and followed by an attempt to highlight the teachers' views regarding its validity.
Educator (E): Who would want to perform the multiplication's verification?
Teacher (T1): (He made the cross and began to supplement it explaining every step that he followed).

$$\begin{array}{c|c} 1 & 5 \\ \hline 5 & 5 \end{array}$$

Top left corner: We add the digits of the first factor until one-digit number results: 4+7+8+9=29, 2+8=10, 1+0=1. We write this number on this corner.
Top right corner: We add the digits of the second factor until one-digit number results: 6+3+5=13 and 1+4=5. We write number 5 on this corner.
Bottom left corner: we calculate the product of the two numbers and we find $1 \cdot 5 = 5$. We write it in the bottom left corner.
Bottom right corner: we calculate the sum of the digits of the two numbers, we find 3+0+4+1+0+1+5=14 and 1+4=5 . We write it on the bottom right corner.
Educator (E): Is the multiplication correct?
Teacher (T10): Yes!
Educator (E): When a multiplication- if checked with this method- is correct?
Teacher (T11): The multiplication is correct if the numbers on the second line of the cross are the same.
Educator (E): Is this always true?
The following discussion took place:
Teacher (T12): This method does not always ensure that the multiplication is correct.
Educator (E): Why?
Teachers (T): For many reasons most of the teachers answered simultaneously.
Educator (E): Who would like to discuss some and then try to analyze them?
Teacher (T13): One possibility is that the digits that are presented are not placed in the correct position. That is, instead of the number 3041015 that is the correct result, the number 3041015 is written which results by reversing two of its digits.
Teacher (T14): Another possibility is that the digits that are presented in the last product are different (due to an error in the addition of the partial sums) resulting in the same sum. That is to say, instead of the number 3041015 which expresses the correct result and number 5 being the final one-digit sum of digits, the number written is 3041915 which has also the same sum of digits.
Teacher (T15): Another case is when an additional 0 is interposed between the digits of the correct number, meaning instead of the number 3041015, the number 30410015 is written.
Teacher (T16): Or a 0 is added at the end of the number. Thus, instead of the number 3041015, the number 30410150 is written.
Teacher (T17): A 0 is skipped either between the digits of the number or before its end, so for example instead of the number 3041015 we have number

304115.
Teacher (T18): A badly written 0 can be considered as 9 or vice versa, for example instead of the number 3041015 we have number 3941015.
Teacher (T2): With this method many errors can occur.
Educator (E): Based on the above cases or others, by this verification the multiplication seems "correct", but it is not. So, what do you think about this verification, does it always show if the multiplication is correct or not?
Teacher (T9): If the multiplication is correctly performed this method verifies it. If, however, the multiplication is not correctly performed, it is not certain that this will be shown by this verification.
Teacher (T9): Why does this happen and how is it explained?
Educator (E): Do you think that this verification is similar to the verifications of the other three operations?
Teacher (T9): Does this verification is a "necessary condition" for the multiplication to be correct but not sufficient?
Educator (E): A condition can only be necessary as it happens with the cross verification.
Afterwards the educator presented the basics from the equal remainder numbers' theory. He proved the method of the particular verification of multiplication and it was applied to the example that was previously discussed.
If two numbers $x, y \in R$, then

$$S_{x.y} \equiv S_x \cdot S_y (mod 9)$$

Proof
We know that: $x \equiv S_x(\text{mod}9)$ and $y \equiv S_y(\text{mod}9)$, therefore $x \cdot y \equiv S_x \cdot S_y(\text{mod}9)$. However $x \cdot y \equiv S_{x.y}(mod 9)$, therefore $S_{x.y} \equiv S_x \cdot S_y(\text{mod}9)$.
Example of the above proof:
Consider the numbers x = 4789 and y = 635. Then $S_x = 4+7+8+9 = 28 \equiv 1$ (mod9) and $S_y = 6 + 3 + 5 = 14 \equiv 5$ (mod9), therefore $S_{x.y} \equiv 1.5(mod 9) \equiv 5(mod 9)$.
However $x.y = 4789.635 = 3041015$, $S_{x.y} = 3+0+4+1+0+1+5 = 14 \equiv 5$. Hence, $S_{x.y} \equiv S_x \cdot S_y(mod 9)$.
With this proposal it is proved that the necessary but not the sufficient condition exists in order for the multiplication to be correct. This method, as it was shown from the discussion, is used by all the teachers in their practice, without being as easy and understandable as the verifications of the other three operations. This resulted from teachers' comments during the interpretation of the cross method, which monopolized the discussion. Regarding the clarification of the phrase 'necessary and sufficient condition'

it was shown that this was not always clear to teachers. However this was not the subject of this study.

4 Conclusion

The standard algorithm for the multiplication of large numbers is brought to Europe by the Arabic-speaking people of Africa and requires the memorization of the basic multiplication facts. Multiplication is an important tool not only for constructing a firm foundation for proportional reasoning and the algebraic thinking, but also for solving real-life problems [6].
Teachers' mathematical knowledge includes not only Mathematics but also their teaching. The framework of the conceptual field may help them organize appropriate didactic situations and interventions [1]. It is essential that the improvement of the teaching of Mathematics regarding teachers' explanations, the representations and the examples they use and also of the method with which all the above are developed in addition to the way they themselves interact with their students, something that is achieved with their continuous training [9].
In the framework of this particular training it was found that schoolteachers use different ways in order to verify if a multiplication is correct. In conclusion: A. they reverse the multiplier with the multiplicand using the commutative property, B. they calculate partial products and they sum them up afterwards, c. they perform multiplication without using held, d. they calculate the partial products writing analytical numbers in thousands, hundreds, tens and ones using the distributive property, e. they use various informal forms for the execution of multiplication and g. they use the cross method.
The justification of the first four ways of verification was complete and understandable by most of the teachers; however, the third way was not used as much as the others.
For the verification of the cross method the results showed that all teachers use it in their daily practice at school, even if it is not included in the school textbooks. Most of the teachers that participated in this study they used but didn't empirically consider it reliable since they could not explain it adequately.
It is probable that the use of this method of multiplication's verification is related to teachers' age, which in our case teachers had at least ten years of professional experience. Further research which will include candidate schoolteachers with different curriculum of undergraduate studies would be of special interest.

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On weights of 2-repeated bursts

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Abstract

It is generally seen that the behavior of the bursts depend upon the nature of the channel. In a very busy communication channel bursts repeat themselves. In this communication we are exploring the idea of weight consideration of 2-repeated bursts of length b(fixed). Some results on weights of 2-repeated bursts of length b(fixed) are derived and some combinatorial results with weight constraint for 2-repeated bursts of length b(fixed) are also given.

Key words: repeated burst errors, weight of bursts, burst error correcting codes.

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1 Introduction

In most of the communication channels disturbances due to lightning, break downs and loose connections affect successive digits for some length of the word, causing errors in bursts. Abramson (1959) initiated the idea of such errors and developed a class of error correcting codes which correct all double adjacent errors. Later, a systematic study in this direction was made by Fire (1959), Regier (1960) and Elspas (1960). These studies were based on the assumption that if errors occur in the form of bursts then all digits within a burst may not be corrupted. Easy implementation and efficient functioning are the added advantages with burst error correcting codes. Stone (1961) and Bridwell and Wolf (1970) considered multiple bursts. It was noted by Chien and Tang (1965) hat in several channels errors occur in the form of a burst but not in the end digit of the burst. Channels due to Alexander, Gryb and Nast (1960) belong to this category. In the view of this Chien and Tang modified the definition of a burst which in literature is known as CT burst. Although, this definition was further modified by Dass (1980).

In general communication the messages are long and the strings of bursts may be short repeating in a vector itself. The notion of repeated burst was introduced by Berardi, Dass and Verma (2009). They defined 2-repeated bursts and obtained results for correction and detection of such type of errors. Dass, Garg and Zannetti (2008) introduced a different type of repeated burst, termed as repeated burst of length b(fixed). Later on Dass and Garg (2009) defined 2-repeated burst of length b(fixed) and gave codes for correcting and detecting such type of errors. Sharma and Dass (1976) were first to study bursts in terms of weight. The area of 2-repeated burst of length b(fixed) with weight w was explored by Dass and Garg (2011).

In this paper, we obtain results regarding the weight of all vectors having 2-repeated bursts of length b(fixed). The paper has been organized as follows: In section 2 basic definitions are stated with some examples. In section 3 some results on weights of 2-repeated bursts of length b(fixed) are derived.

In this correspondence, we shall consider the space of all n-tuples whose nonzero components are taken from the field of q code characters with elements $0, 1, 2, \ldots, q-1$. The weight of a vector is considered in Hamming sense as the number of non-zero entries.

2 Preliminaries

We give definition of a burst, defined by Fire (1959):

Definition 2.1. *A burst of length b* is a vector all of whose nonzero components are confined to some b consecutive components, the first and the last of which is nonzero.

A vector may have not just one cluster of errors, but more than one. Lumping them into one burst, amounts to neglecting the nature of communication and unnecessarily considering longer burst which may have a part, which is not of cluster in-between. For example in a very busy communication channel, sometimes, bursts repeat themselves. Berardi, Dass and Verma (2009) introduced the idea of repeated bursts. In particular they defined '2-*repeated burst*'.

A 2-repeated burst of length b may be defined as follows:

Definition 2.2. *A* 2*-repeated burst of length* b is a vector of length n whose only nonzero components are confined to two distinct sets of b consecutive components, the first and the last component of each set being nonzero.

Example 2.1. (0001204100300) is a 2-repeated burst of length 4 over GF(5).

Chien and Tang (1965) defined a burst of length b which is called as CT burst of length b and may be defined as follows:

Definition 2.3. *A CT burst of length b is a vector whose only non-zero components are confined to some b consecutive positions, the first of which is non-zero.*

This definition was further modified by Dass (1980) as follows:

Definition 2.4. *A burst of length b(fixed) is an n-tuple whose only non-zero components are confined to b consecutive positions, the first of which is non-zero and the number of its starting positions in an n-tuple is among the first $n - b + 1$ components.*

Following is the definition of a 2-repeated burst of length b(fixed) as given by Dass and Garg (2009):

Definition 2.5. *A* 2*-repeated burst of length b(fixed) is an n-tuple whose only non-zero components are confined to two distinct sets of b consecutive digits, the first component of each set is non-zero and the number of its starting positions is amongst the first $n - 2b + 1$ components.*

For example, (10000010000) is a 2-repeated burst of length up to 5(fixed) whereas (0000100100) is a 2-repeated burst of length at most 3 (fixed).

Dass and Garg (2011) defined a 2-repeated burst of length b(fixed) with weight w as follows:

Definition 2.6. *A* 2*-repeated burst of length b(fixed) with weight w or less is an n-tuple whose only non-zero components are confined to two distinct sets of b consecutive components the first component of each set is non-zero where each set can have at most w non-zero components ($w \leq b$), and the number of its starting positions is among the first $n - 2b + 1$ components.*

For example, (001111000000100000) is a 2-repeated burst of length up to 6(fixed) with weight 4 or less.

