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# Rough Set Theory Applied To Hyper $BCK$-Algebra

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**Abstract**

The aim of this paper is to introduce the notions of lower and upper approximation of a subset of a hyper $BCK$-algebra with respect to a hyper $BCK$-ideal. We give the notion of rough hyper subalgebra and rough hyper $BCK$-ideal, too, and we investigate their properties.

**Key words**: rough set, rough (weak, strong) hyper $BCK$-ideal, rough hyper subalgebra, regular congruence relation.

**MSC 2010**: 20N20, 20N25.

## 1 Introduction

In 1966, Y. Imai and K. Iseki [2] introduced a new notion, called a $BCK$-algebra. The hyper structure theory (called also multi algebras ) was introduced in 1934 by F. Marty [6] at the 8th Congress of Scandinavian Mathematicians. In [3], Y. B. Jun, M. M. Zahedi, X. L. Xin, R. A. Borzooei applied the hyper structures to $BCK$-algebras and they introduced the notion of hyper $BCK$-algebra (resp. hyper K-algebra) which is a generalization of $BCK$-algebra (resp. hyper $BCK$-algebra). They also introduced the notion of hyper $BCK$-ideal, weak hyper $BCK$-ideal, hyper K-ideal and weak hyper K-ideal and gave relations among them. In 1982, Pawlak introduced the concept of rough set (see [7]). Recently Jun [5] applied rough set theory to $BCK$-algebras. In this paper, we apply the rough set theory to hyper $BCK$-algebras.

# 2 Preliminaries

Let $U$ be a universal set. For an equivalence relation $\Theta$ on $U$, the set of elements of $U$ that are related to $x \in U$, is called the *equivalence class* of $x$ and is denoted by $[x]_\Theta$. Moreover, let $U/\Theta$ denote the family of all equivalence classes induced on $U$ by $\Theta$. For any $X \subseteq U$, we write $X^c$ to denote the complement of $X$ in $U$, that is the set $U\backslash X$. A pair $(U, \Theta)$ where $U \neq \phi$ and $\Theta$ is an equivalence relation on $U$ is called an *approximation space.*
The interpretation in rough set theory is that our knowledge of the objects in $U$ extends only up to membership in the class of $\Theta$ and our knowledge about a subset $X$ of $U$ is limited to the class of $\Theta$ and their unions. This leads to the following definition.

**Definition 2.1.** [7] For an approximation space $(U, \Theta)$, by a rough approximation in $(U, \Theta)$ we mean a mapping $Apr : P(U) \longrightarrow P(U) \times P(U)$ defined for every $X \in P(U)$ by $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where

$$\underline{Apr}(X) = \{x \in U | [x]_\Theta \subseteq X\},$$
$$\overline{Apr}(X) = \{x \in U | [x]_\Theta \cap X \neq \phi\}.$$

$\underline{Apr}(X)$ is called a *lower rough approximation* of $X$ in $(U, \Theta)$, whereas $\overline{Apr}(X)$ is called an *upper rough approximation* of $X$ in $(U, \Theta)$.

**Definition 2.2.** [7] Given an approximation space $(U, \Theta)$, a pair $(A, B) \in P(U) \times P(U)$ is called a *rough set* in $(U, \Theta)$ if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$.

**Definition 2.3.** ([7]) Let $(U, \Theta)$ be an approximation space and $X$ be a non-empty subset of $U$.

(i) If $\underline{Apr}(X) = \overline{Apr}(X)$, then $X$ is called *definable.*

(ii) If $\underline{Apr}(X) = \phi$, then $X$ is called *empty interior.*

(iii) If $\overline{Apr}(X) = U$, then $X$ is called *empty exterior*.

Let $H$ be a non-empty set endowed with a hyper operation "$\circ$", that is $\circ$ is a function from $H \times H$ to $P^*(H) = P(H) - \{\phi\}$. For two subsets $A$ and $B$ of $H$, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

**Definition 2.4.** ([3]) By a *hyper $BCK$-algebra* we mean a non- empty set $H$ endowed with a hyper operation "$\circ$"and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$"the *hyper order* in $H$.

**Theorem 2.5.** ([3]) In any hyper $BCK$-algebra $H$, the following hold:

(a1) $0 \circ 0 = \{0\}$,

(a2) $0 \ll x$,

(a3) $x \ll x$,

(a4) $A \ll A$,

(a5) $A \ll 0$ implies $A = \{0\}$,

(a6) $A \subseteq B$ implies $A \ll B$,

(a7) $0 \circ x = \{0\}$,

(a8) $x \circ y \ll x$,

(a9) $x \circ 0 = \{x\}$,

(a10) $y \ll z$ implies $x \circ z \ll x \circ y$,

(a11) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$ and $x \circ z \ll y \circ z$,

(a12) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$,

for all $x, y, z \in H$ and for all non-empty subsets $A$ and $B$ of $H$.

**Definition 2.6.** ([3]) Let $H$ be a hyper $BCK$-algebra and let $S$ be a subset of $H$ containing 0. If $S$ be a hyper $BCK$-algebra with respect to the hyper operation "$\circ$"on $H$, we say that $S$ is a *hyper subalgebra* of $H$.

**Theorem 2.7.** ([3]) Let $S$ be a non-empty subset of hyper $BCK$-algebra $H$. Then $S$ is a hyper subalgebra of $H$ if and only if $x \circ y \subseteq S$, for all $x, y \in S$.

**Definition 2.8.** ([3]) Let $I$ be a non-empty subset of hyper $BCK$-algebra $H$ and $0 \in I$.

(i) $I$ is said to be a *hyper $BCK$-ideal* of $H$ if $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

(ii) $I$ is said to be a *weak hyper $BCK$-ideal* of $H$ if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

(iii) $I$ is called a *strong hyper $BCK$-ideal* of $H$ if $(x \circ y) \cap I \neq \phi$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

**Theorem 2.9.** ([3]) If $H$ be a hyper $BCK$-algebra, then

(i) every hyper $BCK$-ideal of $H$ is a weak hyper $BCK$-ideal of $H$.

(ii) every strong hyper $BCK$-ideal of $H$ is a hyper $BCK$-ideal of $H$.

**Definition 2.10.** ([4]) Let $H$ be a hyper $BCK$-algebra. A hyper $BCK$-ideal $I$ of $H$ is called *reflexive* if $x \circ x \subseteq I$ for all $x \in H$.

**Definition 2.11.** ([1]) Let $\Theta$ be an equivalence relation on hyper $BCK$-algebra $H$ and $A, B \subseteq H$. Then,

(i) $A\Theta B$ means that, there exist $a \in A$ and $b \in B$ such that $a\Theta b$,

(ii) $A\bar{\Theta}B$ means that, for all $a \in A$ there exists $b \in B$ such that $a\Theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\Theta b$,

(iii) $\Theta$ is called a *congruence relation* on $H$, if $x\Theta y$ and $x'\Theta y'$ imply $x \circ x'\bar{\Theta}y \circ y'$ for all $x, y, x', y' \in H$.

(iv) $\Theta$ is called a *regular relation* on $H$, if $x \circ y\Theta\{0\}$ and $y \circ x\Theta\{0\}$ imply $x\Theta y$ for all $x, y \in H$.

**Example 2.12.** Let $H_1 = \{0, 1, 2\}$, $H_2 = \{0, a, b\}$ and hyper operations "$\circ_1$"and "$\circ_2$"on $H_1$ and $H_2$ are defined respectively, as follow:

| $\circ_1$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{1\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0, 2\}$ |

| $\circ_2$ | 0 | a | b |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| a | $\{a\}$ | $\{0, a\}$ | $\{0, a\}$ |
| b | $\{b\}$ | $\{a, b\}$ | $\{0, b\}$ |

Then $(H_1, \circ_1)$ and $(H_2, \circ_2)$ are hyper $BCK$-algebras. Define the equivalence relation $\Theta_1$ and $\Theta_2$ on $H_1$ and $H_2$, respectively, as

$$\Theta_1 = \{(0, 0), (1, 1), (2, 2), (0, 2), (2, 0)\},$$

and

$$\Theta_2 = \{(0, 0), (a, a), (b, b), (0, a), (a, 0)\}.$$

It is easily checked that $\Theta_1$ is a congruence relation on $H_1$. But $\Theta_2$ is not a congruence relation on $H_2$, since $b\Theta_2 b$ and $0\Theta_2 a$ but $b \circ 0\bar{\Theta}_2 b \circ a$ is not true.

**Example 2.13.** Let $(H_1, \circ_1)$ be a hyper $BCK$-algebra as Example 2.12. Let $H_2 = \{0, a, b, c\}$ and define the hyper operation "$\circ_2$"on $H_2$ as follow:

| $\circ_2$ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| a | $\{a\}$ | $\{0, a\}$ | $\{0\}$ | $\{a\}$ |
| b | $\{b\}$ | $\{b\}$ | $\{0, a\}$ | $\{b\}$ |
| c | $\{c\}$ | $\{c\}$ | $\{c\}$ | $\{0, c\}$ |

Then $(H_2, \circ_2)$ is a hyper $BCK$-algebra. Define the congruence relation $\Theta_1$ and $\Theta_2$ on $H_1$ and $H_2$, respectively, by

$$\Theta_1 = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\},$$

and

$$\Theta_2 = \{(0, 0), (a, a), (b, b), (c, c), (0, b), (b, 0)\}.$$

It is easily checked that $\Theta_1$ is a regular congruence relation on $H_1$, but $\Theta_2$ is not a regular relation on $H_2$, since $a \circ b\Theta_2\{0\}$ and $b \circ a\Theta_2\{0\}$ but $(a, b) \notin \Theta_2$.

**Theorem 2.14.** ([1]) Let $\Theta$ be a regular congruence relation on hyper $BCK$-algebra $H$. Then $[0]_\Theta$ is a hyper $BCK$-ideal of $H$.

**Theorem 2.15.** ([1]) Let $\Theta$ be a regular congruence relation on $H, I = [0]_\Theta$ and $\frac{H}{I} = \{I_x : x \in H\}$, where $I_x = [x]_\Theta$ for all $x \in H$. Then $\frac{H}{I}$ with hyper operation "$\circ$"and hyper order "$<$"which is defined as follow, is a hyper $BCK$-algebra which is called *quotient hyper $BCK$-algebra*,

$$I_x \circ I_y = \{I_z : z \in x \circ y\},$$

and

$$I_x < I_y \Longleftrightarrow I \in I_x \circ I_y.$$

**Theorem 2.16.** ([1]) Let $I$ be a reflexive hyper $BCK$-ideal of $H$ and relation $\Theta$ on $H$ be defined as follow:

$$x\Theta y \Longleftrightarrow x \circ y \subseteq I \text{ and } y \circ x \subseteq I$$

for all $x, y \in H$. Then $\Theta$ is a regular congruence relation on $H$ and $I = [0]_\Theta$.

# 3 Rough hyper $BCK$-ideals

Throughout this section $H$ is a hyper $BCK$-algebra. In this section first we define lower and upper approximation of the subset $A$ of $H$ with respect to hyper $BCK$-ideal of $H$ and prove some properties. Then we give the definition of (weak, strong) rough hyper $BCK$-ideals and investigate the relation between them and (weak, strong) hyper $BCK$-ideals of $H$.

**Definition 3.1.** Let $\Theta$ be a regular congruence relation on hyper $BCK$-algebra $H$, $I = [0]_\Theta$, $I_x = [x]_\Theta$ and $A$ be a non-empty subset of $H$. Then the sets

$$\underline{Apr}_I(A) = \{x \in H | I_x \subseteq A\},$$
$$\overline{Apr}_I(A) = \{x \in H | I_x \cap A \neq \phi\}.$$

are called *lower and upper approximation* of the set $A$ with respect to the hyper $BCK$-ideal $I$, respectively.

**Proposition 3.2.** For every approximation space $(H, \Theta)$ and every subsets $A, B \subseteq H$, we have:

(1) $\underline{Apr}_I(A) \subseteq A \subseteq \overline{Apr}_I(A)$,

(2) $\underline{Apr}_I(\phi) = \phi = \overline{Apr}_I(\phi)$,

(3) $\underline{Apr}_I(H) = H = \overline{Apr}_I(H)$,

(4) if $A \subseteq B$, then $\underline{Apr}_I(A) \subseteq \underline{Apr}_I(B)$ and $\overline{Apr}_I(A) \subseteq \overline{Apr}_I(B)$,

(5) $\underline{Apr}_I(\underline{Apr}_I(A)) = \underline{Apr}_I(A)$,

(6) $\overline{Apr}_I(\overline{Apr}_I(A)) = \overline{Apr}_I(A)$,

(7) $\overline{Apr}_I(\underline{Apr}_I(A)) = \underline{Apr}_I(A)$,

(8) $\underline{Apr}_I(\overline{Apr}_I(A)) = \overline{Apr}_I(A)$,

(9) $\underline{Apr}_I(A) = (\overline{Apr}_I(A^c))^c$,

(10) $\overline{Apr}_I(A) = (\underline{Apr}_I(A^c))^c$,

(11) $\overline{Apr}_I(A \cap B) \subseteq \overline{Apr}_I(A) \cap \overline{Apr}_I(B)$,

(12) $\underline{Apr}_I(A \cap B) = \underline{Apr}_I(A) \cap \underline{Apr}_I(B)$,

(13) $\overline{Apr}_I(A \cup B) = \overline{Apr}_I(A) \cup \overline{Apr}_I(B)$,

(14) $\underline{Apr}_I(A \cup B) \supseteq \underline{Apr}_I(A) \cup \underline{Apr}_I(B)$,

(15) $\underline{Apr}_I(I_x) = H = \overline{Apr}_I(I_x)$ for all $x \in H$.

*Proof.* (1), (2) and (3) are straightforward.

(4) For any $x \in \underline{Apr}_I(A)$ we have $I_x \subseteq A \subseteq B$ and so $x \in \underline{Apr}_I(B)$. Now, suppose that $x \in \overline{Apr}_I(A)$. Then $I_x \cap A \neq \phi$ and so $I_x \cap B \neq \phi$. Hence $x \in \overline{Apr}_I(B)$.

(5) Since $\underline{Apr}_I(A) \subseteq A$, by (4) we have $\underline{Apr}_I(\underline{Apr}_I(A)) \subseteq \underline{Apr}_I(A)$. Now, let $x \in \underline{Apr}_I(A)$. Then $I_x \subseteq A$. Since for any $y \in I_x$, we have $I_x = I_y$, then $I_y \subseteq A$ and so $y \in \underline{Apr}_I(A)$. Therefore, $I_x \subseteq \underline{Apr}_I(A)$ and then we obtain $x \in \underline{Apr}_I(\underline{Apr}_I(A))$.

(6) By (1) and (4), $\overline{Apr}_I(A) \subseteq \overline{Apr}_I(\overline{Apr}_I(A))$. On the other hand, we assume that $x \in \overline{Apr}_I(\overline{Apr}_I(A))$. Then we have $I_x \cap \overline{Apr}_I(A) \neq \phi$ and so there exist $a \in I_x$ and $a \in \overline{Apr}_I(A)$. Hence $I_a = I_x$ and $I_a \cap A \neq \phi$ which imply $I_x \cap A \neq \phi$. Therefore, $x \in \overline{Apr}_I(A)$.

(7) By (1), we have $\underline{Apr}_I(A) \subseteq \overline{Apr}_I(\underline{Apr}_I(A))$. Now, let $x \in \overline{Apr}_I(\underline{Apr}_I(A))$. Then $I_x \cap \underline{Apr}_I(A) \neq \phi$ and so there exist $a \in I_x$ and $a \in \underline{Apr}_I(A)$. Hence $I_a = I_x$ and $I_a \subseteq A$ which imply $I_x \subseteq A$. Therefore, $x \in \underline{Apr}_I(A)$.

(8) By (1), we have $\underline{Apr}_I(\overline{Apr}_I(A)) \subseteq \overline{Apr}_I(A)$. Now, we assume that $x \in \overline{Apr}_I(A)$. Then $I_x \cap A \neq \phi$. For every $y \in I_x$, we have $I_y = I_x$ and so $I_y \cap A \neq \phi$. Hence $y \in \overline{Apr}_I(A)$ which implies $I_x \subseteq \overline{Apr}_I(A)$. Therefore, $x \in \underline{Apr}_I(\overline{Apr}_I(A))$.

(9) For any subset $A$ of $H$ we have:

$$\begin{aligned}
(\overline{Apr}_I(A^c))^c &= \{x \in H : x \notin \overline{Apr}_I(A^c)\} \\
&= \{x \in H : I_x \cap A^c = \phi\} \\
&= \{x \in H : I_x \subseteq A\} \\
&= \{x \in H : x \in \underline{Apr}_I(A)\} \\
&= \underline{Apr}_I(A).
\end{aligned}$$

(10) For any subset $A$ of $H$ we have:

$$\begin{aligned}
(\underline{Apr}_I(A^c))^c &= \{x \in H : x \notin \underline{Apr}_I(A^c)\} \\
&= \{x \in H : I_x \not\subseteq A^c\} \\
&= \{x \in H : I_x \cap A \neq \phi\} \\
&= \{x \in H : x \in \overline{Apr}_I(A)\} \\
&= \overline{Apr}_I(A).
\end{aligned}$$

(11) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by (4), $\overline{Apr}_I(A \cap B) \subseteq \overline{Apr}_I(A)$ and $\overline{Apr}_I(A \cap B) \subseteq \overline{Apr}_I(B)$. Hence $\overline{Apr}_I(A \cap B) \subseteq \overline{Apr}_I(A) \cap \overline{Apr}_I(B)$.

(12) For any subset $A$ and $B$ of $H$ we have:

$$\begin{aligned} x \in \underline{Apr}_I(A \cap B) &\Longleftrightarrow I_x \subseteq A \cap B \\ &\Longleftrightarrow I_x \subseteq A \text{ and } I_x \subseteq B \\ &\Longleftrightarrow x \in \underline{Apr}_I(A) \text{ and } x \in \underline{Apr}_I(B) \\ &\Longleftrightarrow x \in \underline{Apr}_I(A) \cap \underline{Apr}_I(B). \end{aligned}$$

(13) For any subset $A$ and $B$ of $H$ we have

$$\begin{aligned} x \in \overline{Apr}_I(A \cup B) &\Longleftrightarrow I_x \cap (A \cup B) \neq \phi \\ &\Longleftrightarrow (I_x \cap A) \cup (I_x \cap B) \neq \phi \\ &\Longleftrightarrow I_x \cap A \neq \phi \text{ or } I_x \cap B \neq \phi \\ &\Longleftrightarrow x \in \overline{Apr}_I(A) \text{ or } x \in \overline{Apr}_I(B) \\ &\Longleftrightarrow x \in \overline{Apr}_I(A) \cup \overline{Apr}_I(B). \end{aligned}$$

(14) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by (4), $\underline{Apr}_I(A) \subseteq \underline{Apr}_I(A \cup B)$ and $\underline{Apr}_I(B) \subseteq \underline{Apr}_I(A \cup B)$, which imply that $\underline{Apr}_I(A) \cup \underline{Apr}_I(B) \subseteq \underline{Apr}_I(A \cup B)$.

(15) The proof is straightforward. $\square$

**Corollary 3.3.** Let $(H, \Theta)$ be an approximation space. Then

(i) for every $A \subseteq H$, $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are definable sets,

(ii) for every $x \in H, I_x$ is definable set.

*Proof.* (i) By proposition 3.2 (5) and (7), we have $\underline{Apr}_I(\underline{Apr}_I(A)) = \underline{Apr}_I(A) = \overline{Apr}_I(\underline{Apr}_I(A))$. Hence $\underline{Apr}_I(A)$ is a definable set. On the other hand by proposition 3.2 (6) and (8), we have $\overline{Apr}_I(\overline{Apr}_I(A)) = \overline{Apr}_I(A) = \underline{Apr}_I(\overline{Apr}_I(A))$. Therefore $\overline{Apr}_I(A)$ is a definable set.

(ii) By proposition 3.2 (15) the proof is clear. $\square$

**Theorem 3.4.** Let $\Theta$ be a regular congruence relation on $H$, $I = [0]_\Theta$ be a hyper $BCK$-ideal of $H$ and $A, B$ are non-empty subsets of $H$. Then

(i) $\overline{Apr}_I(A) \circ \overline{Apr}_I(B) = \overline{Apr}_I(A \circ B)$,

(ii) $\underline{Apr}_I(A) \circ \underline{Apr}_I(B) \subseteq \underline{Apr}_I(A \circ B)$.

*Proof.* (i) Let $z \in \overline{Apr}_I(A) \circ \overline{Apr}_I(B)$. Then there exist $a \in \overline{Apr}_I(A)$ and $b \in \overline{Apr}_I(B)$ such that $z \in a \circ b$. Hence $I_a \cap A \neq \phi$ and $I_b \cap B \neq \phi$ and so there exist $c \in I_a \cap A$ and $d \in I_b \cap B$ such that $a\Theta c$ and $b\Theta d$. Since $\Theta$ is a congruence relation on $H$, then we have $a \circ b\bar{\Theta} c \circ d$ and because $z \in a \circ b$, then there exist $y \in c \circ d$ such that $z\Theta y$. Hence $y \in I_z$. On the other hand, $y \in c \circ d \subseteq A \circ B$ which implies $I_z \cap (A \circ B) \neq \phi$ and so $z \in \overline{Apr}_I(A \circ B)$. Therefore $\overline{Apr}_I(A) \circ \overline{Apr}_I(B) \subseteq \overline{Apr}_I(A \circ B)$. Now, suppose that $x \in \overline{Apr}_I(A \circ B)$. Then $I_x \cap (A \circ B) \neq \phi$. Let $z \in I_x \cap (A \circ B)$, then there exist $a \in A$ and $b \in B$ such that $z \in a \circ b$ and $I_x = I_z$. Thus we have $I_z \in I_a \circ I_b$ and so $I_x \in I_a \circ I_b$. Hence $x \in a \circ b \subseteq A \circ B \subseteq \overline{Apr}_I(A) \circ \overline{Apr}_I(B)$. Therefore, $\overline{Apr}_I(A \circ B) \subseteq \overline{Apr}_I(A) \circ \overline{Apr}_I(B)$. □

(ii) Let $z \in \underline{Apr}_I(A) \circ \underline{Apr}_I(B)$. Then there exist $a \in \underline{Apr}_I(A)$ and $b \in \underline{Apr}_I(B)$ such that $z \in a \circ b$, $I_a \subseteq A$ and $I_b \subseteq B$. For every $y \in I_z$, we have $I_z = I_y \in I_a \circ I_b$ and so $y \in a \circ b \subseteq A \circ B$. Then $y \in A \circ B$ and so $I_z \subseteq A \circ B$. Therefore $z \in \underline{Apr}_I(A \circ B)$. □

**Example 3.5.** Let $H = \{0, 1, 2\}$ and define the hyper operation "$\circ$"on $H$ as follow:

| $\circ$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0\}$ | $\{1\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0, 2\}$ |

Then $(H, \circ)$ is a hyper $BCK$-algebra. Define the equivalence relation $\Theta$ by

$$\Theta = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}.$$

Then $\Theta$ is a regular congruence relation on $H$ and so we have:

$$I = [0]_\Theta = \{0, 1\}, I_1 = [1]_\Theta = \{0, 1\}, I_2 = [2]_\Theta = \{2\}.$$

Now, if we let $A = \{1, 2\}$ and $B = \{0, 2\}$, then we have $A \circ B = \{0, 1, 2\}$ and so

$$\begin{aligned}
&\underline{Apr}_I(A) = \{x \in H | I_x \subseteq A\} = \{2\},\\
&\overline{Apr}_I(A) = \{x \in H | I_x \cap A \neq \phi\} = \{0, 1, 2\},\\
&\underline{Apr}_I(B) = \{x \in H | I_x \subseteq B\} = \{2\},\\
&\overline{Apr}_I(B) = \{x \in H | I_x \cap B \neq \phi\} = \{0, 1, 2\},\\
&\underline{Apr}_I(A \circ B) = \{x \in H | I_x \subseteq A \circ B\} = \{0, 1, 2\},\\
&\overline{Apr}_I(A \circ B) = \{x \in H | I_x \cap (A \circ B) \neq \phi\} = \{0, 1, 2\},\\
&\overline{Apr}_I(A) \circ \overline{Apr}_I(B) = \{0, 1, 2\},\\
&\underline{Apr}_I(A) \circ \underline{Apr}_I(B) = \{0, 2\}.
\end{aligned}$$

Therefore, we see that $\underline{Apr}_I(A) \circ \underline{Apr}_I(B) \neq \underline{Apr}_I(A \circ B)$ but $\overline{Apr}_I(A) \circ \overline{Apr}_I(B) = \overline{Apr}_I(A \circ B)$.

**Definition 3.6.** Let $\Theta$ be a regular congruence relation on $H$, $I = [0]_\Theta$ be a hyper $BCK$-ideal of $H$ and $A$ be a non-empty subset of $H$. If $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are hyper subalgebra of $H$, then $A$ is called a *rough hyper subalgebra* of $H$.

**Theorem 3.7.** If $I$ be a hyper $BCK$-ideal and $J$ be a hyper subalgebra of $H$, then

(i) $\overline{Apr}_I(J)$ is a hyper subalgebra of $H$.

(ii) If $I \subseteq J$, then $\underline{Apr}_I(J)$ is a hyper subalgebra of $H$.

*Proof.* (i) Since $0 \in J \subseteq \overline{Apr}_I(J)$, then $\overline{Apr}_I(J) \neq \phi$. Now, we assume that $x, y \in \overline{Apr}_I(J)$. We must prove that $x \circ y \subseteq \overline{Apr}_I(J)$. Since $I_x \cap J \neq \phi$ and $I_y \cap J \neq \phi$, we can let $t \in I_x \cap J$, $s \in I_y \cap J$ and $z \in x \circ y$. Hence $I_z \in I_x \circ I_y = I_t \circ I_s$ and so $z \in t \circ s \subseteq J$. Thus we have $z \in J$ and $z \in I_z$ and so $I_z \cap J \neq \phi$. Therefore, $z \in \overline{Apr}_I(J)$ and so $x \circ y \subseteq \overline{Apr}_I(J)$.

(ii) Since $I = I_0 \subseteq J$, we have $0 \in \underline{Apr}_I(J) \neq \phi$. Now, suppose that $a, b \in \underline{Apr}_I(J)$. Then $I_a \subseteq J$ and $I_b \subseteq J$. For every $z \in a \circ b$ and every $y \in I_z$, we have $I_z = I_y \in I_a \circ I_b$ and so $y \in a \circ b \subseteq J$. Hence $I_z \subseteq J$, which implies that $z \in \underline{Apr}_I(J)$. Therefore, $a \circ b \subseteq \underline{Apr}_I(J)$. $\square$

**Theorem 3.8.** Let $\Theta$ and $\Phi$ be two regular congruence relations on $H$ and $I = [0]_\Theta$, $J = [0]_\Phi$ be two hyper $BCK$-ideals of $H$ such that $I \subseteq J$. Then for any nonempty subset $A$ of $H$, we have:

(i) $\underline{Apr}_J(A) \subseteq \underline{Apr}_I(A)$,

(ii) $\overline{Apr}_I(A) \subseteq \overline{Apr}_J(A)$.

*Proof.* (i) First we show that if $I \subseteq J$, then $I_x \subseteq J_x$. Let $y \in I_x$. Then $x\Theta y$. Since $\Theta$ is a congruence relation on $H$ and $x\Theta x$, then $x \circ x \bar{\Theta} x \circ y$. Since $0 \in x \circ x$, then there exist $t \in x \circ y$ such that $0\Theta t$ and so $t \in [0]_\Theta = I \subseteq J = [0]_\Phi$. Thus by hypothesis, $t \in [0]_\Phi$ and so $x \circ y \Phi \{0\}$. By the similar way, we can show that $y \circ x \Phi \{0\}$. Since $\Phi$ is a regular congruence relation, we get $x\Phi y$ and so $y \in [x]_\Phi = J_x$. Therefore, $I_x \subseteq J_x$. Now, let $x \in \underline{Apr}_J(A)$. Then $J_x \subseteq A$ and so $I_x \subseteq A$ which implies $x \in \underline{Apr}_I(A)$.

(ii) Assume that $x \in \overline{Apr}_I(A)$. Then $I_x \cap A \neq \phi$. Since $I_x \subseteq J_x$, we have $J_x \cap A \neq \phi$. Therefore, $x \in \overline{Apr}_J(A)$. □

**Corollary 3.9.** Let $\Theta$ and $\Phi$ are two regular congruence relations on $H$, $I = [0]_\Theta$, $J = [0]_\Phi$ be two hyper $BCK$-ideals of hyper $BCK$-algebra $H$ and $A$ be a non-empty subset of $H$. Then

(i) $\underline{Apr}_I(A) \cap \underline{Apr}_J(A) \subseteq \underline{Apr}_{I\cap J}(A)$,

(ii) $\overline{Apr}_{I\cap J}(A) \subseteq \overline{Apr}_I(A) \cap \overline{Apr}_J(A)$.

*Proof.* By theorem 3.8, the proof is clear. □

**Definition 3.10.** Let $\Theta$ be a regular congruence relation on $H$, $I = [0]_\Theta$ be a hyper $BCK$-ideal of $H$, $A$ be a non-empty subset of $H$ and $Apr_I(A) = (\underline{Apr}_I(A), \overline{Apr}_I(A))$ be a rough set in the approximation space $(H, \Theta)$. If $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are hyper $BCK$-ideals (resp, weak, strong) of $H$, then $A$ is called a *rough hyper $BCK$-ideal* (resp, weak, strong) of $H$.

**Example 3.11.** Let $H = \{0, 1, 2, 3\}$ and hyper operation "$\circ$"on $H$ is defined as follow:

| $\circ$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{0\}$ | $\{1\}$ |
| 2 | $\{2\}$ | $\{2\}$ | $\{0,1\}$ | $\{2\}$ |
| 3 | $\{3\}$ | $\{3\}$ | $\{3\}$ | $\{0,3\}$ |

Then $(H, \circ, 0)$ is a hyper $BCK$-algebra. We define the regular congruence relation on $H$ as follow:

$$\Theta = \{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0)\}.$$

So we have:

$$I = I_0 = I_1 = \{0,1\}, I_2 = \{2\}, I_3 = \{3\}.$$

Now, let $A = \{0, 1, 3\}$ be a subset of $H$, then

$$\underline{Apr}_I(A) = \{x \in H | I_x \subseteq A\} = \{0, 1, 3\},$$
$$\overline{Apr}_I(A) = \{x \in H | I_x \cap A \neq \phi\} = \{0, 1, 3\}.$$

Easily we give that $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are hyper $BCK$-ideals. Therefore, $A$ is a rough hyper $BCK$-ideal of $H$.

**Example 3.12.** Let $H = \{0, a, b, c\}$. By the following table $(H, \circ)$ is a hyper $BCK$-algebra.

| $\circ$ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| a | $\{a\}$ | $\{0,a\}$ | $\{0\}$ | $\{a\}$ |
| b | $\{b\}$ | $\{b\}$ | $\{0,a\}$ | $\{b\}$ |
| c | $\{c\}$ | $\{c\}$ | $\{c\}$ | $\{0,c\}$ |

Now, let relation $\Theta$ on $H$ is defined as follow:

$$\Theta = \{(0,0), (a,a), (b,b), (c,c), (0,b), (b,0), (0,a), (a,0), (a,b), (b,a)\}.$$

Then,

$$I_0 = I_a = I_b = \{0, a, b\}, I_c = \{c\}.$$

Let $J_1 = \{0, c\}$, $J_2 = \{0, b\}$ and $J_3 = \{c\}$. Then,

$$\underline{Apr}_I(J_1) = \{c\}, \overline{Apr}_I(J_1) = \{0, a, b, c\},$$
$$\underline{Apr}_I(J_2) = \{\}, \overline{Apr}_I(J_2) = \{0, a, b\},$$
$$\underline{Apr}_I(J_3) = \{c\}, \overline{Apr}_I(J_3) = \{c\}.$$

Hence we can see that $J_1$ is a hyper $BCK$-ideal of $H$ but $\underline{Apr}_I(J_1)$ is not a hyper $BCK$-ideal. Moreover $J_2$ is not a hyper $BCK$-ideal but $\overline{Apr}_I(J_2)$ is a hyper $BCK$-ideal of $H$. In follows, $J_3$ is not a hyper $BCK$-ideal and neither $\underline{Apr}_I(J_3)$ nor $\overline{Apr}_I(J_3)$ is a hyper $BCK$-ideal of $H$.

**Theorem 3.13.** Let $\Theta$ be a regular congruence relation on $H$ and $I = [0]_\Theta$ be a hyper $BCK$-ideal of $H$. Then

(i) If $J$ be a weak hyper $BCK$-ideal of $H$ containing $I$, then $\underline{Apr}_I(J)$ is a weak hyper $BCK$-ideal of $H$,

(ii) If $J$ be a hyper $BCK$-ideal of $H$ containing $I$, then $\underline{Apr}_I(J)$ is a hyper $BCK$-ideal of $H$,

(iii) If $J$ be a strong hyper $BCK$-ideal of $H$ containing $I$, then $\underline{Apr}_I(J)$ is a strong hyper $BCK$-ideal of $H$.

*Proof.* (i) Since $I = I_0 \subseteq J$, then $0 \in \underline{Apr}_I(J)$. Now, Let $x, y \in H$ be such that $x \circ y \subseteq \underline{Apr}_I(J)$ and $y \in \underline{Apr}_I(J)$. We must prove that $I_x \subseteq J$. Let $a \in I_x$ and $b \in I_y$. Then $a\Theta x$ and $b\Theta y$. Since $\Theta$ is a congruence relation on $H$, we have $a \circ b\overline{\Theta}x \circ y$ and so for every $z \in a \circ b$, there exist $t \in x \circ y$ such that $z\Theta t$. Since $x \circ y \subseteq \underline{Apr}_I(J)$, we have $t \in \underline{Apr}_I(J)$ and so $I_t = I_z \subseteq J$ which implies $z \in J$. Thus $a \circ b \subseteq J$. On the other hand, $b \in I_y \subseteq J$. Since $J$ is a weak hyper $BCK$-ideal, we have $a \in J$ and so $I_x \subseteq J$. Hence $x \in \underline{Apr}_I(J)$. Therefore, $\underline{Apr}_I(J)$ is a weak hyper $BCK$-ideal of $H$.

(ii) Let $x, y \in H$ be such that $x \circ y \ll \underline{Apr}_I(J)$ and $y \in \underline{Apr}_I(J)$. We must prove that $I_x \subseteq J$. Let $a \in I_x$ and $b \in I_y$. Then $a\Theta x$ and $b\Theta y$. Since $\Theta$ is a congruence relation on $H$, we have $a \circ b\overline{\Theta}x \circ y$ and so for every $z \in a \circ b$, there exist $z' \in x \circ y$ such that $z\Theta z'$. Since $z' \in x \circ y \ll \underline{Apr}_I(J)$, then there exists $t \in \underline{Apr}_I(J) \subseteq J$ such that $z' \ll t$ and so from $z\Theta z'$, we have $I_0 \in I_{z'} \circ I_t = I_z \circ I_t$. Hence $0 \in z \circ t$ and then $z \ll t$. Thus we have proved that for every $z \in a \circ b$, there exist $t \in J$ such that $z \ll t$ which means that $a \circ b \ll J$. On the other hand we have $b \in I_y \subseteq J$. Since $J$ is a hyper $BCK$-ideal of $H$, we have $a \in J$. Thus $I_x \subseteq J$ which implies that $x \in \underline{Apr}_I(J)$. Therefore, $\underline{Apr}_I(J)$ is a hyper $BCK$-ideal of $H$.

(iii) Suppose that $x, y \in H$ be such that $(x \circ y) \cap \underline{Apr}_I(J) \neq \phi$ and $y \in \underline{Apr}_I(J)$. Let $a \in I_x$ and $b \in I_y$. Then $a\Theta x$ and $b\Theta y$. Since $\Theta$ is a congruence relation on $H$, we have $a \circ b\overline{\Theta} x \circ y$. Since $(x \circ y) \cap \underline{Apr}_I(J) \neq \phi$, then there exist $t \in H$ such that $t \in x \circ y$ and $t \in \underline{Apr}_I(J)$. Now, $t \in x \circ y\overline{\Theta} a \circ b$ implies that there exist $z \in a \circ b$ such that $z\Theta t$ and so $I_t = I_z \subseteq J$. Hence $z \in J$ and so $(a \circ b) \cap J \neq \phi$. On the other hand, we have $b \in I_y \subseteq J$. Since $J$ is a strong hyper $BCK$-ideal of $H$, then we have $a \in J$ which implies $I_x \subseteq J$ that means $x \in \underline{Apr}_I(J)$. Therefore, $\underline{Apr}_I(J)$ is a strong hyper $BCK$-ideal of $H$. □

**Theorem 3.14.** Suppose that $I$ be a hyper $BCK$-ideal of $H$ and $\Theta$ be a regular congruence relation on $H$ which is defined as follow:

$$x\Theta y \Leftrightarrow x \circ y \subseteq I \text{ and } y \circ x \subseteq I.$$

(i) If $J$ be a weak hyper $BCK$-ideal of $H$ containing $I$, then $\overline{Apr}_I(J)$ is a weak hyper $BCK$-ideal of $H$,

(ii) If $J$ be a hyper $BCK$-ideal of $H$ containing $I$, then $\overline{Apr}_I(J)$ is a hyper $BCK$-ideal of $H$,

(iii) If $J$ be a strong hyper $BCK$-ideal of $H$ containing $I$, then $\overline{Apr}_I(J)$ is a strong hyper $BCK$-ideal of $H$.

*Proof.* (i) Since $I \subseteq J \subseteq \overline{Apr}_I(J)$, then we have $0 \in \overline{Apr}_I(J)$. Let $x, y \in H$ be such that $x \circ y \subseteq \overline{Apr}_I(J)$ and $y \in \overline{Apr}_I(J)$. Then $I_y \cap J \neq \phi$ and for every $z \in x \circ y$, we have $z \in \overline{Apr}_I(J)$ which means $I_z \cap J \neq \phi$. Thus there exist $a, b \in H$ such that $a \in I_y \cap J$ and $b \in I_z \cap J$ which imply that $a\Theta y$, $b\Theta z$ and $a, b \in J$. Thus $y \circ a \subseteq I \subseteq J$ and $z \circ b \subseteq I \subseteq J$ and so we get $y, z \in J$, since $J$ is a weak hyper $BCK$-ideal. Thus we have proved that for any $z \in x \circ y$, we have $z \in J$ and so $x \circ y \subseteq J$. Since $J$ is a weak hyper $BCK$-ideal and $y \in J$, obviously we have $x \in J$. Since $x \in I_x$, then $I_x \cap J \neq \phi$. Therefore $x \in \overline{Apr}_I(J)$ and so $\overline{Apr}_I(J)$ is a weak hyper $BCK$-ideal of $H$.

(ii) Let $x, y \in H$ be such that $x \circ y \ll \overline{Apr}_I(J)$ and $y \in \overline{Apr}_I(J)$. Then $I_y \cap J \neq \phi$ and for every $z \in x \circ y$, there exist $t \in \overline{Apr}_I(J)$ such that $z \ll t$ and $I_t \cap J \neq \phi$. Thus, there exist $c, d \in H$ such that $c \in I_t \cap J$ and $d \in I_y \cap J$ and so $c\Theta t$, $d\Theta y$ and $c, d \in J$. Hence $t \circ c \subseteq I \subseteq J$ and $y \circ d \subseteq I \subseteq J$. Since $J$ is a hyper $BCK$-ideal and $c, d \in J$, we have $y, t \in J$. Thus, we have proved that for every $z \in x \circ y$, there exist $t \in J$ such that $z \ll t$ which means that $x \circ y \ll J$ and so from $y \in J$ we get $x \in J$. Consequently, $I_x \cap J \neq \phi$ and so $x \in \overline{Apr}_I(J)$. Therefore, $\overline{Apr}_I(J)$ is a hyper $BCK$-ideal.

(iii) Let $x, y \in H$ be such that $(x \circ y) \cap \overline{Apr}_I(J) \neq \phi$ and $y \in \overline{Apr}_I(J)$. Then $I_y \cap J \neq \phi$ and so there exist $z \in H$ such that $z \in x \circ y$ and $z \in \overline{Apr}_I(J)$. Hence $I_z \cap J \neq \phi$ and so there exist $c, d \in H$ such that $c \in I_z \cap J$ and $d \in I_y \cap J$. Hence $c\Theta z$ and $d\Theta y$ where $c, d \in J$. Thus we have $z \circ c \subseteq I \subseteq J$ and $y \circ d \subseteq I \subseteq J$. Since $J$ is a strong hyper $BCK$-ideal and $c, d \in J$, we have $z \in J$ and $y \in J$. Thus we have proved that $(x \circ y) \cap J \neq \phi$ and $y \in J$. Since $J$ is a strong hyper $BCK$-ideal, we have $x \in J$ and so $I_x \cap J \neq \phi$ which means that $\overline{Apr}_I(J)$ is a strong hyper $BCK$-ideal of $H$. $\square$

# 4 Conclusion

This paper is intend to built up connection between rough sets and hyper $BCK$-algebras. We have presented a definition of the lower and upper approximation of a subset of a hyper $BCK$-algebra with respect to a hyper $BCK$-ideal. This definition and main results are easily extended to other algebraic structures such as hyper $K$-algebra, hyper $I$-algebra, etc.

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# On multiplication $\Gamma$-modules

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**Abstract**

In this article, we study some properties of multiplication $M_\Gamma$-modules and their prime $M_\Gamma$-submodules. We verify the conditions of ACC and DCC on prime $M_\Gamma$-submodules of multiplication $M_\Gamma$-module.

**Key words**: $\Gamma$-ring, multiplication $M_\Gamma$-module, prime $M_\Gamma$-submodule, prime ideal.

**MSC 2010**: 13A15, 16D25, 16N60.

## 1 Introduction

The notion of a $\Gamma$-ring was first introduced by Nobusawa [17]. Barnes [5] weakened slightly the conditions in the definition of $\Gamma$-ring in the sense of Nobusawa. After the $\Gamma$-ring was defined by Barnes and Nobusawa, a lot of researchers studied on the $\Gamma$-ring. Barnes [5], Kyuno [15] and Luh [16] studied the structure of $\Gamma$-rings and obtained various generalizations analogous of corresponding parts in ring theory. Recently, Dumitru, Ersoy, Hoque, *Öztürk*, Paul, Selvaraj, have studied on several aspects in gamma-rings (see [10, 8, 12, 14, 18, 19, 20]).

McCasland and Smith [14] showed that any Noetherian module $M$ contains only finitely many minimal prime submodules. D. D. Anderson [2] generalized the well-known counterpart of this result for commutative rings, i.e., he abandoned the Noetherianness and showed that if every prime ideal minimal over an ideal I is finitely generated, then R contains only finitely many prime ideals minimal over $I$. Behboodi and Koohy [7] showed that this result of Anderson was true for any associative ring (not necessarily commutative) and also, they extended it to multiplication modules, i.e., if M is a multiplication module such that every prime submodule minimal over a submodule $K$ is finitely generated, then $M$ contains only finitely many prime submodules minimal over $K$.

In this paper, we study some properties of multiplication left $M_\Gamma$-modules and their prime $M_\Gamma$-submodules. This paper is organized as follows: In Section 2, we review some basic notions and properties of $\Gamma$-rings. In Section 3, the concept of a moltiplication $M_\Gamma$-module is introduced and its basic properties are discussed. Also, we show that If $L$ is a left operator ring of the $\Gamma$-ring $M$ and $A$ is a multiplication unitary left $M_\Gamma$-module, then $A$ is a multiplication left $L$-module. In Section 4, we proved that in fact this result was true for $\Gamma$-rings and $M_\Gamma$-modules.

# 2 Preliminaries

In this section we recall certain definitions needed for our purpose.

Recall that for additive abelian groups $M$ and $\Gamma$ we say that $M$ is a $\Gamma$-ring if there exists a mapping

$$\begin{aligned} \cdot : M \times \Gamma \times M &\longrightarrow M \\ (m, \gamma, m') &\longrightarrow m\gamma m' \end{aligned}$$

such that for every $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following hold:

1. $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$ and $a\alpha(b+c) = a\alpha b + a\alpha c$;
2. $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Note that any ring $R$, can be regarded as an $R$-ring. A $\Gamma$-ring $M$ is called commutative, if for any $x, y \in M$ and $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$. $M$ is called a $\Gamma$-ring with unit, if there exists elements $1 \in M$ and $\gamma_0 \in \Gamma$ such that for any $m \in M$, $1\gamma_0 m = m = m\gamma_0 1$.

If $A$ and $B$ are subsets of a $\Gamma$-ring $M$ and $\Theta \subseteq \Gamma$, we denote $A\Theta B$, the subset of $M$ consisting of all finite sums of the form $\sum a_i\gamma_i b_i$, where $(a_i, \gamma_i, b_i) \in A \times \Theta \times B$. For singleton subsets we abbreviate this notation for example, $\{a\}\Theta B = a\Theta B$.

A subset $I$ of a $\Gamma$-ring $M$ is said to be a right ideal of $R$ if $I$ is an additive subgroup of $M$ and $I\Gamma M \subseteq I$. A left ideal of $M$ is defined in a similar way. If $I$ is both a right and left ideal, we say that $A$ is an ideal of $M$.

For each subset $S$ of a $\Gamma$-ring $M$, the smallest right ideal containing $S$ is called the right ideal generated by $S$ and is denoted by $|S\rangle$. Similarly we define $\langle S|$ and $\langle S\rangle$, the left and two-sided (respectively) ideals generated by $S$. For each $a$ of a Γ-ring $M$, the smallest right ideal containing $a$ is called the principal right ideal generated by $a$ and is denoted by $|a\rangle$. We similarly define $\langle a|$ and $\langle a\rangle$, the principal left and two-sided (respectively) ideals generated by $a$. We have $|a\rangle = Za + a\Gamma M$, $\langle a| = Za + M\Gamma a$, and $\langle a\rangle = Za + a\Gamma M + M\Gamma a + M\Gamma a\Gamma M$, where $Za = \{na : n \text{is an integer}\}$.

Let $I$ be an ideal of Γ-ring $M$. If for each $a+I$, $b+I$ in the factor group $M/I$, and each $\gamma \in \Gamma$, we define $(a+I)\gamma(b+I) = a\gamma b + I$, then $M/I$ is a Γ-ring which we shall call the difference Γ-ring of $M$ with respect to $I$.

Let $M$ be a Γ-ring and $F$ the free abelian group generated by $\Gamma \times M$. Then $A = \{\sum_i n_i(\gamma_i, x_i) \in F : a \in M \Rightarrow \sum_l n_i a\gamma_i x_i = 0\}$ is a subgroup of $F$. Let $R = F/A$, the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. It can be verified easily that $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ and $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$ for all $\alpha$, $\beta \in \Gamma$ and $x$, $y \in M$. We define a multiplication in $R$ by $\sum_i[\alpha_i, x_i]\sum_J[\beta_j, y_j] = \sum_{iJ}[\alpha_i, x_i\beta_j y_j]$. Then $R$ forms a ring. If we define a composition on $M \times R$ into $M$ by $a\sum_l[\alpha_i, x_i] = \sum_i a\alpha_i x_i$ for $a \in M$, $\sum_i[\alpha_i, x_i] \in R$, then $M$ is a right R-module, and we call $R$ the right operator ring of the Γ -ring $M$. Similarly, we may construct a left operator ring $L$ of $M$ so that $M$ is a left $L$-module. Clearly $I$ is a right (left) ideal of $M$ if and only if $I$ is a right $R$-module (left L- module) of $M$. Also if $A$ is a right (left) ideal of $R(L)$, then MA$(AM)$ is an ideal of $M$. For subsets $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i[\gamma_i, x_i]$ in $R$, where $\gamma_i \in \Phi$, $x_i \in N$, and we denote by $[(\Phi, N)]$ the set of all elements $[\varphi, x]$ in $R$, where $\varphi \in \Phi$, $x \in N$. Thus, in particular, $R = [\Gamma, M]$.

An ideal $P$ of $M$ is prime if, for any ideals $U$ and $V$ of $M$, $U\Gamma U \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$. A subset $S$ of $M$ is an $m$-system in $M$ if $S = \emptyset$ or if $a, b \in S$ implies $< a > \Gamma < b > \cap S \neq \emptyset$. The prime radical $\mathcal{P}(A)$ is the set of $x$ in $M$ such that every $m$-system containing $x$ meets $A$. The prime radical of the zero ideal in a Γ-ring $M$ is called the prime radical of the Γ-ring $M$ which we denote by $\mathcal{P}(M)$.

An ideal $Q$ of $M$ is semi-prime if, for any ideals $U$ of $M$, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$.

**Proposition 2.1.** *[15] If $Q$ is an ideal in a commutative Γ-ring with unit $M$, then $P(Q)$ is the smallest semi-prime ideal in $M$ which contains $Q$, i.e.*

$$\mathcal{P}(Q) = \bigcap P$$

*where $P$ runs over all the semi-prime ideals of $M$ such that $Q \subseteq P$.*

Let $P$ be a proper ideal in a commutative Γ-ring with unit $M$. It is clear that the following conditions are equivallent.

1. $P$ is semi-prime.
2. For any $a \in M$, if $a\gamma_0 a \in P$, then $a \in P$.
3. For any $a \in M$ and $n \in \mathbb{N}$, if $(a\gamma_0)^n a \in P$, then $a \in P$.

**Proposition 2.2.** *[13] Let $Q$ be an ideal in a commutative $\Gamma$-ring with unit $M$ and $A$ be the set of all $x \in M$ such that $(x\gamma_0)^n x \in Q$ for some $n \in \mathbb{N}\cup\{0\}$, where $(x\gamma_0)^0 x = x$. Then $A = \mathcal{P}(Q)$.*

# 3 $M_\Gamma$-module

Let $M$ be a $\Gamma$-ring. A left $M_\Gamma$-module is an additive abelian group $A$ together with a mapping $\cdot : M \times \Gamma \times A \longrightarrow A$ ( the image of $(m, \gamma, a)$ being denoted by $m\gamma a$), such that for all $a, a_1, a_2 \in A$, $\gamma, \gamma_1, \gamma_2 \in \Gamma$, and $m, m_1, m_2 \in M$ the following hold:

1. $m\gamma(a_1 + a_2) = m\gamma a_1 + m\gamma a_2$;
2. $(m_1 + m_2)\gamma a = m_1\gamma m + m_2\gamma a$;
3. $m_1\gamma_1(m_2\gamma_2 a) = (m_1\gamma_1 m_2)\gamma_2 a$.

A right $M_\Gamma$-module is defined in analogous manner. If $I$ is a left ideal of a $\Gamma$-ring $M$, then $I$ is a left $M_\Gamma$-module with $r\gamma a$ ($r \in M$,$\gamma \in \Gamma$,$a \in I$) being the ordinary product in $M$. In particular, $\{0\}$ and $M$ are $M_\Gamma$-modules.

Let $A$ be a left $M_\Gamma$-module and $B$ a nonempty subset of $A$. $B$ is a $M_\Gamma$-submodule of $A$, which we denote by $B \leq A$, provided that $B$ is an additive subgroup of $A$ and $m\gamma b \in B$, for all $(m, \gamma, b) \in M \times \Gamma \times B$.

**Definition 3.1.** *Let $A$ be a left $M_\Gamma$-module and $X$ a subset of $A$. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be the family of all $M_\Gamma$-submodule of $A$ which contain $X$. Then $\bigcap_{\lambda\in\Lambda} A_\lambda$ is called the $M_\Gamma$-submodule of $A$ generated by the set $X$ and denoted $\langle X|$.*

If $B \subseteq A$, $N \subseteq M$ and $\Theta \subseteq \Gamma$, we denote $N\Theta B$, the subset of $A$ consisting of all finite sums of the form $\sum n_i\gamma_i b_i$ where $(n_i, \gamma_i, b_i) \in N \times \Theta \times B$. For singleton subsets we abbreviate this notation for example, $\{n\}\Theta B = n\Theta B$.

If $X = \{a_1, \ldots, a_n\}$, we write $\langle a_1, \ldots, a_n|$ in place of $\langle X|$. If $A = \langle a_1, \ldots, a_n|$, ($a_i \in A$), $A$ is said to be finitely generated. If $a \in A$, the $M_\Gamma$-submodule $\langle a|$ of $A$ is called the cyclic $M_\Gamma$-submodule generated by $a$. We have $\langle X| = ZX + M\Gamma X$, where $ZS = \{\sum_{i=1}^k n_i x_i : n_i \in Z, x_i \in S$ and $k$ is an integer$\}$.

Finally, if $\{B_\lambda\}_{\lambda\in\Lambda}$ is a family of $M_\Gamma$-submodules of $A$, then the $M_\Gamma$-submodule generated by $X = \bigcup_{\lambda\in\Lambda} B_\lambda$ is called the sum of the $M_\Gamma$-modules

$B_\lambda$ and usually denoted $\langle X| = \sum_{\lambda\in\Lambda} B_\lambda$. If the index set $\Lambda$ is finite, the sum of $B_1, \ldots, B_k$ is denoted $B_1 + B_2 + ... + B_k$. It is clear that if $\{B_\lambda\}_{\lambda\in\Lambda}$ is a family of $M_\Gamma$-submodules of $A$, then $\sum_{\lambda\in\Lambda} B_\lambda$ consists of all finite sums $b_{\lambda_1} + \ldots + b_{\lambda_k}$ with $b_{\lambda_j} \in B_{\lambda_l}$.

**Proposition 3.1.** *Let $M$ be a $\Gamma$-ring and $\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of left ideals of $M$. If $A$ is a left $M_\Gamma$-module, then*

$$(\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A = \sum_{\lambda\in\Lambda}(I_\lambda\Gamma A).$$

*Proof.* Let $x \in (\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A$. Then there exists $a_1, \ldots, a_k \in A$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$ and $x_1, \ldots, x_k \in \sum_{\lambda\in\Lambda} I_\lambda$ such that $x = \sum_{t=1}^{k} x_t\gamma_t a_t$, it follows that for $1 \leq t \leq k$, $x_t = \sum_{j=1}^{k_t} i_{\lambda_{jt}}$ with $i_{\lambda_{jt}} \in I_{\lambda_{jt}}$. Hence $x = \sum_{t=1}^{k}\sum_{j=1}^{k_t} i_{\lambda_{jt}}\gamma_t a_t \in \sum_{\lambda\in\Lambda}(I_\lambda\Gamma A)$. Therefore $(\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A \subseteq \sum_{\lambda\in\Lambda}(I_\lambda\Gamma A)$. Also, Since for every $\lambda \in \Lambda$, $I_\lambda\Gamma A \subseteq (\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A$, we conclude that $\sum_{\lambda\in\Lambda}(I_\lambda\Gamma A) \subseteq (\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A$. Hence $(\sum_{\lambda\in\Lambda} I_\lambda)\Gamma A = \sum_{\lambda\in\Lambda}(I_\lambda\Gamma A)$. □

**Definition 3.2.** *If $A$ is a left $M_\Gamma$-module and $\mathcal{S}$ is the set of all $M_\Gamma$-submodules $B$ of $A$ such that $B \neq A$, then $\mathcal{S}$ is partially ordered by set-theoretic inclusion. $B$ is a maximal $M_\Gamma$-submodule if and only if $B$ is a maximal element in the partially ordered set $\mathcal{S}$.*

**Proposition 3.2.** *If $A$ is a non-zero finitely generated left $M_\Gamma$-module, then the following statements are hold.*

1. *If $K$ is a proper $M_\Gamma$-submodule of $A$, then there exists a maximal $M_\Gamma$-submodule of $A$ such that contain $K$.*

2. *$A$ has a maximal $M_\Gamma$-submodule.*

*Proof.* (1) Let $A = \langle a_1, \ldots, a_n|$ and

$$\mathcal{S} = \{L : K \subseteq L \text{ and } L \text{ is a proper } M_\Gamma\text{-submodule of } A\}.$$

$\mathcal{S}$ is partially ordered by inclusion and note that $\mathcal{S} \neq \emptyset$, since $K \in \mathcal{S}$. If $\{L_\lambda\}_{\lambda\in\Lambda}$ is a chain in $\mathcal{S}$, then $L = \bigcup_{\lambda\in\Lambda} L_\lambda$ is a $M_\Gamma$-submodule of $A$. We show that $L \neq A$. If $L = A$, then for every $1 \leq i \leq n$, there exists $\lambda_i \in \Lambda$ such that $a_i \in L_{\lambda_i}$. Since $\{L_\lambda\}_{\lambda\in\Lambda}$ is a chain in $\mathcal{S}$, we conclude that there exists $1 \leq j \leq n$ such that $a_1, \ldots, a_n \in L_{\lambda_j}$. Therefore $A = L_{\lambda_j} \in \mathcal{S}$ which contradicts the fact that $A \notin \mathcal{S}$. It follows easily that $L$ is an upper bound $\{L_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{S}$. By Zorn's Lemma there exists a proper $M_\Gamma$-submodule $B$ of $A$ that is maximal in $\mathcal{S}$. It is a clear that $B$ a maximal $M_\Gamma$-submodule of $A$ such that contain $K$.

(2) By part (1), it suffices we put $K = \langle 0|$. □

**Definition 3.3.** *A left $M_\Gamma$-module $A$ is unitary if there exists an element, say 1 in $M$ and an element $\gamma_0 \in \Gamma$, such that, $1\gamma_0 a = a$ and $1\gamma_0 m = m = m\gamma_0 1$ for every $(a, m) \in A \times M$.*

**Corolary 3.1.** *If $M$ is a unitary left (right) $M_\Gamma$-module, then $M$ has a left (right) maximal ideal.*

*Proof.* It is evident by Proposition 3.2. □

Let $A$ be a left $M_\Gamma$-module. let $X \subseteq A$ and let $B \leq A$. Then the set $(B : X) := \{m \in M : m\Gamma X \subseteq B\}$ is a left ideal of $M$. In particular, if $a \in A$, then $(0 : a) := ((0) : \{a\})$ is called the left annihilator of $a$ and $(0 : A) := ((0) : A)$ is an ideal of $M$ called the annihilating ideal of $A$. Furthermore $A$ is said to be faithful if and only if $(0 : A) = (0)$.

**Definition 3.4.** *A left $M_\Gamma$-module $A$ is called a multiplication left $M_\Gamma$-module if each $M_\Gamma$-submodule of $A$ is of the form $I\Gamma A$, where $I$ is an ideal of $M$.*

**Proposition 3.3.** *Let $B$ be a $M_\Gamma$-submodule of multiplication left $M_\Gamma$-module $A$. Then $B = (B : A)\Gamma A$.*

*Proof.* It is a clear that $(B : A)\Gamma A \subseteq B$. Since $A$ is a multiplication left $M_\Gamma$-module, we conclude that there exists ideal $I$ of $\Gamma$-ring $M$ such that $B = I\Gamma A$, it follows that $B = I\Gamma A \subseteq (B : A)\Gamma A \subseteq B$. Therefore $B = (B : A)\Gamma A$. □

**Proposition 3.4.** *Let $A$ be a left $M_\Gamma$-module. $A$ is multiplication if and only if for every $a \in A$, there exists ideal $I$ in $M$ such that $\langle a| = I\Gamma A$.*

*Proof.* In view of Definition 3.4, it is enough to show that if for every $a \in A$, there exists ideal $I$ in $M$ such that $\langle a| = I\Gamma A$, then $A$ is multiplication. Let $B$ be an $M_\Gamma$-submodule of $A$. Then for every $b \in B$, there exists ideal $I_b$ in $M$ such that $\langle b| = I_b\Gamma A$. By Proposition 3.1, $(\sum_{b\in B} I_b)\Gamma A = \sum_{b\in B}(I_b\Gamma A) = \sum_{b\in B}\langle b| = B$, it follows that $A$ is multiplication. □

**Proposition 3.5.** *Let $M$ be a $\Gamma$-ring which has a unique maximal ideal $Q$ and $A$ be a unitary multiplication left $M_\Gamma$-module. If every ideal $I$ in $M$ is contained in $Q$, then for every $a \in A \setminus Q\Gamma A$, $\langle a| = A$.*

*Proof.* Suppose that $a \in A \setminus Q\Gamma A$. Since $A$ is multiplication left $M_\Gamma$-module, we conclude that there exists ideal $I$ in $M$ such that $\langle a| = I\Gamma A$. Clearly $I \not\subseteq Q$ and hence $I = M$, which implies $\langle a| = M\Gamma A = A$. □

**Corolary 3.2.** *Let $\Gamma$-ring $M$ be a unitary left $M_\Gamma$-module which has a unique maximal ideal $Q$ and $A$ be a unitary multiplication left $M_\Gamma$-module. Then for every $a \in A \setminus Q\Gamma A$, $\langle a| = A$.*

*Proof.* By Propositions 3.2 and 3.5, it is evident. □

**Proposition 3.6.** *Let $L$ be a left operator ring of the $\Gamma$-ring $M$ and let $A$ be a unitary left $M_\Gamma$-module. If we define a composition on $L \times A$ into $A$ by $(\sum_l [x_i, \alpha_i])a = \sum_i x_i \alpha_i a$ for $a \in A$, $\sum_i [x_i, \alpha_i] \in L$, then $A$ is a left $L$-module. Also, for every $B \subseteq A$, $B$ is a $M_\Gamma$-submodule of $A$ if and only if $B$ is a $L$-submodule of $A$.*

*Proof.* Suppose that $1 \in M$ and $\gamma_0 \in \Gamma$ such that for every $(a, m) \in A \times M$, $1\gamma_0 a = a$ and $1\gamma_0 m = m = m\gamma_0 1$. Let $\sum_{i=1}^t [x_i, \alpha_i] = \sum_{j=1}^s [y_j, \beta_j] \in L$ and $a = b \in A$. By definition of left operator ring of the $\Gamma$-ring $M$, we conclude that $\sum_{i=1}^t x_i \alpha_i 1 = \sum_{j=1}^s y_j \beta_j 1$, it follows that

$$\begin{array}{rcl} (\sum_{i=1}^t [x_i, \alpha_i])a & = & \sum_{i=1}^t x_i \alpha_i a \\ & = & \sum_{i=1}^t (x_i \alpha_i (1\gamma_0 a)) \\ & = & \sum_{i=1}^t (x_i \alpha_i 1)\gamma_0 a \\ & = & (\sum_{i=1}^t x_i \alpha_i 1)\gamma_0 a \\ & = & (\sum_{j=1}^s y_j \beta_j 1)\gamma_0 b \\ & = & \sum_{j=1}^s y_j \beta_j b \\ & = & (\sum_{j=1}^s [y_j, \beta_j])b \end{array}$$

Hence composition on $L \times A$ into $A$ is a well-defined. Let $r = \sum_{i=1}^t [x_i, \alpha_i]$ and $s = \sum_{j=1}^s [y_j, \beta_j]$. Then for every $a \in A$,

$$\begin{array}{rcl} (rs)a & = & (\sum_{i,j} [x_i \alpha_i y_j, \beta_j])a \\ & = & \sum_{i,j} (x_i \alpha_i y_j)\beta_j a \\ & = & \sum_{i,j} x_i \alpha_i (y_j \beta_j a) \\ & = & \sum_{i=1}^t x_i \alpha_i (\sum_{j=1}^s y_j \beta_j a) \\ & = & (\sum_{i=1}^t [x_i, \alpha_i])(\sum_{j=1}^s y_j \beta_j a) \\ & = & r((\sum_{j=1}^s [y_j, \beta_j])a) \\ & = & r(sa) \end{array}$$

The remainder of the proof is now easy. □

**Proposition 3.7.** *Let $L$ be a left operator ring of the $\Gamma$-ring $M$. If $A$ is a multiplication unitary left $M_\Gamma$-module, then $A$ is a multiplication left $L$-module.*

*Proof.* Let $B$ be a $L$-submodule of $A$. By Proposition 3.6, $B$ is a $M_\Gamma$-submodule of $A$ and there exists ideal $I$ of $\Gamma$-ring $M$ such that $B = I\Gamma A$. It well known that $[\Gamma, I]$ is an ideal of $L$. We show that $B = [I, \Gamma]A$. Suppose that $a_1, \ldots, a_t \in A$, and for every $1 \leq i \leq t$, $\sum_{j=1}^{k_i} [x_{i_j}, \alpha_{i_j}] \in [I, \Gamma]$. Then we have $\sum_{i=1}^{t}(\sum_{j=1}^{k_i}[x_{i_j},\alpha_{i_j}])a_i = \sum_{i=1}^{t}\sum_{j=1}^{k_i} x_{i_j}\alpha_{i_j}a_i) \in B$ and it follows that $[I,\Gamma]A \subseteq B$. Also, if $b \in B$, then there exists $x_1,\ldots,x_t \in I$,$\gamma_1,\ldots,\gamma_t \in \Gamma$, and $a_1,\ldots,a_t \in A$ such that $b = \sum_{i=1}^{t} x_i\gamma_i a_i = \sum_{i=1}^{t}[x_i,\gamma_i]a_i \in [I,\Gamma]A$ and we conclude that $B = [I,\Gamma]A$. □

**Proposition 3.8.** *Let $A$ be a unitary cyclic left $M_\Gamma$-module. If $L$ is a left operator ring of the $\Gamma$-ring $M$ and for every $l,l' \in L$, there exists $l'' \in L$ such that $ll' = l''l$, then $A$ is a multiplication left $L$-module.*

*Proof.* Let $B$ be a $L$-submodule of $A$ and $I = \{l \in L : lA \subseteq B\}$, then $IA \subseteq B$. Since $A$ is a unitary cyclic left $M_\Gamma$-module, we conclude that there exists $a \in A$ such that $A = M\Gamma a$. Let $b \in B$. Hence there exists $m_1,\ldots,m_t \in M$ and $\gamma_1,\ldots,\gamma_t \in \Gamma$ such that $b = \sum_{i=1}^{t} m_i\gamma_i a$. In view of operations of addition and multiplication in left $L$-module $A$, we have $b = \sum_{i=1}^{t}[m_i,\gamma_i]a = (\sum_{i=1}^{t}[m_i,\gamma_i])a$. We put $l = \sum_{i=1}^{t}[m_i,\gamma_i]$ and it follows that $b = la$. If $a' \in A$, then a similar argument shows that there exists $l' \in L$ such that $a' = l'a$. By hypothesis, there exists $l'' \in L$ such that $ll' = l''l$. Therefore $la' = ll'a = l''la = l''b \in B$ and it follows that $l \in I$, this is $b = la \in IA$. Hence $B = IA$ and the proof is now complete. □

**Definition 3.5.** *Let $A$ be a unitary left $M_\Gamma$-module and $B$ be a $M_\Gamma$-submodule in $A$ and $P \in Max(M)$. $A$ is called $P$-cyclic if there exist $p \in P$ and $b \in B$ such that $(1-p)\gamma_0 B \subseteq M\Gamma b$ and also, it is clear that $(1-p)\gamma_0 B = (1-p)\Gamma B$. Define $T_P B$ as the set of all $b \in B$ such that $(1-p)\gamma_0 b = 0$, for some $p \in P$.*

**Lemma 3.1.** *Let $A$ be a unitary left $M_\Gamma$-module and $B$ be a $M_\Gamma$-submodule in $A$ and $P \in Max(M)$. If $M$ is a commutative $\Gamma$-ring, then $T_P B$ is a $M_\Gamma$-submodule in $A$.*

*Proof.* Suppose $b_1,b_2 \in T_P B$. So there exist $p_1,p_2 \in P$ such that $b_1 = p_1\gamma_0 b_1$ and $b_2 = p_2\gamma_0 b_2$. Let $p_0 = p_1+p_2-p_1\gamma_0 p_2$. It is clear that $(1-p_0)\gamma_0(b_1-b_2) = 0$. Hence $b_1 - b_2 \in T_P B$. Let $x \in M\Gamma(T_P B)$. So $x = \sum_{i=1}^{n} m_i\gamma_i b_i$, where $n \in \mathbb{N}$, $b_i \in T_P B$, $\gamma_i \in \Gamma$ and $m_i \in M$ $(1 \le i \le n)$. Suppose $i \in \{1,\cdots,n\}$. Since $b_i \in T_P N$, there exists $p_i \in P$ such that $(1-p_i)\gamma_0 m_i\gamma_i b_i = 0$. Hence $m_i\gamma_i b_i \in T_P N$. Thus $x \in T_P B$. Hence $M\Gamma T_P B = T_P B$. □

**Proposition 3.9.** *Let $M$ be a commutative $\Gamma$-ring and let $A$ be a unitary left $M_\Gamma$-module. $A$ is multiplication $M_\Gamma$-module if and only if for any ideal $P \in Max(M)$, either $A = T_P A$ or $A$ is $P$-cyclic.*

*Proof.* Let $A$ be a multiplication ideal and $P \in Max(M)$. First suppose that $A = P\Gamma A$. Since $A$ is multiplication ideal, we conclude that for every $a \in A$, there exists an ideal $I$ in $M$ such that $< a >= I\Gamma A$. Hence $< a >= P\Gamma <$

$a>$. So there exists $p \in P$ such that $(1-p)\gamma_0 a = 0$, it follows that $a \in T_P B$ and then $A = T_P A$.