Weight structure being of quite some interest, in the next section, we present some results on weights of 2-repeated bursts of length b(fixed).

3 Results on Weights of 2-repeated bursts

Let W_{2b} denotes the total weight of all vectors having 2-repeated bursts of length b in the space of all n-tuples. Before obtaining W_{2b} in terms of n and b we give two results in the lemmas below, on counting the 2-repeated bursts.

Lemma 3.1. *The total number of* 2*-repeated bursts length* $b > 1$*(fixed), in the space of all* n*-tuple over* GF(q)*, is*

$$\frac{(n-2b+1)(n-2b+2)}{2}(q-1)^2 q^{2(b-1)}\,. \tag{1}$$

Proof. Total number of 2-repeated bursts of length b(fixed) in the space of all n-tuples over $GF(q)$ is, refer Theorem 1 of Dass, Garg and Zannetti (2008),

$$1+\binom{b}{1}(q-1)q^{b-1}+\sum_{i=1}^{n-2b+1}(q-1)q^{b-1}\left[1+\binom{n-2b-i+2}{1}(q-1)q^{b-1}\right]. \tag{2}$$

Eqn. (2) includes the cases when all vectors are zero and when in the last $2b-1$ position there remains only a single burst of length b(fixed).

As we are counting the number of 2-repeated bursts of length b(fixed) only, eqn. (2) reduces to the following form

$$\sum_{i=1}^{n-2b+1}(q-1)q^{b-1}\left[1+\binom{n-2b-i+2}{1}(q-1)q^{b-1}\right]$$

or

$$\frac{(n-2b+1)(n-2b+2)}{2}(q-1)^2 q^{2(b-1)}.$$

This proves the result. □

Next we impose weight restriction on 2-repeated bursts and count their numbers. The results are given in the lemma below.

Lemma 3.2. *The total number of vectors having* 2*-repeated bursts of length* b*(fixed) with weight* w $(1 \le w \le b)$ *in the space of all* n*-tuples is:*

$$\frac{(n-2b+1)(n-2b+2)}{2}[L_{w,q}^{b-1}]^2, \tag{3}$$

where

$$\left[\sum_{s=1}^{w}\binom{b-1}{s-1}(q-1)^s\right] = L_{w,q}^{b-1} \tag{4}$$

is the incomplete binomial expansion of $[1+(q-1)]^{b-1}$ *up to the* $(q-1)^w$ *in the ascending powers of* $(q-1)$*,* $w \le b$*.*

Proof. Let us consider a vector having 2-repeated bursts of length b(fixed) with weight w. Its only nonzero components are confined to two distinct sets of consecutive components, the first component of each set is nonzero, where each set can have at most w non-zero components ($w \leq b$), and the number of its starting positions is among the first $n-2b+1$ components. Now, each of these, the first component of each set may be any of the $q-1$ nonzero field elements. As we are considering only 2-repeated bursts of length b(fixed) with weight w, in a vector of length n, this will have non-zero positions as follows:

(i) First position of first burst.

(ii) First position of second burst.

(iii) Some $r-1$ amongst the $b-1$ in-between positions of first burst ($1 \leq r \leq w$) and then some $s-1$ in the in-between $b-1$ positions of the second burst ($1 \leq s \leq w$).

(iv) Other positions have the value 0.

Thus analyzing in combinatorial ways, in the earlier counting factor $[(q-1)q^{b-1}]^2$ replacing one factor q^{b-1} by $\binom{b-1}{r-1}(q-1)^{r-1}$ and the other by $\binom{b-1}{s-1}(q-1)^{s-1}$ each 2-repeated burst will give its number by:

$$\begin{aligned}
&(q-1)(q-1)\sum_{r=1}^{w}\binom{b-1}{r-1}(q-1)^{r-1}\sum_{s=1}^{w}\binom{b-1}{s-1}(q-1)^{s-1} \\
&= \sum_{r=1}^{w}\binom{b-1}{r-1}(q-1)^{r}\left[\sum_{s=1}^{w}\binom{b-1}{s-1}(q-1)^{s}\right].
\end{aligned}$$

Then from eqn. (4) the number of each 2-repeated burst of length b(fixed) with weight w is given by,

$$[L_{w,q}^{b-1}]^2.$$

Therefore, the total number of 2-repeated bursts of length b(fixed) and weight w, with sum of their starting position $\dfrac{(n-2b+1)(n-2b+2)}{2}$ is

$$\frac{(n-2b+1)(n-2b+2)}{2}[L_{w,q}^{b-1}]^2\,.$$

This proves the lemma. $\square$

Now we return to finding an expression for W_{2b}, the total weight of all vectors having 2-repeated bursts of length b(fixed) in the space of all n-tuples.

Theorem 3.1. *For $n \geq b$*

$$W_2 = \frac{n(n-1)}{2}(q-1)^2 \tag{5}$$

and

$$W_{2b} = \frac{(n-2b+1)(n-2b+2)}{2} w^2 [L_{w,q}^{b-1}]^2 \,. \tag{6}$$

Proof. The value of W_2 follows simply by considering all vectors having any two non-zero entries out of n. Their number clearly is given by

$$\binom{n}{2}(q-1)^2 = \frac{n(n-1)}{2}(q-1)^2 \,.$$

This gives the value of W_2 as stated.

Next, for $b > 1$, using the Lemma 3.2, the total weight of all vectors having 2-repeated bursts of length b(fixed) each with weight of each burst at most w, is given by

$$\sum_{i=1}^{w}\sum_{j=1}^{w} \frac{(n-2b+1)(n-2b+2)}{2} i[L_{w,q}^{b-1}] \cdot j[L_{w,q}^{b-1}]$$
$$= \frac{(n-2b+1)(n-2b+2)}{2} w^2 [L_{w,q}^{b-1}]^2 \,.$$

This completes the proof of the theorem. □

Further, in coding theory, an important criterion is to look for minimum weight in a group of vectors. Our following theorem is a result in that direction.

Theorem 3.2. *The minimum weight of a vector having 2-repeated burst of length $b > 1$(fixed) in the space of all n-tuples is at most*

$$\left[\frac{wL_{w,q}^{b-1}}{(q-1)q^{b-1}}\right]^2 \,. \tag{7}$$

Proof. From Lemma 3.1, it is clear that the number of 2-repeated bursts of length b(fixed) in the space of all n-tuples with symbols taken from the field of q elements is

$$[q^{(b-1)}(q-1)]^2 \frac{(n-2b+1)(n-2b+2)}{2} \,.$$

Also from Theorem 3.1, their total weight is

$$\frac{(n-2b+1)(n-2b+2)}{2} w^2 [L_{w,q}^{b-1}]^2 .$$

Since the minimum weight element can at most be equal to the average weight, an upper bound on minimum weight of a 2-repeated burst of length b(fixed) is given by

$$\frac{(n-2b+1)(n-2b+2)}{2} w^2 [L_{w,q}^{b-1}]^2 \cdot \frac{2}{(n-2b+1)(n-2b+2)(q-1)^2 q^{2(b-1)}}$$
$$= \left[\frac{w L_{w,q}^{b-1}}{(q-1)\, q^{(b-1)}} \right]^2 .$$

This proves the result. □

4 Concluding remarks

Here we have considered vectors having two bursts of equal lengths b(fixed), with or without weight constraints. Studies generalizing these considerations have also attracted our attention that will be reported separately. With these bursts as error patterns in block-wise manner will be a part of later study as codes capable of correcting such type of error patterns will improve the communication rate.

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The sum of the series of reciprocals of the cubic polynomials with triple non-positive integer root

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Abstract

This contribution, which is a follow-up to author's paper [1] dealing with the sums of the series of reciprocals of some quadratic polynomials, deals with the series of reciprocals of the cubic polynomials with triple non-positive integer root. Three formulas for the sum of this kind of series expressed by means of harmonic numbers are derived and presented, together with one approximate formula, and verified by several examples evaluated using the basic programming language of the computer algebra system Maple 16. This contribution can be an inspiration for teachers who are teaching the topic Infinite series or as a subject matter for work with talented students.

Key words: telescoping series, harmonic numbers, CAS Maple, Riemann zeta function.

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1 Introduction

Let us recall some basic terms. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum $\displaystyle\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$. The *sequence of partial sums* $\{s_n\}$ associated to a series $\displaystyle\sum_{k=1}^{\infty} a_k$ is defined for each n

as the sum $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$. The series $\sum_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence of partial sums $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series $\sum_{k=1}^{\infty} a_k$ has a *sum* s and write $\sum_{k=1}^{\infty} a_k = s$.

The *nth harmonic number* is the sum of the reciprocals of the first n natural numbers: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$. The *generalized harmonic numbers* of order n in power r is the sum

$$H_{n,r} = \sum_{k=1}^{n} \frac{1}{k^r}\,, \tag{1}$$

where $H_{n,1} = H_n$ are harmonic numbers. Every generalized harmonic number of order n in power m can be written as a function of generalized harmonic number of order n in power $m-1$ using formula (see [2]):

$$H_{n,m} = \sum_{k=1}^{n-1} \frac{H_{k,m-1}}{k(k+1)} + \frac{H_{n,m-1}}{n}\,, \tag{2}$$

whence

$$H_{n,2} = \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n}\,, \qquad H_{n,3} = \sum_{k=1}^{n-1} \frac{H_{k,2}}{k(k+1)} + \frac{H_{n,2}}{n}\,.$$

Therefore

$$H_{n,3} = \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \left(\sum_{i=1}^{k-1} \frac{H_i}{i(i+1)} + \frac{H_k}{k} \right) + \frac{1}{n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n^2}\,,$$

thus

$$H_{n,3} = \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \left(\sum_{i=1}^{k-1} \frac{H_i}{i(i+1)} + \frac{H_k}{k} + \frac{H_k}{n} \right) + \frac{H_n}{n^2}\,. \tag{3}$$

From formula (1), where $r = 1, 2, 3$ and $n = 1, 2, \ldots, 8$, we get this table:

n	1	2	3	4	5	6	7	8
H_n	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$
$H_{n,2}$	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	$\frac{5269}{3600}$	$\frac{5369}{3600}$	$\frac{266681}{176400}$	$\frac{1077749}{705600}$
$H_{n,3}$	1	$\frac{9}{8}$	$\frac{251}{216}$	$\frac{2035}{1728}$	$\frac{256103}{216000}$	$\frac{28567}{24000}$	$\frac{9822481}{8232000}$	$\frac{78708473}{65856000}$