Now suppose that $A \neq P\Gamma A$ and $x \in A \setminus P\Gamma A$. Then there exists an ideal $I$ in $M$ such that $<x>= I\Gamma A$ and $P + I = M$. Obviously, if we assume that $p \in P$, then $(1-p)\gamma_0 A \subseteq M\Gamma x$. Therefore $A$ is $P$-cyclic.

Conversely, suppose that $B$ is a $M_\Gamma$-submodule in $A$. Define $I$ as the set of all $m \in M$, where $m\gamma_0 a \in B$ for any $a \in A$. Clearly $I$ is an ideal in $M$ and $I\Gamma A \subseteq B$. Let $b \in B$. Define $K$ as the set of all $m \in M$, where $m\gamma_0 b \in I\Gamma A$. We claim $K = M$. Assume that $K \subset M$. Hence by Zorns Lemma there exists $Q \in Max(M)$ such that $K \subseteq Q \subset M$. By hypothesis $A = T_Q A$ or $A$ is $Q$-cyclic. If $A = T_Q A$, then there exists $s \in Q$ such that $(1-s)\gamma_0 b = 0$. Hence $(1-s) \in K \subseteq Q$, it follows that $1 \in Q$, a contradiction. If $A$ is $Q$-cyclic, then there exist $t \in Q$ and $c \in A$ such that $(1-t)\gamma_0 A \subseteq M\Gamma c =< c >$. Define $L$ as the set of all $m \in M$ such that $m\gamma_0 c \in (1-t)\gamma_0 B$. Clearly $L$ is an ideal in $M$ and $L\gamma_0 c \subseteq (1-t)\gamma_0 B \subseteq< c >$. Hence $(1-t)\gamma_0 B \subseteq L\gamma_0 c$. So $(1-t)\gamma_0 B = L\gamma_0 c$, it follows that $(1-t)\gamma_0 L\gamma_0 A \subseteq (1-t)\gamma_0 B \subseteq B$ and $(1-t)\gamma_0 L \subseteq I$. Therefore $(1-t)\gamma_0(1-t)\gamma_0 B \subseteq I\Gamma A$. Hence $(1-t)\gamma_0(1-t) \in K \subseteq Q$. Thus $1-t \in Q$, it follows that $1 \in Q$, a contradiction. Hence $K = M$ and $b \in I\Gamma A$. Thus $A$ is a multiplication ideal. $\square$

Let $A$ be a left $M_\Gamma$-module. $A$ is said to have the intersection property provided that for every non-empty collection of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of $M$,

$$\bigcap_{\lambda\in\Lambda} I_\lambda \Gamma A = (\bigcap_{\lambda\in\Lambda} I_\lambda)\Gamma A.$$

If left $M_\Gamma$-module of $A$ has intersection property, then for every non-empty collection of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of $M$,

$$\bigcap_{\lambda\in\Lambda} I_\lambda \Gamma A = (\bigcap_{\lambda\in\Lambda}(I_\lambda + Ann(A)))\Gamma A.$$

**Proposition 3.10.** *Let $M$ be a commutative $\Gamma$-ring and let $A$ be a unitary left $M_\Gamma$-module.*

1. *If $A$ has intersection property and for any $M_\Gamma$-submodule $N$ in $A$ any ideal $I$ in $M$ which $N \subset I\Gamma A$, there exists ideal $J$ in $M$ such that $J \subset I$ and $N \subseteq J\Gamma A$, then $A$ is multiplication left $M_\Gamma$-module.*

2. *If $A$ is faithful left multiplication $M_\Gamma$-module, then $A$ has intersection property and for any $M_\Gamma$-submodule $N$ in $A$ any ideal $I$ in $M$ which $N \subset I\Gamma A$, there exists ideal $J$ in $M$ such that $J \subset I$ and $N \subseteq J\Gamma A$.*

*Proof.* (1) Let $N$ be a $M_\Gamma$-submodule in $A$ and

$$\mathcal{S} = \{I : \ I \text{ is an ideal of } M \text{ and } N \subseteq I\Gamma A\}.$$

Clearly $M \in \mathcal{S}$. Since $A$ has intersection property, we conclude from Zorns Lemma that $\mathcal{S}$ has a minimal member $I$ (say). Since $N \subseteq I\Gamma A$ and $I$ is minimal element of $\mathcal{S}$, we can conclude that $N = I\Gamma A$. It follows that $A$ is a multiplication ideal.

(2) Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a nonempty collection of ideal in $M$ and $I = \bigcap_{\lambda\in\Lambda} I_\lambda$. Clearly $I\Gamma A \subseteq \bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma A)$. Let $x \in \bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma A)$ and we put $L = \{m \in M : m\gamma_0 x \in I\Gamma A\}$. We claim $L = M$. Assume that $L \subset M$. By Proposition 3.2, there exists $P \in Max(M)$ such that $L \subseteq P$. It is clear that $x \notin T_P A$. Hence $T_P A \neq A$ and by Proposition 3.9, $A$ is $P$-cyclic. Hence there exist $a \in A$ and $p \in P$ such that $(1-p)\gamma_0 A \subseteq M\Gamma a =< a >$. Thus $(1-p)\gamma_0 x \in \bigcap_{\lambda\in\Lambda}(I_\lambda\gamma_0 a)$ and so for any $\lambda \in \Lambda$, $(1-p)\gamma_0 x \in I_\lambda\gamma_0 a$. It is clear that $(1-p)\gamma_0(1-p) \in L \subseteq P$, in view of the fact that $A$ is faithful. Hence $1 \in P$, a contradiction. Therefore $L = M$, it follows that $x = 1\gamma_0 x \in I\Gamma A$ and $A$ has intersection property. Now suppose $N$ be a $M_\Gamma$-submodule in $A$ and $I$ be an ideal in $M$ which $N \subset I\Gamma A$. Since $A$ is multiplication $M_\Gamma$-module, there exists an ideal $J$ in $M$ such that $N = J\Gamma A$. Let $K = I \cap J$. Clearly, $K \subset I$ and since $A$ has intersection property, we conclude that $N \subseteq K\Gamma A$. The proof is now complete. $\square$

**Proposition 3.11.** *Let $A$ be a faithful multiplication $M_\Gamma$-module and $I, J$ be two ideals in $M$. $I\Gamma A \subseteq J\Gamma A$ if and only if either $I \subseteq J$ or $A = [J : I]\Gamma A$.*

*Proof.* Let $I \nsubseteq J$. Note that $[J : I] = \bigcap_{i\in X}[J :< i >]$ where $X$ is the set of all elements $i \in I$ with $i \notin J$. By Proposition 3.10,

$$[J : I]\Gamma A = \bigcap_{i\in X}([J :< i >]\Gamma A)$$

If for every $i \in X$, $A = [J :< i >]\Gamma A$, then $A = [J : I]\Gamma A$, which finishes the proof. Let $i \in X$ and $Q = [J :< i >]$. It is clear that $Q \neq M$. Let $\Omega$ denote the collection of all semi-prime ideals $P$ in $M$ containing $Q$. Suppose that there exists $P \in \Omega$ such that $A \neq P\Gamma A$ and $x \in A \setminus P\Gamma A$. Since $A$ is a multiplication $M_\Gamma$-module, we conclude that there exists ideal $D$ in $M$ such that $< x >= D\Gamma A$ and $D \nsubseteq P$. Thus $c\Gamma A \subseteq< x >$ for some $c \in D \setminus P$. Now we have $c\Gamma a\Gamma A \subseteq J\Gamma < x >$. It is easily to show that for any $\gamma \in \Gamma$, there exists $\gamma_1 \in \Gamma$ and $b \in J$ such that $(c\gamma a - 1\gamma_1 b)\gamma_0 x = 0$, it follows that $(c\gamma a - 1\gamma_1 b)\Gamma c\Gamma A = 0$. Hence $c\gamma c \in [J :< i >] = Q$. Since $P$ is a semi-prime ideal containing $Q$, we conclude that $c \in P$, a contradiction. Therefore for every $P \in \Omega$, $A = P\Gamma A$ and by Propositions 2.1 and 3.10,

$A = P(Q)\Gamma A$. Let $j \in A$. It is easily to show that $< j >= P(Q)\Gamma < j >$. Then there exists $s \in P(Q)$ such that for every $n \in \mathbb{N}$, $j = (s\gamma_0)^n j$. By Proposition 2.2, there exists $t \in \mathbb{N} \cup \{0\}$ such that $(s\gamma_0)^t s \in Q$, it follows that $j = (s\gamma_0)^t s\gamma_0 j \in Q\Gamma A$, i.e., $A \subseteq Q\Gamma A$. Hence $Q\Gamma A = A$. The converse is evident. □

# 4 Prime $M_\Gamma$-submodule

Through this section $M$ and $A$ will denote a commutative Γ-ring with unit and an unitary left $M_\Gamma$-module, respectively.

**Definition 4.1.** *A prime ideal $P$ in $M$ is called a minimal prime ideal of the ideal $I$ if $I \subseteq P$ and there is no prime ideal $P'$ such that $I \subseteq P' \subset P$. Let $Min(I)$ denote the set of minimal prime ideals of $I$ in Γ-ring $M$, and every element of $Min((0))$ is called minimal prime ideal.*

**Proposition 4.1.** *If an ideal $I$ of Γ-ring $M$ is contained in a prime ideal $P$ of $M$, then $P$ contains a minimal prime ideal of $I$.*

*Proof.* Let

$$\mathcal{A} = \{Q : Q \text{ is prime ideal of } M \text{ and } I \subseteq Q \subseteq P\}.$$

By Zorn's Lemma, there is a prime ideal $Q$ of $R$ which is minimal member with respect to inclusion in $\mathcal{A}$. Therefore $Q \in Min(I)$ and $I \subseteq Q \subseteq P$. □

**Lemma 4.1.** *Let Γ be a finitely generated group. If $I$ and $J$ are finitely generated ideals of $M$, then $I\Gamma J$ is finitely generated ideal of $M$.*

*Proof.* Let $I = \langle a_1, \dots, a_n\rangle$, $J = \langle b_1, \dots, b_m\rangle$, and $\Gamma = \langle \gamma_1, \dots, \gamma_k\rangle$. It is clear that $I\Gamma J = \langle a_i\gamma_t b_j : 1 \leq i \leq n, 1 \leq t \leq k, 1 \leq j \leq m\rangle$. □

**Proposition 4.2.** *Let Γ be a finitely generated group. If $I$ is a proper ideal of $M$ and each minimal prime ideal of $I$ is finitely generated, then $Min(I)$ is finite set.*

*Proof.* Consider the set

$$\mathcal{S} = \{P_1\Gamma P_2 \dots P_n; n \in \mathbb{N} \text{ and } P_i \in Min(I), \text{for each } 1 \leq i \leq n\}$$

and set

$$\Delta = \{K; K \text{ is an ideal of } M \text{ and } Q \nsubseteq K, \text{ for each } Q \in \mathcal{S}\}$$

which is the non-empty set, since $I \in \Delta$. $(\Delta, \subseteq)$ is the partial ordered set. Suppose $\{K_\lambda\}_{\lambda\in\Lambda}$ is the chain of $\Delta$ in which $\Lambda \neq \emptyset$ and set $K = \bigcup_{\lambda\in\Lambda} K_\lambda$. It is clear that $K$ is an ideal of $M$. Also, if there exits $Q \in \mathcal{S}$ such that $Q \subseteq K$, then by Lemma 4.1, $Q = P_1\Gamma P_2...P_n$ is finitely generated ideal of $M$, i.e., $Q = \langle x_1, \dots, x_n\rangle$. But $Q \subseteq K$ implies that $x_1, x_2, ..., x_n \in K$. Thus there exists $\lambda \in \Lambda$ such that $x_1, x_2, ..., x_n \in K_\lambda$ and so $Q \subseteq K_\lambda$, contradiction. Hence, for each $Q \in \mathcal{S}$, $Q \nsubseteq K$ and $K \in \Delta$ is the upper band of this chain.

By Zorhn's lemma $\Delta$ has maximal element such as $Q$. Now if $a \notin Q$ and $b \notin Q$ for $a, b \in M$, then $Q \subseteq \langle Q \cup \{a\}\rangle$ and $Q \subseteq \langle Q \cup \{b\}\rangle$. Maximality of $Q$ implies that $\langle Q \cup \{a\}\rangle$, $\langle Q \cup \{b\}\rangle \notin \Delta$. So there exists $Q_1$ and $Q_2$ in $\mathcal{S}$ such that $Q_1 \subseteq \langle Q \cup \{a\}\rangle$ and $Q_2 \subseteq \langle Q \cup \{b\}\rangle$. It is clear that $Q_1\Gamma Q_2 \subseteq Q$ which is contradiction, since $Q_1\Gamma Q_2 \in \mathcal{S}$. Therefore $\langle a\rangle\Gamma\langle b\rangle \nsubseteq Q$ and $Q$ is a prime ideal of $M$ contained $I$. By Proposition 4.1, there exists a minimal prime ideal $P \subseteq Q$. But $P \in \mathcal{S}$, contradictory with $Q \in \Delta$. Above contradicts show that there exists $Q' = P_1\Gamma P_2...P_m \in \mathcal{S}$ such that $Q' \subseteq I$.

Now for each $P \in Min(I)$ we have $Q' \subseteq I \subseteq P$ and $P_1\Gamma P_2...P_m \subseteq P$. It is clear that $P_j \subseteq P$ for some $1 \leq j \leq m$. Thus $P_j = P$, since $P$ is minimal. Hence $Min(I) = \{P_1, P_2, ..., P_m\}$ is finite. □

**Proposition 4.3.** *For proper $M_\Gamma$-submodule $B$ of $A$, the following statements equivalent:*

1. *For every $M_\Gamma$-submodule $C$ of $A$, if $B \subset C$, then $(B : A) = (B : C)$.*
2. *For every $(m, a) \in M \times A$, if $m\Gamma a \subseteq B$, then $a \in B$ or $m \in (B : A)$.*

*Proof.* (1) $\Rightarrow$ (2) Let $(m, a) \in M \times A$ such that $m\Gamma a \subseteq B$ and $a \notin B$. It is clear that $B \subset B + M\Gamma a$. Since $m\Gamma(B + M\Gamma a) \subseteq m\Gamma B + m\Gamma(M\Gamma a) = m\Gamma B + M\Gamma(m\Gamma a) \subseteq B$, we conclude from statement (1) that $m \in (B : B + M\Gamma a) = (B : A)$ and the proof is now complete.

(2) $\Rightarrow$ (1) Let $C$ be a $M_\Gamma$-submodule of $A$ such that $B \subset C$. It is clear that $(B : A) \subseteq (B : C)$. Now, suppose that $m \in (B : C)$. Since $B \subset C$, we infer that there exists $a \in C \setminus B$ such that $m\Gamma a \subseteq B$. By statement (2), $m \in (B : A)$ and the proof is now complete.

□

**Definition 4.2.** *A proper $M_\Gamma$-submodule $B$ of $A$ is said to be prime if $m\Gamma a \subseteq B$ for $(m, a) \in M \times A$ implies that either $a \in B$ or $m \in (B : A)$.*

**Proposition 4.4.** *If $B$ is a prime $M_\Gamma$-submodule of $A$, then $(B : A)$ is a prime ideal of $\Gamma$-ring $M$.*

*Proof.* Let $x, y \in M$ such that $\langle x\rangle\Gamma\langle y\rangle \subseteq (B : A)$ and $x \notin (B : A)$. Then there exists $\gamma \in \Gamma$ and $a \in A$ such that $x\gamma a \notin B$, and also, $y\Gamma(x\gamma a) = (y\Gamma x)\gamma a = (x\Gamma y)\gamma a \subseteq B$. Since $B$ is a prime $M_\Gamma$-submodule of $A$ and $x\gamma a \notin B$, we conclude that $y\Gamma A \subseteq B$, i. e., $y \in (B : A)$. The proof is now complete. □

**Proposition 4.5.** *Let $A$ be a multiplication left $M_\Gamma$-module, and $B$, $B_1$, ..., $B_k$ be $M_\Gamma$-submodules of $A$. If $B$ is a prime $M_\Gamma$-submodule of $A$, then the following statements are equivalent.*

1. *$B_j \subseteq B$ for some $1 \leq j \leq k$.*

2. *$\bigcap_{i=1}^k B_i \subseteq B$.*

*Proof.* (1) ⇒ (2) It is clear.

(2) ⇒ (1) We have $B_i = I_i\Gamma A$ for some ideals $I_i$, $(1 \leq i \leq k)$ of Γ-ring $M$. Then $(\bigcap_{i=1}^k I_i)\Gamma A \subseteq \bigcap_{i=1}^k(I_i\Gamma A) = \bigcap_{i=1}^k B_i \subseteq B$ and so $\bigcap_{i=1}^k I_i \subseteq (B : A)$. Since $M$ is a commutative Γ-ring, we infer that for every permutations $\theta$ of $\{1, 2, \ldots, k\}$, $I_1\Gamma I_2 \cdots I_k = I_{\theta(1)}\Gamma I_{\theta(2)} \cdots I_{\theta(k)}$, it follows that $I_1\Gamma I_2 \cdots I_k \subseteq \bigcap_{i=1}^k I_i \subseteq (B : A)$. Since by Proposition 4.4, $(B : A)$ is prime ideal of Γ-ring $M$, we conclude that $I_j \subseteq (B : A)$ for some $1 \leq j \leq k$. Therefore, by Proposition 3.3, $B_j = I_j\Gamma A \subseteq B$ for some $1 \leq j \leq k$. □

**Proposition 4.6.** *If $A$ is a multiplication left $M_\Gamma$-module, then for $M_\Gamma$-submodule $B$ of $A$, the following statements are equivalent.*

1. *$B$ is prime $M_\Gamma$-submodule of $A$.*

2. *$(B : A)$ is prime ideal of Γ-ring $M$.*

3. *There exists prime ideal $P$ of Γ-ring $M$ such that $B = P\Gamma A$ and for every ideal $I$ of $M$, $I\Gamma A \subseteq B$ implies that $I \subseteq P$.*

*Proof.* (1) ⇒ (2) By Proposition 4.4, It is evident.

(2) ⇒ (3) We put

$$\mathcal{M} = \{P : B = P\Gamma A \text{ and } P \text{ is an ideal of } \Gamma\text{-ring } M\ \}$$

Since $A$ is multiplication left $M_\Gamma$-module, we conclude that $(\mathcal{M}, \subseteq)$ is a non-empty partial order set. Let $\{P_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{M}$ be a chain. By Proposition 3.10, $\bigcap_{\lambda\in\Lambda} P_\lambda \in \mathcal{M}$ is an upper bound of $\{P_\lambda\}_{\lambda\in\Lambda}$. By Zorn's Lemma $\mathcal{M}$ has a maximal element. Thus, we can pick a $P$ to be maximal element of $\mathcal{M}$. Let $x, y \in M$ and $\langle x\rangle\Gamma\langle y\rangle \subseteq P$. Hence $(\langle x\rangle\Gamma\langle y\rangle)\Gamma A \subseteq P\Gamma A = B$ and we infer that $\langle x\rangle\Gamma\langle y\rangle \subseteq (B : A)$. Now, by statement (2), $x \in (B : A)$ or $y \in (B : A)$. Since $A$ is multiplication left $M_\Gamma$-module, we conclude from the Proposition

3.3 that $B = (B : A)\Gamma A$, it follows that $(B : A) \in \mathcal{M}$. On the other hand, clearly $P \subseteq (B : A)$ and so $P = (B : A)$, i.e., $x \in P$ or $y \in P$, Thus $P$ is prime ideal of $\Gamma$-ring $M$.

(3) $\Rightarrow$ (1) Let prime ideal $P$ of $\Gamma$-ring $M$ such that $B = P\Gamma A$ and for every ideal $I$ of $\Gamma$-ring $M$, $I\Gamma A \subseteq B$ implies that $I \subseteq P$. It is clear that $P = (B : A)$. Let $m \in M$ and $a \in A$ such that $m\Gamma a \subseteq B$. Since $A$ is a multiplication $S$-act, we conclude that there exists an ideal $I$ of $\Gamma$-ring $M$ such that $\langle a\rangle = I\Gamma A$, it follows that $(m\Gamma I)\Gamma A = m\Gamma(I\Gamma A) = m\Gamma(M\Gamma a) = (m\Gamma M)\Gamma a = (M\Gamma m)\Gamma a = M\Gamma(m\Gamma a) \subseteq B$. Therefore $m\Gamma I \subseteq (B : A) = P$ and it is easy to see directly that $\langle m\rangle\Gamma I \subseteq (B : A)$. Then $m\Gamma A \subseteq B$ or $a \in I\Gamma A \subseteq B$ and the proof is now complete. □

**Lemma 4.2.** *Let $A$ be a finitely generated left $M_\Gamma$-module. If $I$ is an ideal of $M$ such that $A = I\Gamma A$, then there exists $i \in I$ such that $(1 - i)\gamma_0 A = 0$.*

*Proof.* If $A =< a_1, \dots, a_n >$, then for every $1 \le i \le n$, there exists $y_{i1}, \dots, y_{in} \in I$ such that $a_i = \sum_{j=1}^{n} y_{ij}\gamma_0 a_j$, it follows that

$$-y_{i1}\gamma_0 a_1 - \cdots - y_{i(i-1)}\gamma_0 a_{i-1} + (1 - y_{ii})\gamma_0 a_i - y_{i(i+1)}\gamma_0 a_{i+1} - \cdots - y_{in}\gamma_0 a_n = 0.$$

If

$$B = \begin{bmatrix} 1 - y_{11} & -y_{12} & \cdots & -y_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ -y_{n1} & -y_{n2} & \cdots & 1 - y_{nn} \end{bmatrix},$$

then there exists $y \in I$ such that $det_\Gamma(B) = (1 + y)$, where

$$det_\Gamma(B) = \sum sign(\sigma) b_{1,\sigma(1)}\gamma_0 b_{2,\sigma(2)}\gamma_0 \cdots \gamma_0 b_{n,\sigma(n)}$$

and $\sigma$ runs over all the permutation on $\{1, 2, \dots, n\}$ (see [13]). Since for every $1 \le i \le n$, $det_\Gamma(B)\gamma_0 a_i = 0$, we conclude that $(1 + y)\gamma_0 A = 0$ and by setting $i = -y$ the proof will be completed. □

**Proposition 4.7.** *Let $A$ be a finitely generated faitfull multiplication left $M_\Gamma$-module. For proper ideal of $P$ in $M$, the following statements are equivalent.*

1. *$P$ is a prime ideal of $M$.*

2. *$P\Gamma A$ is a prime $M_\Gamma$-submodule of $A$.*

*Proof.* (1) $\Rightarrow$ (2) Let $I$ be an ideal of $M$ such that $I\Gamma A \subseteq P\Gamma A$. Then by Proposition 3.11, either $I \subseteq P$ or $A = [P : I]\Gamma A$. If $A = [P : I]\Gamma A$, then by Lemma 4.2, there exists $i \in [P : I]$ such that $(1 - i)\gamma_0 A = 0$. Since $A$ is a faitfull $M_\Gamma$-module, we conclude that $i = 1$ and $I \subseteq P$. Hence by Proposition 4.6, $P\Gamma A$ is a prime $M_\Gamma$-submodule of $A$.

(2) $\Rightarrow$ (1) Since $A$ is a faitfull $M_\Gamma$-module and $[P\Gamma A : A]\Gamma A \subseteq P\Gamma A$, we conclude from the Proposition 3.11 and Lemma 4.2 that $[P\Gamma A : A] \subseteq P$. Hence $[P\Gamma A : A] = P$ and by Proposition 4.6, $P$ is a prime ideal of $M$. □

**Proposition 4.8.** *Let $A$ be a multiplication left $M_\Gamma$-module. Then*

1. *If $M$ satisfies $ACC$ $(DCC)$ on prime ideals, then $A$ satisfies $ACC$ $(DCC)$ on prime $M_\Gamma$-submodules.*

2. *If $A$ is faitfull $M_\Gamma$-module and $(B : A)$ is a minimal prime ideal in $M$, then $B$ is a minimal prime $M_\Gamma$-submodule of $A$.*

*Proof.* (1) Assume that $B_1 \subseteq B_2 \subseteq \ldots$ is a chain of prime $M_\Gamma$-submodule of $A$. By Proposition 4.4, $(B_1 : A) \subseteq (B_2 : A) \subseteq \ldots$ is a chain of prime ideal of Γ-ring $M$. By hypothesis there exists $k \in \mathbb{N}$ such that for every $i \geq k$, $(B_i : A) = (B_k : A)$. It follows from Proposition 3.3 that $B_i = (B_i : A)\Gamma A = (B_k : A)\Gamma A = B_k$. Thus $A$ satisfies $ACC$ on prime $M_\Gamma$-submodules.

(2) assume that $B'$ is a prime $M_\Gamma$-submodule of $A$ such that $B' \subseteq B$. By Proposition 4.6, $(B' : A) \subseteq (B : A)$ is a chain of prime ideal of Γ-ring $M$. By hypothesis $(B' : A) = (B : A)$, it follows from Proposition 3.3 that $B' = (B' : A)\Gamma A = (B : A)\Gamma A = B$. Thus $B$ is a minimal prime $M_\Gamma$-submodule of $A$. □

**Proposition 4.9.** *Let $A$ be a finitely generated faitfull multiplication left $M_\Gamma$-module. Then*

1. *If $A$ satisfies $ACC$ $(DCC)$ on prime $M_\Gamma$-submodules, then Γ-ring $M$ satisfies $ACC$ $(DCC)$ on prime ideals.*

2. *If $B$ is a minimal prime $M_\Gamma$-submodule of $A$, then $(B : A)$ is a minimal prime ideal of Γ-ring $M$.*

*Proof.* (1) Assume that $P_1 \subseteq P_2 \subseteq \ldots$ is a chain of prime ideals of Γ-ring $M$. By Proposition 4.7, $P_1\Gamma A \subseteq P_2\Gamma A \subseteq \ldots$ is a chain of prime $M_\Gamma$-submodule of $A$. By hypothesis there exists $k \in \mathbb{N}$ such that for every $i \geq k$, $P_k\Gamma A = P_i\Gamma A$. Since $A$ is a finitely generated faitfull multiplication $M_\Gamma$-module, we conclude from the Proposition 3.11 and Lemma 4.2 that $P_k = P_i$.

(2) By Proposition 4.6, $(B : A)$ is a prime ideal of Γ-ring $M$. Assume that $P$ is a prime ideal of Γ-ring $M$ such that $P \subseteq (B : A)$. Hence by Proposition 3.3, $P\Gamma A \subseteq (B : A)\Gamma A = B$. Since by Proposition 4.7, $P\Gamma A$ is a prime $M_\Gamma$-submodule of $A$, we conclude from our hypothesis that $P\Gamma A = (B : A)\Gamma A$.

Since $A$ is a finitely generated faitfull multiplication $M_\Gamma$-module, we conclude from the Proposition 3.11 and Lemma 4.2 that $P = (B : A)$. The proof is now complete. $\square$

**Proposition 4.10.** *Let $\Gamma$ be a finitely generated group. Let $A$ be a finitely generated faitfull multiplication left $M_\Gamma$-module.*

1. *If every prime ideal of $\Gamma$-ring $M$ is finitely generated, then $A$ contains only a finitely many minimal prime $M_\Gamma$-submodule.*

2. *If every minimal prime $M_\Gamma$-submodule of $A$ is finitely generated, then $\Gamma$-ring $M$ contains only a finite number of minimal prime ideal.*

*Proof.* (1) Assume that $\{B_\lambda\}_{\lambda\in\Lambda}$ is the family of minimal prime $M_\Gamma$-submodules of $A$. Set $I_\lambda = (B_\lambda : A)$ for $\lambda \in \Lambda$. By Proposition 4.9, each $I_\lambda$ is a minimal prime ideal of $\Gamma$-ring $M$. On the other hand, by Proposition 4.2, $M$ contains only a finite number of minimal prime ideal as $\{I_1, I_2, \ldots I_n\}$. Now suppose that $\lambda \in \Lambda$. So $I_\lambda = I_i$, for some $1 \leq i \leq n$ and by Proposition 3.3, $B_\lambda = I_\lambda \Gamma A = I_i \Gamma A$. Thus $\{I_1\Gamma A, I_2\Gamma A, \ldots, I_n\Gamma A\}$ is the finite family of minimal prime $M_\Gamma$-submodule of $A$.

(2) Suppose that $I$ and $J$ are two distinct minimal prime ideal of $\Gamma$-ring $M$. By Proposition 3.11 and Lemma 4.2, $A \neq I\Gamma A \neq J\Gamma A$ and also, by Proposition 4.7, $I\Gamma A$ and $J\Gamma A$ are prime $M_\Gamma$-submodules of $A$. Assume that $B_1$ and $B_2$ are two prime $M_\Gamma$-submodules of $A$ such that $B_1 \subseteq I\Gamma A$ and $B_2 \subseteq J\Gamma A$. By Proposition 3.3, $B_1 = (B_1 : A)\Gamma A$ and $B_2 = (B_2 : A)\Gamma A$. By Proposition 3.11 and Lemma 4.2, $(B_1 : A) \subseteq I$ and $(B_2 : A) \subseteq J$. Since $I$ and $J$ are two distinct minimal prime ideal of $\Gamma$-ring $M$, we conclude from the Proposition 4.4 that $(B_1 : A) = I$ and $(B_2 : A) = J$. This says that $I\Gamma A$ and $J\Gamma A$ are two distinct minimal prime $M_\Gamma$-submodules of $A$. Now if $\Gamma$-ring $M$ contains infinite many minimal prime ideals, then $A$ must have infinitely many minimal prime $M_\Gamma$-submodules which is contradiction. $\square$

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# Fuzzy bi-objective optimization model for multi-echelon distribution network

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**Abstract**

It is important for modern businesses to search the ways for continuous improvement in performance of their supply chains. The effective coordination and integrated decision making across the supply chain enhances the performance among its various partners in a multi stage network. The partners considered in this paper are product suppliers, processing points (PP), distribution centres (DC) and retail outlets (RO). The network addresses an uncertain environment threatened by different sources in order to captivate the real world conditions. The uncertain demand of deteriorating products and its dependent costs develop uncertainties in the environment. On the other hand, suppliers and processing points have restricted capacities for the retail outlets' order amount happened in each period. A bi-objective non-linear fuzzy mathematical model is developed in which the uncertainties are represented by the fuzzy set theory. The proposed model shows cost minimization and best supplier selection coordination under the conditions of capacity constraints, uncertain parameters and product's deteriorating nature. The fish and fish products give good examples for the proposed model. To solve, the model is converted into crisp form and solved with the help of fuzzy goal programming.

**Key words**: multi stage, supplier selection, processing point, fuzzy goal programming, supply chain, Bi-objective.

# 1 Introduction

With the growing importance of supply chain management (SCM) in enterprise development and in the operation of socio-economic systems, cost management has become a strategic business issue in recent years. It involves not only the financial flows but also the associated material flows and information flows among supply chain partners. Moreover, it plays an indispensable role in bringing profits and competitive advantage to firms, and consequently receives increasing attention from both supply chain managers and academics. Activities in supply chain system consist of transforming natural resources, raw materials and components into finished product and their final delivery to the end customers. Most of these economic activities form an integral part of the value chain. From this view point, cost management in supply chains is not limited to individual enterprises, but extends to all the purchasing, warehousing, production and distribution activities along the chain. Its goal is to provide a management tool and method to design the integrated chain, to promote its development and to reduce the total cost of supply chain system. However, a lot more complexity is involved in effectively integrating all the supply chain activities in a cost efficient manner owing to shorter life cycle of products and increased competition among suppliers who are offering different opportunities to the retailer. The uncertain demand of deteriorating products and their dependent costs creates uncertainty in the environment and consequently results in an indecisive and unsure environment for the decision makers. Choosing high level of procured quantity and inventory to avoid shortages will definitely lead to an immense increase in the cost of purchase and inventory holding. In this regard, operations management practices and mathematical models provide a sound framework for effective and integrative decision making across supply chain. For minimizing the cost and improving the overall performance, major functions considered are economic ordered quantity decisions, supplier selection decisions, inventory & capacity decisions and transportation policies in multi periods and for multi products. While economic ordered quantity decisions aim to minimize the cost of procurement, inventory and transportation, the intent of supplier selection and transportation policy selection decisions is to maximize inbound logistics performance by attaining a high degree of quality and delivery performance. Due to the inherent interdependency among these decisions, a firm cannot optimize them separately. Hence the main purpose of this paper is to develop a model addressing above issues i.e. to characterize the optimal decisions that each partner in supply chain should adopt to motivate the chain partners to coordinate so that everyone benefits from the improved performance of the system.

Though procurement functions need to consider cost minimization objective, yet in doing so one cannot compromise on quality and delivery related criteria. Nowadays, quality and delivery related objectives are being given higher priority than cost criterion during procurement decisions. Suppliers' performance on quality and delivery criteria has a significant influence on the ordered quantity and the total transportation costs. Taking into account the above observations, in this study we develop a fuzzy bi-objective non-linear programming model for an integrated economic ordered quantity, supplier selection and transportation policy problem. We investigate a problem in which multi products are procured from multiple suppliers in multiple periods considering limitations on capacity at supplier point and processing point for deteriorating products. We also incorporate cost of inventory at distribution centres & retail outlets and transportation cost and policy concepts in one stage to another. Imprecise demand and other uncertain known parameters make the environment of model uncertain and fuzzy. To summarize the above discussions, the present work shows (1) a fuzzy bi-objective multi stage non-linear optimization model that includes computation of cost of procurement, processing, holding and transportation as first objective and the other objective shows the process to choose best supplier on the basis of delivery and quality; (2) the coordination among multi stages, i.e. (i) procurement stage; (ii) processing stage constituted of (a) Receiving & Scanning, (b) Sorting & Packaging & (c) Scanning & Dispatching; (iii) distribution centres and (iv) retail outlets; (3) transportation policies and minimum cost per weight from processing stage to distribution centres and transportation cost per unit from distribution centre to retail outlet; (4) fuzzy set theory to coordinate uncertain parameters; (5) coordination in procurement, demand and inventory so the zero shortage is ensured.

# 2 Literature Review

There are vast researches working on supplier selection problems with different approaches. One of the most important decisions related to procurement operations is supplier evaluation and selection. There are several factors involved such as price offered by the supplier, lead time, the quality of items, the capacity of supplier and the geographical location of supplier while making supplier evaluation and selection decisions (Ho et al., 2010). Ho et al. (2010), the three most important criteria considered while selecting suppliers are product quality, delivery lead time and price. Hassini (2008) studies a lot sizing and supplier selection problem when supplier capacity reservation dependent on lead time. Ravindran, Bilsel, Wadhwa, and Yang (2010) study supplier selection and order allocation considering incremental price breaks. Liao and Rittscher (2007) propose a multi objective programming model for supplier selection, procurement lot sizing and carrier selection decisions. Razmi and Maghool (2010) propose a fuzzy bi-objective model for multiple items, multiple period, supplier selection and purchasing problem under capacity constraint and budget limitation. Zhang and Zhang (2011) formulate a mixed integer programming model for selecting suppliers and allocating the ordering quantity properly among the selected suppliers to minimize the selection, purchase and inventory costs. Jolai, Yazdian, Shahanaghi, and Khojasteh (2011) proposed a two-phase approach for supplier selection and order allocation problem under fuzzy environment for multiple products from multiple suppliers in multiple periods. Pal, Sana, and Chaudhuri (2012) addressed a multi-echelon suppler chain with two suppliers in which the main supplier may face supply disruption and the secondary supplier is reliable but more expensive, and the manufacturer may produce defective items. Kilic (2013) discussed an integrated approach including fuzzy Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and a mixed integer linear programming model is developed to select the best supplier in a multi-item/multi-supplier environment.

Few of the studies have addressed problems having multi objectives and with fuzziness. Madronero, Peidro, and Vassant (2010) used S-curve membership functions for Fuzzy aspiration levels for objective functions, maximum capacity of the vendors as RHS, budget amount allocated to vendors as RHS with Fuzzy programming by using modified Werner's fuzzy or operator. Wu, Zhang, Wu and Olson (2010) used Trapezoidal membership functions for Fuzzy model parameters as objective function coefficients and right hand side (RHS) constants with Sequential quadratic programming. Arikan (2011) used Triangular and Right triangular membership functions for Fuzzy aspiration levels for objective functions and demand level as RHS with Lai and Hwang's augmented max–min model. Concerning with multi-choice goals, decision-making behaviour and limit of resources, Lee, Kang, and Chang (2009) develop a fuzzy multiple goal programming model to help downstream companies to select thin film transistor liquid display suppliers for cooperation. They used triangular membership functions for fuzzy aspiration levels for objective functions. Further, a multi-objective model for supplier selection in multi-service outsourcing is developed by Feng, Fan, and Li (2011). A multi objective mathematical model has been discussed by Seifbarghy and Esfandiari (2013), which includes minimizing the transaction costs of purchasing from suppliers as well as other objectives as minimizing the purchasing cost, rejected units, and late delivered units, and maximizing the evaluation scores of the selected suppliers. The problem is converted into single objective using weighting method and solved using meta-heuristics. Aghai, Mollaverdi and Saddagh (2014), outlined a fuzzy multi-objective programming model to propose supplier selection taking quantitative, qualitative, and risk factors into consideration. Also quantity discount has been considered to determine the best suppliers and to place the optimal order quantities among them.

From the literature, it is evident that most studies have not paid much attention to uncertainty in supplier's information and many problematic criteria in the conditions of multi product, transportation modes and multiple sourcing. The main purpose of this paper has been outlined as (1) to propose a fuzzy bi-objective mathematical model to choose the supplier with best performance on the basis of quality & delivery percentages and to keep the cost optimum while procurement, processing of products and transportation, the ideal number of inventory items so that shortages does not take place, and optimum quantity from suppliers subject to the constraints pertaining to demand, suppliers capacity, processing capacity and inspection, (2) the objectives are conflicting in nature as minimization of cost and performance maximization of the supplier. Because of uncertain parameters the environment of the problem becomes fuzzy, for which, fuzzy goal programming method has been used to solve the mathematical model of cost minimization and suppliers selection with maximum performance.

# 3 Problem Definition

To manage different entities to minimize their cost and simultaneously measuring the suppliers' performances in the environment of uncertainty, the current paper presents a fuzzy bi-objective mixed integer non-liner model. The first objective of the proposed model minimizes the cost of integration of procurement and distribution. This comprises of multi source (suppliers), two processing points, multi distribution centres & multi retail outlets and incorporating transportation costs and policies. The second objective focuses on performance and selection of suppliers on the bases of on-time delivery percentage and acceptance percentage of the ordered quantity.

The first stage of first objective explains procurement cost as per optimum procured quantity from the active suppliers, processing cost per unit in three levels at processing point. At this point receiving, scanning, sorting and packing of goods takes time, hence holding cost is included in the processing cost. The second stage shows the fuzzy cost of holding at distribution centres and cost of transportation of goods from processing points to distribution centres which is completed through two modes of transportation as full truck load (TL) mode and truck load (TL) & less than truck load (LTL) mode. In truck load transportation mode, the cost is fixed of one truck up to a given capacity. In this mode, the company may use less than the capacity available but cost per truck will not be reduced. However, sometimes the weighted quantity may not be large enough to corroborate the cost associated with a TL mode. In such situation, a LTL mode may be used. LTL is defined as a shipment of weighted quantity which does not fill a truck. In such a case, transportation cost is taken on the bases of per unit weight. The third stage includes inspection, fuzzy holding cost at retail outlet and transportation cost per unit in the account from distribution centres to retail outlet. The second objective is to find best suppliers with the combination of fuzzy on-time delivery percentage and fuzzy acceptance percentage of the ordered quantity.

The model integrates inventory, procurement and transportation mechanism to minimize all costs discussed above and also chooses the best supplier. In the model, all the co-ordinations among supply chain partners are being managed under one buyer who is taking care of processing points, distribution centres and retail outlets but not sources (suppliers) directly. The total cost of the model becomes fuzzy due to fuzzy holding cost and demand. On the other hand, performance level is also fuzzy as percentage of on-time delivery and acceptances are fuzzy. Hence, the model discussed above is fuzzy bi-objective mixed integer non-linear model. In the solution process, the fuzzy model is converted into crisp and further fuzzy goal programming approach is employed where each objective could be assigned a different weight.

# 4 Proposed Model Formulation

The model is based on following assumptions:

- Finite planning horizon
- Demand at retail outlet is uncertain and no shortages are allowed
- Initial inventory at the beginning of planning horizon is zero
- Inventory at retail outlet deteriorates at constant rate
- Inspection cost of received goods at retail out is fixed
- No transportation cost is discussed as it is considered as part of purchasing cost
- Holding cost is part of processing cost at processing point

## 4.1 Sets

| Set | Cardinality | Index |
|---|---|---|
| Product | $P$ | $i$ |
| Supplier | $J$ | $j$ |
| Processing Point | $Z$ | $z$ |
| Distribution Centre | $M$ | $m$ |
| Retail outlet | $O$ | $o$ |
| Time period | $T$ | $t$ |

## 4.2 Parameters

$\widetilde{C}$ : Fuzzy total cost

$C_0$ & $C_0^*$ : Aspiration & Tolerance level of fuzzy total cost

$\widetilde{PR}$ : Fuzzy performance of supplier

$PR_0$ & $PR_0^*$ : Aspiration & Tolerance level of fuzzy performance of supplier

$\underset{imt}{\widetilde{HD}}$ & $\overline{HD}_{imt}$ : Fuzzy & Defuzzified holding cost per unit of product $i$ for $t^{th}$ period at $m^{th}$ distribution centre

$\varphi_{ijzt}$ : Unit purchase cost for $i^{th}$ product in $t^{th}$ period from supplier j for $z^{th}$ processing point

$A$: Cost per weight of transportation in LTL policy

$K_{zmt}$ : Fixed freight cost for each truck load in period $t$from processing point z to distribution centre m

$TC_{imot}$ : Transportation cost for unit in period $t$from distribution centre m to retail outlet o

$\underset{iot}{\widetilde{HR}}$ & $\overline{HR}_{iot}$ : Fuzzy & defuzzified holding cost per unit of product $i$ for $t^{th}$ period at retail outlet o

$\lambda_{iot}$ : Inspection cost per unit of product $i$ in period $t$ at retail outlet o

$\underset{iot}{\widetilde{D}}$ & $\overline{D}_{iot}$ : Fuzzy & defuzzified demand at retail outlet o for product $i$ in period $t$

$IN_{izt}$: Initial Inventory processing point z in beginning of planning horizon for product $i$

$\eta$ : Deterioration percentage of $i^{th}$ product at retail outlet

$w_i$: Per unit weight of product $i$

$\omega$ : Weight transported in each full truck

$\underset{ijzt}{\widetilde{DT}}$ & $\overline{DT}_{ijzt}$ : Fuzzy & defuzzified percentage of on-time delivery time for product $i$ in period $t$ for supplier $j$ for processing point z

$\underset{ijzt}{\widetilde{AC}}$ & $\overline{AC}_{ijzt}$ : Fuzzy & defuzzified percentage of acceptance for product $i$ in period $t$ for supplier $j$ for processing point z

$\delta_{ijz}$: Capacity at supplier $j$ for product $i$for $z^{th}$ processing point

$\alpha_{izrt}$ : Capacity of Receiving & Scanning level ($r$) at $z^{th}$ processing point for product i in period t

$C_{izrt}$ : Cost of Receiving & Scanning level ($r$) at $z^{th}$ processing point for product i in period t

$\beta_{izst}$ : Capacity of Sorting & Packing level ($s$) at $z^{th}$ processing point for product i in period t

$C_{izst}$ : Cost of Sorting & Packing ($s$) at $z^{th}$ processing point for product i in period t

$\gamma_{izdt}$ : Capacity of Scanning & Dispatching level ($d$) at $z^{th}$ processing point for product i in period t

$C_{izdt}$ : Cost of Scanning & Dispatching ($d$) at $z^{th}$ processing point for product i in period t

## 4.3 Decision Variable

$X_{ijzt}$ : Optimum ordered quantity of product $i$ ordered in period $t$from supplier j transported to processing point z

$V_{ijt}$: If ordered quantity is procured by active supplier $j$ for product $i$ in period $t$then the variable takes value 1 otherwise zero

$u_{zmt}$: Usage of modes, either TL & LTL mode (value is 1) or only TL mode (value is 0)

## 4.4 Operating Variables

$Y_{izt}$ : Procured quantity reached at Receiving & Scanning level of zth processing point from all the active suppliers

$A_{izt}$ : Goods moved to Sorting & Packaging from Receiving & Scanning level at $\mathrm{z}^{\mathrm{th}}$ processing point

$E_{imt}$ : Goods reaching at $m^{th}$ distribution centre from all processing points

$j_{zmt}$ : Total number of truck loads in period $t$ from processing point z to distribution centre m

$Q_{zmt}$ : Weighted quantity in excess of truckload capacity

$G_{iot}$ : Total quantity reached at retail outlet o from all distribution centres

$I_{izt}$ : Inventory at processing point in period t for product i

$I_{imt}$ : Inventory at distribution centre in period t for product i

$I_{iot}$ : Inventory at retail outlet in period t for product i

$B_{izmt}$ : Quantity of product i shipped from $z^{th}$ processing point to $m^{th}$ distribution centre in period t

$F_{imot}$ : Quantity of product i shipped from $m^{th}$ distribution centre to $o^{th}$ retail outlet in period t

$L_{zmt}$ : Weighted quantity transported from $z^{th}$ processing point to $m^{th}$ distribution centre in period t

## 4.5 Fuzzy Optimization Model Formulation

Fuzzy dependent environment with respect to uncertain independent variables cannot be quantified by Crisp mathematical programming approaches. Fuzzy optimization approach permits adequate solutions of real problems in the presence of vague information by defining the mechanisms to quantify uncertainties directly. Therefore, we formulate fuzzy optimization model for vague aspiration levels on cost, demand, on-time delivery percentage and acceptance percentage the decision maker may decide the aspiration and tolerance levels on the basis of past experience and knowledge.

### 4.5.1 Formulation of objectives

Initially a bi-objective fuzzy model is formulated which discusses about fuzzy total cost and performance of the suppliers. The first objective of the model minimizes the total cost, consisting of procurement cost of goods from supplier, processing cost, holding cost at distribution centres, transportation cost from processing point to distribution centres and further to retail outlets, holding cost at retail outlets and finally inspection cost of the reached quantity at retail outlets.

$$
\begin{aligned}
Minimize,\ \widetilde{C} = & \sum_{t=1}^{T}\sum_{z=1}^{Z}\sum_{j=1}^{J}\sum_{i=1}^{P}\varphi_{ijzt}X_{ijzt}V_{ijzt} \\
& + \sum_{t=1}^{T}\sum_{z=1}^{Z}\sum_{i=1}^{P}\left[\sum_{r=1}^{R}C_{izrt}Y_{izt} + \left(\sum_{s=1}^{S}C_{izst} + \sum_{d=1}^{D}C_{izdt}\right)A_{izt}\right] \\
& + \sum_{t=1}^{T}\sum_{m=1}^{M}\sum_{i=1}^{P}\underset{imt}{\widetilde{HD}}\,E_{imt}
\end{aligned}
$$

$$+\sum_{t=1}^{T}\sum_{m=1}^{M}\sum_{z=1}^{Z}\left[\left(AQ_{zmt}+j_{zmt}K_{zmt}\right)u_{zmt}+\left(j_{zmt}+1\right)K_{zmt}\left(1-u_{zmt}\right)\right]$$

$$+\sum_{t=1}^{T}\sum_{o=1}^{O}\sum_{m=1}^{M}\sum_{i=1}^{P}TC_{imot}F_{imot}+\sum_{t=1}^{T}\sum_{o=1}^{O}\sum_{i=1}^{P}\underset{iot}{\widetilde{HR}}\,I_{iot}+\sum_{t=1}^{T}\sum_{o=1}^{O}\sum_{i=1}^{P}\lambda_{iot}G_{iot}$$

The second objective discusses the performance of suppliers and maximizes the performance percentage of supplier as per on-delivery time percentage and acceptance percentage of ordered quantity.

$$Maximize\ \widetilde{PR}=\sum_{t=1}^{T}\sum_{z=1}^{Z}\sum_{j=1}^{J}\sum_{i=1}^{P}\left(\underset{ijzt}{\widetilde{DT}}+\underset{ijzt}{\widetilde{AC}}\right)V_{ijzt}$$

### 4.5.2 Constraint Formulation

All the suppliers must have enough capacity to fulfil the orders. The following equation ensures that the active supplier shall have enough capacity to complete the orders from processing point.

$$X_{ijzt}\leq\delta_{ijz}V_{ijzt}\quad\forall\, i,j,\,z,t$$

Next equation ensures that only one supplier can be active for a particular product in a period. However, same supplier can be active again in next period.

$$\sum_{j=1}^{J}V_{ijzt}=1\quad\forall\ i,\,t,z$$

Goods are reaching at z$^{th}$ processing point from all the suppliers.

$$Y_{izt}=\sum_{j=1}^{J}X_{ijzt}\quad\forall\, i,\,t,z$$

At Receiving & Scanning level in processing point, 2% from each lot is rejected and removed.

$$A_{izt}=0.98Y_{izt}\quad\forall\, i,\,t,z$$

Quantity dispatched from z$^{th}$ processing point is being transported to all distribution centres.

$$A_{izt}=\sum_{m=1}^{M}B_{izmt}\quad\forall i,\ z,\ t$$

Goods reaching at m$^{th}$ distribution centre are transported from all the processing points.

$$E_{imt} = \sum_{z=1}^{Z} B_{izmt} \quad \forall i,\, m,\, t$$

Goods are transported from m$^{th}$ distribution centre to all the retail outlets.

$$E_{imt} = \sum_{o=1}^{O} F_{imot} \quad \forall i,\, m,\, t$$

Goods reaching at o$^{th}$ retail outlets Eiot are transported from all the distribution centres

$$G_{iot} = \sum_{m=1}^{M} F_{imot} \quad \forall i,\, o,\, t$$

Following three equations explain the capacities in processing point at all the levels respectively i.e. Receiving and Scanning level, Sorting & Packaging level and Scanning and Dispatching level.

$$Y_{izt} \leq \alpha_{izrt} \quad \forall\, i,\, z, t, r$$

$$A_{izt} \leq \beta_{izst} \quad \forall\, i,\, t, z, s$$

$$A_{izt} \leq \gamma_{izdt} \quad \forall\, i,\, t, z, d$$

Next three equations show balancing equations at Processing Point, which also takes care of no shortages assumption. First two equations of the set calculate inventory at end of the period with respect to quantity reached at receiving and scanning level from the supplier and quantity sent to sorting & packaging level. The third equitation takes care of the shortages by balancing the quantity between the two levels discussed above.

$$I_{izt} = I_{izt-1} + Y_{izt} - A_{izt} \quad \forall\, i,\, t > 1, z$$

$$I_{izt} = IN_{izt} + Y_{izt} - A_{izt} \quad \forall\, i,\, t = 1, z$$

$$\sum_{t=1}^{T} I_{izt} + \sum_{t=1}^{T} Y_{izt} \geq \sum_{t=1}^{T} A_{izt} \quad \forall\, i, z$$

Balancing at distribution centres have been discussed in next three equation, where assumption of no shortages has also been taken care of.

$$I_{imt} = I_{imt-1} + E_{imt} - \sum_{o=1}^{O} F_{imot} \quad \forall\, i,\, t > 1, m$$

$$I_{im1} = 0 \quad \forall\, i, m$$

$$\sum_{t=1}^{T} I_{imt} + \sum_{t=1}^{T} E_{imt} \geq \sum_{t=1}^{T} \sum_{o=1}^{O} F_{imot} \quad \forall\, i, m$$

At retail outlets also, inventory has been balanced with respect to the received quantity and demand.

$$I_{iot} = I_{iot-1} + G_{iot} - \underset{iot}{\tilde{D}} - \eta I_{iot} \quad \forall\, i,\, t > 1, o$$

$$I_{io1} = 0 \quad \forall\, i, o$$

$$(1-\eta) \sum_{t=1}^{T} I_{iot} + \sum_{t=1}^{T} G_{iot} \underset{\sim}{\geq} \sum_{t=1}^{T} \underset{iot}{\tilde{D}} \quad \forall\, i, o$$

Following equation is an integrator and calculates the weighted quantity which is to be transported from processing point to distribution centres.

$$L_{zmt} = \sum_{i=1}^{P} \omega_i B_{izmt} \quad \forall\, z, t, m$$

The next equation finds out transportation policy as per the weighted quantity. Here, the costs of TL policy and TL&LTL policy are compared as per the weight.

$$L_{zmt} \leq \left(Q_{zmt} + j_{zmt} w\right) u_{zmt} + \left(j_{zmt} + 1\right) w \left(1 - u_{zmt}\right) \quad \forall\, z, m, t$$

The calculation of overhead quantity in TL&LTL policy is calculated by comparing total weighted quantity with total number of full truck loads as per weight is discussed in following equation.

$$L_{zmt} = Q_{zmt} + j_{zmt} w \quad \forall z, m,\ t$$

Lastly, describing the nature of decision variables and enforcing the binary and non-negative restrictions to them.

$X_{ijzt},\ Y_{izt},\ A_{izt},\ E_{imt},\ F_{imot},\ G_{iot},\ L_{zmt} \geq 0$; $V_{ijzt},\ u_{zmt} \in [0,\ 1]$;
$I_{imt},\ I_{iot},\ I_{izt},\ Q_{zmt},\ j_{zmt}$ are integer.

### 4.5.3 Formulated Model

$$
\begin{aligned}
Minimize\ \widetilde{C} = {} & \sum_{t=1}^{T}\sum_{z=1}^{Z}\sum_{j=1}^{J}\sum_{i=1}^{P}\varphi_{ijzt}X_{ijzt}V_{ijzt} \\
& + \sum_{t=1}^{T}\sum_{z=1}^{Z}\sum_{i=1}^{P}\left[\sum_{r=1}^{R}C_{izrt}Y_{izt} + \left(\sum_{s=1}^{S}C_{izst} + \sum_{d=1}^{D}C_{izdt}\right)A_{izt}\right] \\
& + \sum_{t=1}^{T}\sum_{m=1}^{M}\sum_{i=1}^{P}\underset{imt}{\widetilde{HD}}\,E_{imt} \\
& + \sum_{t=1}^{T}\sum_{m=1}^{M}\sum_{z=1}^{Z}\left[\left(AQ_{zmt} + j_{zmt}K_{zmt}\right)u_{zmt} + \left(j_{zmt}+1\right)K_{zmt}\left(1-u_{zmt}\right)\right] \\
& + \sum_{t=1}^{T}\sum_{o=1}^{O}\sum_{m=1}^{M}\sum_{i=1}^{P}TC_{imot}F_{imot} + \sum_{t=1}^{T}\sum_{o=1}^{O}\sum_{i=1}^{P}\underset{iot}{\widetilde{HR}}\,I_{iot} + \sum_{t=1}^{T}\sum_{o=1}^{O}\sum_{i=1}^{P}\lambda_{iot}G_{iot}
\end{aligned}
$$

$$
Maximize\ \widetilde{PR} = \sum_{t=1}^{T}\sum_{z=1}^{Z}\sum_{j=1}^{J}\sum_{i=1}^{P}\left(\underset{ijzt}{\widetilde{DT}} + \underset{ijzt}{\widetilde{AC}}\right)V_{ijzt}.
$$

Subject to $X_{ijzt} \le \delta_{ijz}V_{ijzt} \quad \forall\, i, j,\, z, t \quad \sum_{j=1}^{J}V_{ijzt} = 1 \quad \forall\ i,\, t, z$

$$
\begin{aligned}
& Y_{izt} = \sum_{j=1}^{J}X_{ijzt} \quad \forall\, i,\, t, z \\
& A_{izt} = 0.98Y_{izt} \quad \forall\, i,\, t, z \\
& A_{izt} = \sum_{m=1}^{M}B_{izmt} \quad \forall i,\, z,\, t \\
& E_{imt} = \sum_{z=1}^{Z}B_{izmt} \quad \forall i,\, m,\, t \\
& E_{imt} = \sum_{o=1}^{O}F_{imot} \quad \forall i,\, m,\, t \\
& G_{iot} = \sum_{m=1}^{M}F_{imot} \quad \forall i,\, o,\, t \\
& Y_{izt} \le \alpha_{izrt} \quad \forall\, i,\, z, t, r \\
& A_{izt} \le \beta_{izst} \quad \forall\, i,\, t, z, s \\
& A_{izt} \le \gamma_{izdt} \quad \forall\, i,\, t, z, d \\
& I_{izt} = I_{izt-1} + Y_{izt} - A_{izt} \quad \forall\, i,\, t > 1, z \\
& I_{izt} = IN_{izt} + Y_{izt} - A_{izt} \quad \forall\, i,\, t = 1, z \\
& \sum_{t=1}^{T}I_{izt} + \sum_{t=1}^{T}Y_{izt} \ge \sum_{t=1}^{T}A_{izt} \quad \forall\, i, z
\end{aligned}
$$

$$I_{imt} = I_{imt-1} + E_{imt} - \sum_{o=1}^{O} F_{imot} \quad \forall\, i,\, t > 1, m$$
$$I_{im1} = 0 \quad \forall\, i, m$$
$$\sum_{t=1}^{T} I_{imt} + \sum_{t=1}^{T} E_{imt} \geq \sum_{t=1}^{T}\sum_{o=1}^{O} F_{imot} \quad \forall\, i, m$$
$$I_{iot} = I_{iot-1} + G_{iot} - \underset{iot}{\tilde{D}} - \eta I_{iot} \quad \forall\, i,\, t > 1, o$$
$$I_{io1} = 0 \quad \forall\, i, o$$
$$(1-\eta)\sum_{t=1}^{T} I_{iot} + \sum_{t=1}^{T} G_{iot} \underset{\sim}{\geq} \sum_{t=1}^{T} \underset{iot}{\tilde{D}} \quad \forall\, i, o$$
$$L_{zmt} = \sum_{i=1}^{P} \omega_i B_{izmt} \quad \forall\, z, t, m$$
$$L_{zmt} \leq (Q_{zmt} + j_{zmt} w)\, u_{zmt} + (j_{zmt} + 1)\, w\, (1 - u_{zmt}) \quad \forall\, z, m, t$$
$$L_{zmt} = Q_{zmt} + j_{zmt} w \quad \forall z, m,\ t$$
$$X_{ijzt},\ Y_{izt},\ A_{izt},\ E_{imt},\ F_{imot},\ G_{iot},\ L_{zmt} \geq 0;\ V_{ijzt},\ u_{zmt} \in [0,\ 1]\,;$$
$I_{imt},\ I_{iot},\ I_{izt},\ Q_{zmt},\ j_{zmt}$ are integer.

# 5 Solution Algorithm

## 5.1 Fuzzy Solution Algorithm

In following algorithm by Zimmermann (1976) specifies the sequential steps to solve the fuzzy mathematical programming problems.

**Step 1.** Compute the crisp equivalent of the fuzzy parameters using a defuzzification function. Here, ranking technique is employed to defuzzify the parameters as

$$F_2(A) = (a_l + 2a_m + a_u)/4,$$

where $a_l, a_m, a_u$ are the Triangular Fuzzy Numbers (TFN).

Let $\underset{iot}{\bar{D}}$ be the defuzzified value of $\underset{iot}{\tilde{D}}$ and $(D^1_{iot}, D^2_{iot}, D^3_{iot})$ for each $i, o$ & $t$ be triangular fuzzy numbers then, $\overline{D}_{iot} = (D^1_{iot} + 2D^2_{iot} + D^3_{iot})/4$. Similarly, $\underset{imt}{\bar{HD}}$ and $\underset{iot}{\bar{HR}}$ are defuzzified aspired holding cost at warehouse and destination.

**Step 2.** Since industry is highly volatile and customer demand changes in every short span, a precise estimation of cost and performance aspirations is a major area of discussion. Hence, a better way to come out of such situation is to incorporate tolerance and aspiration level with the main objectives. The model discussed in section 4.5.3 can thus be re-written as follows:

Find X,

$X \in S$

$(1-\eta)\sum_{t=1}^{T} I_{iot} + \sum_{t=1}^{T} G_{iot} \underset{\sim}{\geq} \sum_{t=1}^{T} \bar{D}_{iot} \quad \forall\, i, o$
$C(X) \underset{\sim}{\leq} C_0$
$PR \underset{\sim}{\geq} PR_0$
$X_{ijzt},\ Y_{izt},\ A_{izt},\ E_{imt},\ F_{imot},\ G_{iot},\ L_{zmt} \geq 0;\ V_{ijzt},\ u_{zmt} \in [0,\ 1]\,;$
$I_{imt},\ I_{iot},\ I_{izt},\ Q_{zmt},\ j_{zmt}$ are integer.

**Step3.** Define appropriate membership functions for each fuzzy inequalities as well as constraint corresponding to the objective functions.

$$\mu_C(X) = \begin{cases} 1\ ;\ \mathrm{C}(X) \leq C_0 \\ \frac{C_0^* - C(X)}{C_0^* - C_0}\ ;\ \mathrm{C}_0 \leq C(X) < C_0^* \\ 0\ ;\ \mathrm{C}(X) > C_0^* \end{cases},$$

$$\mu_{PR}(X) = \begin{cases} 1\ ;\ PR \geq PR_0 \\ \frac{PR - PR_0^*}{PR_0 - PR_0^*}\ ; PR_0^* \leq PR < PR_0 \\ 0\ ;\ PR < PR_0^* \end{cases}$$

$$\mu_{I_{iot}}(X) = \begin{cases} 1\ ;\ I_{iot}(X) \geq \overline{D}_0 \\ \frac{I_{iot}(X) - \overline{D}_0^*}{\overline{D}_0 - \overline{D}_0^*}\ ;\ \overline{D}_0^* \leq I_{iot}(X) < \overline{D}_0 \\ 0\ ;\ I_{iot}(X) > \overline{D}_0^* \end{cases}$$

Where $\overline{D}_0 = \sum_{t=1}^{T}\sum_{o=1}^{O} \overline{D}_{iot}$is the aspiration and $\overline{D}_0^*$is the tolerance level to inventory constraints.

**Step4.** Employ extension principle to identify the fuzzy decision, which results in a crisp mathematical programming problem given by

Maximize $\alpha$
Subject to $\mu_c(X) \geq \alpha,$
$\mu_{PR}(X) \geq \alpha,$
$\mu_{I_{iot}}(X) \geq \alpha,$
$X \in S$

Where $\alpha$ represents the degree up to which the aspiration of the decision-maker is met. The above problem can be solved by the standard crisp mathematical programming algorithms.

**Step5.** Following Bellman and Zadeh (1970), while solving the problem following steps 1-4, the objective of the problem is also treated as a constraint. Each constraint is considered to be an objective for the decision-maker and the problem can be looked as a fuzzy bi-objective mathematical programming problem. Further, each objective can have a different level of importance and can be assigned weight to measure the relative importance. The resulting problem can be solved by the weighted min max approach. On substituting the values for $\mu_{PR}(x)$ and $\mu_C(x)$the problem becomes

*Maximize* $\alpha$
*subject to*
$PR(x) \geq PR_0 - (1 - w_1\alpha)(PR_0 - PR_0^*)$
$C(x) \leq C_0 + (1 - w_2\alpha)(C_0^* - C_0)$ (P1)
$\mu_{I_{iot}}(X) \geq \alpha$
$X \in S$
$w_1 \geq 0,\ w_2 \geq 0,\ \ w_1 + w_2 = 1, \alpha \in [0, 1]$

**Step6.** If a feasible solution is not obtained for the problem in Step 5, then we can use the fuzzy goal programming approach to obtain a compromised solution given by Mohamed (1997). The method is discussed in detail in the next section.

## 5.2 Fuzzy Goal Programming Approach

On solving the problem, we found that the problem (P1) is not feasible; hence the management goal cannot be achieved for a feasible value of $\alpha$[0,1]. Then, we use the fuzzy goal programming technique to obtain a compromised solution. The approach is based on the goal programming technique for solving the crisp goal programming problem given by Mohamed (1997). The maximum value of any membership function can be 1; maximization of $\alpha$[0,1] is equivalent to making it as close to 1 as best as possible. This can be achieved by minimizing the negative deviational variables of goal programming (i.e., $\eta$) from 1. The fuzzy goal programming formulation for the given problem (P1) introducing the negative and positive deviational variables $\eta_j$ & $\rho_j$ is given as

Minimize $u$
subject to $\mu_{PR}(X) + \eta_1 - \rho_1 = 1$
$\mu_C(X) + \eta_2 - \rho_2 = 1$
$u \geq w_j * \eta_j \quad j = 1, 2$
$\eta_j * \rho_j = 0 \quad j = 1, 2$
$w_1 + w_2 = 1$
$\alpha = 1 - u$
$\eta_j, \rho_j \geq 0; X \in S; u \in [0, 1]; w_1, w_2 \geq 0$

# 6 Case Study

Fish is a highly perishable food which needs proper handling and preservation if it is to have a long shelf life and also retain a desirable quality and its nutritional value. The central concern of fish processing is to prevent fish from deterioration. When fish are captured or harvested for commercial purposes, they need some pre-processing so they can be delivered to the next part of the supply chain in a fresh and undamaged condition. This means, for example, that fish caught by a fishing vessel need handling so they can be stored safely until the boat lands the fish on shore. Some of the methods to preserve and process fish and fish products include control of temperature using ice, refrigeration or freezing, sorting and grading, chilling, storing the chilled fish. The model is validated for the case on fish and fish products. Case is taken for two suppliers, two processing points, three distribution centres and three retail outlets for three time periods. Each processing point has its own internal three stages i.e. Receiving & Scanning, Sorting & Packing and Scanning & Dispatching. At processing point, fish products are received and scanned, which have been pre-processed to reduce the deterioration percentage. Afterwards, they are sorted as per quality checks and packed and further sent to the next stage for final scanning before dispatching to the distribution centres. The objectives include minimizing the cost of procurement, processing, transportation and inventory by obtaining the optimal ordered quantity, transportation weights & minimum inventory and maximizing the performance of procurement by choosing the best supplier on the basis of delivery and quality. The data on cost of procurement from suppliers, processing cost, transportation cost from one stage to another, cost of inspection and inventory carrying cost has been discussed.