2 The sum of the series of reciprocals of the cubic polynomials with triple non-positive integer root

We deal with the problem to determine the sum $s(a,a,a)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^3}$$

for non-positive integers a, i.e. to determine the sum $s(0,0,0)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots\,, \tag{4}$$

the sum $s(-1,-1,-1)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)^3} = \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots = s(0,0,0) - s_1(0,0,0) = s(0,0,0) - 1\,,$$

the sum $s(-2,-2,-2)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)^3} = \frac{1}{3^3} + \frac{1}{4^3} + \cdots = s(0,0,0) - s_2(0,0,0) = s(0,0,0) - \frac{9}{8}$$

etc. Clearly, we get the formula

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^3} = s(0,0,0) - s_{-a}(0,0,0)\,, \tag{5}$$

where $s_{-a}(0,0,0)$ is the $(-a)$th partial sum of the series (4). Several values of the nth partial sums $s_n(0,0,0)$, briefly denoted by s_n, are: $s_{100} \doteq 1.2020074$, $s_{1000} \doteq 1.2020564$, $s_{10000} \doteq 1.2020569$, $s_{100000} \doteq 1.2020569$. Let us note that the series $s(0,0,0)$ converges to the *Apéry's constant* $1.202056903159\ldots$, which represents the value $\zeta(3)$ of the *Riemann zeta function*

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots\,.$$

The partial sums $s_n(0,0,0)$ so present the generalized harmonic numbers $H_{n,3}$. According to formula (5) is

$$s(a,a,a) = \zeta(3) - H_{-a,3}\,, \tag{6}$$

then using formula (3) we get

Theorem 2.1. *The series* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k-a)^3}$, *where* a *is a negative integer, has the sum*

$$s(a,a,a)=\zeta(3)-\sum_{k=1}^{-a-1}\frac{1}{k(k+1)}\left(\sum_{i=1}^{k-1}\frac{H_i}{i(i+1)}+\frac{H_k}{k}-\frac{H_k}{a}\right)-\frac{H_{-a}}{a^2}\,. \tag{7}$$

Now, we express formula (3) in another form. We have

$$\begin{aligned}
H_{n,3}=&\frac{1}{2}\left(\frac{H_1}{1}+\frac{H_1}{n}\right)+\frac{1}{6}\left(\frac{H_1}{2}+\frac{H_2}{2}+\frac{H_2}{n}\right)+\\
&+\frac{1}{12}\left(\frac{H_1}{2}+\frac{H_2}{6}+\frac{H_3}{3}+\frac{H_3}{n}\right)+\frac{1}{20}\left(\frac{H_1}{2}+\frac{H_2}{6}+\frac{H_3}{12}+\frac{H_4}{4}+\frac{H_4}{n}\right)+\\
&+\frac{1}{30}\left(\frac{H_1}{2}+\frac{H_2}{6}+\frac{H_3}{12}+\frac{H_4}{20}+\frac{H_5}{5}+\frac{H_5}{n}\right)+\cdots\\
\cdots+&\frac{1}{(n-3)(n-2)}\left(\frac{H_1}{2}+\frac{H_2}{6}+\cdots+\frac{H_{n-4}}{(n-4)(n-3)}+\frac{H_{n-3}}{n-3}+\frac{H_{n-3}}{n}\right)+\\
&+\frac{1}{(n-2)(n-1)}\left(\frac{H_1}{2}+\frac{H_2}{6}+\cdots+\frac{H_{n-3}}{(n-3)(n-2)}+\frac{H_{n-2}}{n-2}+\frac{H_{n-2}}{n}\right)+\\
&+\frac{1}{(n-1)n}\left(\frac{H_1}{2}+\frac{H_2}{6}+\cdots+\frac{H_{n-2}}{(n-2)(n-1)}+\frac{H_{n-1}}{n-1}+\frac{H_{n-1}}{n}\right)+\frac{H_n}{n^2}\,,
\end{aligned}$$

i.e.

$$\begin{aligned}
H_{n,3}=&\frac{H_1}{1\cdot 2}\left(\frac{1}{1}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\\
&+\frac{H_2}{2\cdot 3}\left(\frac{1}{2}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\\
&+\frac{H_3}{3\cdot 4}\left(\frac{1}{3}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\cdots\\
\cdots+&\frac{H_{n-3}}{(n-3)(n-2)}\left(\frac{1}{n-3}+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\\
+&\frac{H_{n-2}}{(n-2)(n-1)}\left(\frac{1}{n-2}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\frac{H_{n-1}}{(n-1)n}\left(\frac{1}{n-1}+\frac{1}{n}\right)+\frac{H_n}{n^2}\,.
\end{aligned} \tag{8}$$

Because $\dfrac{1}{k(k+1)}=\dfrac{1}{k}-\dfrac{1}{k+1}$, then the nth partial sum t_n of the telescoping series $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k(k+1)} = \frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\cdots$ is

$$t_n = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{1}{2}-\frac{1}{n+2}\,,$$

for the expressions in the first three parentheses of formula (8) we get

$$\begin{aligned}
\frac{1}{1}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n} &= 1+t_{n-2}+\frac{1}{n} = \\
&= 1+\left(\frac{1}{2}-\frac{1}{n}\right)+\frac{1}{n} = \frac{3}{1\cdot 2}\,,\\
\frac{1}{2}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n} &= \frac{1}{2}+t_{n-2}-t_1+\frac{1}{n} = \\
&= \frac{1}{2}+\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{5}{2\cdot 3}\,,\\
\frac{1}{3}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n} &= \frac{1}{3}+t_{n-2}-t_2+\frac{1}{n} = \\
&= \frac{1}{3}+\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{7}{3\cdot 4}
\end{aligned}$$

and analogously for the expressions in the last three parentheses of formula (8) we get

$$\begin{aligned}
\frac{1}{n-3}+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n} &= \frac{1}{n-3}+t_{n-2}-t_{n-4}+\frac{1}{n} = \\
&= \frac{1}{n-3}+\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{n-2}\right) = \frac{2n-5}{(n-3)(n-2)}\,,\\
\frac{1}{n-2}+\frac{1}{(n-1)n}+\frac{1}{n} &= \frac{1}{n-2}+t_{n-2}-t_{n-3}+\frac{1}{n} = \\
&= \frac{1}{n-2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{n-1} = \frac{2n-3}{(n-2)(n-1)}\,,\\
\frac{1}{n-1}+\frac{1}{n} &= \frac{2n-1}{(n-1)n}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
H_{n,3} &= \frac{H_1}{1\cdot 2}\cdot\frac{3}{1\cdot 2}+\frac{H_2}{2\cdot 3}\cdot\frac{5}{2\cdot 3}+\frac{H_3}{3\cdot 4}\cdot\frac{7}{3\cdot 4}+\cdots\\
&\cdots+\frac{H_{n-2}}{(n-2)(n-1)}\cdot\frac{2n-3}{(n-2)(n-1)}+\frac{H_{n-1}}{(n-1)n}\cdot\frac{2n-1}{(n-1)n}+\frac{H_n}{n^2}\,,
\end{aligned}$$

hence

$$H_{n,3} = \frac{3H_1}{(1\cdot 2)^2} + \frac{5H_2}{(2\cdot 3)^2} + \cdots + \frac{(2n-3)H_{n-2}}{[(n-2)(n-1)]^2} + \frac{(2n-1)H_{n-1}}{[(n-1)n]^2} + \frac{H_n}{n^2}\,,$$

thus

$$H_{n,3} = \sum_{k=1}^{n-1} \frac{(2k+1)H_k}{[k(k+1)]^2} + \frac{H_n}{n^2}\,. \tag{9}$$

From formulas (6) and (9) we obtain

Theorem 2.2. *The series* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k-a)^3}$*, where a is a negative integer, has the sum*

$$s'(a,a,a) = \zeta(3) - \sum_{k=1}^{-a-1} \frac{(2k+1)H_k}{[k(k+1)]^2} - \frac{H_{-a}}{a^2}\,. \tag{10}$$

Remark 2.1. In [4] it is derived that a good approximation for the partial sum $s_n(0,0,0)$ is the expression

$$\sum_{k=1}^{n} \frac{1}{k^3} \approx \zeta(3) - \frac{1}{4}\left(\frac{2}{n^2} - \frac{2}{n^3} + \frac{1}{n^4}\right). \tag{11}$$

It is stated that for small n, say, $n = 5$, the relative error in the above approximation is vanishingly small, i.e. about 0.03%, and that for larger $n \sim 1000$, the error is swamped by machine precision.

If we use formulas (11) and (5), where a is a negative integer, we get

$$\begin{aligned} s(a,a,a) &= s(0,0,0) - s_{-a}(0,0,0) = \\ &= \zeta(3) - \sum_{k=1}^{-a} \frac{1}{k^3} \approx \zeta(3) - \left[\zeta(3) - \frac{1}{4}\left(\frac{2}{(-a)^2} - \frac{2}{(-a)^3} + \frac{1}{(-a)^4}\right)\right], \end{aligned}$$

so we have an approximate formula $s(a,a,a) \approx \dfrac{1}{4}\left(\dfrac{2}{a^2} + \dfrac{2}{a^3} + \dfrac{1}{a^4}\right)$ and an approximate sum

$$\overline{s}(a,a,a) = \frac{2a^2 + 2a + 1}{4a^4}\,. \tag{12}$$

Example 2.1. Evaluate the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+5)^3}$$

by means of formula: **i)** (7), **ii)** (10), **iii)** (5), **iv)** (12) and compare obtained results.

Solution:

i) The series has by Theorem 2.1, where $a=-5$, the sum

$$s(-5,-5,-5)=\zeta(3)-\sum_{k=1}^{4}\frac{1}{k(k+1)}\left(\sum_{i=1}^{k-1}\frac{H_i}{i(i+1)}+\frac{H_k}{k}+\frac{H_k}{5}\right)-\frac{H_5}{25}\,.$$

The last summand $\dfrac{H_5}{25}=\dfrac{137/60}{25}=\dfrac{137}{1500}$. Now, we evaluate the middle summand:

$$\sum_{k=1}^{4}\frac{1}{k(k+1)}\left(\sum_{i=1}^{k-1}\frac{H_i}{i(i+1)}+\frac{H_k}{k}+\frac{H_k}{5}\right)=\frac{1}{1\cdot 2}\left(\frac{H_1}{1}+\frac{H_1}{5}\right)+$$
$$+\frac{1}{2\cdot 3}\left(\frac{H_1}{1\cdot 2}+\frac{H_2}{2}+\frac{H_2}{5}\right)+\frac{1}{3\cdot 4}\left(\frac{H_1}{1\cdot 2}+\frac{H_2}{2\cdot 3}+\frac{H_3}{3}+\frac{H_3}{5}\right)+$$
$$+\frac{1}{4\cdot 5}\left(\frac{H_1}{1\cdot 2}+\frac{H_2}{2\cdot 3}+\frac{H_3}{3\cdot 4}+\frac{H_4}{4}+\frac{H_4}{5}\right).$$

If we denote this summand as S and use the values of the first five harmonic numbers from the table above, we get

$$S=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{5}\right)+\frac{1}{6}\left(\frac{1}{2}+\frac{3/2}{2}+\frac{3/2}{5}\right)+\frac{1}{12}\left(\frac{1}{2}+\frac{3/2}{6}+\frac{11/6}{3}+\frac{11/6}{5}\right)+$$
$$+\frac{1}{20}\left(\frac{1}{2}+\frac{3/2}{6}+\frac{11/6}{12}+\frac{25/12}{4}+\frac{25/12}{5}\right)+$$
$$=\frac{1}{2}\cdot\frac{6}{5}+\frac{1}{6}\cdot\frac{31}{20}+\frac{1}{12}\cdot\frac{311}{180}+\frac{1}{20}\cdot\frac{265}{144}=\frac{1891}{1728}\,.$$

Altogether we have

$$s(-5,-5,-5)=\zeta(3)-\frac{1891}{1728}-\frac{137}{1500}=\zeta(3)-\frac{256103}{216000}\doteq 0.016394866122\,.$$

ii) By Theorem 2.2 we get an easy and effective way how to obtain the required sum:

$$s'(-5,-5,-5)=\zeta(3)-\sum_{k=1}^{4}\frac{(2k+1)H_k}{[k(k+1)]^2}-\frac{H_5}{25}=$$
$$=\zeta(3)-\frac{3H_1}{(1\cdot 2)^2}-\frac{5H_2}{(2\cdot 3)^2}-\frac{7H_3}{(3\cdot 4)^2}-\frac{9H_4}{(4\cdot 5)^2}-\frac{H_5}{25}\,.$$

By means of the first five values of the harmonic numbers we have

$$s'(-5,-5,-5) = \zeta(3) - \frac{3}{4}\cdot 1 - \frac{5}{36}\cdot\frac{3}{2} - \frac{7}{144}\cdot\frac{11}{6} - \frac{9}{400}\cdot\frac{25}{12} - \frac{1}{25}\cdot\frac{137}{60} = \\ = \zeta(3) - \frac{256103}{216000} \doteq 0.016394866122\,.$$

iii) The third and in this case much more easily way, how to determine the sum $s(-5,-5,-5)$, is to use formula (5) and the value of $s_5(0,0,0) = H_{5,3}$ from the table above. So we immediately obtain the required result:

$$s(-5,-5,-5) = s(0,0,0) - s_5(0,0,0) = \zeta(3) - \frac{256103}{216000} \doteq 0.016394866122\,.$$

iv) If we use formula (12), we get the approximate sum

$$\overline{s}(-5,-5,-5) = \frac{2(-5)^2+2(-5)+1}{4(-5)^4} = \frac{2\cdot 5^2 - 2\cdot 5+1}{4\cdot 5^4} = \frac{41}{2500} = 0.0164\,.$$

Formulas (7), (10), and (5) give identical result 0.016394866122, while formula (12) gives approximate result 0.0164. The relative error of the fourth approximate result is $3.13\cdot 10^{-4} \sim 0.03\,\%$.

3 Numerical verification

We solve the problem to determine the values of the sum $s(a,a,a)$ of the series $\sum_{k=1}^{\infty} \frac{1}{(k-a)^3}$ for $a = -1, -2, \ldots, -10, -99, -100, -500, -999, -1000$. We use on the one hand an approximate evaluation of the sum

$$s(a,a,a,t) = \sum_{k=1}^{t} \frac{1}{(k-a)^3},$$

where $t = 10^6$, and formula (12) for approximate evaluation sum $\overline{s}(a,a,a)$, and on the other hand formulas (7) and (10) for evaluation the sum $s(a,a,a)$ We compare 15 quadruplets of the sums $s(a,a,a)$, $s'(a,a,a)$, $s(a,a,a,10^6)$, and $\overline{s}(a,a,a)$ to verify formulas (7) and (10) and to determine the relative error of two approximate sums $s(a,a,a,10^6)$ and $\overline{s}(a,a,a)$. We use procedures `hnum` and `rp3aaaneg` written in the basic programming language of the CAS Maple 16 and one **for** statement:

```
hnum:=proc(n)
    local m,h; h:=0;
    for m from 1 to n do
        h:=h+1/m;
    end do;
end proc;

rp3aaaneg:=proc(a,t)
    local i,k,A,A2,s,s1,s2,s3,saaa,s2aaa,sumaaa,sumaaaline,z3;
    A:=-a; A2:=A*A; s:=0; saaa:=0; s2aaa:=0; sumaaa:=0;
    z3:=1.2020569031595942854O;
    for k from 1 to A-1 do
        s1:=0; s2:=0; s3:=0;
        if k-1=0 then s2:=0 else
        for i from 1 to k-1 do
            s2:=s2+hnum(i)/(i*(i+1));
        end do;
        end if;
        s2:=s2+hnum(k)/k+hnum(k)/A;
        s1:=s1+s2/(k*(k+1));
        s:=s+s1;
        s3:=s3+((2*k+1)*hnum(k))/(k*k*(k+1)*(k+1));
    end do;
    saaa:=z3-s-hnum(A)/A2; s2aaa:=z3-s3-hnum(A)/A2;
    print("a=",a,":saaa=",evalf[20](saaa),s2aaa=",evalf[20](s2aaa));
    for k from 1 to t do
        sumaaa:=sumaaa+1/((k-a)*(k-a)*(k-a));
    end do;
    print("sumaaa(",t,")=",evalf[20](sumaaa));
    sumaaaline:=(2*A2+2*a+1)/(4*A2*A2);
    print("sumaaaline=",evalf[20](sumaaaline));
    print("rerrsumaaa=",evalf[20]((abs(sumaaa-saaa))/saaa));
    print("rerrsumaaaline=",evalf[20]((abs(sumaaaline-saaa))/saaa));
end proc:

A:=[-1,-2,-3,-4,-5,-6,-7,-8,-9,-10,-99,-100,-500,-999,-1000];
for a in A do
    rp3aaaneg(a,1000000);
end do;
```

The approximate values of the sums $s(a,a,a)$ and $s(a,a,a,10^6)$, denoted briefly s and $s(10^6)$, and the sum $\overline{s}(a,a,a)$, denoted $\overline{s}$, obtained by the procedures above and rounded to 9 decimals, are written into the following table (the sums $s'(a,a,a)$ give identically values as the sums $s(a,a,a)$) :

a	-1	-2	-3	-4	-5
s	0.202056903	0.077056903	0.040019866	0.024394866	0.016394866
$s(10^6)$	0.202056903	0.077056903	0.040019866	0.024394866	0.016394866
$\overline{s}$	0.250000000	0.078125000	0.040123457	0.024414063	0.016400000
a	-6	-7	-8	-9	-10
s	0.011765236	0.008849784	0.006896659	0.005524917	0.004524917
$s(10^6)$	0.011765236	0.008849785	0.006896660	0.005524917	0.004524917
$\overline{s}$	0.011766975	0.008850479	0.006896973	0.005525072	0.004525000
a	-99	-100	-500	-999	-1000
s	0.000050502	0.000049502	0.000001996	0.000000501	0.000000500
$s(10^6)$	0.000050502	0.000049502	0.000001996	0.000000500	0.000000499
$\overline{s}$	0.000050503	0.000049503	0.000001996	0.000000501	0.000000500

Computation of 15 quadruplets of the sums $s(a,a,a)$, $s'(a,a,a)$, $s(a,a,a,10^6)$ and $\overline{s}(a,a,a)$ took about 21 hours and 30 minutes. The relative errors of the approximate sums $s(a,a,a,10^6)$, i.e. the ratios

$$\left|[s(a,a,a,10^6) - s(a,a,a)]/s(a,a,a)\right|,$$

range from 10^{-10} (for $a=-1$) to 10^{-5} (for $a=-1000$), and the relative errors of the approximate sums $\overline{s}(a,a,a)$, i.e. the ratios

$$\left|[\overline{s}(a,a,a) - s(a,a,a)]/s(a,a,a)\right|,$$

range from 10^{-1} (for $a=-1$) to 10^{-5} (for $a=-1000$).

4 Conclusion

We dealt with the sum of the series of reciprocals of the cubic polynomials with triple non-positive integer root a, i.e. with the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k-a)^3}$. We stated that its sum clearly can be for great number of members t (we used $t=10^6$) approximately computed by formula

$$s(a,a,a,t) = \sum_{k=1}^{t}\frac{1}{(k-a)^3},$$

we derived that the approximate value of its sum is for a negative a given by simple formula

$$\overline{s}(a,a,a) = \frac{2a^2+2a+1}{4a^4},$$

and we derived that the precise value of the sum is for a negative a given by formula $s(a,a,a) = \zeta(3) - H_{-a,3}$, i.e. by formula

$$s(a,a,a) = \zeta(3) - \sum_{k=1}^{-a-1} \frac{1}{k(k+1)} \left(\sum_{i=1}^{k-1} \frac{H_i}{i(i+1)} + \frac{H_k}{k} - \frac{H_k}{a} \right) - \frac{H_{-a}}{a^2} \,,$$

and also by easier formula

$$s'(a,a,a) = \zeta(3) - \sum_{k=1}^{-a-1} \frac{(2k+1)H_k}{[k(k+1)]^2} - \frac{H_{-a}}{a^2} \,.$$

We verified these results by computing 15 quadruplets of the four sums above for $a = -1, -2, \ldots, -10,\ -99, -100, -500, -999, -1000$ by using the CAS Maple 16 and compared their values. The series of reciprocals of the cubic polynomials with triple non-positive integer root so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a formula which can be expressed in closed form.

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Some applications of linear difference equations in finance with wolfram|alpha and maple

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Abstract

The principle objective of this paper is to show how linear difference equations can be applied to solve some issues of financial mathematics. We focus on the area of compound interest and annuities. In both cases we determine appropriate recursive rules, which constitute the first order linear difference equations with constant coefficients, and derive formulas required for calculating examples. Finally, we present possibilities of application of two selected computer algebra systems Wolfram|Alpha and Maple in this mathematical area.

Key words: linear difference equation, compound interest, future value of an annuity, periodic payment, computer algebra systems.

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1 Introduction

The values of most economic variables are given as a sequence of values observed at discrete time intervals or periods. These sequences are often specified by recursion with some initial elements. But it is preferable to know a rule in the form of an equation for the n-th element to calculate the values of sequence elements. The recursive rule of a sequence represents a difference equation and the functional notation for the n-th element can be obtained by solving this difference equation (see [7]).

Many formulas used in financial mathematics can be derived from the recursive rules between two consecutive elements which constitute difference equations of the first order. This includes for example simple and compound interest calculation, the present and future value of an annuity and loan amortization.

2 Compound Interest

2.1 Derivation of Formula

A sum of money deposited in a bank earns interest which is added to the principal at regular intervals and the new amount is used for calculating the interest for the next conversion period. We shall develop a formula for the total amount of money that is accumulated by a given principal after a certain number of conversion periods, see also [3], [4].