Three types of fish have been discussed in the case are Rohu, Katle and Pomfret which are ranging from Rs.80 to Rs.190 per kg. In the case, uncertain parameters are performance parameters, holding cost and demand. Further, defuzzified holding costs at all distribution centres and retail outlets are Rs.14, Rs.8 and Rs.8 for three fish types respectively in all the periods. The capacity at both the suppliers is 300 and 380 packets for fish type 'Rohu', 370 and 390 packets for fish type 'Katle' and 360 and 380 packets for fish type 'Pomfret'. In processing stage, the costs of receiving & scanning, sorting & packing and scanning & dispatching are Rs.1, Rs.2 and Rs.2.5 respectively per packet. Inspection cost per packet is Rs.2 and deterioration percentage is constant with 3% deterioration cost.

| | Product Type | | |
|---|---|---|---|
| Supplier | Rohu | Katle | Pomfret |
| Supplier 1 | 134 | 90 | 190 |
| Supplier 2 | 185 | 85 | 185 |

Table 1: Purchase Cost in all periods and at all processing points

| | Product Type | | |
|---|---|---|---|
| Processing Point | Rohu | Katle | Pomf |
| PP 1 | 320 | 310 | 300 |
| PP 2 | 355 | 275 | 245 |

Table 2: Capacity at all stages in processing point for all periods

| | Supplier 1 to PP1 & PP2 | | | | | |
|---|---|---|---|---|---|---|
| Product Type | Period 1 | | Period 2 | | Period 3 | |
| | AC | DT | AC | DT | AC | DT |
| Rohu | 0.93 | 0.98 | 0.93 | 0.98 | 0.93 | 0.98 |
| Katle | 0.99 | 0.98 | 0.99 | 0.98 | 0.99 | 0.98 |
| Pomfret | 0.95 | 0.98 | 0.95 | 0.98 | 0.95 | 0.98 |
| | Supplier 2 to PP1 & PP2 | | | | | |
| Product Type | Period 1 | | Period 2 | | Period 3 | |
| | AC | DT | AC | DT | AC | DT |
| Rohu | 0.95 | 0.99 | 0.95 | 0.99 | 0.95 | 0.99 |
| Katle | 0.93 | 0.97 | 0.93 | 0.97 | 0.93 | 0.97 |
| Pomfret | 0.95 | 0.97 | 0.95 | 0.97 | 0.95 | 0.97 |

Table 3: De-fuzzified Delivery time (DT) and Acceptance (AC) Probabilities

| | Distribution Centre | | |
|---|---|---|---|
| Processing Point | DC 1 | DC 2 | DC 3 |
| PP 1 | 2000 | 2500 | 2500 |
| PP 2 | 2200 | 2900 | 2400 |

Table 4: Transportation cost per truck

| | Retail Outlet | | |
|---|---|---|---|
| Distribution Centre | RO 1 | RO 2 | RO 3 |
| DC 1 | 2 | 2.2 | 1.9 |
| DC 2 | 2.2 | 2.5 | 2.1 |
| DC 3 | 1.9 | 1.8 | 2 |

Table 5: Transportation cost per packet from DC to RO

| | Product Type | | |
|---|---|---|---|
| Retail Outlet | Rohu | Katle | Pomfret |
| RO 1 | 100 | 160 | 140 |
| RO 2 | 110 | 150 | 135 |
| RO 3 | 105 | 170 | 150 |

Table 6: De-fuzzified demand in all time periods

Truckload per truck is 250kg. Overhead quantity transportation cost is Rs.9 per packet.

## 6.1 Results and Managerial Implications

The model helps company to provide minimum total cost incurred co-ordinating all the entities. Rs. 1085767 is the total cost which consists of holding cost at distribution centres as Rs.65758, procurement cost of Rs.856600, processing cost of Rs.33001, cost of transportation from processing point to distribution centres of Rs.76588, holding cost at retail outlets of Rs.28015.63, cost of transportation from distribution centres to retail outlets of Rs.13848.80 and finally inspection cost of Rs.11956. It is observed from the results that highest proportion is of the cost of procurement, which clearly validates the requirement of supplier selection. Further, keeping a valid track of transportation polices is equally important as the second highest portion in the cost is due to the transportation cost only. Next observation is towards the impact of the product's nature as holding cost at distribution centre contributes towards the third highest portion in the cost. To prevent the over valuation of cost, the aspiration and tolerance level have been considered as Rs.950000 and Rs.1220000. As validated with the help of cost, the suppliers' performance is second objective of the model which is a combination of on-time delivery and acceptance percentage of the suppliers. The higher the performance of the supplier, better the performance of the company. Keeping the aspiration level of suppliers' performance as 39 and tolerance as 30, the performance level of suppliers obtained is 35.04. The model tries to activate the high performers to procure ordered quantity so that uncertainty in the environment can be managed. Nearby 78% of the aspiration level of cost and performance has been attained which makes the environment more certain and crisp for future decisions.

| | Processing Point 1 | | | | | |
|---|---|---|---|---|---|---|
| | Per. 1 | | Per. 2 | | Per. 3 | |
| Pr.T. | S1 | S2 | S1 | S2 | S1 | S2 |
| Rohu | 0 | 350 | 0 | 350 | 0 | 350 |
| Katle | 350 | 0 | 350 | 0 | 350 | 0 |
| Pomfret | 350 | 0 | 350 | 0 | 350 | 0 |
| | Processing Point 2 | | | | | |
| | Per. 1 | | Per. 2 | | Per. 3 | |
| Pr.T. | S1 | S2 | S1 | S2 | S1 | S2 |
| Rohu | 0 | 350 | 0 | 350 | 0 | 150 |
| Katle | 350 | 0 | 350 | 0 | 350 | 0 |
| Pomfret | 350 | 0 | 350 | 0 | 350 | 0 |

Table 7: Optimum ordered quantity from supplier (S1-S2)

In Table 7, the positive ordered quantity indicates the active supplier to supply goods as he has the highest performance percentage between the two suppliers on the bases of on-time delivery, acceptance percentage and capacity. It can help in reducing the procurement cost and making the process smooth in further echelon.

Tables 8 and 9 shows ending inventory at processing points and retail outlets, which ensures no shortages in the case of unexpected demand. It is observed that at second retail outlet, storage capacity and infrastructure is better as well as the cost of holding is also low, hence inventory is higher at this outlet in comparison to others. Inventory at distribution is not discussed as no inventory was leftover at any of the distribution centres.

While transporting weighted quantity to distribution centres, the policy type, number of trucks and overhead weights are to be checked as each of them incurs cost. In the Table 10 it is observed that while transporting

| | Processing Point | | | | | |
|---|---|---|---|---|---|---|
| | Period 1 | | Period 2 | | Period 3 | |
| Product Type | PP1 | PP2 | PP1 | PP2 | PP1 | PP2 |
| Rohu | 7 | 7 | 14 | 14 | 21 | 21 |
| Katle | 7 | 7 | 14 | 14 | 21 | 21 |
| Pomfret | 3 | 7 | 10 | 14 | 17 | 21 |

Table 8: Inventory at processing points (in packets)

| | Retail Outlet | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Period 1 | | | Period 2 | | | Period 3 | | |
| ProductType | RO1 | RO2 | RO3 | RO1 | RO2 | RO3 | RO1 | RO2 | RO3 |
| Rohu | 0 | 0 | 0 | 112 | 171 | 78 | 11 | 698 | 1 |
| Katle | 0 | 0 | 0 | 131 | 69 | 0 | 2 | 317 | 75 |
| Pomfret | 0 | 0 | 0 | 144 | 58 | 51 | 5 | 487 | 8 |

Table 9: Inventory at retail outlets (in packets)

from processing point 1 to distribution centre 1 in period 2, only Truckload (T*) policy is used as 250kg can be transported by 1 truck. In this case, LTL policy will become expensive. On the other side, transporting from processing point 1 to distribution centre 1 in period 1, TL & LTL$^?$ policy is used as 49kg should be transported as per unit weight. In the case of TL&LTL policy, if overhead weighted quantity is transported through full truckload, the cost of transportation will become much higher than using LTL policy.

Where TL & LTL is indicated as TLT and only TL is indicated as T.

Some more operational variables who helped in smooth process of goods from one level to other are as follows:

# 7 Conclusion

In the emerging business scenario, the concepts of time, volume and capacity become even more essential to the managerial decision-making. Customers are more sensitive to delivery times and service quality. The coordination among the members of the chain helps them to make a cost-effective procurement and distribution network as well as better response to the cus-

| | Distribution Centre 1 | | | | | |
|---|---|---|---|---|---|---|
| | Period 1 | | Period 2 | | Period 3 | |
| | PP1 | PP2 | PP1 | PP2 | PP1 | PP2 |
| Tpt Quantity | 49 | 7 | 250 | 0 | 329 | 250 |
| No. of Trucks | 0 | 0 | 1 | 0 | 1 | 0 |
| Tpt Mode | TLT? | TLT | T* | T | TLT | T |
| Qty Overhead | 49 | 7 | 0 | 0 | 79 | 0 |
| | Distribution Centre 2 | | | | | |
| | Period 1 | | Period 2 | | Period 3 | |
| | PP1 | PP2 | PP1 | PP2 | PP1 | PP2 |
| Tpt Quantity | 749 | 761 | 752 | 1000 | 686 | 500 |
| No. of Trucks | 2 | 3 | 3 | 4 | 2 | 2 |
| Tpt Mode | TLT | TLT | TLT | T | T | T |
| Qty Overhead | 249 | 11 | 2 | 0 | 186 | 0 |
| | Distribution Centre 3 | | | | | |
| | Period 1 | | Period 2 | | Period 3 | |
| | PP1 | PP2 | PP1 | PP2 | PP1 | PP2 |
| Tpt Quantity | 35 | 261 | 27 | 29 | 14 | 279 |
| No. of Trucks | 0 | 1 | 0 | 0 | 0 | 1 |
| Tpt Mode | TLT | TLT | TLT | TLT | TLT | TLT |
| Qty Overhead | 35 | 11 | 27 | 29 | 14 | 29 |

Table 10: Transported quantity, no. of trucks, transportation mode, overhead quantity

| $\boldsymbol{E}_{imt}$ | Period 1 | | | Period 2 | | | Period 3 | | |
|---|---|---|---|---|---|---|---|---|---|
| Dis.C. | Rohu | Katle | Pomf | Rohu | Katle | Pomf | Rohu | Katle | Pomf |
| DC 1 | 18 | 16 | 22 | 94 | 144 | 12 | 1 | 0 | 578 |
| DC 2 | 451 | 634 | 425 | 544 | 534 | 674 | 657 | 421 | 108 |
| DC 3 | 217 | 36 | 43 | 48 | 8 | 0 | 28 | 265 | 0 |

Table 11:

| $\boldsymbol{G}_{iot}$ | Period 1 | | | Period 2 | | | Period 3 | | |
|---|---|---|---|---|---|---|---|---|---|
| R.O. | Rohu | Katle | Pomf | Rohu | Katle | Pomf | Rohu | Katle | Pomf |
| RO 1 | 0 | 35 | 0 | 215 | 295 | 288 | 0 | 31 | 1 |
| RO 2 | 641 | 647 | 426 | 286 | 221 | 195 | 658 | 408 | 578 |
| RO 3 | 45 | 4 | 64 | 185 | 170 | 203 | 28 | 247 | 107 |

Table 12:

tomers' demand. The authors explain the coordination among many entities of supply chain. As mentioned in the objectives of this study, the main plan of this research is to find optimum quantity from the best suppliers under fuzzy environment to develop an optimum coordination among multi supplier, multi processing points, multi distribution centres and multiple number of retail outlets. To attain the objective, a fuzzy bi-objective mathematical model is formulated with objective functions of cost and combination of timely delivery & acceptance of lot, keeping the constraints as supplier capacity, processing capacity, deteriorating nature of the product and truck capacity. The parameters in study as holding cost, consumption, delivery time and acceptance percentage are fuzzy in nature. To handle the issues of uncertainty and fuzziness, the model is converted into crisp form with the help of membership functions of fuzzy modeling. The parameters are also converted into crisp form by using triangular fuzzy numbers. To obtain the solutions, a fuzzy goal programming is employed. Hence, the current study is able to find a balance between minimum cost and best performed supplier. The proposed model was validated by applying to the real case study data.

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# Recognizability in Stochastic Monoids

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**Abstract**

Stochastic monoids and stochastic congruences are introduced and the syntactic stochastic monoid $M_L$ associated to a subset $L$ of a stochastic monoid $M$ is constructed. It is shown that $M_L$ is minimal among all stochastic epimorphisms $h : M \to M'$ whose kernel saturates $L$. The subset $L$ is said to be stochastically recognizable whenever $M_L$ is finite. The so obtained class is closed under boolean operations and inverse morphisms.

**Key words**: recognizability, stochastic monoids, minimization.

**MSC 2010**: 68R01, 68Q10, 20M32.

# 1 Introduction

A stochastic subset of a set $M$ is a function $F : M \to [0, 1]$ with the additional property $\Sigma_{m \in M} F(m) = 1$, i.e., $F$ is a discrete probability distribution. The corresponding class is denoted by $Stoc(M)$. Our subject of study, in the present paper, are stochastic monoids which were introduced in [4]. A stochastic monoid is a set $M$ equipped with a stochastic multiplication $M \times M \to Stoc(M)$ which is associative and unitary. It can be viewed as a nondeterministic monoid (cf. [1, 2, 3]) with multiplication $M \times M \to \mathcal{P}(M)$ such that for all $m_1, m_2 \in M$ a discrete probability distribution is assigned on the set $m_1 \cdot m_2$.

A congruence on a stochastic monoid $M$ is an equivalence $\sim$ on $M$ such that $m_1 \sim m_1'$ and $m_2 \sim m_2'$ imply

$$\sum_{n\in C}(m_1 \cdot m_2)(n) = \sum_{n\in C}(m_1' \cdot m_2')(n)$$

for all $\sim$-classes $C$. The quotient $M/\sim$ admits a stochastic monoid structure rendering the canonical function $m \mapsto [m]$ an epimorphism of stochastic monoids. The classical Isomorphism Theorem of Algebra still holds in the stochastic setup, namely

> for any epimorphism of stochastic monoids $h : M \to M'$ and every stochastic congruence $\sim$ on $M'$ its inverse image $h^{-1}(\sim)$ defined by
>
> $$m_1 h^{-1}(\sim) m_2 \quad \text{iff} \quad h(m_1) \sim h(m_2),$$
>
> is again a stochastic congruence and the quotient stochastic monoids $M/h^{-1}(\sim)$ and $M'/\sim$ are isomorphic. In particular if $\sim$ is the equality, then $h^{-1}(=)$ is the kernel congruence of $h$ (denoted by $\sim_h$)
>
> $$m_1 \sim_h m_2 \quad \text{iff} \quad h(m_1) = h(m_2),$$
>
> and the stochastic monoids $M/\sim_h$ and $M'$ are isomorphic.

We show that stochastic congruences are closed under the join operation. This allows us to construct the greatest stochastic congruence included in an equivalence $\sim$. It is the join of all stochastic congruences on $M$ included into $\sim$ and it is denoted by $\sim^{stoc}$. The quotient stochastic monoid $M/\sim^{stoc}$ is denoted by $M^{stoc}$ and has the following universal property:

> given an epimorphism of stochastic monoids $h : M \to M'$ whose kernel $\sim_h$ saturates the equivalence $\sim$ there exists a unique epimorphism of stochastic monoids $h' : M' \to M^{stoc}$ such that $h' \circ h = h^{stoc}$, where $h^{stoc} : M \to M^{stoc}$ is the canonical epimorphism into the quotient.

This result states that $h^{stoc}$ is minimal among all epimorphisms saturating $\sim$.

Let $M$ be a stochastic monoid and $L \subseteq M$. Denote by $\sim_L$ the greatest congruence of $M$ included in the partition (equivalence) $\{L, M - L\}$, i.e., $\sim_L = \{L, M - L\}^{stoc}$. The quotient stochastic monoid $M_L = M/\sim_L$ will be called the syntactic stochastic monoid of $L$ and it is characterized by the following universal property.

> For every stochastic monoid $M$ and every epimorphism $h : M \to M'$ verifying $h^{-1}(h(L)) = L$, there exists a unique epimorphism $h' : M' \to M_L$ such that $h' \circ h = h_L$ where $h_L : M \to M_L$ is the canonical projection into the quotient.

A subset $L$ of a stochastic monoid $M$ is stochastically recognizable if there exist a finite stochastic monoid $M'$ and a morphism $h : M \to M'$ such that $h^{-1}(h(L)) = L$. By taking into account the previous result we get that $L$ is recognizable if and only if its syntactic stochastic monoid is finite. Moreover stochastically recognizable subsets are closed under boolean operations and inverse morphisms.

## 2 Stochastic Subsets

Some useful elementary facts are displayed. Let $(x_i)_{i\in I}$, $(x_{ij})_{i\in I, j\in J}$, $(y_j)_{j\in J}$ be families of nonnegative reals, then

$$\underset{i\in I, j\in J}{sup}\, x_{ij} = \underset{i\in I}{sup}\, \underset{j\in J}{sup}\, x_{ij} = \underset{j\in J}{sup}\, \underset{i\in I}{sup}\, x_{ij}, \qquad \underset{i\in I, j\in J}{sup}\, x_i y_j = \underset{i\in I}{sup}\, x_i \cdot \underset{j\in J}{sup}\, y_j,$$

provided that the above suprema exist. If $\underset{I'\subseteq_{fin} I}{sup}\, \Sigma_{i\in I'} x_i$ exists, then we say that the sum $\Sigma_{i\in I} x_i$ exists and we put

$$\underset{i\in I}{\Sigma} x_i = \underset{I'\subseteq_{fin} I}{sup}\, \underset{i\in I'}{\Sigma} x_i$$

where the notation $I' \subseteq_{fin} I$ means that $I'$ is a finite subset of $I$.

It holds

$$\sum_{i\in I, j\in J} x_{ij} = \sum_{i\in I}\sum_{j\in J} x_{ij} = \sum_{j\in J}\sum_{i\in I} x_{ij}, \qquad \sum_{i\in I, j\in J} x_i y_j = \sum_{i\in I} x_i \sum_{j\in J} y_j.$$

Let $M$ be a non empty set and $[0,1]$ the unit interval, a *stochastic subset* of $M$ is a function $F : M \to [0,1]$ with the additional property that the sum of its values exists and is equal to 1

$$\sum_{m\in M} F(m) = 1.$$

We denote by $Stoc(M)$ the set of all stochastic subsets of $M$.

Let $F_i : M \to \mathbb{R}_+$, $i \in I$, be a family of functions such that for every $m \in M$ the sum $\sum_{i\in I} F_i(m)$ exists. Then the assignment

$$m \mapsto \sum_{i\in I} F_i(m)$$

defines a function from $M$ to $\mathbb{R}_+$ denoted by $\sum_{i\in I} F_i$, i.e.,

$$\Big(\sum_{i\in I} F_i\Big)(m) = \sum_{i\in I} F_i(m), \quad m \in M.$$

Now let $(\lambda_i)_{i\in I}$ be a family in $[0,1]$ such that $\sum_{i\in I}\lambda_i = 1$ and $F_i \in Stoc(M)$, $i \in I$. For any finite subset $I'$ of $I$ and any $m \in M$, we have

$$\sum_{i\in I}\lambda_i F_i(m) = \sup_{I'\subseteq_{fin} I}\sum_{i\in I'}\lambda_i F_i(m) \leq 1.$$

Thus $\sum_{i\in I}\lambda_i F_i$ is defined and belongs to $Stoc(M)$ because

$$\begin{aligned}\sum_{m\in M}\Big(\sum_{i\in I}\lambda_i F_i\Big)(m) =& \sum_{m\in M}\sum_{i\in I}\lambda_i F_i(m) = \sum_{i\in I}\sum_{m\in M}\lambda_i F_i(m)\\ =&\Big(\sum_{i\in I}\lambda_i\Big)\Big(\sum_{m\in M}F_i(m)\Big) = 1\cdot 1 = 1.\end{aligned}$$

Thus we can state:

**Strong Convexity Lemma (SCL).** *The set $Stoc(M)$ is a strongly convex set, i.e., for any stochastic family*

$$\lambda_i \in [0,1],\quad F_i \in Stoc(M),\ i \in I$$

*the function $\sum_{i\in I}\lambda_i F_i$ is in $Stoc(M)$.*

For arbitrary sets $M, M'$ any function $h : M \to Stoc(M')$ can be extended into a function $\bar{h} : Stoc(M) \to Stoc(M')$ by setting

$$\bar{h}(F) = \sum_{m\in M} F(m)\cdot h(m).$$

In particular, any function $h : M \to M'$ is extended into a function $\bar{h} : Stoc(M) \to Stoc(M')$ by the same as above formula. This formula is legitimate since by the strong convexity lemma

$$\sum_{m\in M} F(m) = 1$$

and $h(m)$ is a stochastic subset of $M$.

Hence, for any stochastic subset $F : M \to [0,1]$ we have the expansion formula

$$F = \sum_{m\in M} F(m)\hat{m}$$

where $\hat{m} : M \to [0,1]$ stands for the singleton function

$$\hat{m}(n) = \begin{cases} 1, & \text{if } n = m;\\ 0, & \text{if } n \neq m.\end{cases}$$

Often $\hat{m}$ is identified with $m$ itself.

# 3 Stochastic Congruences

Our main interest is focused on equivalences in the stochastic setup. Any equivalence relation $\sim$ on the set $M$, can be extended into an equivalence relation $\approx$ on the set $Stoc(M)$ as follows: for $F, F' \in Stoc(M)$ we set $F \approx F'$ if and only if for each $\sim$-class $C$ it holds

$$\sum_{m \in C} F(m) = \sum_{m \in C} F'(m),$$

that is both $F, F'$ behave stochastically on $C$ in similar way. The above sums exist because $F, F'$ are stochastic subsets of $M$:

$$\sum_{m \in C} F(m) \leq \sum_{m \in M} F(m) = 1.$$

The equivalence $\approx$ has a fundamental property, it is compatible with strong convex combinations.

**Proposition 3.1.** *Assume that $(\lambda_i)_{i \in I}$ is a stochastic family of numbers in $[0,1]$ and $F_i, F_i' \in Stoc(M)$, for all $i \in I$. Then*

$$F_i \approx F_i', \text{ for all } i \in I, \quad \text{implies} \quad \sum_{i \in I} \lambda_i F_i \approx \sum_{i \in I} \lambda_i F_i'.$$

*Proof.* By hypothesis we have

$$\sum_{m \in C} F_i(m) = \sum_{m \in C} F_i'(m)$$

for any $\sim$-class $C$ in $M$, and thus

$$\begin{aligned} \sum_{m \in C} \left( \sum_{i \in I} \lambda_i F_i \right)(m) &= \sum_{m \in C} \sum_{i \in I} \lambda_i F_i(m) = \sum_{i \in I} \lambda_i \sum_{m \in C} F_i(m) \\ &= \sum_{i \in I} \lambda_i \sum_{m \in C} F_i'(m) = \sum_{m \in C} \sum_{i \in I} \lambda_i F_i'(m) \\ &= \sum_{m \in C} \left( \sum_{i \in I} \lambda_i F_i' \right)(m) \end{aligned}$$

that is

$$\sum_{i \in I} \lambda_i F_i \approx \sum_{i \in I} \lambda_i F_i'$$

as wanted. $\square$

# 4 Stochastic Monoids

A stochastic monoid is a set $M$ equipped with a stochastic multiplication, i.e. a function

$$M \times M \to Stoc(M), \quad (m_1, m_2) \mapsto m_1 m_2$$

which is associative

$$\sum_{n \in M} (m_1 m_2)(n)(n m_3) = \sum_{n \in M} (m_2 m_3)(n)(m_1 n)$$

and unitary i.e. there is an element $e \in M$ such that

$$me = m = em, \text{ for all } m \in M.$$

For instance any ordinary monoid can be viewed as a stochastic monoid. In the present study it is important to have a congruence notion. More precisely, let $M$ be a stochastic monoid and $\sim$ an equivalence relation on the set $M$, such that: $m_1 \sim m_1'$ and $m_2 \sim m_2'$ implies

$$\sum_{m \in C} (m_1 m_2)(m) = \sum_{m \in C} (m_1' m_2')(m)$$

for all $\sim$-classes $C$, then $\sim$ is called a *stochastic congruence* on $M$. This condition can be reformulated as follows: $m_1 \sim m_1'$ and $m_2 \sim m_2'$ implies

$$m_1 m_2 \approx m_1' m_2'.$$

**Proposition 4.1.** *The quotient set $M/\sim$ is structured into a stochastic monoid by defining the stochastic multiplication via the formula*

$$([m_1][m_2])([n]) = \sum_{m \in [n]} (m_1 m_2)(m).$$

*Proof.* First observe that the above multiplication is well defined. Next for every $\sim$-class $[b]$ we have

$$\begin{aligned}
(([m_1][m_2])\,[m_3])\,([b]) &= \sum_{[n] \in M/\sim} ([m_1][m_2])([n])([n][m_3])([b]) \\
&= \sum_{[n] \in M/\sim} \sum_{n_1 \in [n]} (m_1 m_2)(n_1) \sum_{b' \in [b]} (n m_3)(b')
\end{aligned}$$

Since $n \sim n_1$ we get

$$\begin{aligned}
&= \sum_{[n]\in M/\sim} \sum_{n_1\in[n]} (m_1m_2)(n_1) \sum_{b'\in[b]} (n_1m_3)(b') \\
&= \sum_{[n]\in M/\sim} \sum_{b'\in[b]} \sum_{n_1\in[n]} (m_1m_2)(n_1)(n_1m_3)(b') \\
&= \sum_{b'\in[b]} \sum_{n_1\in M} (m_1m_2)(n_1)(n_1m_3)(b').
\end{aligned}$$

By taking into account the associativity of $M$ we obtain:

$$\begin{aligned}
&= \sum_{b'\in[b]} \sum_{n_1\in M} (m_2m_3)(n_1)(m_1n_1)(b') \\
&= ([m_1]([m_2][m_3]))([b]). \qquad \square
\end{aligned}$$

Congruences on an ordinary monoid $M$ coincide with stochastic congruences when $M$ is viewed as a stochastic monoid. The first question arising is whether stochastic congruence is a good algebraic notion. This is checked by the validity of the known isomorphism theorems in their stochastic variant.

Given stochastic monoids $M$ and $M'$, a strict morphism from $M$ to $M'$ is a function $h : M \to M'$ preserving stochastic multiplication and units, i.e.,

$$\bar{h}(m_1m_2) = h(m_1)h(m_2), \quad h(e) = e',$$

for all $m_1, m_2 \in M$, where $e$, $e'$ are the units of $M$, $M'$ respectively, and $\bar{h} : Stoc(M) \to Stoc(M')$ the canonical extension of $h$ defined in Section 2.

**Theorem 4.1.** *Given an epimorphism of stochastic monoids $h : M \to M'$ and a stochastic congruence $\sim$ on $M'$, its inverse image $h^{-1}(\sim)$ defined by*

$$m_1h^{-1}(\sim)m_2 \quad \textit{if} \quad h(m_1) \sim h(m_2)$$

*is also a stochastic congruence and the stochastic quotient monoids $M/h^{-1}(\sim)$ and $M'/\sim$ are isomorphic.*

*Proof.* Assume that

$$m_1h^{-1}(\sim)m_1' \text{ and } m_2h^{-1}(\sim)m_2'$$

that is

$$h(m_1) \sim h(m_1') \text{ and } h(m_2) \sim h(m_2').$$

Then

$$\bar{h}(m_1m_2) = h(m_1)h(m_2) \approx h(m_1')h(m_2') = \bar{h}(m_1'm_2'),$$

that is for all $C \in M'/\sim$, we have

$$\sum_{c\in C}\bar{h}(m_1m_2)(c) = \sum_{c\in C}\bar{h}(m_1'm_2')(c),$$

but

$$\sum_{c\in C}\bar{h}(m_1m_2)(c) = \sum_{c\in C}\sum_{m\in M}(m_1m_2)(m)h(m)(c) = \sum_{m\in M}(m_1m_2)(m)\sum_{c\in C}h(m)(c)$$
$$= \sum_{m\in h^{-1}(C)}(m_1m_2)(m).$$

Recall that all $h^{-1}(\sim)$-classes are of the form $h^{-1}(C)$, $C \in M'/\sim$. Consequently,

$$= \sum_{m\in h^{-1}(C)}(m_1m_2)(m) = \sum_{m\in h^{-1}(C)}(m_1'm_2')(m)$$

which shows that $h^{-1}(\sim)$ is indeed a congruence of the stochastic monoid $M$. The desired isomorphism $\hat{h} : M/h^{-1}(\sim) \to M'/\sim$ is given by

$$\hat{h}([m]_{h^{-1}(\sim)}) = [h(m)]_{\sim}. \qquad \square$$

**Corolary 4.1.** *Let $h : M \to M'$ be an epimorphism of stochastic monoids. Then the kernel equivalence*

$$m_1 \sim_h m_2 \text{ if } h(m_1) = h(m_2)$$

*is a congruence on $M$ and the stochastic quotient monoid $M/\sim_h$ is isomorphic to $M'$.*

Given stochastic monoids $M_1, \ldots, M_k$ the stochastic multiplication

$$[(m_1, \ldots, m_k) \cdot (m_1', \ldots, m_k')](n_1, \cdots, n_k) = (m_1m_1')(n_1)\cdots(m_km_k')(n_k)$$

structures the set $M_1\times\cdots\times M_k$ into a stochastic monoid so that the canonical projection

$$\pi_i : M_1 \times \cdots \times M_k \to M_i, \quad \pi_i(m_1, \ldots, m_k) = m_i$$

becomes a morphism of stochastic monoids. Notice that the above multiplication is stochastic because

$$\sum_{\substack{n_i\in M_i\\1\le i\le k}}(m_1m_1')(n_1)\cdots(m_km_k')(n_k) = \sum_{n_1\in M_1}(m_1m_1')(n_1)\cdots\sum_{n_k\in M_k}(m_km_k')(n_k)$$
$$= 1\cdots 1 = 1.$$

**Theorem 4.2.** *Let $\sim_i$ be a stochastic congruence on the stochastic monoid $M_i$ $(1 \leq i \leq k)$. Then $\sim_1 \times \cdots \times \sim_k$ is a stochastic congruence on the stochastic monoid $M_1 \times \cdots \times M_k$ and the stochastic monoids $M_1 \times \cdots \times M_k / \sim_1 \times \cdots \times \sim_k$ and $M_1/\sim_1 \times \cdots \times M_k/\sim_k$ are isomorphic.*

# 5 Greatest Stochastic Congruence Saturating an Equivalence

First observe that, due to the symmetric property which an equivalence relation satisfies, the sumability condition in the definition of a congruence can be replaced by the weaker condition: $m_1 \sim m_1'$ and $m_2 \sim m_2'$ implies

$$\sum_{m \in C} (m_1 m_2)(m) \leq \sum_{m \in C} (m_1' m_2')(m)$$

for all $\sim$-classes $C$.

**Lemma 5.1.** *The equivalence $\sim$ on the stochastic monoid $M$ is a congruence if and only if the following condition is fulfilled: $m \sim m'$, implies*

$$\sum_{b \in C} (m \cdot n)(b) \leq \sum_{b \in C} (m' \cdot n)(b) \quad \text{and} \quad \sum_{b \in C} (n \cdot m)(b) \leq \sum_{b \in C} (n \cdot m')(b).$$

*Proof.* One direction is immediate whereas for the opposite direction we have: $m_1 \sim m_1'$ and $m_2 \sim m_2'$ imply

$$\sum_{b \in C} (m_1 \cdot m_2)(b) \leq \sum_{b \in C} (m_1' \cdot m_2)(b) \leq \sum_{b \in C} (m_1' \cdot m_2')(b). \qquad \square$$

Next we demonstrate that stochastic congruences are closed under the join operation. We recall that the join $\bigvee_{i \in I} \sim_i$ of a family of equivalences $(\sim_i)_{i \in I}$ on a set $A$ is the reflexive and transitive closure of their union:

$$\bigvee_{i \in I} \sim_i = \left( \bigcup_{i \in I} \sim_i \right)^* .$$

**Theorem 5.1.** *If $(\sim_i)_{i \in I}$ is a family of stochastic congruences on $M$, then their join $\bigvee_{i \in I} \sim_i$ is also a stochastic congruence.*

*Proof.* Let $\sim_1, \sim_2$ be two congruences on $M$ and $\sim = \sim_1 \vee \sim_2$. First we show that $m \sim_1 m'$ implies

$$\sum_{b \in C} (m \cdot n)(b) \leq \sum_{b \in C} (m' \cdot n)(b),$$

for all $\sim$-classes $C$. From the inclusion $\sim_1 \subseteq \sim$ we get that $C$ is the disjoint union

$$C = \bigcup_{j=1}^{m} C_j^1$$

where $C_j^1$ denote $\sim_1$-classes. Then

$$\sum_{b\in C}(m\cdot n)(b) = \sum_{j=1}^{m}\sum_{b\in C_j^1}(m\cdot n)(b) \le \sum_{j=1}^{m}\sum_{b\in C_j^1}(m'\cdot n)(b) = \sum_{b\in C}(m'\cdot n)(b).$$

By a similar argument we show that $m \sim_2 m'$ implies

$$\sum_{b\in C}(m\cdot n)(b) \le \sum_{b\in C}(m'\cdot n)(b),$$

for all $\sim$-classes $C$. Now, if $m \sim m'$, without any loss we may assume that

$$m \sim_1 m_1 \sim_2 m_2 \sim_1 \cdots \sim_1 m_{2\lambda-1} \sim_2 m'$$

for some elements $m_1, \ldots, m_{2\lambda-1} \in M$. Applying successively the previous facts, we obtain

$$\sum_{b\in C}(m\cdot n)(b) \le \sum_{b\in C}(m_1\cdot n)(b) \le \cdots \le \sum_{b\in C}(m_{2\lambda-1}\cdot n)(b) \le \sum_{b\in C}(m'\cdot n)(b).$$

For an arbitrary set of congruences we proceed in a similar way. $\square$

The previous result leads us to introduce the greatest stochastic congruence included into an equivalence $\sim$ of $M$. It is the join of all stochastic congruences on $M$ included into $\sim$ and it is denoted by $\sim^{stoc}$. The quotient stochastic monoid $M/\sim^{stoc}$ is denoted by $M^{stoc}$ and has the following universal property

**Theorem 5.2.** *Given an epimorphism of stochastic monoids $h : M \to M'$ whose kernel $\sim_h$ saturates the equivalence $\sim$ there exists a unique epimorphism of stochastic monoids $h' : M' \to M^{stoc}$ rendering commutative the triangle*

$$\begin{tikzcd} & M \arrow[dl, "h"'] \arrow[dr, "h^{stoc}"] & \\ M' \arrow[rr, "h'"'] & & M^{stoc} \end{tikzcd}$$

*where $h^{stoc} : M \to M^{stoc}$ is the canonical projection $m \mapsto [m]_{stoc}$ sending every element $m \in M$ on its $\sim^{stoc}$-class.*

*Proof.* By virtue of the Isomorphism Theorem the stochastic monoid $M'$ is isomorphic to the quotient $M/\sim_h$. Since by assumption $\sim_h \subseteq \sim^{stoc}$, $h'$ is the following composition

$$M' \xrightarrow{\sim} M/\sim_h \xrightarrow{f} M/\sim^{stoc} = M^{stoc},$$

with $f([m]_h) = [m]_{stoc}$, $[m]_h$ being the $\sim_h$-class of $m$. $\square$

The previous result states that $h^{stoc}$ is minimal among all epimorphisms saturating $\sim$.