Let r stand for the annual interest rate and k denote the number of conversion periods in a year. Let n be equal to the number of conversion periods in the term of the deposit. Let y_n represent the amount on deposit at the end of n conversion periods and P the initial sum deposited (i.e. principal). We obtain the following recursive rule

$$y_{n+1} = y_n + \frac{r}{k}\, y_n = \left(1 + \frac{r}{k}\right) y_n, \quad n = 0, 1, 2, \ldots$$

with $y_0 = P$, where the fraction $\frac{r}{k}$ stands for the interest rate per conversion period and $\frac{r}{k} y_n$ is the interest generated during $(n+1)$st period. The previous formula represents the first order homogeneous linear difference equation with constant coefficients

$$y_{n+1} - \left(1 + \frac{r}{k}\right) y_n = 0 \tag{1}$$

with initial condition

$$y_0 = P. \tag{2}$$

The above problem (1), (2) can be solved by using the properties of a geometric sequence, see [1] and [8]. But our approach will be different due to the use of a difference equation. The characteristic equation of (1) takes form

$$z - \left(1 + \frac{r}{k}\right) = 0$$

with real root

$$z = 1 + \frac{r}{k}.$$

According to [5], the general solution is a geometric sequence

$$y_n = C\left(1+\frac{r}{k}\right)^n, \quad C \in \mathbb{R}.$$

A constant C can be specified from the initial condition (2) for period $n = 0$, hence

$$C = P.$$

Thus the general solution is given by

$$y_n = P\left(1+\frac{r}{k}\right)^n \tag{3}$$

and represents the compound interest formula. This formula gives the amount y_n into which principal P grows when it earns compound interest for n conversion periods at an interest rate of $\frac{r}{k}$ per conversion period.

2.2 Illustrative Examples

Example 2.1. *An amount of EUR* $1,000$ *is deposited into a savings account at an annual interest rate of* 2.5%*, compounded yearly. What will the value of the account be worth after 20 years?*

To find the amount we use formula (3). We have principal $P = 1,000$, annual interest rate $r = 0.025$, number of conversion periods per year $k = 1$ and total number of conversion periods $n = 20$. After plugging those figures into the formula, we get

$$y_{20} = 1000\left(1+\frac{0.025}{1}\right)^{20} \doteq 1638.62$$

Example 2.2. *Find the number of years required for a given sum of money to double itself if the interest rate is* 3%*, compounded quarterly.*

Substituting $y_n = 2P$ in the compound interest formula (3), we have

$$2P = P\left(1+\frac{r}{k}\right)^n$$

which implies

$$2 = \left(1+\frac{r}{k}\right)^n. \tag{4}$$

Taking natural logarithms on both sides and using properties of logarithms gives

$$n = \frac{\ln 2}{\ln\left(1+\frac{r}{k}\right)}.$$

To calculate the number of years N we have to divide the total number of conversion periods n by their number in a year k

$$N = \frac{1}{k} \frac{\ln 2}{\ln\left(1 + \frac{r}{k}\right)}. \tag{5}$$

Setting $r = 0.03$, $k = 4$ we get the required number of years

$$N = \frac{1}{4} \frac{\ln 2}{\ln\left(1 + \frac{0.03}{4}\right)} \doteq 23.19$$

3 Future Value of an Annuity

3.1 Derivation of Formula

An annuity is essentially a sequence of periodic payments, usually equal in amount, payable at equal intervals of time over the course of a fixed time period. The future value of an annuity is the total value of its periodic payments enhanced at interest rate for given number of conversion periods. It is defined as the sum of the amounts of all payments and the total compound interest earned on these payments to the time of the last payment. See for example [1], [4].

Suppose the constant sum R is deposited at the end of each conversion period in a bank which credits interest at the annual rate r. The deposits are made k times each year over n conversion periods. Let y_n denote the total amount in the account at the end of n conversion periods. We shall find the total worth of an annuity after n deposits.

The recursive rule for the future value of an annuity can be written as

$$y_{n+1} = y_n + \frac{r}{k} y_n + R = \left(1 + \frac{r}{k}\right) y_n + R, \quad n = 0, 1, 2, \ldots$$

with $y_0 = 0$, where $\frac{r}{k}$ is the interest rate per conversion period.

This equation constitutes the first order nonhomogeneous linear difference equation with constant coefficients

$$y_{n+1} - \left(1 + \frac{r}{k}\right) y_n = R \tag{6}$$

with initial condition

$$y_0 = 0. \tag{7}$$

In financial mathematics, the above problem (6), (7) is solved by using the properties of a geometric sequence. But we will proceed by means of difference equations like in the case of compound interest. To solve nonhomogeneous difference equation (6) we consider the corresponding homogeneous difference equation

$$y_{n+1} - \left(1 + \frac{r}{k}\right) y_n = 0 \tag{8}$$

which is the same as (1) in the case of compound interest. Hence the general solution $\bar{y}_n$ of this homogeneous difference equation is given by

$$\bar{y}_n = C\left(1 + \frac{r}{k}\right)^n, \quad C \in \mathbb{R}. \tag{9}$$

The right-hand side of the nonhomogeneous difference equation (6) is a constant R which is a polynomial of degree zero. Thus a particular solution Y_n can be estimated by

$$Y_n = b, \quad b \in \mathbb{R}.$$

For more details see [5]. Using the method of undetermined coefficients (see [2]) we substitute the above estimate into (6). We get

$$b - \left(1 + \frac{r}{k}\right) b = R$$

and solving for b we obtain

$$b = -R\,\frac{k}{r}.$$

Therefore the particular solution of (6) takes the form

$$Y_n = -R\,\frac{k}{r}. \tag{10}$$

Using (9), (10) according to the superposition principle (see [6]), the general solution of the nonhomogeneous linear difference equation (6) is the sum

$$y_n = Y_n + \bar{y}_n = -R\,\frac{k}{r} + C\left(1 + \frac{r}{k}\right)^n, \quad C \in \mathbb{R}.$$

A constant C can be specified from the initial condition (7) for period $n = 0$. Hence we obtain

$$C = R\,\frac{k}{r}.$$

Consequently, the general solution of (6) takes the form

$$y_n = -R\,\frac{k}{r} + R\,\frac{k}{r}\left(1 + \frac{r}{k}\right)^n$$

which can be written as

$$y_n = R\,\frac{\left(1+\frac{r}{k}\right)^n - 1}{\frac{r}{k}}. \tag{11}$$

The above relation represents the future value of an annuity formula which gives the amount of an annuity of n payments of R at the compound rate $\frac{r}{k}$ per conversion period under the assumption that the payment interval equals the conversion period. The future value of an annuity formula is used to calculate what value at a future date would be for a series of periodic payments.

In financial mathematics, it is common to use the following form of the formula (11) setting $i = \frac{r}{k}$ where i represents the interest rate per compounding interval (see [8] and [10])

$$y_n = R\,\frac{(1+i)^n - 1}{i}. \tag{12}$$

3.2 Illustrative Examples

Example 3.1. *Suppose EUR 500 is deposited at the end of every six-month period in a bank, whose annual rate is* 3.4%*, compounded semiannually. How much will this account be worth after 7 years?*

We get the solution using (11), where $R = 500$, $r = 0.034$, $k = 2$, $n = 14$. Then we obtain

$$y_{14} = 500\,\frac{\left(1+\frac{0.034}{2}\right)^{14} - 1}{\frac{0.034}{2}} \doteq 7828.64$$

Example 3.2. *Find the payment amount that you should deposit at the end of each month in a bank so that EUR* 35,000 *will be available after* 10 *years if the interest rate is* 1.6%*, compounded monthly after each deposit.*

Solving for R from the future value of an ordinary annuity formula (11) we get

$$R = y_n\,\frac{\frac{r}{k}}{\left(1+\frac{r}{k}\right)^n - 1}.$$

In our case we have $r = 0.016$, $k = 12$, $n = 120$, $y_{120} = 35,000$. Hence the monthly payment is calculated as follows

$$R = 35,000\,\frac{\frac{0.016}{12}}{\left(1+\frac{0.016}{12}\right)^{120} - 1} \doteq 269.149$$

4 Solving with Computer Algebra Systems

In mathematics of finance, Excel is commonly used for calculations. In this paper the quoted calculations of compound interest and annuity are completed by computational tool Wolfram|Alpha and mathematical software Maple, respectively.

4.1 Wolfram|Alpha

We will demonstrate the computation of compound interest (3) and Example 2.2 through the free online service Wolfram|Alpha, which is available via any web browser at http://wolframalpha.com. This tool provides mathematical computations based on software Mathematica and accepts completely free-form input, commands are specified by the name of operation in English.

To solve the difference equation (1) with the initial condition (2) we type both equations together separated by comma into an input field writing indexes in parentheses. The provided general solution (3) is shown in Figure 1.

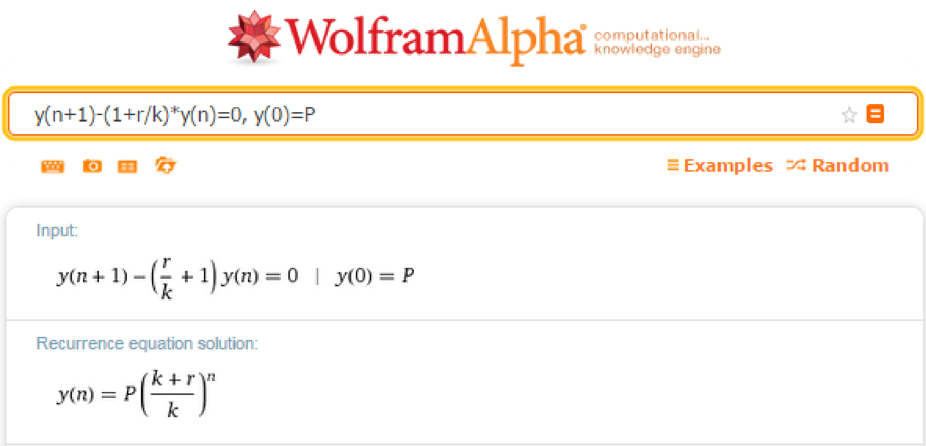

Fig. 1. Compound interest formula

Further, in Figure 2 you can see calculation of number of years from Example 2.2 using derived formula (5). We get the same result by solving equation (4) and using the command `solve` and the reserved word `for` as shown in Figure 3.

Fig. 2. Calculation of number of years by using derived formula

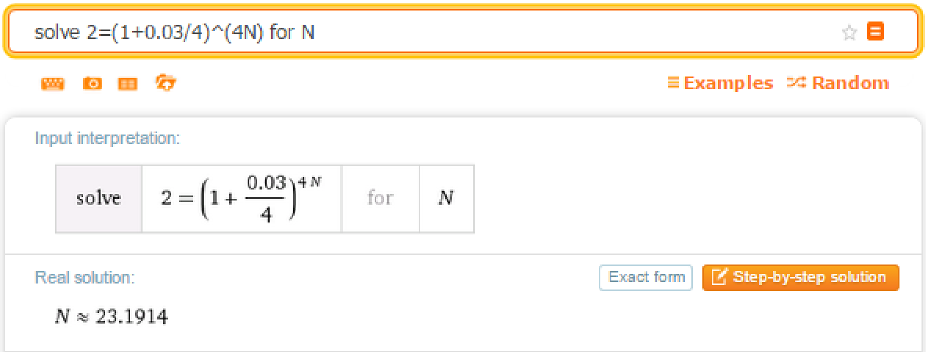

Fig. 3. Calculation of number of years by solving equation

4.2 Maple

Now we show the computation of the future value of an annuity (11) and illustrative Example 3.2.

We assign the recurrence relation (6) to the name `REq`.

```
> REq:=y(n+1)-y(n)*(1+r/k)=R;
```

Maple returns the output:

$$REq := y(n+1) - y(n)\left(1 + \frac{r}{k}\right) = R$$

Then we make the assignment of the initial condition (7) to the name `IC`.

```
> IC:=y(0)=0:
```

To solve the given difference equation we execute the command **`rsolve`** which solves among others the first order linear difference equations. A single recurrence relation and a boundary condition are the first argument, the second argument indicates the function that is solved for. Indexes are written in parentheses.

```
> rsolve({REq,IC},y(n));
```

$$\frac{Rk\left(\frac{k+r}{k}\right)^n}{r} - \frac{kR}{r}$$

The above obtained expression corresponds to the future value of an annuity (11).

For determining the payment R we type the following command, where FV equals to the total amount y_n in the account upon the last deposit (i.e the future value of an annuity).

```
> isolate(%=FV,R):simplify(%);
```

$$R = \frac{FV\,r}{k\left(\left(\frac{k+r}{k}\right)^n - 1\right)}$$

To make the calculation of Example 3.2 we use command `subs`.

```
> subs(k=12,r=0.016,FV=35000,n=120,%);
```

$$R = 269.1493510$$

4.3 Comparison of Used Systems

The professional Maple is very powerful tool which enables to make new procedures and modules, save and read them or together with other data store in a library. On the other hand, it requires certain programming skills.

In comparison with Maple, Wolfram|Alpha does not allow to save and reload the results of computations and make own procedures, also its performance is rather slow. But its significant advantage is that it is free online and very simple to use. Moreover, Wolfram|Alpha provides a variety of computations from other fields, for example from money and finance.

5 Conclusion

This paper has discussed linear difference equations and their applications in economics (see also [9]). These equations are frequently used especially in financial mathematics and some of their typical applications have been presented here.

Our main aim was to show relationship between some formulas of financial topics and mathematical knowledge which is required for their deriving. We have focused on derivation of the compound interest and the future value of an annuity formula by means of solution of difference equations. The simultaneous application of mathematical software has been demonstrated, the supplementary computations have been performed through Maple and Wolfram|Alpha.

Finally, the paper emphasizes the need for mathematics in economic subjects. The presented approach can be used in teaching of mathematics at economic universities and helps to provide students with the opportunities to apply their mathematics in relevant economics contexts.

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The intertemporal choice behavior: the role of emotions in a multiagent decision problem

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Abstract

Traditional Discounted Utility Model assumes an exponential delay discount function, with a constant discount rate: this implies dynamic consistency and stationary intertemporal preferences. Contrary to the normative setting, decision neuroscience stresses a lack of rationality, i.e., inconsistency, in some intertemporal choice behaviors. We deal with both models are dealt with in the framework of some relevant decision problems.

Key words: time preference, exponential discounting, hyperbolic discounting.

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1 Introduction

The traditional Discounted Utility model (DU model) (Samuelson, 1937) [20] fails in being both normative and descriptive. Indeed several studies, especially carried out in psychology and neuroeconomics, reveal the existence of relevant anomalies violating the axioms of the traditional model (Section 3).

Bechara and colleagues [2] show that decision making processes are guided by emotional signaling, which allow people to choose advantageously before they realized the strategy that worked best. This fact justifies the presence of anomalies in intertemporal choice and the use of hyperbolic delay discounting (declining as the length of the delay increases), so, people have the tendency to increasingly choose a smaller-sooner reward over a larger-later reward as the delay occurs sooner in time. This entails intertemporal inconsistency and preferences reversal. Even so, an impatient behavior not necessarily can be considered incoherent (Section 4).

The results of some studies by Shiv et al. [21] and Naqvi et al. [17] have demonstrated that patients with lesions in specific components of a neural circuitry critical for the processing of emotions will make more advantageous decisions than normal subjects when faced with the types of positive-expected-value gambles that most people routinely shun (Section 5).

Recent neuroeconomic and econophysical studies have explored neurobiological and psychological factors, e.g. impulsivity and inconsistency that determined individual differences in intertemporal choice. Takahashi et al. [25] attempt to dissociate impulsivity and inconsistency in their econophysical studies proposing a quasi-exponential delay discount function. Other behavioral economists propose *multiple selves* models attempting to measure the strength of the internal conflict within the decision maker, best known as *quasi-hyperbolic discount* model (Laibson, 1997) [11] (Section 6).

To fight impulsivity Strotz [23] proposed two strategies that might be adopted by a person who foresees how her preferences will change over time; Thaler and Shefrin [26] built a structure in which the individual is treated as if he contained two distinct psyches denoted as *planner* and *doer* (Section 7).

In a multiagent decision context the objective for a decision group is to choose a common decision, that is an alternative which is judged the best by the majority of the decision makers. So in most strategic decisions, it is important to be able to estimate the characteristics and behaviors of others. If the characteristics of other players are unknown, estimating them is a critical task (Section 8). Moreover, psychological evidence suggests people own beliefs, values, and habits tend to bias their perceptions of how widely they are shared (*false consensus effect*). This effect demonstrates an inability of individuals to process information rationally (Section 9). Therefore when we use the aggregation of the agent preferences to assess consensus, we obtain a coefficient which includes the false consensus effect that depends on the subjectivity and also increases the degree of consensus. In order to eliminate the component of human judgment vagueness a procedure defined by ordered weighted averaging (OWA) operators, introduced by Yager [29], can be applied (Section 10).

An experiment performed by Engelmann and Strobel [8] demonstrates that a false consensus effect is present only if information about decision of other members of the group is implicit. So the consensus effect is not always false but only when people, forming expectations concerning decisions of others, weight their own decision more heavily than that of a randomly selected person from the same population (see [6], [7]), (Section 11). The result is linked with the analysis of false consensus effect in cooperative and non-cooperative decision problem. Indeed, in a cooperative decision problem, agents know choices of other members, while in a non-cooperative one they have to judge choices of others (Section 12).

2 Traditional discounting model and decision neuroscience

The standard economic model of discounted utility assumes that economic agents make intertemporal choices over consumption profiles $(c_t, \dots, c_T)$ and such preferences can be represented by an intertemporal utility function $U^t(c_t, \dots, c_T)$, which can be described by the following special functional form:

$$U^t(c_t, \dots, c_T) = \sum_{k=0}^{T-t} D(k)u(c_{t+k})$$

where

$$D(k) = \left(\frac{1}{1+\rho}\right)^k$$

So the *DU* model assumes an exponential temporal discounting function and a constant discount rate (ρ), which represents the pure rate of time preference of the individual.

An important implication of constant discount rate and exponential discounting function is that intertemporal preferences of the individual are time-consistent: if at time t a person prefers c_2 at $t+2$ to c_1 at $t+1$, then at time $t+1$ she must prefer c_2 at $t+2$ to c_1 instantly. So, with the same temporal options and the same information, later preferences confirm earlier preferences.

However, several empirical studies have documented various inadequacies of the *DU* model as a descriptive model of behavior. Behavioral economic theories on decision process have found that there are a number of behavior patterns that violate the rational choice theory [27].

Decision neuroscience is an emerging area of research whose goal is to integrate research in neuroscience and behavioral decision making. It calls into question the theories of choice that assume decisions derive from an assessment of the future outcomes of various options and alternatives through some type of cost-benefit analysis, which ignore influence of emotions on decision-making.

This investigation explores the neural road map for the physiological processes intervening between knowledge and behavior, and the potential interruptions that lead to a disconnection between what one knows and what one decides to do. Decision making studies in neurological patients, who can no longer process emotional information, normally suggest that people make judgments not only by evaluating the consequences and their probability of occurring, but also and even sometimes primarily at a gut or emotional level (see [1]).

3 Behavioral finance: empirical anomalies violating DU model

Some studies concerning the individual behavior from the psychological perspective, e.g. related with discounting real or hypothetical rewards, show the existence of violations of the DU model. A first empirical remark is that discount rates are not constant over time, but appear to decline - a pattern often referred to as *hyperbolic discounting* ([22], [23]). Furthermore, even for a given delay, discount rates vary across different locations of intertemporal choices [28].

Delay effect, *magnitude effect*, *sign effect* and *sequence effect* are among the relevant anomalies in intertemporal choice, we will deal with.

The *delay effect* rests on the evidence that as waiting time increases, the discount rates tend to be higher in the short intervals than in the longer ones. Prelec and Loewenstein [18] define this anomaly as *common difference effect* and *immediacy effect*. We can set out delay effect as:

$$(x, s) \sim (y, t) \quad \text{but} \quad (x, s + h) < (y, t + h)$$

for

$$y > x, s < t \quad \text{and} \quad h > 0$$

If two capitals, (x, s) and (y, t), are indifferent, $(x, s) \sim (y, t)$, their projections onto a common instant p have to coincide:

$$xA(s, p) = yA(t, p) \quad \text{if and only if} \quad \frac{x}{y} = \frac{A(t, p)}{A(s, p)} = v(s, t, p)$$

being $A(t, p)$ the discount function which represents the amount available at p instead of one euro available at t, and $v(s, t, p)$ the corresponding financial factor. In the same way, if $(x, s+h) \sim (y, t+h)$, this implies that

$$xA(s+h,p) = yA(t+h,p)$$

if and only if

$$\frac{x}{y} = \frac{A(t+h,p)}{A(s+h,p)} = v(s+h,t+h,p)$$

Then:

$$v(s,t,p) < v(s+h,t+h,p)$$

The magnitude effect can be described as follows. Larger outcomes are discounted at a lower rate than smaller outcomes. Let us suppose that the instantaneous discount rate is inversely proportional to the discounted amount:

$$A(c,z) = ce^{-\int_0^z \frac{k}{c} dx} = ce^{-\frac{k}{c}z}$$

Prelec and Loewenstein [18] formulate the magnitude effect as follows:

$$(x,s) \sim (y,t) \quad \text{implies} \quad (ax,s) < (ay,t)$$

for $y > x > 0$, $s < t$ and

$$(-x,s) \sim (-y,t) \quad \text{implies} \quad (-ax,s) > (-ay,t)$$

The sign effect. Gains are discounted at a higher rate than losses of the same magnitude. Prelec and Loewenstein [18] proposed the amplification loss property implying that, changing the sign of an amount from gains to losses, the weight of this amount increases:

$$(x,s) \sim (y,t) \quad \text{implies} \quad (-x,s) > (-y,t)$$

for $y > x > 0$, $s < t$.

Increasing sequences of consumption are preferred over decreasing ones even if the total amount is the same. In general, when subjects choose among different sequences of two events people tend to save the better thing for last, contradicting the standard assumption of a positive interest rate. In the *improving sequence effect*, for all s and t, and $s < t$, there is a c_0 such that, for all $y > x > c_0$, the following preference holds

$$\{(x,s),(y,t)\} >_p \{(y,s),(x,t)\}$$

in the instant p ([16], [26]).