# 6 Syntactic Stochastic Monoids

Let $M$ be a stochastic monoid and $L \subseteq M$. Denote by $\sim_L$ the greatest congruence of $M$ included in the partition (equivalence) $\{L, M - L\}$, i.e.,

$$\sim_L = \{L, M - L\}^{stoc}.$$

The quotient stochastic monoid $M_L = M/\sim_L$ will be called the *syntactic stochastic monoid* of $L$ and it is characterized by the following universal property.

**Theorem 6.1.** *For every stochastic monoid $M$ and every epimorphism $h : M \to M'$ verifying $h^{-1}(h(L)) = L$, there exists a unique epimorphism $h' : M' \to M_L$ rendering commutative the triangle*

$$\begin{tikzcd} & M \arrow[dl, "h"'] \arrow[dr, "h_L"] & \\ M' \arrow[rr, "h'"'] & & M_L \end{tikzcd}$$

*where $h_L$ is the canonical morphism sending every element $m \in M$ to its $\sim_L$-class.*

*Proof.* The hypothesis $h^{-1}(h(L)) = L$ means that $\sim_h$ saturates $L$ and so the statement follows immediately by Theorem 5.2. $\square$

Given stochastic monoids $M, M'$ we write $M' < M$ if there is a stochastic monoid $\bar{M}$ and a situation

$$M' \xleftarrow{h} \bar{M} \xrightarrow{i} M$$

where $i$ (resp. $h$) is a monomorphism (resp. epimorphism).

**Theorem 6.2.** *Given subsets $L_1, L_2, L$ of a stochastic monoid $M$ it holds*

*i)* $M_{L_1 \cap L_2} < M_{L_1} \times M_{L_2}$,

*ii)* $M_L = M_{\bar{L}}$, *where $\bar{L}$ designates the set theoretic complement of $L$,*

*iii)* $M_{L_1 \cup L_2} < M_{L_1} \times M_{L_2}$,

*iv) If $h : M \to N$ is an epimorphism of ND-monoids and $L \subseteq N$, then $M_{h^{-1}(L)} = M_L$.*

*Proof.* The proof follows by applying Theorem 6.1. □

A subset $L$ of a stochastic monoid $M$ is *stochastically recognizable* if there exist a finite stochastic monoid $M'$ and a morphism $h : M \to M'$ such that $h^{-1}(h(L)) = L$. The class of stochastically recognizable subsets of $M$ is denoted by $StocRec(M)$. By taking into account Theorem 6.1 we get

**Proposition 6.1.** *$L \subseteq M$ is recognizable if and only if its syntactic stochastic monoid is finite, $card(M_L) < \infty$.*

Putting this result together with Theorem 6.2 we yield

**Proposition 6.2.** *The class $StocRec(M)$ is closed under boolean operations and inverse morphisms.*

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# Dynamique du problème $3x+1$ sur la droite réelle

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**Abstract**

The $3x+1$ *problem* is a difficult conjecture dealing with quite a simple algorithm on the positive integers. A possible approach is to go beyond the discrete nature of the problem, following M. Chamberland who used an analytic extension to the half-line $\mathbb{R}^+$. We complete his results on the dynamic of the critical points and obtain a new formulation the $3x+1$ problem. We clarify the links with the question of the existence of wandering intervals. Then, we extend the study of the dynamic to the half-line $\mathbb{R}^-$, in connection with the $3x-1$ *problem.* Finally, we analyze the mean behaviour of real iterations near $\pm\infty$. It follows that the average growth rate of the iterates is close to $(2+\sqrt{3})/4$ under a condition of uniform distribution modulo 2.

**Key words** : $3x+1$ problem, one-dimensional dynamics, attracting cycles, asymptotic analysis.

**MSC 2010** : 37E05.

## 1 Introduction

Gènéralement attribué à Lothar Collatz, le *problème* $3x+1$ est aussi appelé *conjecture de Syracuse*, en référence à l'Université du màme nom. Il se rapporte à la fonction $T$ définie sur les entiers positifs par

$$T(n) := \begin{cases} (3n+1)/2 & \text{si n est impair,} \\ n/2 & \text{sinon.} \end{cases} \tag{1.1}$$

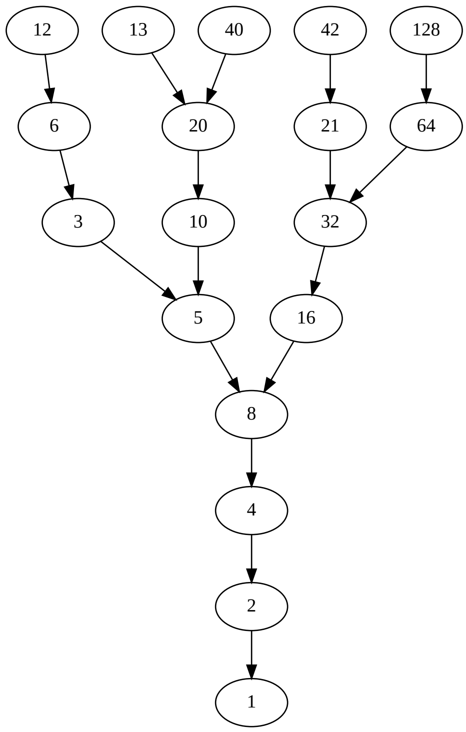


FIG. 1 – Arbre inverse du problème $3x + 1$ représentant l'ensemble des antécédents de 1 sur sept itérations.

Il s'agit de prouver que toute itération de $T$ à partir d'un entier positif $n$ arbitraire conduit nécessairement à la valeur 1. Cette valeur est cyclique de période 2 : $T(T(1)) = 1$.

**Conjecture 1.1.** ***Problème*** $3x + 1$
*Pour tout entier $n > 0$, il existe un entier $k \geq 0$ tel que $T^k(n) = 1$.*[1]

La figure 1 représente toutes les orbites qui aboutissent à 1 en un maximum de sept itérations.

Le problème $3x + 1$ se ramène entièrement aux deux conjectures 1.2 et 1.3 sur la dynamique de la fonction $T$.

**Conjecture 1.2.** ***Absence de trajectoires divergentes***
*Tout entier positif $n$ a une orbite $\{T^i(n)\}_{i=0}^{\infty}$ bornée.*

**Conjecture 1.3.** ***Absence de cycles non-triviaux***
*Il n'existe pas d'entiers $n > 2$ et $k > 0$ tels que $T^k(n) = n$.*

[1]On note $T^k(n)$ le $k^{\text{Àme}}$ itéré de $T$.

La conjecture 1.2 implique que tout entier positif a une orbite cyclique à partir d'un certain rang par itération de $T$. La conjecture 1.3 stipule que le seul cycle possible est le cycle $(1,2)$.

Généralement, on convient de stopper les itérations lorsque la valeur 1 est atteinte. Ainsi on appelle *temps de vol* de $n$ le plus petit entier $k$ tel que $T^k(n)=1$.

T. Oliveira e Silva a vérifié par des calculs sur ordinateur que tout entier positif $n<5\cdot 2^{60}$ a un temps de vol fini [7].

Les conjectures 1.2 et 1.3, bien qu'abondamment étudiées, ne sont toujours pas résolues. On pourra se référer aux ouvrages de J. Lagarias [7] et G.J. Wirsching [10] pour une synthèse détaillée des résultats partiels relatifs au problème $3x+1$ et diverses variantes.

R. E. Crandall [4] a avancé un argument heuristique basé sur l'idée de promenade aléatoire : si l'on considère uniquement la sous-suite des itérés impairs d'un entier $n$ assez grand, on s'attend à ce que l'ensemble des rapports possibles entre deux termes successifs impairs, à savoir $3/2$, $3/4$, $3/8$, $\ldots$, aient pour probabilités respectives les valeurs $1/2$, $1/4$, $1/8$, $\ldots$. On obtient comme rapport moyen la valeur $3/4$. Ceci découle de l'égalité

$$\left(\frac{3}{2}\right)^{\frac{1}{2}}\cdot\left(\frac{3}{4}\right)^{\frac{1}{4}}\cdot\left(\frac{3}{8}\right)^{\frac{1}{8}}\cdots=\frac{3}{4}. \tag{1.2}$$

Cet argument plaide fortement en faveur de la conjecture 1.2.

Dans le cadre de notre étude, nous appellerons *vitesse moyenne* d'une séquence finie $\left\{n,T(n),\cdots,T^k(n)\right\}$ la quantité $\left(T^k(n)/n\right)^{1/k}$.

Un raisonnement analogue [2] à celui de Crandall suggère que la vitesse moyenne d'une séquence arbitraire non-cyclique a statistiquement une valeur proche de $\sqrt{3}/2\simeq 0.866\ldots$, moyenne géométrique de $1/2$ et $3/2$. En effet, la croissance d'une séquence dépend principalement de la parité des itérés successifs. Or, on s'attend à ce que les parités soient équiréparties sur un grand nombre d'itérations.

Ainsi le temps de vol $k$ d'un entier $n$ serait tel que $(1/n)^{1/k}\approx\sqrt{3}/2$ et l'on obtiendrait la valeur moyenne

$$k\approx\frac{2\ln n}{\ln\left(\frac{4}{3}\right)}$$

en l'absence de cycle [7, p. 7].

Ces estimations sont confortées par les calculs numériques. Il semble donc qu'un tel raisonnement permette de saisir l'essentiel de la dynamique asymptotique du problème $3x+1$.

# 2 Extension sur les réels positifs

Une approche possible du problème $3x+1$ est de sortir du cadre discret et d'étendre $T$ par une fonction analytique sur l'ensemble des nombres réels [3] ou complexes [5, 8]. Nous opterons pour l'extension réelle[2] qui nous parait la plus naturelle, définie par l'équation (2.1) ci-après, et nous expliciterons les liens étroits qu'entretiennent la dynamique sur les réels et le problème $3x+1$.

Chamberland [3] a étudié la dynamique sur la demi-droite $\mathbb{R}^+$ de la fonction analytique

$$f(x) := x + \frac{1}{4} - \left(\frac{x}{2} + \frac{1}{4}\right)\cos(\pi x) \tag{2.1}$$

qui vérifie $f(n) = T(n)$ pour tout entier $n$, et $f\left(\mathbb{R}^+\right) = \mathbb{R}^+$ . Il a ainsi obtenu plusieurs résultats significatifs :

(2.2) Le point fixe 0 est attractif ainsi que les cycles $\mathcal{A}_1 := \{1, 2\}$ et $\mathcal{A}_2 := \{1.192\ldots, 2.138\ldots\}$ de période 2.

(2.3) La dérivée *Schwartzienne* de $f$ est négative sur $\mathbb{R}^+$.

(2.4) Les intervalles $[0, \mu_1]$ et $[\mu_1, \mu_3]$ sont invariants par $f$, où $\mu_1 = 0.277\ldots$ et $\mu_3 = 2.445\ldots$ sont des points fixes répulsifs.

(2.5) Tout cycle d'entiers positifs est attractif.

(2.6) Il existe des orbites monotones non-bornées sur $\mathbb{R}^+$.

Par ailleurs, il énonce la conjecture "Stable Set" [3] ci-dessous :

**Conjecture 2.1.** ***Cycles attractifs sur*** $\mathbb{R}^+$
*La fonction $f$ n'admet aucun cycle attractif sur l'intervalle* $[\mu_3, +\infty)$.

Une conséquence immédiate de (2.5) est que la conjecture 2.1 entraîne la conjecture 1.3 du problème $3x+1$.

Puis, il définit l'ensemble des orbites non-bornées

$$U_f^\infty := \left\{ x \in \mathbb{R}^+ : \limsup_{k\to\infty} f^k(x) = \infty \right\}. \tag{2.7}$$

[2]Le deuxième auteur (O. Rozier) avait antérieurement suggéré l'étude de l'extension (2.1) dans le plan complexe et obtenu des représentations graphiques des bassins d'attraction [1].

Le résultat (2.6) prouve que $U_f^\infty$ est infini, et l'on démontre que $U_f^\infty$ contient un ensemble de Cantor dans chaque intervalle $[n, n+1]$ pour tout entier $n \geq 2$ [8]. Il suit que $U_f^\infty$ n'est pas dénombrable.

**Conjecture 2.2.** ***Orbites non-bornées sur*** $\mathbb{R}^+$
*L'ensemble* $U_f^\infty$ *est d'intérieur vide.*

La conjecture 2.2 est une formulation faible de la conjecture "Unstable Set" [3]. Nous allons montrer qu'elle a des liens logiques avec le problème $3x + 1$.

**Lemme 2.1.** *Soit* $\{c_n\}_{n=0}^\infty$ *l'ensemble des points critiques de* $f$ *dans* $\mathbb{R}^+$*, ordonnés de telle sorte que* $0 < c_1 < c_2 < \ldots$.
*Alors on a*

$$n - \frac{1}{\pi^2 n} < c_n < n, \text{ si } n \text{ est pair;}$$

$$n < c_n < n + \frac{3}{\pi^2 n}, \text{ si } n \text{ est impair.}$$

*Démonstration.* (indications) Soit $n$ un entier positif. On a

$$f'(x) = 1 - \frac{1}{2}\cos(\pi x) + \pi\left(\frac{x}{2} + \frac{1}{4}\right)\sin(\pi x)$$

et on vérifie facilement que $n - \frac{1}{2} < c_n < n$ si $n$ est pair, et $n < c_n < n + \frac{1}{2}$ si $n$ est impair.

De plus, on a toujours $f'(n) > 0$ et on montre que

$$f'\left(n - \frac{1}{\pi^2 n}\right) < \frac{(20 - 6\pi^2 n)\, n + 1}{24\pi^2 n^3} < 0, \text{ si } n \text{ est pair,}$$

$$f'\left(n + \frac{3}{\pi^2 n}\right) < \frac{(18 - 6\pi^2 n)\, n + 9}{8\pi^2 n^3} < 0, \text{ si } n \text{ est impair,}$$

en utilisant les encadrements $1 - \frac{t^2}{2} < \cos t < 1$ et $t - \frac{t^3}{6} < \sin t < t$ pour $0 < t < 1$. $\square$

**Lemme 2.2.** *On considère la famille d'intervalles* $J_n^a := \left[n, n + \frac{a}{\pi^2 n}\right]$ *pour tout entier* $n > 0$ *et tout réel* $a$ *tel que* $\frac{27}{8} < a < 6$.
*Alors on a* $f\left(J_n^a\right) \subset J_{f(n)}^a$ *pour tout entier* $n$ *assez grand.*
*Si de plus* $a = \frac{7}{2}$, *alors l'inclusion est vraie pour tout* $n > 0$.

*Démonstration.* Soit un entier $n > 0$ et un réel $a$ tel que $\frac{27}{8} < a < 6$.

1[er] cas : $n$ est pair, $f(n) = \frac{n}{2}$ et $f$ est croissante sur $J_n^a$. On vérifie alors que

$$f\left(n + \frac{a}{\pi^2 n}\right) \leq f(n) + \frac{a}{\pi^2 f(n)} + A \cdot B$$

avec

$$A = \frac{a}{8\pi^4 n^3} \text{ et } B = \pi^2 n \left(2\left(a-6\right)n + a\right) + 2a^2$$

en utilisant l'inégalité $1 - \cos t < \frac{t^2}{2}$ pour $0 < t < 1$. Comme $a - 6 < 0$, il est clair que $A \cdot B < 0$ pour $n$ suffisamment grand.

Si de plus $a = \frac{7}{2}$, alors $B \leq \frac{49}{2} - 13\pi^2 < 0$ pour tout $n$.

2[e] cas : $n$ est impair, $f(n) = \frac{3n+1}{2}$ et $f$ est croissante sur $[n, c_n]$ et décroissante sur $\left[c_n, n + \frac{a}{\pi^2 n}\right]$. On vérifie alors que

$$f\left(n + \frac{a}{\pi^2 n}\right) \geq f(n) - A \cdot B$$

$A$ et $B$ étant défini comme précédemment, donc $A \cdot B < 0$ pour $n$ suffisamment grand. Si de plus $a = \frac{7}{2}$, alors $A \cdot B < 0$ pour tout $n \geq 3$, et dans le cas $n = 1$, on a

$$f\left(1 + \frac{7}{2\pi^2}\right) = 2.013\ldots > f(1).$$

D'après le lemme 2.1, on a $c_n = n + \frac{b}{\pi^2 n}$ avec $0 < b < 3$. Il vient

$$f(c_n) - f(n) \leq \frac{3b}{2\pi^2 n} - \frac{n}{2}\left(1 - \cos\left(\frac{b}{\pi n}\right)\right)$$

puis en utilisant l'inégalité $1 - \cos t > \frac{t^2}{2} - \frac{t^4}{24}$ pour $0 < t < 1$,

$$f(c_n) - f(n) < \frac{b(6-b)}{4\pi^2 n} + \frac{b^4}{48\pi^4 n^3} \leq \frac{9}{4\pi^2 n} + \frac{27}{16\pi^4 n^3}.$$

On obtient

$$f(c_n) < f(n) + \frac{a}{\pi^2 f(n)} + \frac{C}{D}$$

avec

$$C = 4\pi^2 n^2 \left((27 - 8a)n + 9\right) + 81n + 27 \text{ et } D = 16\pi^4 n^3 (3n+1).$$

On voit que $C < 0$ pour $n$ suffisamment grand. Si de plus $a = \frac{7}{2}$ et $n \geq 11$, on a alors

$$C = 4\pi^2 n^2 (9 - n) + 81n + 27 < 0$$

et dans les cas où $n = 1, 3, 5, 7$ ou $9$, on vérifie numériquement que

$$f(c_n) - f(n) - \frac{7}{(3n+1)\pi^2} < 0$$

en utilisant les valeurs $c_1 = 1.180938\ldots$, $c_3 = 3.084794...$, $c_5 = 5.054721...$, $c_7 = 7.040311...$ et $c_9 = 9.031889....$ □

On déduit du lemme 2.2 un lien logique entre les conjectures 1.2 et 2.2 :

**Théorème 2.1.** *La conjecture 2.2 implique la conjecture 1.2 (absence d'orbites non-bornées) du problème* $3x + 1$ *.*

*Démonstration.* Supposons que la conjecture 2.2 soit vraie et que la conjecture 1.2 soit fausse. Alors il existe un entier positif $n_0$ tel que

$$\limsup_{k\to\infty} f^k(n_0) = \infty.$$

D'après le lemme 2.2, une simple récurrence donne

$$f^k\left(J_{n_0}^{\frac{7}{2}}\right) \subset J_{f^k(n_0)}^{\frac{7}{2}}$$

pour tout entier $k \geq 0$. Donc l'ensemble $U_f^\infty$ contient l'intervalle $J_{n_0}^{\frac{7}{2}}$, ce qui est en contradication avec notre hypothèse que $U_f^\infty$ soit d'intérieur vide. □

# 3 Dynamique des points critiques

Les résultats (2.3) et (2.5) entrainent que le bassin d'attraction immédiat de tout cycle d'entiers strictement positifs contient au moins un point critique [3]. Pour cette raison, Chamberland a effectué des calculs numériques relatifs aux orbites des points critiques $c_n$ pour $n \leq 1000$. Il énonce la conjecture "Critical Points" ci-dessous :

**Conjecture 3.1.** ***Points critiques***
*Tous les points critiques* $c_n$, $n > 0$, *sont attirés par l'un des cycles* $\mathcal{A}_1$ *ou* $\mathcal{A}_2$*.*

Nous complétons ici les résultats numériques de Chamberland. Une précision de 1500 chiffres décimaux en virgule flottante est requise pour le calcul de certaines orbites ($c_{646}$ par exemple). Nous avons vérifié nos résultats avec deux logiciels différents, Mathematica et Maple.

D'après nos calculs, les cycles $\mathcal{A}_1$ et $\mathcal{A}_2$ attirent tous les points critiques $c_n$ pour $n \leq 2000$. Plus précisément, $c_n$ est attiré par $\mathcal{A}_2$ pour [3] $n = \mathbf{1}$,

[3]En gras les valeurs déjà obtenues par Chamberland.

**3**, **5**, 382, 496, **502**, 504, 508, 530, 550, 644, 646, **656**, 666, 754, 830, 874, 1078, 1150, 1214, 1534, 1590, 1598, 1614, 1662, 1854, et par $\mathcal{A}_1$ pour toutes les autres valeurs de $n \leq 2000$. Nous avons observé que l'orbite de $c_n$ est toujours proche de l'orbite de $n$, sauf pour $n \equiv -2 \pmod{64}$ et pour $n$=54, 334, 338, 366, 390, 442, 444, 470, 484, 486, 496, 500, ….

Les résultats numériques suggèrent la conjecture suivante[4] :

**Conjecture 3.2.** ***Points critiques d'ordre impair***
*Les points critiques $c_n$ sont attirés par le cycle $\mathcal{A}_1 = \{1, 2\}$ pour tout entier $n \geq 7$ impair.*

Nous montrons à présent que la conjecture 3.2 suffit pour reformuler complètement le problème $3x + 1$.

**Théorème 3.1.** *Soit un entier impair $n \geq 7$ dont l'orbite contient 1. Alors le point critique $c_n$ est attiré par le cycle $\mathcal{A}_1$.*

*Démonstration.* Considérons un entier impair $n \geq 7$ dont l'orbite contient 1. La construction de l'arbre des orbites inverses de 1, représenté sur la figure 1, montre que l'orbite de $n$ contient l'un des entiers 12, 13, 16 ou 40. On déduit de règles itératives modulo 3 sur les entiers que les antécédents de 12 sont des entiers pairs. Il vient que $f^k(n) = 13$, 16 ou 40 pour un entier $k \geq 0$. Les lemmes 2.1 et 2.2 entraînent que $c_n$ appartient à $J_n^{\frac{7}{2}}$ et $f^k(c_n)$ se trouve dans $J_{13}^{\frac{7}{2}} \cup J_{16}^{\frac{7}{2}} \cup J_{40}^{\frac{7}{2}}$.

1[er] cas : $f^k(n) = 13$, $f^k(c_n) \in J_{13}^{\frac{7}{2}}$. La séquence des itérés de $f^k(n)$ est $13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1$.

Soit $m$ un entier pris dans cette séquence. La fonction $f$ est unimodale sur $J_m^{\frac{7}{2}}$ avec un maximum en $c_m$ lorsque $m$ est impair, et strictement croissante lorsque $m$ est pair. Ce comportement permet de déterminer les images successives de $J_{13}^{\frac{7}{2}}$ en fonction de $c_{13} = 13.022478\ldots$.

$$f\left(J_{13}^{\frac{7}{2}}\right) = [20, f(c_{13})]$$

$$f^3\left(J_{13}^{\frac{7}{2}}\right) = \left[5, f^3(c_{13})\right]$$

avec $f^3(c_{13}) = 5.0249\ldots < c_5 = 5.0547\ldots$.

$$f^7\left(J_{13}^{\frac{7}{2}}\right) = \left[1, f^7(c_{13})\right]$$

[4]Dans [5], une conjecture analogue avec davantage d'hypothèses est formulée relativement à une autre extension de la fonction $T$ sur les réels.

avec $f^7(c_{13}) = 1.0184\ldots$.

De plus la fonction $f^2$ est strictement croissante sur l'intervalle $(1, c_1)$ avec une unique point fixe $x_1 = 1.023686\ldots$ qui est répulsif. Il suit que l'intervalle $[1, x_1)$ fait partie du bassin d'attraction immédiat du cycle $\mathcal{A}_1$ et que $c_n$ est attiré par $\mathcal{A}_1$.

2$^{\text{e}}$ cas : $f^k(n) = 16$, $f^k(c_n) \in J_{16}^{\frac{7}{2}}$. On a la séquence $16 \to 8 \to 4 \to 2 \to 1$. Comme précédemment, on obtient l'image

$$f^4\left(J_{16}^{\frac{7}{2}}\right) = \left[1, f^4\left(16 + \frac{7}{32\pi^2}\right)\right]$$

avec $f^4\left(16 + \frac{7}{32\pi^2}\right) = 1.0227\ldots < x_1$. Donc $c_n$ est attiré par $\mathcal{A}_1$.

3$^{\text{e}}$ cas : $f^k(n) = 40$, $f^k(c_n) \in J_{40}^{\frac{7}{2}}$, et la séquence des itérés est $40 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1$. De la màme manière, on itère les images successives

$$f^3\left(J_{40}^{\frac{7}{2}}\right) = \left[5, f^3\left(40 + \frac{7}{80\pi^2}\right)\right]$$

avec $f^3\left(40 + \frac{7}{80\pi^2}\right) = 5.0118\ldots < c_5 = 5.0547\ldots$,

$$f^7\left(J_{40}^{\frac{7}{2}}\right) = \left[1, f^7\left(40 + \frac{7}{80\pi^2}\right)\right]$$

avec $f^7\left(40 + \frac{7}{80\pi^2}\right) = 1.0047\ldots < x_1$. Ainsi $c_n$ est attiré par $\mathcal{A}_1$ dans tous les cas. □

**Remarque 3.1.** *Dans cette démonstration, il n'est pas possible de fusionner les cas 1 et 3 en partant de l'entier 20 car* $f^6\left(J_{20}^{\frac{7}{2}}\right) = \left[1, f^6\left(20 + \frac{7}{40\pi^2}\right)\right] = [1, 1.023691\ldots]$ *n'est pas inclus (de très peu) dans le bassin d'attraction de* $\mathcal{A}_1$ *délimité par* $x_1 = 1.023686\ldots$.

**Corollaire 3.1.** *La conjecture 3.2 est logiquement équivalente au problème* $3x + 1$.

*Démonstration.* Une conséquence immédiate du théorème 3.1 est que la conjecture 1.1 (problème $3x + 1$) implique la conjecture 3.2 sur la dynamique des points critiques d'ordre impair. On démontre à présent la réciproque.

Considérons un entier $n > 0$. Son orbite contient au moins un entier impair $f^{k_1}(n)$, $k_1 \geq 0$. Si $f^{k_1}(n) \leq 5$, alors l'orbite de $n$ contient le point 1 (cf. figure 1). On considère à présent le cas $f^{k_1}(n) \geq 7$.

Supposons que la conjecture 3.2 soit vraie. Alors il existe un entier positif $k_2$ tel que

$$f^{k_2}\left(c_{f^{k_1}(n)}\right) < 2.$$

De plus, le lemme 2.2 donne par récurrence l'inclusion

$$f^{k_2}\left(c_{f^{k_1}(n)}\right) \in J^{\frac{7}{2}}_{f^{k_1+k_2}(n)}.$$

Il découle l'égalité

$$f^{k_1+k_2}(n) = 1. \quad \square$$

# 4 Intervalles errants

L'existence d'*intervalles errants* [9] dans la dynamique de l'extension $f$ est une question ouverte avec d'importantes implications pour le problème $3x+1$.

**Conjecture 4.1.** ***Absence d'intervalles errants***
*La fonction $f$ n'admet pas d'intervalles errants dans $\mathbb{R}^+$.*

Elle est au cœur du théorème ci-dessous.

**Théorème 4.1.** *On a les relations suivantes entre conjectures :*
*(a) la conjecture 2.2 entraîne la conjecture 4.1,*
*(b) la conjecture 4.1 entraîne la conjecture 1.2.*

*Démonstration.* Par l'absurde.
(a) Supposons que la conjecture 2.2 soit vraie et que la conjecture 4.1 soit fausse. Cela implique que la fonction $f$ admette une famille d'intervalles errants sur une partie bornée de $\mathbb{R}^+$. Or ce serait en contradiction avec la propriété (2.3) : la dérivée Schwartzienne de $f$ est négative sur $\mathbb{R}^+$.

(b) Supposons que la conjecture 1.2 soit fausse. Alors il existe un entier positif $n$ tel que $\lim_{i\to\infty} f^i(n) = +\infty$. D'après le lemme 2.2, les intervalles $\left\{f^i\left(J_n^{7/2}\right)\right\}_{i=0}^{\infty}$ sont inclus dans les intervalles $\left\{J^{7/2}_{f^i(n)}\right\}_{i=0}^{\infty}$, deux à deux disjoints. Il s'agit d'une famille d'intervalles errants. $\square$

Une synthèse des liens logiques entre conjectures est donnée en annexe.

# 5 Extension sur les réels négatifs

L'ensemble $\mathbb{R}^-$ des réels négatifs est également invariant par la fonction $f$ définie par (2.1). La dynamique sur les entiers négatifs est alors identique, au signe près, à celle de la fonction "$3x-1$", notée $U$ et définie sur les entiers positifs par

$$U(n) := \begin{cases} (3n-1)/2 & \text{si n est impair,} \\ n/2 & \text{sinon.} \end{cases} \tag{5.1}$$

En effet, on a la relation de conjugaison $f(-n) = -U(n)$ pour tout entier $n$ positif. La fonction $U$ admet le point fixe 1 et a deux cycles connus : $\{5, 7, 10\}$ de période 3 et $\{17, 25, 37, 55, 82, 41, 61, 91, 136, 68, 34\}$ de période 11. Cela conduit à formuler le *"problème $3x - 1$"* :

**Conjecture 5.1.** ***Problème*** $3x - 1$
*Pour tout entier $n > 0$, il existe un entier $k \geq 0$ tel que $U^k(n) = 1, 5$ ou $17$.*

Les valeurs de $f$ sur $\mathbb{R}^+$ et $(-\infty, -1]$ sont liées pas l'équation fonctionnelle

$$f(x) - f(-1-x) = 2x + 1 \tag{5.2}$$

de sorte que les points fixes de $f$ sur $(-\infty, -1]$ sont exactement les points $\nu_i := -1 - \mu_i$, où $\{\mu_i\}_{i=0}^{\infty}$ désigne l'ensemble des points fixes de $f$ sur $\mathbb{R}^+$, $\mu_0 = 0 < \mu_1 < 1 < \mu_2 < 2 < \ldots$.

Néanmoins, la dynamique de $f$ sur $\mathbb{R}^-$ diffère partiellement de celle que l'on a pu décrire sur $\mathbb{R}^+$, comme le montrent les propriétés (5.3) à (5.7).

(5.3) Les points fixes 0 et $\nu_1 = -1.277\ldots$ sont attractifs, ainsi que les cycles
$\mathcal{B}_1 := \{x, f(x), f^2(x)\}$ où $x = -5.046002\ldots$,
$\mathcal{B}_2 := \{x, f(x), f^2(x)\}$ où $x = -4.998739\ldots$,
$\mathcal{B}_3 := \{x, f(x), \ldots, f^{10}(x)\}$ où $x = -17.002728\ldots$,
$\mathcal{B}_4 := \{x, f(x), \ldots, f^{10}(x)\}$ où $x = -16.999991\ldots$.

(5.4) La dérivée Schwartzienne de $f$ n'est pas partout négative sur $\mathbb{R}^-$.

(5.5) Les intervalles $[-1, 0]$ et $[\nu_1, -1]$ sont invariants par $f$.

(5.6) Tout cycle d'entiers négatifs est répulsif.

(5.7) Il existe des orbites monotones non-bornées sur $\mathbb{R}^-$.

| Point ou cycle attractif | Période | Multiplicateur |
|---|---|---|
| 0 | 1 | 0.5 |
| $\nu_1$ | 1 | 0.385708... |
| $\mathcal{B}_1$ | 3 | 0.036389... |
| $\mathcal{B}_2$ | 3 | 0.866135... |
| $\mathcal{B}_3$ | 11 | 0.003773... |
| $\mathcal{B}_4$ | 11 | 0.926287... |

TAB. 1 – Coefficients multiplicateurs des points et cycles attractifs sur les réels négatifs.

*Démonstration.* (indications)

Propriété (5.3) : Les vitesses d'attraction sont données dans le tableau 1.

Propriété (5.4) : La dérivée Schwartzienne est positive sur un intervalle contenant le point -0.2. On a en effet $Sf(-0.2) = 39.961\ldots$, où

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Propriété (5.5) : La fonction $f$ est strictement croissante sur l'intervalle $[\nu_1, 0]$ contenant le point fixe répulsif -1.