4 Anticipation of future events and hyperbolic discounting

In contrast with the historically dominant view of emotions as a negative influence in human behavior, recent research in neuroscience and psychology has highlighted the positive roles played by emotions in decision making (Bechara et al. [2]; Damasio [5]; Loewenstein and Lerner [12]). Although strong negative emotions can lead destructive patterns of behavior, some Authors (see [2]; [5]; [21]) have shown that individuals with emotional dysfunction tend to perform poorly compared with those who have intact emotional processes.

An experiment exhibited in [2] leads to the conclusion that decision making is guided by emotional signaling generated in anticipation of future events. Without the ability to generate these emotional signals, the patients fail to avoid choices that lead to losses, and instead continue to sample from the disadvantageous choices until they go broke in a manner that is akin to how they behave in real life. In normal individuals, unconscious biases guide behavior before conscious knowledge does. Without the help of such biases, overt knowledge may be insufficient to ensure advantageous behavior.

Decision maker preferences are inconsistent and change over time, because normal people possess anticipatory indices of somatic states, that represent unconscious biases that are linked to prior experiences with reward and punishment. These biases alarm the normal subject about selecting a disadvantageous course of action, even before the subject becomes aware of the goodness or badness of the choice he is about to make [1]. Indeed, when normal people won or lost money on an investment round, they adopted a conservative strategy and became more reluctant to invest on the subsequent round [21].

Furthermore the preference for more immediate rewards per se is not always irrational, because there are opportunity costs and risk associated with non-gaining in delaying the rewards.

As a consequence there is considerable agreement among psychologists and economists that the notion of exponential discounting should be replaced by some form of hyperbolic discounting, which can represent the tendency of the individuals to increasingly choose a smaller-sooner reward over a larger-later reward as the delay occurs sooner in time (*delay effect*).

Many authors proposed different hyperbolic discount functions, in which temporal discount function δ increases with the delay to an outcome. Loewenstein and Prelec [13] proposed the form:

$$d(t) = \left(\frac{1}{1+\alpha t}\right)^{\frac{\beta}{\alpha}}$$

where $\beta > 0$ is the degree of discounting and $\alpha > 0$ is the departure from exponential discounting.

Hyperbolic discounting has been applied to a wide range of phenomena, including consumption-saving behavior. Consistent with hyperbolic discounting, people's investment behavior exhibits patience in the long run and impatience in the short run [28].

A second type of empirical support for hyperbolic discounting comes from experiments on dynamic inconsistency. Studies and empirical evidences show that delay effect can derive in preference reversal between two rewards as the time distance to these rewards diminishes. A hyperbolic discount model can demonstrate this; indeed, non-exponential time preference curves can cross [23] and consequently the preference for one future reward over another may change in time [28].

5 The negative side of emotions: impulsivity

The positive roles played by emotions when making decisions are in contrast with some contexts in which individuals deprived of normal emotional reactions might actually make better decisions than normal individuals. For instance, consider the case of a patient with ventromedial prefrontal damage (which involves severe impairments in judgment and emotion) who was driving under hazardous road conditions [5]. When other drivers reached an icy patch, they hit their brakes in panic, causing their vehicles to skid out of control, but the patient crossed the icy patch unperturbed, gently pulling away from a tailspin and driving ahead safely. The patient remembered the fact that not hitting the brakes was the appropriate behavior, and his lack of fear allowed him to perform optimally [21].

Other evidences suggest that even relatively mild negative emotions that do not result in a loss of self-control can play a counterproductive role among normal individuals in some situations. When gambles that involve some possible loss are presented one at a time, most people display extreme levels of risk aversion toward the gambles, a condition known as myopic loss aversion [3]. If myopic loss aversion does indeed have an emotional basis, then any dysfunction in neural systems subserving emotion ought to result in reduced levels of risk aversion and, thus, lead to more advantageous decisions in cases in which risk taking is rewarded. Furthermore individuals deprived of normal emotional reactions might, in certain situations, make more advantageous decisions than those not deprived of such reactions; so the lack of emotional reactions may lead to more advantageous decisions [21].

Indeed in many cases, indeed, temptations induce disadvantageous behavior, and when temptation becomes too great, what the person knows to be his best long run interests conflicts with his short run desires. Sociologists and psychologists have persistently studied impulsivity relative to its resultant behaviors such as drug addiction, suicide, aggression and violence. These studies suggests that individuals who frequently engage in impulsive behavior may fail to appropriately evaluate the consequences of their behavior [28].

6 Neuroeconomics: impulsivity and inconsistency in intertemporal choice

The greatest contradiction to rational theory, in intertemporal choice, is inconsistent preference, usually manifested as temporary preference for options that are extremely costly or harmful in the long run. This behavior can be typically seen in psychiatric disorders (alcoholism, drug abuse), but also in more ordinary phenomena (overeating, credit card debt) [28].

Some investigations in neuroeconomics, a specialized field of decision neuroscience, have found that addicts are more myopic, i.e., they have large time-discount rates, in comparison with non-addict populations [4]. It results that hyperbolic discounting may explain various human problematic behaviors [11]: loss of self-control, failure in planned abstinence from addictive substances and relapse, a deadline rush due to procrastination, failure in saving enough before retirement and risky sexual behavior. Addiction and financial mismanagement frequently co-occur, and elevated delay discounting may be a common mechanism contributing to both of these problematic behaviors.

We have noted that the preference for more immediate rewards per se is not always irrational or inconsistent (Section 4); therefore, impulsivity in intertemporal choice is rationalizable for several categories of persons. The behaviors of addicts are clinically problematic, but economically rational when their choices are time-consistent, if they have large discount rates with an exponential discount function. However, it is known that addicts also discount delayed outcomes hyperbolically, suggesting the intertemporal choices of addicts are time-inconsistent, resulting in a loss of self-control [4]:

they act more impulsively at the moment of the choice, against their own previously-intended plan. Moreover if large discount rates are due to habitual drug intake, it is expected that discount rates decreased after long term abstinence. However, recent studies on alcoholics and smokers report that abstinence does not dramatically reduce discount rates of former alcoholics and smokers [24].

Behavioral neuroeconomic and econophysical studies have proposed two discount models, in order to clarify the neural and behavioral correlates of impulsivity and inconsistency in intertemporal choice, namely, a quasi-exponential discount model and a quasi-hyperbolic discount model.

Quasi-exponential discount model. Takahashi et al. [25] have proposed and examined the following function for subjective value $V(D)$ of delayed reward:

$$V(D) = \frac{A}{\exp_q(k_q D)} = \frac{A}{[1+(1-q)k_q D]^{\frac{1}{1-q}}}$$

where D denotes a delay until receipt of a reward, A the value of a reward at $D = 0$, and k_q a parameter of impulsivity at delay $D = 0$ (q-exponential discount rate) and the q-exponential function is defined as:

$$\exp_q(x) = (1 + (1-q)x)^{\frac{1}{1-q}}$$

This function can distinctly parametrize impulsivity and inconsistency [28].

Quasi-hyperbolic discount model. Behavioral economists have proposed that the inconsistency in intertemporal choice may be attributed to an internal conflict between multiple selves within a decision maker. As a consequence, there are at least two exponential discounting selves (with two exponential discount rates) in a single individual; and when delayed rewards are at the distant future (> 1 year), the self with a smaller discount rate wins, while delayed rewards approach to the near future (within a year), the self with a larger discount rate wins, resulting in preference reversal over time. This intertemporal choice behavior can be parametrized in a quasi-hyperbolic discount model (also as a $\beta - \delta$ model). For discrete time τ (the unit assumed is one year) the quasi-hyperbolic discount factor is defined [11] as:

$$F(\tau) = \beta\delta^t$$

for $\tau = 1, 2, 3\ldots$ and $F(0) = 1$, $0 < \beta < \delta < 1$.

A discount factor between the present and one-time period later β is smaller than that between two future time-periods δ.

In the continuous time, the proposed model is equivalent to the linearly-weighted two-exponential functions (generalized quasi-hyperbolic discounting):

$$V(D) = A[w \exp(-k_1 D) + (1-w)\exp(-k_2 D)]$$

where $0 < w < 1$, is a weighting parameter and k_1 and k_2 are two exponential discount rates ($k_1 < k_2$). Note that the larger exponential discount rate of the two k_2, corresponds to an impulsive self, while the smaller discount rate k_1 corresponds to a patient self [28].

7 Self-control against impulsivity: Strotz model and Thaler and Shefrin model

A number of mechanisms of self-control are predicted by hyperbolic discounting. Strotz proposed two strategies that might be adopted by a person who foresees how her preferences will change over time.

1. The *strategy of precommitment*. A person commits himself to perform a plan of action. For instance, consider a consumer with an initial endowment K_0 of consumer goods which has to be allocated over the finite interval $(0, T)$. At time t he wishes to maximize his utility function:

   $$J_0 = \int_0^T \lambda(t-0)U[\bar{c}(t), t]\,\mathrm{dt}$$

   subject to $\int_0^T c(t)\,\mathrm{dt} = K_0$, where $[\bar{c}(t), t]$ is the instantaneous rate of consumption at time period t, and $\lambda(t-0)$ is a discount factor, whose value depends on the elapsing time between a past or future date and present. This implies that the discounted marginal utility of consumption should be the same for all periods. But, at a later date, the consumer may reconsider his consumption plan. Then the problem is to maximize

   $$J_0 = \int_0^T \lambda(t-\tau)U[c(t), t]\,\mathrm{dt}$$

   subject to $\int_\tau^T c(t)\,\mathrm{dt} = K_\tau = K_0 - \int_0^\tau c(t)\,\mathrm{dt}$.

   The optimal pattern of consumption will change with changes in τ and if the original plan is altered, the individual is said to display *dynamic inconsistency*. Strotz showed that individuals will not alter the original plan only if $\lambda(t, \tau)$ is an exponential in $|t-\tau|$.

2. The *strategy of consistent planning*. Since precommitment is not always a feasible solution to the problem of intertemporal conflict, an individual may adopt a different strategy: take into account future changes in the utility function and reject any plan that he will not follow through. His problem is then to find the best plan among those he will actually follow.

   In the setting of multiple selves models, in order to control impulsivity, Thaler and Shefrin [26] proposed a planner-doer model which draws upon principal agent theory. They treat an individual as if he contained two distinct psyches: one *planner*, which pursue longer-run results, and multiple *doers*, which are concerned only with short-term satisfactions, so they care only about their own immediate gratification (and have no affinity for future or past doers). For instance, consider an individual with a fixed income stream $y = [y_1, y_2, \dots, y_T]$, where

$$\sum_t y_t = Y$$

   has to be allocated over the finite interval $(0, T)$. The planner would choose a consumption plan to maximize his utility function

$$V(Z_1, Z_2 \dots, Z_T)$$

   subject to $\sum_t c_t \leq Y$, where Z_t is a utility function of level consumption in t (c_t).