Propriété (5.6) : Voir les indications dans [3, p.16].

Propriété (5.7) : La démonstration est similaire à celle de (2.6). □

**Remarque 5.1.** *Les cycles $\mathcal{B}_2$ et $\mathcal{B}_4$ sont très faiblement attractifs car leur multiplicateur est proche de 1 (cf. tableau 1). On vérifie également que les cycles contenant les points -5 et -17 sont très faiblement répulsifs, avec pour multiplicateurs respectifs les rationnels 9/8 et 2187/2048.*

Comme précédemment, on note $c_n$ les points critiques proches des entiers $n < 0$, et on peut montrer que les itérés successifs de $c_n$ pour $n$ impair négatif restent proches des itérés de $n$, par valeurs inférieures. Nous avons vérifié numériquement pour tout entier $n$, $-1000 < n < 0$, que

– si $n$ est impair et $f^k(n) = -1$ (resp. -5, -17) pour un entier $k$, alors l'orbite de $c_n$ converge vers $\nu_1$ (resp. $\mathcal{B}_1$, $\mathcal{B}_3$) ;
– si $n$ est pair et $f^k(n) = -1$ (resp. -5, -17) pour un entier $k$, alors l'orbite de $c_n$ converge vers 0 (resp. $\mathcal{B}_2$, $\mathcal{B}_4$), sauf pour $n$=-34, -66, -98, -130, -132, -162, -174, -194, -202, -226, ... où l'orbite de $c_n$ converge vers $\mathcal{B}_3$, $\mathcal{B}_1$, $\mathcal{B}_3$, $\nu_1$, $\mathcal{B}_3$, $\mathcal{B}_1$, $\nu_1$, $\mathcal{B}_1$, $\nu_1$, $\nu_1$, ... respectivement. On note que les entiers $n \equiv -2 \pmod{32}$ semblent toujours faire partie des exceptions.

Le plus souvent, lorsque $n < 0$ est pair, l'orbite de $c_n$ reste proche de l'orbite de $n$, par valeurs supérieures. Pour $n$=-34, -98, -132, -162, -202, ... les itérés de $c_n$ finissent pas àtre inférieurs aux itérés de $n$, sans s'en éloigner pour autant. Pour $n$=-66, -130, -174, -194, -258, ... les orbites de $n$ et de $c_n$ sont décorrélées après un nombre fini d'itérations. Dans ce dernier cas, on observe une répartition des orbites de $c_n$ dans chacun des six bassins d'attraction de $\mathbb{R}^-$ : 0, $\nu_1$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ et $\mathcal{B}_4$.

**Conjecture 5.2.** ***Points critiques d'ordre négatif impair***
*Les points critiques $c_n$ sont attirés soit par le point fixe $\nu_1$, soit par l'un des cycles $\mathcal{B}_1$ ou $\mathcal{B}_3$, pour tout entier $n < 0$ impair.*

# 6 Dynamique asymptotique

Dans cette partie, nous étudions le comportement moyen de séquences finies ou infinies d'itérations de $f$, afin de déterminer la vitesse moyenne asymptotique (i.e. au voisinage de $\pm\infty$).

Nous dirons ainsi qu'une séquence infinie $S = \{f^i(x)\}_{i=0}^{\infty}$ est uniformément distribuée modulo 2 (u. d. mod 2) si et seulement si la discrépance à l'origine de $\{f^i(x) \bmod 2\}_{i=0}^{n-1}$ dans l'intervalle $[0, 2]$, notée $D_n^*(S \bmod 2)$, vérifie[5]

$$\lim_{n\to\infty} D_n^*(S \bmod 2) = 0.$$

Dans le cas d'une séquence finie $S = \{f^i(x)\}_{i=0}^{n}$, nous dirons de manière informelle que $S$ est u. d. mod 2 dès lors que $D_n^*(S \bmod 2) \ll 1$.

On rappelle que la notion de discrépance est une mesure de l'uniformité de la distribution d'une séquence de points $\mathcal{X} = \{x_1, \ldots, x_n\} \in [a, b]^n$ et est définie par

$$D_n^*(\mathcal{X}) := \sup_{a\le c\le b} \left| \frac{|\{x_1, \ldots, x_n\} \cap [a, c)|}{n} - \frac{c-a}{b-a} \right| \tag{6.1}$$

Elle intervient notamment dans l'inégalité de Koksma [6] :

[5]On note $x \bmod 2$ la valeur modulo 2 de tout réel $x$, définie par $x \bmod 2 := x - 2\lfloor \frac{x}{2} \rfloor$.

**Théorème 6.1.** *(Koksma) Soit $f : [a, b] \to \mathbb{R}$ une fonction à variation (totale) $V(f)$ bornée. Alors pour toute séquence $\mathcal{X} = \{x_1, \ldots, x_n\} \in [a, b]^n$, on a*

$$\left| \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \frac{1}{b-a} \int_a^b f(t)\, \mathrm{d}t \right| < V(f) D_n^*(\mathcal{X})$$

Nous considérons dorénavant que la fonction $f$ définie par (2.1) s'applique sur $\mathbb{R}$ tout entier. Comme $f$ ne s'annule qu'en 0, il suit que $f^n(x)$ est de màme signe que $x$ pour tout réel $x \neq 0$ et tout entier $n$.

Notre approche consiste à approximer $f(x)/x$ par son asymptote sinusoïdale

$$g(x) := 1 - \frac{\cos(\pi x)}{2} \tag{6.2}$$

dont on détermine la moyenne géométrique.

**Lemme 6.1.** *La moyenne géométrique $\tau$ de la fonction réelle $g(x) = 1 - \cos(\pi x)/2$ sur $[0, 2]$ est égale à $\alpha/4$, où $\alpha = 2 + \sqrt{3}$ est racine du polynôme $X^2 - 4X + 1$.*

*Démonstration.* On cherche à calculer $\tau := \exp\left(\frac{1}{2}\int_0^2 \ln\left(g(t)\right)\, \mathrm{d}t\right)$ avec

$$g(t) = 1 - \cos(\pi t)/2 = \left(\alpha - e^{i\pi t}\right)\left(\alpha - e^{-i\pi t}\right)/(4\alpha) = \left|\alpha - e^{i\pi t}\right|^2/(4\alpha).$$

On obtient

$$\ln \tau = \int_0^2 \ln \left|\alpha - e^{i\pi t}\right|\, \mathrm{d}t - \ln\left(4\alpha\right).$$

La formule de Jensen relative aux fonctions analytiques sur le disque de centre $\alpha$ et de rayon 1 donne le résultat attendu

$$\ln \tau = 2 \ln \alpha - \ln(4\alpha) = \ln\left(\frac{\alpha}{4}\right). \quad \square$$

On montre à présent qu'au voisinage de $\pm\infty$ toute séquence d'itérations u. d. mod 2 de $f$ décroit avec une vitesse moyenne proche de $\tau = (2+\sqrt{3})/4 \simeq 0.933\ldots$.

**Théorème 6.2.** *Soit une séquence finie d'itérations $S = \{f^i(x)\}_{i=0}^{n}$ telle que $\min\{|f^i(x)|\}_{i=0}^{n-1} \geq M$ pour un réel $M > \frac{1}{3}$. Alors on a*

$$\left| \frac{1}{n} \ln\left(\frac{f^n(x)}{x}\right) - \ln \tau \right| < 2\left(\ln 3\right) D_n^*(S \bmod 2) - \ln\left(1 - \frac{1}{3M}\right).$$

*Démonstration.* On considère la formulation $f(t) = g(t)\,(t + h(t))$ où $h$ est la fonction périodique

$$h(t) := \frac{1-\cos(\pi t)}{4g(t)} = \frac{1-\cos(\pi t)}{4-2\cos(\pi t)}.$$

On a donc

$$\frac{f^n(x)}{x} = \prod_{i=0}^{n-1} \frac{f^{i+1}(x)}{f^i(x)} = \prod_{i=0}^{n-1} g\left(f^i(x)\right)\left(1 + \frac{h\left(f^i(x)\right)}{f^i(x)}\right)$$

Il vient alors

$$\frac{1}{n}\ln\left(\frac{f^n(x)}{x}\right) - \ln\tau = A + B$$

avec

$$A = \frac{1}{n}\sum_{i=0}^{n-1}\ln\left(g\left(f^i(x)\right)\right) - \ln\tau$$

et

$$B = \frac{1}{n}\sum_{i=0}^{n-1}\ln\left(1 + \frac{h\left(f^i(x)\right)}{f^i(x)}\right)$$

D'après le lemme 6.1,

$$\ln\tau = \frac{1}{2}\int_0^2 \ln\left(g(t)\right)\,\mathrm{d}t$$

On applique l'inégalité de Koksma :

$$|A| \leq V(\phi) D_n^*(S \bmod 2)$$

où $V(\phi)$ est la variation totale de la fonction $\phi(t) := \ln\left(g(t)\right)$ sur $[0,2]$, soit $V(\phi) = 2\phi(1) - \phi(2) - \phi(0) = 2\ln 3$.

Pour majorer $|B|$, on vérifie que la fonction $h(t)$ est à valeur dans $[0, 1/3]$ avec un maximum en $t = 1$. On en déduit que

$$|B| \leq \max\left(-\ln\left(1 - \frac{1}{3M}\right),\, \ln\left(1 + \frac{1}{3M}\right)\right) = -\ln\left(1 - \frac{1}{3M}\right). \quad \square$$

Le théorème 6.2 est inopérant pour les séquences d'entiers, dont la vitesse moyenne attendue est $\sqrt{3}/2$, strictement inférieure à $\tau$. Il permet toutefois d'établir un lien entre la vitesse moyenne et la distribution modulo 2 des itérations.

**Théorème 6.3.** *Soit $x$ un réel d'orbite $\{f^i(x)\}_{i=0}^{\infty}$ telle que*

$$\liminf_{i\to\infty} |f^i(x)| > \frac{1}{3(1-\tau)} \simeq 4.97\ldots$$

*Alors l'orbite de $x$ n'est pas uniformément distribuée modulo 2.*

*Démonstration.* Il existe un entier positif $N$ et un réel $a > 1$ tels que

$$|f^i(x)| \geq \frac{a}{3(1-\tau)}$$

pour tout $i \geq N$.

On considère les séquences finies $S_n = \{f^i(x)\}_{i=N}^{n+N}$ pour tout $n$ entier positif, et on pose $M_n := \min\{|f^i(x)|\}_{i=N}^{n+N}$.

D'après le théorème 6.2,

$$\frac{1}{n}\ln\left(\frac{f^{n+N}(x)}{f^N(x)}\right) - \ln\tau < 2(\ln 3)D_n^*(S_n \bmod 2) - \ln\left(1 - \frac{1}{3M_n}\right).$$

Il vient

$$2(\ln 3)D_n^*(S_n \bmod 2) > A_n + B_n$$

avec

$$A_n = \frac{1}{n}\ln\left(\frac{f^{n+N}(x)}{f^N(x)}\right)$$

et

$$B_n = -\ln\tau + \ln\left(1 - \frac{1}{3M_n}\right).$$

D'une part, on vérifie aisément que $\liminf_{n\to\infty} A_n \geq 0$. D'autre part, on a

$$B_n \geq -\ln\tau + \ln\left(1 - \frac{1-\tau}{a}\right) = \ln\left(1 + \frac{(a-1)(1-\tau)}{a\tau}\right) > 0.$$

On obtient donc le résultat souhaité :

$$\liminf_{n\to\infty} D_n^*(S_n \bmod 2) \geq \frac{\ln\left(1 + \frac{(a-1)(1-\tau)}{a\tau}\right)}{2\ln 3} > 0. \quad \square$$

L'existence d'orbites tendant vers l'infini a été prouvée par Chamberland pour la fonction $f$ et le corollaire 6.1 donne une condition nécessaire sur l'ensemble des valeurs modulo 2 d'une telle orbite.

**Corollaire 6.1.** *Soit $x$ un réel d'orbite $\{f^i(x)\}_{i=0}^{\infty}$ divergente telle que*

$$\lim_{i\to\infty} |f^i(x)| = +\infty.$$

*Alors l'orbite de $x$ n'est pas u. d. mod 2.*

Ce résultat renforce la conjecture 2.2. En effet, on peut s'attendre à ce que la condition de distribution uniforme modulo 2 des itérations de $f$ soit le plus souvent valide au voisinage de $\pm\infty$, compte tenu des propriétés suivantes :
– le diamètre et la densité des zones contractantes tend vers 0,
– l'amplitude des oscillations devient infiniment grande.

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# Annexe

La figure 2 ci-dessous résume quelques-uns des principaux résultats de cet article sous la forme de liens logiques entre diverses conjectures.

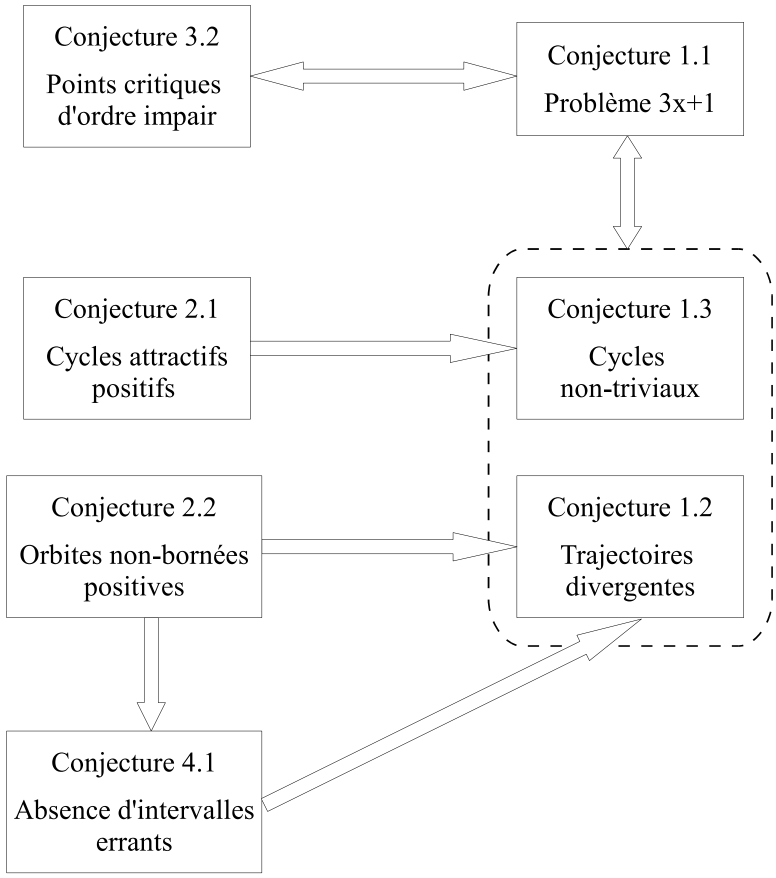


FIG. 2 – Liens logiques entre conjectures. La partie gauche concerne le cadre continu $\mathbb{R}^+$ et la partie droite le cadre discret $\mathbb{Z}^+$.

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# A multiobjective optimization model for optimal supplier selection in multiple sourcing environment

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**Abstract**

Supplier selection is an important concern of a firm's competitiveness, more so in the context of the imperative of supply-chain management. In this paper, we use an approach to a multiobjective supplier selection problem in which the emphasis is on building supplier portfolios. The supplier evaluation and order allocation is based upon the criteria of expected unit price, expected score of quality and expected score of delivery. A fuzzy approach is proposed that relies on nonlinear $S$-shape membership functions to generate different efficient supplier portfolios. Numerical experiments conducted on a data set of a multinational company are provided to demonstrate the applicability and efficiency of the proposed approach to real-world applications of supplier selection.

**Key words**: Multiobjective optimization, Fuzzy supplier selection, Nonlinear optimization, Membership functions.

**MSC 2010**: 90C30, 90C70.

## 1 Introduction

Supplier selection or vendor selection is a multi-criteria decision making (MCDM) problem. One of the well known studies on supplier selection by Dickson [10] discusses 23 important evaluation criteria for supplier selection. It has been pointed out that quality, delivery, and performance history are the three most important criteria. Other important studies that highlights the importance of evaluation criteria for supplier selection includes the works of Ghodsypour and O'Brien [13], Ho et al. [16], Weber et al. [35]. Many authors have discussed optimization models of supplier selection problem. Parthiban et al. [26] developed an integrated model based on 10 criteria including quality, delivery, productivity, service, costs for the supplier selection problem. Punniyamoorthy et al. [27] applied 10 criteria for supplier evaluation including quality, technical capability, financial position. Karpak et al. [19] used a goal programming model to minimize costs and maximize delivery reliability and quality in supplier selection when assigning order quantities to each supplier. Weber and Current [36] used multi-objective linear programming for supplier selection to systematically analyze the trade-off between conflicting factors. Recently, Feng et al. [12] proposed a multiobjective model to select desired suppliers and also developed a multiobjective algorithm based on Tabu search for solving it. Reviews of supplier selection criteria and methods can be found in studies carried out by Aissaoui et al. [1] and Chai et al. [8].

In real-world, for supplier selection problem, decision makers do not have exact and complete information related to various input parameters. In such cases the fuzzy set theory (FST) [38] is considered one of the best tools to handle uncertainty. The supplier selection formulations have benefited greatly from the FST in terms of integrating quantitative and qualitative information, subjective preferences and knowledge of the decision maker. A review of literature on applications of FST in supplier selection shows that a variety of approaches are being used. Kumar et al. [20] presented fuzzy goal programming models to capture uncertainty related to the supplier selection problem. Amid et al. [2, 3] developed a weighted additive fuzzy model for supplier selection problem. Bayrak et al. [6] presented a fuzzy multi-criteria group decision making approach to supplier selection based on fuzzy arithmetic operation. Chen et al. [9] extended the concept of TOPSIS method to develop a methodology for solving supplier selection problems in fuzzy environment. Erol et al. [11] and Li et al. [24] discussed the applications of FST in supplier selection. Kwang et al. [21] introduced a combined scoring method with fuzzy expert systems approach for determination of best supplier. Kahraman et al. [18] developed a fuzzy AHP model to select the best supplier firm providing the most satisfaction for the criteria determined. Shaw et al. [30] proposed an integrated approach that combines fuzzy AHP and fuzzy multiobjective linear programming for selecting the appropriate supplier. Toloo and Nalchigar [32] proposed a new integrated data envelopment analysis model which is able to identify most appropriate supplier in presence of both cardinal and ordinal data. Tsai and Hung [33] proposed a fuzzy goal programming approach that integrates activity-based costing and performance evaluation in a value-chain structure for optimal green supply chain supplier selection and flow allocation. Yücel and Güneri [37] developed a weighted additive fuzzy programming approach for multi-criteria supplier selection. Recently, Amid et al. [4] developed a weighted maxmin fuzzy model to handle effectively the vagueness of input data and different weights of criteria in a supplier selection problem. Arikan [5] proposed a fuzzy mathematical model and a novel solution approach to satisfy the decision maker's aspirations for fuzzy goals.

In all the studies mentioned thus far, supplier selection is driven by non-portfolio based approaches only. This type of framework is restrictive as it does not provide the decision maker with an opportunity to leverage the supplier diversity with reference to preferences in respect of cost, quality and delivery. Recently, Guu et al. [15] discussed supplier selection problem with interval coefficients using portfolio based approach. In this paper, we consider three supplier's selection criteria, namely, expected unit price, expected score of quality and expected score of delivery. The proposed fuzzy optimization model simultaneously minimize the expected unit cost and maximize the expected score of quality and expected score of delivery. The model is constrained by several realistic constraints, namely, demand constraint, maximal and minimal fraction of the total order allocation to a single supplier, number of suppliers held in the portfolio. Note that in comparison to the approach used in Guu et al. [15], the proposed approach is capable of generating many efficient supplier portfolios using different shape parameters of the nonlinear $S$-shape membership functions from which the decision maker may choose the one according to his/her preferences.

The paper is organized as follows. In Section 2, we present multiobjective programming model of supplier selection based on portfolio theory. In Section 3, we present fuzzy optimization models of supplier selection using nonlinear $S$-shape fuzzy membership functions. The proposed models are test-run in Section 4. This section also includes a discussion of the results obtained. Finally in Section 5, we submit our concluding observations.

# 2 The supplier selection problem

Here, we assume that the decision maker allocate orders among $n$ suppliers offering different price, quality and delivery. We use the following variables and parameters in the supplier selection model:

$x_i$: the proportion of total order allocated to $i$-th supplier,

$p_i$: the per unit net purchase price from $i$-th supplier,

$q_i$: the percentage of quality level of $i$-th supplier,

$d_i$: the percentage of on-time-delivery level of $i$-th supplier ,
$y_i$: the binary variable indicating whether the $i$-th supplier is contained in the supplier portfolio or not, i.e.,

$$y_i = \begin{cases} 1, & \text{if } i\text{-th supplier is contained in the supplier portfolio,} \\ 0, & \text{otherwise,} \end{cases}$$

$u_i$: the maximal fraction of the total order allocated to the $i$-th supplier ,
$l_i$: the minimal fraction of the total order allocated to the $i$-th supplier .

## 2.1 Objectives

• **Expected unit price**
The expected unit cost is the weighted average of the prices quoted by different suppliers, the fractions of the overall quantity ordered to them serving as the respective weights. Here, we consider the overall demand as 1 which overcomes the dependence of supplier selection problem on the units of measurement of the commodities [15].
The expected unit price of the supplier portfolio is expressed as

$$f_1(x) = \sum_{i=1}^{n} p_i x_i \,.$$

• **Expected score of quality**
Quality of the supplies is measured in terms of the extent of satisfaction (fraction) with quality. We use the expected score of quality which in effect is the average of the satisfaction of the established standards by different suppliers as an objective of supplier selection [15]. The expected score of quality of the supplier portfolio is expressed as

$$f_2(x) = \sum_{i=1}^{n} q_i x_i \,.$$

• **Expected score of delivery**
A supplier's compliance (fraction of 1) with on-time-delivery schedule is regarded as his/her score of delivery. Using the fraction of quantity allocated to different suppliers as weight [15], the expected score of delivery of the supplier portfolio is expressed as

$$f_3(x) = \sum_{i=1}^{n} d_i x_i \,.$$

## 2.2 Constraints

• Total order constraint on the suppliers:

$$\sum_{i=1}^{n} x_i = 1\,.$$

• Maximal fraction of the total order that can be allocated to a single supplier:

$$x_i \leq u_i y_i\,, \quad i = 1, 2, \ldots, n\,.$$

• Minimal fraction of the total order that can be allocated to a single supplier:

$$x_i \geq l_i y_i\,, \quad i = 1, 2, \ldots, n\,.$$

The constraints corresponding to lower bounds $l_i$ and upper bounds $u_i$ on the allocation to individual suppliers ($0 \leq l_i, u_i \leq 1$, $l_i \leq u_i\,, \forall i$) are included to avoid a large number of very small allocations (lower bounds) and at the same time to ensure a sufficient diversification of the allocation (upper bounds) [15].

• Number of suppliers held in a supplier portfolio:

$$\sum_{i=1}^{n} y_i = h$$

where $h$ is the number of suppliers that the decision maker chooses to include in the supplier portfolio [15]. Of all the suppliers from a given set, the decision maker would pick up the ones that are likely to yield the desired satisfaction of his/her preferences. It is not necessary that all the suppliers from a given set may configure in the supplier portfolio as well.

• No negative proportions of total orders:

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,.$$

### 2.3 The decision problem

The mixed-integer model for purchasing a single item in multiple sourcing networks is presented as follows:

$$
\begin{aligned}
\text{(P1)} \qquad & \min f_1(x) = \sum_{i=1}^{n} p_i x_i \\
& \max f_2(x) = \sum_{i=1}^{n} q_i x_i \\
& \max f_3(x) = \sum_{i=1}^{n} d_i x_i \\
& \text{subject to} \\
& \sum_{i=1}^{n} x_i = 1\,, && (1)\\
& \sum_{i=1}^{n} y_i = h\,, && (2)\\
& x_i \le u_i y_i\,, \quad i = 1, 2, \ldots, n\,, && (3)\\
& x_i \ge l_i y_i\,, \quad i = 1, 2, \ldots, n\,, && (4)\\
& x_i \ge 0\,, \quad i = 1, 2, \ldots, n\,, && (5)\\
& y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n\,. && (6)
\end{aligned}
$$

It may be noted that the basic framework of the supplier selection model (P1) is similar to the one used in [15]; however, instead of using interval coefficients for an uncertain environment as in [15], we rely on fuzzy membership functions to generate supplier selection strategies that meets the preferences of the decision maker.

## 3 Supplier portfolio selection models based on fuzzy set theory

Operationally, formulating an supplier portfolio requires estimation of distributions of price, quality and delivery for the various suppliers. Distributed randomly as they are over the chosen time horizon, such estimates, at best, represent decision maker's subjective interpretation of the information available at the time of decision making. Note that the same information may be interpreted differently by different decision makers. Under such circumstances, the issue of constructing a supplier portfolio becomes the one of a choice from a 'fuzzy' set of subjective interpretations, the term 'fuzzy' being suggestive of the diversity of both the decision maker's objective functions as well as that of the constraints.

Here, we formulate fuzzy multiobjective supplier portfolio selection problem based on vague aspiration levels of decision makers to determine a satisfying supplier portfolio selection strategy. We assume that decision makers indicate aspiration levels on the basis of their prior experience and knowledge. As the aspiration levels are vague, we may refer to the fuzzy membership functions, for example, linear [39, 40], piecewise linear [17], exponential [23], tangent [22]. A linear membership function is most commonly used because it is simple and it is defined by fixing two points: the upper and lower levels of acceptability. However, there are some difficulties in using linear membership functions as pointed out by Watada [34]. Further, if the membership function is interpreted as fuzzy utility of the decision maker, describing the behavior of indifference, preference or aversion towards uncertainty, then a nonlinear membership function provides a better representation. It may also be noted that nonlinear membership functions are much more desirable for real-world decision making, as unlike linear membership functions, for nonlinear membership functions, the marginal rate of increase (or decrease) of membership values as a function of model parameters is not constant-a technique that reflects reality better than the linear case.

In this paper, we use logistic function [34], i.e., a nonlinear $S$-shape membership function to express vague aspiration levels of decision makers. This function has several advantages over other nonlinear membership functions and is considered an appropriate choice in portfolio selection, see Gupta et al. [14].

We now define the following nonlinear $S$-shape membership function of the goal of net price:

- $$\mu_p(x) = \frac{1}{1 + \exp\left(\alpha_p \left(\sum_{i=1}^{n} p_i x_i - p_m\right)\right)},$$

where $p_m$ is the mid-point (middle aspiration level for the net price) at which the membership function value is 0.5 and $\alpha_p$ is provided by decision makers based on their degree of satisfaction of the goal (see Fig. 1).

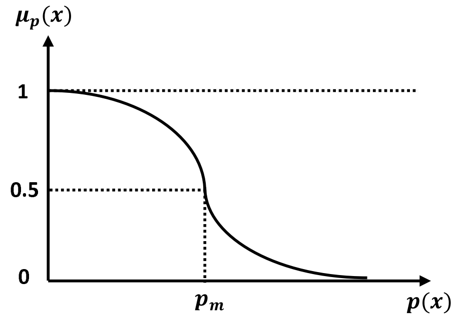


**Figure 1.** Membership function of the goal of net price

The membership function of the goal of quality is given by

- $$\mu_q(x) = \frac{1}{1 + \exp\left(-\alpha_q\left(\sum_{i=1}^{n} q_i x_i - q_m\right)\right)},$$

where $q_m$ is the mid-point and $\alpha_q$ is provided by decision makers based on their degree of satisfaction regarding the level of quality (see Fig. 2).

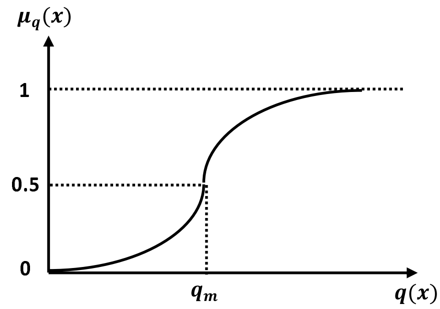


**Figure 2.** Membership function of the goal of quality

Similarly, we define membership functions of the goal of delivery as follows:

- $$\mu_d(x) = \frac{1}{1 + \exp\left(-\alpha_d\left(\sum_{i=1}^{n} d_i x_i - d_m\right)\right)},$$

where $d_m$ is the respective mid-point and $\alpha_d$ is provided by decision makers. Note that the membership function of the goal of delivery as described above, have shape similar to that of the membership function defining the goal of quality.

Using Bellman and Zadeh's maximization principle [7] with the above defined fuzzy membership functions, the fuzzy supplier portfolio selection problem for selecting suppliers is formulated as follows:

$$
\begin{aligned}
\text{(P2)} \qquad & \max\ \eta \\
& \text{subject to} \\
& \eta \leq \mu_p(x)\,, \\
& \eta \leq \mu_q(x)\,, \\
& \eta \leq \mu_d(x)\,, \\
& 0 \leq \eta \leq 1\,, \\
& \text{and Constraints (1)} - (6)\,.
\end{aligned}
$$

The problem (P2) is a nonlinear programming problem. It can be transformed into a linear programming problem by letting $\theta = \log \dfrac{\eta}{1-\eta}$, so that $\eta = \dfrac{1}{1+\exp(-\theta)}$. Since, the logistic function is monotonically increasing, hence, maximizing $\eta$ makes $\theta$ maximize. Therefore, the problem (P2) can be transformed into the following equivalent linear programming problem:

$$
\begin{aligned}
\text{(P3)} \qquad & \max \theta \\
& \text{subject to} \\
& \theta \leq \alpha_p \left( p_m - \sum_{i=1}^{n} p_i x_i \right)\,, \\
& \theta \leq \alpha_q \left( \sum_{i=1}^{n} q_i x_i - q_m \right)\,, \\
& \theta \leq \alpha_d \left( \sum_{i=1}^{n} d_i x_i - d_m \right)\,, \\
& \text{and Constraints (1)} - (6)\,.
\end{aligned}
$$

Note that $\theta \in ]-\infty, +\infty[$. The fuzzy supplier portfolio selection problem (P2)/(P3) leads to a fuzzy decision that simultaneously satisfies all the fuzzy objectives. Then, we determine the maximizing decision as the maximum degree of membership for the fuzzy decision. In this approach, the relationship between various objectives in a fuzzy environment is considered fully symmetric [40], i.e., all fuzzy objectives are treated equivalent. This approach is efficient in computation but it may provide 'uniform' membership degrees for all fuzzy objectives even when achievement of some objective(s) is more stringently required. Therefore, we use the 'weighted additive model' proposed in [31] to incorporate relative importance of various fuzzy objectives in supplier portfolio selection. The weighted additive model of the fuzzy supplier portfolio selection problem is formulated as follows:

$$
\begin{aligned}
\text{(P4)} \qquad & \max \sum_{r=1}^{3} \omega_r \eta_r \\
& \text{subject to} \\
& \eta_1 \leq \mu_p(x)\,, \\
& \eta_2 \leq \mu_q(x)\,, \\
& \eta_3 \leq \mu_d(x)\,, \\
& 0 \leq \eta_r \leq 1\,, \qquad r = 1, 2, 3 \\
& \text{and Constraints } (1) - (6)\,,
\end{aligned}
$$

where $\omega_r$ is the relative weight of the $r$-th objective given by decision makers such that $\omega_r > 0$ and $\sum_{r=1}^{3} \omega_r = 1$.

The max-min approach used in the formulation of the problems (P2)/(P3) and (P4) possesses good computational properties. However, the approach does not ensure fuzzy-efficient solution. To ensure efficiency of the solution, we take recourse to the two-phase approach proposed in [25]. As a result, it becomes possible to choose explicitly a minimum degree of satisfaction (taken to be equal to the solution of the max-min approach) for each fuzzy objective function and examine whether the same can be improved upon or not. Hence, we solve the problems (P5) and (P6) corresponding to the problems (P3) and (P4) respectively in the second-phase.

$$
\begin{aligned}
\text{(P5)} \qquad & \max \sum_{r=1}^{3} \omega_r \theta_r \\
& \text{subject to} \\
& \log \frac{\mu_p(x^*)}{1 - \mu_p(x^*)} \leq \theta_1 \leq \alpha_p \left( p_m - \sum_{i=1}^{n} p_i x_i \right)\,, \\
& \log \frac{\mu_q(x^*)}{1 - \mu_q(x^*)} \leq \theta_2 \leq \alpha_q \left( \sum_{i=1}^{n} q_i x_i - q_m \right)\,, \\
& \log \frac{\mu_d(x^*)}{1 - \mu_d(x^*)} \leq \theta_3 \leq \alpha_s \left( \sum_{i=1}^{n} d_i x_i - d_m \right)\,, \\
& \text{and Constraints } (1) - (6)\,,
\end{aligned}
$$

where $x^*$ is an optimal solution of (P3), $\omega_1 = \omega_2 = \omega_3$, $\omega_r > 0$, $\sum_{r=1}^{3} \omega_r = 1$ and $\theta_r \in ]-\infty, +\infty[$ $r = 1, 2, 3$.

$$
\begin{aligned}
\text{(P6)} \qquad & \max \sum_{r=1}^{3} \omega_r \eta_r \\
& \text{subject to} \\
& \mu_p(x^{**}) \leq \eta_1 \leq \mu_p(x)\,, \\
& \mu_q(x^{**}) \leq \eta_2 \leq \mu_q(x)\,, \\
& \mu_d(x^{**}) \leq \eta_3 \leq \mu_d(x)\,, \\
& 0 \leq \eta_r \leq 1\,, \qquad r = 1, 2, 3 \\
& \text{and Constraints } (1) - (6)\,,
\end{aligned}
$$

where $x^{**}$ is an optimal solution of (P4), $\omega_r$ is the relative weight of the $r$-th objective given by decision makers such that $\omega_r > 0$ and $\sum_{r=1}^{3} \omega_r = 1$.