   On the other hand, the unrestrained doer 1 would borrow $Y - y_1$ on the capital market and therefore choose $c_1 = Y$; the resulting consequence is naturally $c_2 = c_3 = \dots = c_T = 0$. Such an action would suggest a complete absence of psychic integration.

   Then the model focuses on the strategies employed by the planner to control the behavior of the doers, and it proposes two tools at his disposal.

   (a) He can impose *rules* on the doers behavior, which operate by altering the constraints imposed on any given doer; or

   (b) he can use *discretion* accompanied by some method of altering the incentives or rewards to the doer without any self-imposed constraints [28].

8 Multiagent decision problems: consensus and agreement

In a multiagent decision problem an individual needs to take his intertemporal choice considering others' preferences, in order to achieve a consensus over a common decision. Group decision problems, indeed, consist in finding the best alternative(s) from a set of feasible alternatives $A = \{a_1, \ldots, a_m\}$ according to the preferences provided by a group of agents $E = \{e_1, \ldots, e_n\}$. The objective is to obtain the maximum degree of agreement among the agents overall performance judgements on the alternatives (see [22]).

Specifically, every agent assesses each alternative in his preference system. Furthermore the group of agents has to verify if there is a possibility to rank the alternative set in a way shared by (a majority in the group). If such an operation succeeds, the group has reached a *consensus* about the ranking of the alternative set. In real situations, humans rarely come to a unanimous agreement: this has led to evaluate not only crisp degrees of consensus, but also intermediate degrees between 0 and 1, corresponding to partial agreement among all agents. However, full consensus can be considered not necessarily as a result of unanimous agreement, but it can be obtained ever in the case of agreement among a fuzzy majority of agents (see [9], [10]).

9 False consensus

It is well known, not only in the areas of social sciences, that people are egocentric. As pointed out in several experiments, in a multiagent decision problem each decision maker overestimates his own opinion. Social psychology has founded that people with a certain preference tend to make higher judgements of the popularity of that preference in others, compared to the judgements of those with different preferences. This empirical result has been termed the *false consensus effect* (see [19], [16]). It states that individuals overestimate the number of the people who possess the same attributes as they do. People often believe that others are more like themselves than they really are. Thus, their predictions about others' beliefs or behaviors, based on casual observation, are very likely to err in the direction of their own beliefs or behavior. For example, college students who preferred brown bread estimated that over 50% of all other college students preferred brown bread, while white-bread eaters estimated that 37% showed brown bread preference.

As the consequence, in multi-agent decision problem we often have to deal with different opinions, different importance of criteria and agents, who are not fully impartial objective. In this sense, the false consensus effect produces partial objectivity and incomplete impartiality, which perturbs the agreements over the evaluation.

10 Assessing consensus and false consensus

Different methods to compute a degree of a consensus in fuzzy, or imprecise, environments have been defined, and some approaches have been proposed to measure consensus in the context of fuzzy preference relations (see [9], [10]). However, as we have seen, the false consensus effect can lead to an absence of objectivity in the evaluation process. Then just a numerical indication seems not to be sufficient to synthesize the degree of consensus of agents which incorporate both the true knowledge generated in the agent opinion and the subjective component that produces false consensus outputs. The opinion of each agent is decomposed into two components: a vector, made of the ranking of the alternatives, built by means of a classical procedure, e.g., a hierarchical procedure [14], and a fuzzy component that represents the contribution of the false consensus effect, which we assume to be fuzzy in nature [15]. This allows us to consider aggregation operators, such as OWA operators, useful when synthesis among fuzzy variables is to be built [22].

A formal model considers the set N of decision makers, the set A of the alternatives, and the set C of the criteria. Let any decision maker $I \in N$ be able to assess the relevance of each criterion. Precisely, for every i, a function

$$h_i : C \to [0, 1]$$

with $\sum_{c \in C} h_i(c) = 1$, denoting the evaluation or weight that the decision maker assigns to the criterion c, is defined. Furthermore, the function

$$g_i : A \times C \to [0, 1]$$

is defined, such that $g_i(a, c)$ is the value of the alternative a with respect to the criterion c, in the perspective of i.

Let n, p, and m denote the (positive integer) numbers of the elements of the sets N, C, and A, respectively. The value $h_i(c)_{c \in C}$ denotes the evaluation of the p-tuple of the criteria by the decision maker i and the value $g_i(c, a)_{c \in C, a \in A}$, defines the matrix $p \times m$ whose elements are the evaluations, made by i, of the alternatives with respect to each criterion in C. The function: $A \to [0, 1]$, defined by

$$\big(f_i(a)\big)_{a \in A} = h_i(c)_{c \in C} \cdot g_i(c, a)_{c \in C, a \in A}$$

is the evaluation, made by i, of the alternative $a \in A$.

A Euclidean metric that acts between couples of decision makers i and j, i.e., between individual rankings of alternatives, is defined by

$$d(f_i f_j) = \sqrt{\frac{1}{|A|} \sum_{a \in A} \big(f_i(a) - f_j(a)\big)^2}$$

If the functions h_i, g_i range in $[0, 1]$, then also $0 \leq d(f_i f_j) \leq 1$.

If we set $\delta^\star = \max\{d(f_i, f_j)|i, j \in N\}$, then a degree of consensus $\delta^\star$ can be defined as the complement to one of the maximum distance between two positions of the agents:

$$\delta^\star = 1 - \delta^\star = 1 - \max\{d(f_i, f_j)|i, j \in N\}$$

Now to identify the portion of the false consensus effect internal to the consensus reaching process, we have to consider a vector that represents the *components of the consensus* $p(a)P + q(a)Q$. This polynomial representation of the measure of the effect is composed by a numeric component $p(a)P$, that contains all quantitative information available derived from the consensus reaching process, and $q(a)Q$ that reflects the false consensus effect. Then the measure of the effect is:

$$q(a) = \frac{1}{N(d^\star)^2} \sum_{i=1}^{N} (f_i - f_j)^2$$

with $0 \leq q(a) \leq 1$, $\forall i,\ j \in N$.

This component can be estimated by means of OWA operators (a class of decision support tools for providing heuristic solution to situations where several trade-offs should be taken into consideration). In Yager [29] is introduced an approach for multiple criteria aggregation, based on ordered weighted averaging (OWA) operators. By ranking the alternatives, the operators provide an enhanced methodology for evaluating actions on a qualitative basis [22].

11 Study on false consensus effect under varying information conditions. Engelmann and Strobel experiment

In Section 9, false consensus has been defined as an egocentric bias that occurs when people estimate consensus for their own behaviors. The judgements of each agent, indeed, are frequently based, in part, on intuition or subjective beliefs, rather than detailed data on the preferences of the people being predicted. However such intuitive judgements become more pervasive judgements when people lack necessary data to base their judgements. Therefore, according to Dawes (see [6], [7]), classical definition of false consensus does not justify the attribute false. He argues that it is perfectly rational to use the information about one's own decision in the same way as the information about any other randomly selected from a sample. The effect is only false if too much weight is assigned to one's own decision compared to a randomly selected person from the same population. Engelmann and Strobel [8] refer to the effect as defined above as a consensus effect and affirms that people exhibit a false consensus effect if among those with the same total information (i.e. that includes the information about their own decision) the estimates are biased in the direction of their own decision.

To demonstrate this and investigate whether a false consensus effect depends on the cognitive effort needed to retrieve information, Engelmann and Strobel compared two treatments in a simple one-shot experiment.

Results are in opposite direction to a false consensus effect when in a decision group the agents have explicit information about the choice of other members of their own group, while results are in line with a false consensus effect in all groups in which the information were implicit. This shows that most subjects are unwilling or unable to use information that is not handed to them on a silver platter. It appears to us that in the implicit information treatment it does not occur to many subjects that the other subjects' choices are valuable information and that this information is rather easily available, while the prominent information in the explicit information treatments is recognized as valuable information by virtually all subjects (or leads them to unconsciously update their beliefs).

In conclusion, Engelmann and Strobel affirm that there is no false consensus effect if representative information is highly prominent and retrievable without any effort. Indeed, there is even a significant effect in the opposite direction, indicating that subjects consider others' choices as more informative than their own.

12 False consensus effect and emotions in a multiagent decision problem

Multiagent decision problems are characterized by interplay between intertemporal considerations and strategic interactions: two or more agents could have to take a common decision for a future time and in this process they are influenced by emotional signal, which arise with impulsivity and with false or true consensus effect. Theory of games provide tools for describe strategic interaction. Indeed, in non-cooperative interaction each agent makes decisions independently, without collaboration or communication with any of the others. This can be assimilated to situations in which information about decision of other members of decision group is implicit. In this kind of strategic decision the consensus effect is false. As in Engelmann and Strobel experiment, if members of group decision do not cooperate they do not possess information about the choices of others, so the influence of psychological aspects lead to judge others in the same way that they judge themselves. Then two situations are possible:

1) each agent have the same preference and they will reach a common decision that is given by the unanimous choice,

2) the agents have different preferences and do not assign any weight to the other preferences, so it is not possible to aggregate them (see Section 10).

Then the influence of emotions has no negative consequences if the choices of the agents are unanimous, and then the final decision will be also the best decision in the Paretian sense. If this does not happen, it is impossible to achieve a common strategy without arresting impulsivity, and unanimity becomes increasingly difficult to obtain when the number of agents increases.

On the contrary in a cooperative decision problem the influence of false consensus effect is present at period-one, while the loss of self-control of each agent is fought by the imposition of a rule [26]. The rationality of the equilibrium choice of the cooperative game is saved by the possibility of making an arrangement among agents, which represents a pure rule to maintain self-control at later time in Thaler and Shefrin model (Section 7). Moreover with an arrangement the agents have explicit information about the choices of other members, so the lack of false consensus effect is in line with the result of Engelmann and Strobel experiment.

Consider the classic example of coordination game: the battle-of-the sexes. In this game an engaged couple must choose what to do in the evening: the man prefers to watch a baseball game and the women prefers to attend an opera. In terms of utility the payoff for each strategy is:

		Man	
		Opera (O)	Baseball (B)
Woman	Opera (O)	$3, 1$	$0, 0$
	Baseball (B)	$0, 0$	$1, 3$

In the example there are multiple outcomes that are equilibriums: (B, B) and (O, O). However both players would rather do something together than go to separate events, so no single individual has an incentive to deviate if others are conforming to an outcome: the man would attend the opera if he thinks the woman will be there even though he prefers the other equilibrium outcome in which both attend the baseball game.

In this context, a consensus decision making process can be considered as an instrument to choose the best strategy in a coordination game. The final decision is often not the first preference of each individual in the group and they may not even like the final result. But it is a decision to which they all consent because it is the best for the group.

Consequently, a common final decision is achievable only if the man and the woman have explicit information on the decision of other member, then only if there is cooperation.

If the man and the woman do not decide together where spend their time in the evening, probably, the result of implicit information and consequent false consensus effect will be that the man will go to the opera because he thinks that she decides to go there, and the woman will go to the baseball match to meet the man.

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