The problems (P3) and (P5) are linear programming problems which can be solved using the LINDO software [28]. The problems (P4) and (P6) are nonlinear programming problems. Although, for medium or large-sized problems, one may suspect that solving these nonlinear programming problems could be computationally difficult, this is not the case, as many excellent softwares are available to solve them. We can use LINGO [29] to solve (P4) and (P6).

# 4 Numerical illustration

In this section, we present an illustration of the developed supplier portfolio selection decision procedure for a multinational company. The purchasing manager of the company have identified 10 potential suppliers. The manager will select the most favorable suppliers(s) and allocate various proportion of total order among selected suppliers(s) such that to minimize the net price of purchasing and to maximize total quality and delivery level of purchased items.

## 4.1 Supplier allocation

The 10 suppliers form the population from which we attempt to construct a supplier portfolio comprising 5 suppliers. The suppliers profiles shown in Table 1 represents the estimated values of their net price ($p_i$), quality level ($q_i$) and delivery level ($d_i$) along with the estimated values of lower and upper bounds.

Table 1 Input data of suppliers

| | Price (Rs.) | Quality (%) | Delivery (%) | Lower bound ($l_i$) | Upper bound ($u_i$) |
|---|---|---|---|---|---|
| Supplier 1 | 13 | 0.82 | 0.80 | 0.03 | 0.22 |
| Supplier 2 | 12.5 | 0.78 | 0.75 | 0.06 | 0.33 |
| Supplier 3 | 11.5 | 0.70 | 0.80 | 0.03 | 0.20 |
| Supplier 4 | 14 | 0.88 | 0.90 | 0.027 | 0.22 |
| Supplier 5 | 15 | 0.84 | 0.92 | 0.2 | 1.17 |
| Supplier 6 | 16 | 0.95 | 0.88 | 0.06 | 0.27 |
| Supplier 7 | 14.5 | 0.80 | 0.78 | 0.05 | 0.4 |
| Supplier 8 | 15.5 | 0.92 | 0.84 | 0.017 | 0.17 |
| Supplier 9 | 13.5 | 0.85 | 0.85 | 0.03 | 0.25 |
| Supplier 10 | 12 | 0.75 | 0.78 | 0.06 | 0.30 |

We now present the computational results.
Corresponding to $p_m = 13.3$, $q_m = 0.83$ and $d_m = 0.82$, we obtain supplier portfolio selection strategy by solving the problem (P3). To check efficiency of the solution obtained, we use the two-phase approach and solve the problem (P5). If the purchasing manager is not satisfied with the supplier portfolio obtained, more supplier portfolios can be generated by varying the values of the shape parameters in the problem (P3). The computational results summarized in Table 2 are based on three different sets of values of the shape parameters. Note that all the three solutions obtained are efficient, i.e., their criteria vector are nondominated. Table 3 presents proportions of the total order allocated to suppliers in obtained supplier portfolios

Table 2 Summary results of supplier portfolio selection

| Shape parameters & variables | | | | | Net price | Quality level | Delivery level |
|---|---|---|---|---|---|---|---|
| $\eta$ | $\theta$ | $\alpha_p$ | $\alpha_q$ | $\alpha_d$ | | | |
| 0.85900 | 1.80700 | 200 | 600 | 600 | 13.29095 | 0.83301 | 0.84703 |
| 0.58128 | 0.32803 | 100 | 100 | 100 | 13.29671 | 0.83328 | 0.84720 |
| 0.52087 | 0.08353 | 6 | 30 | 30 | 13.28609 | 0.83278 | 0.84688 |

Table 3 The proportions of the total order allocated to suppliers in obtained supplier portfolios

| Shape parameters | | | Suppliers | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_p$ | $\alpha_q$ | $\alpha_d$ | S1 | S2 | S3 | S4 | S5 | S6 | S7 | S8 | S9 | S10 |
| 200 | 600 | 600 | 0.22 | 0.27635 | 0 | 0.22 | 0 | 0 | 0 | 0.03365 | 0.25 | 0 |
| 100 | 100 | 100 | 0.22 | 0.27443 | 0 | 0.22 | 0 | 0 | 0 | 0.03557 | 0.25 | 0 |
| 6 | 30 | 30 | 0.22 | 0.27797 | 0 | 0.22 | 0 | 0 | 0 | 0.03203 | 0.25 | 0 |

Next, we present computational results considering preferences of the purchasing manager for the three objectives.

• **Case 1**
We consider the following weights of the fuzzy goals of expected net price ($\omega_1$), expected quality level ($\omega_2$) and expected delivery level ($\omega_3$): $\omega_1 = 0.6$, $\omega_2 = 0.25$, $\omega_3 = 0.15$. Corresponding to $p_m = 13.3$, $q_m = 0.81$ and $d_m = 0.88$, we obtain supplier portfolio selection strategy by solving the problem (P4). The efficiency of the solution is verified by solving the problem (P6) in the second phase. The corresponding computational results are listed in Tables 4-5. The achievement levels of the various membership functions are $\eta_1 = 0.95744$, $\eta_2 = 0.41261$, $\eta_3 = 0.31576$. Note that these achievement levels are consistent with the purchasing manager preferences, i.e., ($\eta_1 > \eta_2 > \eta_3$) agrees with ($\omega_1 > \omega_2 > \omega_3$).

• **Case 2**
Here, we consider the weights as $\omega_1 = 0.15$, $\omega_2 = 0.6$, $\omega_3 = 0.25$. By taking $p_m = 13.3$, $q_m = 0.81$ and $d_m = 0.88$, we obtain supplier portfolio selection strategy by solving the problem (P4). The solution is verified for efficiency. The corresponding computational results are listed in Tables 4-5. The achievement levels of the various membership functions are $\eta_1 = 0.00023$, $\eta_2 = 0.90362$, $\eta_3 = 0.70285$ which are consistent with the purchasing manager preferences.

• **Case 3**
As performed above in case 1 and case 2, corresponding to the weights $\omega_1 = 0.15$, $\omega_2 = 0.2$, $\omega_3 = 0.65$ and $p_m = 13.3$, $q_m = 0.81$, $d_m = 0.88$, we obtain portfolio selection strategy by solving the problem (P4). The solution is found to be efficient. The corresponding computational results are listed in Tables 4-5. The achievement levels of the various membership functions are $\eta_1 = 0.00028$, $\eta_2 = 0.77664$, $\eta_3 = 0.78516$ which are consistent with the purchasing manager preferences.

Table 4 Summary results of supplier portfolio selection incorporating purchasing manager preferences

| Case | Shape parameters | | | Price | Quality level | Delivery level |
|---|---|---|---|---|---|---|
| | $\alpha_p$ | $\alpha_q$ | $\alpha_d$ | | | |
| Case 1 | 6 | 30 | 30 | 12.78110 | 0.79823 | 0.85422 |
| Case 2 | 6 | 30 | 30 | 14.69752 | 0.88460 | 0.90870 |
| Case 3 | 6 | 30 | 30 | 14.66650 | 0.85154 | 0.92320 |

Table 5 The proportions of the total order allocated to suppliers in obtained supplier portfolios incorporating purchasing manager preferences

| Class | Suppliers | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | S1 | S2 | S3 | S4 | S5 | S6 | S7 | S8 | S9 | S10 |
| Class 1 | 0.0661 | 0 | 0.2 | 0.22 | 0 | 0 | 0 | 0 | 0.25 | 0.2639 |
| Class 2 | 0 | 0 | 0 | 0.22 | 0.21496 | 0.27 | 0 | 0.04504 | 0.25 | 0 |
| Class 3 | 0 | 0 | 0 | 0.027 | 0.646 | 0.06 | 0 | 0.017 | 0.25 | 0 |

The foregoing analysis of the various decision situations from the stand point of decision makers preferences demonstrates that the supplier portfolio selection models developed in this paper discriminate among decision makers. Thus, it is possible to construct efficient portfolios with reference to the diversity of decision maker preferences.

# 5 Conclusions

This paper proposed a flexible approach to multiobjective supplier selection problems. We used the criteria of expected unit price, expected score of quality and expected score of delivery for supplier evaluation and order allocation. Further, the benefits of supplier diversification using trade-offs among the three chosen criteria have been achieved. The upper bounds and lower bounds are used for fractions of order that may be assigned to a particular supplier in order to ensure supplier diversification as well as to avoid the situations where very small fractions of the ordered quantity are obtained. Recognizing that supplier selection involves MCDM in an environment that befits more fuzzy approximation than deterministic formulation, we have transformed the supplier portfolio selection model into a fuzzy model using nonlinear $S$-shape fuzzy membership functions. Numerical illustrations based on 10-supplier universe have been presented to illustrate the effectiveness of the proposed models. The efficiency of the obtained solutions was verified using the two-phase approach.

The main advantage of the proposed models is that if decision maker is not satisfied with any of the supplier portfolios, more portfolios can be generated by varying the values of the shape parameters. These parameters may be configured to suit decision makers preferences. Thus, the fuzzy supplier portfolio selection models proposed in this paper can provide satisfying portfolio selection strategies according to vague aspiration levels, degrees of satisfaction and relative importance of the various objectives.

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# The Lie-Santilli admissible hyperalgebras of type $A_n$

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**Abstract**

The largest class of hyperstructures is the one which satisfy the weak properties. These are called $H_v$-structures introduced in 1990 and they proved to have a lot of applications on several applied sciences. In this paper we present a construction of the hyperstructures used in the Lie-Santilli admissible theory on square matrices.

**Key words**: hyperstructures, $H_v$-structures, hopes, weak hopes, $\partial$-hopes, e-hyperstructures, admissible Lie-algebras.

**MSC 2010**: 20N20, 17B67, 17B70, 17D25.

## 1 Introduction

We deal with hyperstructures called $H_v$*-structures* introduced in 1990 [30], which satisfy the weak axioms where the non-empty intersection replaces the equality.

Some basic definitions are the following:

In a set H equipped with a hyperoperation (abbreviation *hyperoperation = hope)*

$$\cdot : H \times H \to P(H) - \{\varnothing\},$$

we abbreviate by

*WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by

*COW* the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called an $H_v$*-semigroup* if it is *WASS*, it is called $H_v$*-group* if it is reproductive $H_v$-semigroup, i.e.,

$$xH = Hx = H, \forall x \in H.$$

The hyperstructure $(R, +, \cdot)$ is called an $H_v$*-ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy + xz) \neq \varnothing, \ (x+y)z \cap (xz + yz) \neq \varnothing, \ \forall x, y, z \in R.$$

**Motivations.** The motivation for $H_v$-structures is the following: We know that the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an $H_v$-group. This is the motivation to introduce the $H_v$-structures [24].

In an $H_v$-semigroup the powers of an element $h \in H$ are defined as follows:

$$h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h,$$

where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An $H_v$-semigroup $(H, \cdot)$ is called *cyclic of period s*, if there exists an element h, called *generator*, and a natural number s, the minimum one, such that

$$H = h^1 \cup h^2 ... \cup h^s.$$

Analogously the cyclicity for the infinite period is defined [23]. If there is an element h and a natural number s, the minimum one, such that $H = h^s$, then $(H, \cdot)$ is called *single-power cyclic of period s.*

For more definitions and applications on $H_v$-structures, see the books [2],[8],[24],[4],[1] and papers as [3],[28],[21],[22],[26],[9],[14],[13].

The main tool to study hyperstructures are the *fundamental relations* $\beta^*$, $\gamma^*$ and $\epsilon^*$, which are defined in $H_v$-groups, $H_v$-rings and $H_v$-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space, resp. These relations were introduced by T. Vougiouklis [30],[24],[29]. A way to find the fundamental classes is given by theorems as the following [24],[21],[25],[22],[7],[9],[20]:

**Theorem 1.1.** *Let* $(\boldsymbol{H}, \cdot)$ *be an* $H_v$*-group and denote by* $\boldsymbol{U}$ *the set of all finite products of elements of H. We define the relation* $\beta$ *in H by setting* $x\beta y$ *iff* $\{x, y\} \subset \boldsymbol{u}$ *where* $\boldsymbol{u} \in \boldsymbol{U}$*. Then* $\beta^*$ *is the transitive closure of* $\beta$*.*

Analogous theorems for the relations $\gamma^*$ in $H_v$-rings, $\epsilon^*$ in $H_v$-modules and $H_v$-vector spaces, are also proved. An element is called *single* if its fundamental class is singleton [24].

Fundamental relations are used for general definitions. Thus, an $H_v$-ring $(R, +, \cdot)$ is called $H_v$*-field* if $R/\gamma^*$ is a field.

Let $(H, \cdot), (H, *)$ be $H_v$-semigroups defined on the same set H. The hope $(\cdot)$ is called *smaller* than the hope $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in Aut(H, *) \text{ such that } xy \subset f(x * y),\ \forall x, y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H, *)$ contains $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, *)$ is called $H_b$-structure and $(*)$ is called *b-hope*.

**Theorem 1.2.** *(The Little Theorem). Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

**Definition 1.1.** *[20],[25] Let $(H, \cdot)$ be hypergroupoid. We remove $h \in H$, if we consider the restriction of $(\cdot)$ in the set $H - \{h\}$. $\underline{h} \in H$ absorbs $h \in H$ if we replace $h$ by $\underline{h}$ and $h$ does not appear in the structure. $\underline{h} \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of $x$ with both $h$, $\underline{h}$, and consider $h$ and $\underline{h}$ as one class with representative $\underline{h}$, therefore, $h$ does not appear in the hyperstructure.*

For several definitions and applications of hyperstructures in mathematics or in sciences and social sciences one can see [11],[15],[13],[3].

# 2 The theta ($\partial$) hopes

In [19],[32],[11],[15] a hope, in a groupoid with a map f on it, denoted $\partial_f$, is introduced. Since there is no confusion, we write simply *theta* $\partial$. The symbol "$\partial$" appears in Greek papyrus to represent the letter "theta "usually in middle rather than the beginning of the words.

**Definition 2.1.** *Let H be a set equipped with n operations (or hopes) $\otimes_1, ..., \otimes_n$ and a map (or multivalued map) $f : H \to H$ (or $f : H \to P(H) - \emptyset$, respectively), then n hopes $\partial_1, \partial_2, ..., \partial_n$ on H can be defined, called theta-operations (we rename here theta-hopes and we write $\partial$-hope) by putting*

$$x\partial_i y = \{f(x) \otimes_i y, x \otimes_i f(y)\}, \forall x, y \in H \text{ and } i \in \{1, 2, ..., n\}$$

*or, in case where $\otimes_i$ is hope or $f$ is multivalued map, we have*

$$x\partial_i y = (f(x) \otimes_i y) \cup (x \otimes_i f(y)), \forall x, y \in H \textit{ and } i \in \{1, 2, ..., n\}$$

*if $\otimes_i$ is associative then $\partial_i$ is WASS.*

Analogously one can use several maps f, instead than only one.

Let $(G, \cdot)$ be a groupoid and $f_i : G \to G, i \in I$, be a set of maps on G. Take the map $f_\cup : G \to \mathbf{P}(G)$ such that $f_\cup(x) = \{f_i(x)|i \in I\}$ and we call it *the union* of the $f_i(x)$. We call *union $\partial$-hopes*, on G if we consider the map $f_\cup(x)$. A special case is to take the union of f with the identity, i.e. $\underline{f} = f \cup (id)$, so $\underline{f}(x) = \{x, f(x)\}, \forall x \in G$, which is called *b-$\partial$-hope*. We denote the *b-$\partial$-hope* by $(\underline{\partial})$, so

$$x\underline{\partial}y = \{xy, f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G$$

This hope contains the operation $(\cdot)$ so it is a b-hope. If $f : G \to P(G) - \{\emptyset\}$, then the b-$\partial$-hope is defined by using the map $\underline{f}(x) = \{x\} \cup f(x), \forall x \in G$.

**Motivation** for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions s(x), t(x), we have $s\partial t = \{s't, st'\}$ where $(')$ denotes the derivative.

For several results one can see [19],[32].

**Examples.** (a) Taking the application on the derivative, consider all polynomials of up to first degree $g_i(x) = a_i x + b_i$. We have

$$g_1 \partial g_2 = \{a_1 a_2 x + a_1 b_2, a_1 a_2 x + b_1 a_2\},$$

so this is a hope in the first degree polynomials. Remark that all polynomials x+c, where c be a constant, are units.

(b) *The constant map.* Let $(G, \cdot)$ be group and $f(x) = a$, thus $x\partial y = \{ay, xa\}, \forall x, y \in G$. If $f(x) = e$, then we obtain $x\partial y = \{x, y\}$, the smallest incidence hope.

**Properties.** If $(G, \cdot)$ is a semigroup then:

(a) For every f, the $\partial$-hope is WASS.

(b) For every f, the b-$\partial$-hope $(\underline{\partial})$ is WASS.

(c) If f is homomorphism and projection, then $(\partial)$ is associative.

**Properties.**

*Reproductivity.* If $(\cdot)$ is reproductive then $(\partial)$ is also reproductive.

*Commutativity.* If $(\cdot)$ is commutative then $(\partial)$ is commutative. If f is into the centre of G, then $(\partial)$ is commutative. If $(\cdot)$ is COW then, $(\partial)$ is COW.

*Unit elements.* The elements of the kernel of f, are the units of (G,$\partial$).

*Inverse elements.* For given x, the elements $x' = (f(x))^{-1}u$ and $x' = u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

**Proposition.** Let $(G, \cdot)$ be a group then, for all maps $f : G \to G$, the hyperstructure $(G, \partial)$ is an $H_v$-group.

**Definition 2.2.** *Let $(R, +, \cdot)$ be a ring and $f : R \to R$, $g : R \to R$ be two maps. We define two hopes $(\partial_+)$ and $(\partial_-)$, called both theta-hopes, on R as follows*

$$x\partial_+ y = \{f(x) + y, x + f(y)\} \text{ and } x\partial.y = \{g(x) \cdot y, x \cdot g(y)\}, \forall x, y \in G.$$

*A hyperstructure $(R, +, \cdot)$, where $(+)$, $(\cdot)$ are hopes which satisfy all $H_v$-ring axioms, except the weak distributivity, will be called $H_v$-near-ring.*

**Propositions.**

(a) Let $(R, +, \cdot)$ be a ring and $f : R \to R$, $g : R \to R$ be maps. The $(R, \partial)_+, \partial.)$, called *theta*, is an $H_v$-near-ring. Moreover $(\partial_+)$ is commutative.

(b) Let $(R, +, \cdot)$ be a ring and $f : R \to R$, $g : R \to R$ maps, then $(R, \underline{\partial}_+, \partial.)$, is an $H_v$-ring.

**Properties.** *(Special classes).* The theta hyperstructure $(R, \partial_+, \partial.)$ takes a new form and has some properties in several cases as the following ones:

(a) If f is a homomorphism and projection, then

$$x\partial.(y\partial_+ z) \cap (x\partial.y)\partial_+(x\partial.z) = \{f(x)f(y)+f(x)z, f(x)y+f(x)f(z)\} \neq \emptyset.$$

Therefore, $(R, \partial)_+, \partial.)$ is an $H_v$-ring.

(b) If $f(x) = x, \forall x \in R$, then $(R, +, \partial.)$ becomes a multiplicative $H_v$-ring:

$$x\partial.(y + z) \cap (x\partial.y) + (x\partial.z) = \{g(x)y + g(x)z\} \neq \emptyset.$$

If, moreover, f is a homomorphism, then we have a "more" strong distributivity:

$$x\partial.(y + z) \cap ((x\partial.y) + (x\partial.z)) = \{g(x)y + g(x)z, xg(y) + xg(z)\} \neq \emptyset.$$

Now we can see theta hopes in $H_v$-vector spaces and $H_v$-Lie algebras:

**Theorem 2.1.** *Let $(V, +, \cdot)$ be an algebra over the field $(F, +, \cdot)$ and $f : V \to V$ be a map. Consider the $\partial$-hope defined only on the multiplication of the vectors $(\cdot)$, then $(V, +, \partial)$ is an $H_v$-algebra over $F$, where the related properties are weak. If, moreover $f$ is linear then we have*

$$\lambda(x\partial y) = (\lambda x)\partial y = x\partial(\lambda y).$$

Another well known and large class of hopes is given as follows [23],[24]:

Let $(G, \cdot)$ be a groupoid then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-*hopes*: for all $x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$,$(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P-*hyperstructures*. The most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup but we do not know about $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$. In some cases, depending on the choice of P, the $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes, introduced by Davvaz, Santilli, Vougiouklis in [7],[6] is the following:

**Construction 2.1.** *Let $(G, \cdot)$ be an abelian group and $P$ any subset of $G$ with more than one elements. We define the hope $\times P$ as follows:*

$$x \times_p y = \begin{cases} x \times_P y = x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ and } y = e \end{cases}$$

*we call this hope $P_e$-hope. The hyperstructure $(G, \times_p)$ is an abelian $H_v$-group.*

**Matrix Representations**

$H_v$-structures are used in Representation Theory of $H_v$-groups which can be achieved either by generalized permutations or by $H_v$-matrices [28],[24]. Representations by generalized permutations can be faced by translations. In this theory the single elements are playing a crucial role. $H_v$-matrix is called a matrix if has entries from an $H_v$-ring. The hyperproduct of $H_v$-matrices is defined in a usual manner. In representations of $H_v$-groups by $H_v$-matrices, there are two difficulties: To find an $H_v$-ring and an appropriate set of $H_v$-matrices.

Most of $H_v$-structures are used in Representation (abbreviate by *rep*) Theory. Reps of $H_v$-groups can be considered either by generalized permutations or by $H_v$-matrices [24]. Reps by generalized permutations can be achieved by using translations. In the rep theory the singles are playing a crucial role.

The rep problem by $H_v$-matrices is the following:

$H_v$-matrix is called a matrix if has entries from an $H_v$-ring. The hyperproduct of $H_v$-matrices **A**$= (a_{ij})$ and **B**$= (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ $H_v$-matrices, defined in a usual manner:

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) | (c_{ij}) \in \oplus \sum a_{ik} \cdot b_{kj}\},$$

where $(\oplus)$ denotes the *n-ary circle hope* on the hyperaddition.

**Definition 2.3.** *Let $(H, \cdot)$ be an $H_v$-group,$(R, +, \cdot)$ be an $H_v$-ring $R$ and consider a set $M_R = \{(a_{ij}) | a_{ij} \in R\}$ then any map*

$$T : H \to M_R : h \mapsto T(h) \text{ with } T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

*is called $H_v$-matrix rep. If $T(h_1h_2) \subset T(h_1)T(h_2)$, then $\boldsymbol{T}$ is an inclusion rep, if $T(h_1h_2) = T(h_1)T(h_2)$, then $\boldsymbol{T}$ is a good rep.*

# 3 The general $H_v$-Lie Algebra

**Definition 3.1.** *Let $(F, +, \cdot)$ be an $H_v$-field, $(V, +)$ be a COW $H_v$-group and there exists an external hope*

$$\cdot : F \times V \to P(V) - \{\emptyset\} : (a, x) \to zx$$

*such that, for all a,b in F and x,y in V we have*

$$a(x + y) \cap (ax + ay) \neq \emptyset, (a + b)x \cap (ax + bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset,$$

*then V is called an $H_v$-vector space over F. In the case of an $H_v$-ring instead of an $H_v$-field then the $H_v$-modulo is defined. In these cases the fundamental relation $\epsilon^*$ is the smallest equivalence relation such that the quotient $V/\epsilon^*$ is a vector space over the fundamental field $F/\gamma^*$.*

The general definition of an $H_v$-Lie algebra was given in [31] as follows:

**Definition 3.2.** *Let $(L, +)$ be an $H_v$-vector space over the $H_v$-field $(F, +, \cdot)$, $\phi : F \to F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field $F/\gamma$. Similarly, let $\omega_L$ be the core of the* *canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of $L/\epsilon^*$. Consider the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x, y) \to [x, y]$$

*then **L** is an $H_v$-Lie algebra over F if the following axioms are satisfied:*

*(L1) The bracket hope is bilinear, i.e.* $\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \varnothing$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \varnothing,$$

*(L2)* $[x, x] \cap \omega_L \neq \varnothing, \quad \forall x \in L$

*(L3)* $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \quad \forall x, y \in L$

**Definition 3.3.** *Let $(\mathbf{A}, +, \cdot)$ be an algebra over the field F. Take any map $f : A \to A$, then the $\partial$-hope on the Lie bracket $[x, y] = xy - yx$, is defined as follows*

$$x \partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

*Remark that if we take the identity map $f(x) = x, \forall x \in A$, then $x \partial y = \{xy - yx\}$, thus we have not a hope and remains the same operation.*

**Proposition.** Let $(A, +, \cdot)$ be an algebra F and $f : A \to A$ be a linear map. Consider the $\partial - hope$ defined only on the multiplication of the vectors $(\cdot)$, then $(A, +, \cdot)$ is an $H_v$-algebra over F, with respect to the $\partial$-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

**Proposition.** Let $(A, +, \cdot)$ be an algebra and $f : A \to A : f(x) = a$ be a constant map. Consider the $\partial$-hope defined only on the multiplication of the vectors $(\cdot)$, then $(A, +, \partial)$ is an $H_v$-Lie algebra over F.

In the above theorem if one take a=e, the unit element of the multiplication, then the properties become more strong.

# 4 Santilli's admissibility

The Lie-Santilli isotopies born to solve Hadronic Mechanics problems. Santilli proposed [16] a "lifting" of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit.

The isofields needed correspond to $H_v$-structures called e-hyperfields which are used in physics or biology. Definition: Let $(H_o, +, \cdot)$ be the attached $H_v$-field of the $H_v$-semigroup $(H, \cdot)$. If $(H, \cdot)$ has a left and right scalar unit e then $(H_o, +, \cdot)$ is e-hyperfield, the attached *$H_v$-field* of $(H, \cdot)$.

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a "lifting" of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 [5],[17] and they are called *e-hyperfields.* The $H_v$-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology. We present in the following the main definitions and results restricted in the $H_v$-structures.

**Definition 4.1.** *A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse $x^{-1}$, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. Remark that the inverses are not necessarily unique.*

**Definition 4.2.** *A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following axioms are valid:*

1. *$(F, +)$ is an abelian group with the additive unit 0,*
2. *$(\cdot)$ is WASS,*
3. *$(\cdot)$ is weak distributive with respect to $(+)$,*
4. *0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$,*
5. *exist a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$,*
6. *for every $x \in F$ there exists a unique inverse $x^{-1}$, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield*.

Now we present a general construction which is based on the partial ordering of the $H_v$-structures and on the Little Theorem.

**Definition 4.3.** *The Main e-Construction. Given a group* $(G, \cdot)$*, where e is the unit, then we define in G, a large number of hopes* ($\otimes$) *as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

$g_1, g_2$,... *are not necessarily the same for each pair (x,y). Then* $(G, \otimes)$ *becomes an* $H_v$*-group, actually is an* $H_b$*-group which contains the* $(G, \cdot)$*. The* $H_v$*-group* $(G, \otimes)$ *is an e-hypergroup. Moreover, if for each x,y such that* $xy = e$*, so we have* $x \otimes y = xy$*, then* $(G, \otimes)$ *becomes a strong e-hypergroup*

The proof is immediate since we enlarge the results of the group by putting elements from G and applying the Little Theorem. Moreover one can see that the unit e is a unique scalar and for each x in G, there exists a unique inverse $x^{-1}$, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$. So the hyperstructure $(G, \otimes)$ is a strong e-hypergroup.

# 5 Mathematical Realisation of type $A_n$

The representation theory by matrices gives to researchers a flexible tool to see and handle algebraic structures. This is the reason to see Lie-Santilli's admissibility using matrices or hypermatrices to study the multivalued (hyper) case. Using the well known P-hyperoperations we extend the Lie-Santilli's admissibility into the hyperstructure case. We present the problem and we give the basic definitions on the topic which cover the four following cases:

**Construction 5.1.** *[18] Suppose R, S be sets of square matrices (or hypermatrices). We can define the hyper-Lie bracket in one of the following ways:*

1. $[x, y]_{RS} = xRy - ySx$ *(General Case)*
2. $[x, y]_R = xRy - yx$
3. $[x, y]_S = xy - ySx$
4. $[x, y]_{RR} = xRy - yRx$

*The question is when the conditions, for all square matrices (or hypermatrices) x, y, z,*

$$[x, x]_{RS} \ni 0$$

$$[x, [y, z]_{RS}]_{RS} + [y, [z, x]_{RS}]_{RS} + [z, [x, y]_{RS}]_{RS} \ni 0$$

*of a hyper-Lie algebra are satisfied [18].*
*We apply this generalization on the Lie algebras of the type* $A_n$*.*

We deal with Lie-Algebra of type $A_n$, of traceless matrices M (Tr($M$)=0), which is a graded algebra, using the principal realisation used in Infinite Dimensional Kac Moody Lie Algebras introduced in 1981[10] by Lepowsky and Wilson, Kac [12]. In this special algebra examples on the above described hyperstructure theory are being presented.

Denote as

$$E_{ij}(i, j = 1, ..., n)$$

the $n \times n$ matrix which is 1 in the ij-entry and 0 everywhere else and by

$$e_i = E_{ii} - E_{i+1,i+1}, i = 1, ..., n-1$$

The Simple base of the above type is the following:

Base of Level 0 : $e_i$, $i = 1, 2, ..., n-1$

Base of Level 1 : $E_{i,i+1}$, $i = 1, 2, ..., n$

Base of Level 2 : $E_{i,i+2}$, $i = 1, 2, ..., n$

...

Base of Level n-1 : $E_{i,i+(n-1)}$, $i = 1, 2, ..., n$

Denote that all the subscripts are mod n.

Therefore the levels are in bold as follows:

**Level 0 :**

$$\begin{pmatrix} \boldsymbol{a_{11}} & 0 & 0 & \dots & 0 \\ 0 & \boldsymbol{a_{22}} & 0 & \dots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \boldsymbol{a_{nn}} \end{pmatrix}$$

**Level 1 :**

$$\begin{pmatrix} 0 & \boldsymbol{a_{12}} & 0 & \dots & 0 \\ 0 & 0 & \boldsymbol{a_{23}} & \dots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & 0 & \dots & \boldsymbol{a_{n-1,n}} \\ \boldsymbol{a_{n1}} & 0 & 0 & \dots & 0 \end{pmatrix}$$

**Level 2 :**

$$\begin{pmatrix} 0 & 0 & \mathbf{a_{13}} & \dots & 0 \\ 0 & 0 & 0 & \dots & \mathbf{a_{2n}} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{a_{n-1,1}} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{a_{n2}} & 0 & \dots & 0 \end{pmatrix}$$

................................................

**Level n-1 :**

$$\begin{pmatrix} 0 & 0 & 0 & \dots & \mathbf{a_{1n}} \\ \mathbf{a_{21}} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \mathbf{a_{n,n-1}} & 0 \end{pmatrix}$$

For our examples the Konstant's Cyclic Element E is being used as the sum of First Level's Simple Base [10].

$$E = E_{12} + E_{23} + E_{34} + ... + E_{n-1,n} + E_{n1}$$

This element is shifting every element of level $L$ to the next level $L+1$ [10],[27]. The base of the first level as well as for every level, except zero, has n elements. Level 0 has a $n-1$ dimension because of the limitation of the zero trace. The cyclic element gets different element from the base and goes to different ellement of the next level, creating an 1-1 correspondance. The element E shifts level $n-1$ to the Level 0 and because, as already remarked, Level 0 has $n-1$ elements, contrary with every other level, the 1-1 correspondance is being corrupted.

To summarize, according to the related theory, removing from every level (except Level-0), all the powers of E until $n-1$ $(E, E^2, ..., E^{n-1})$, an one to one complete correspondance between all levels, Level-0 included, is being created.

We denote the first power :

$$[E, E_{n1}]^1 = E \cdot E_{n1} - E_{n1} \cdot E = A_1$$

the second power:

$$[E, E_{n1}]^2 = [E, A_1] = A_2$$

..............................

and inductively by the n-power:

$$[E, E_{n1}]^n = [E, A_{n-1}] = A_n$$

One can prove the following:

**Theorem 5.1.**

$$[E, E_{n1}]^n =$$

$$= diag(\binom{n-1}{0}, (-1)^1\binom{n-1}{1}, (-1)^2\binom{n-1}{2}, ..., (-1)^{n-2}\binom{n-1}{n-2}, (-1)^{n-1}\binom{n-1}{n-1})$$

The above theorem helps as to find the basic element of first Level's base and based on this theorem all the $n^{th}$ powers of the elements of the first level can also be found.

**Theorem 5.2.** *Based on this theory and P-hyperstructures a set P with two elements can be used, either from zero or first level, but only with two elements. In this case the shift is depending on the level, so if we take P from Level-0, the result will not change, although the result will be multivalued. In case of different level insted, the shift will be analogous to the level of P.*

In the general case in Construction 5.1(1), one can notice the possible cardinality of the result, checking the Jacoby identity is very big. Even in the small case when $|R| = |S| = |P| = 2$ in the anticommutativity $xPx - xPx$ could have cardinality 4 and the left side of the Jacoby identity is

$$(xP(yPz - zPy) - (yPz - zPy)Px) + (yP(zPx - xPz)-$$

$$-(zPx - xPz)Py) + (zP(xPy - yPx) - (xPy - yPx)Pz)$$

could have cardinality $2^{18}$. The number is reduced in special cases.

**Theorem 5.3.** *In the case of the Lie-algebra of type $A_n$, of traceless matrices M, we can define a hyper-Lie-Santilli-admissible bracket hope as follows:*

$$[xy]p = xPy - yPx$$

*where $P = \{p, q\}$, with p,q elements of the zero level. Then we obtain a hyper-Lie-Santilli-algebra.*

**Proof**

We need only to proof the anticommutativity and the Jacobi identity as in the hyperstructure case. Therefore we have

(a) $[xy]p = xPy - yPx = \{0, xpx - xqx, xqx - xpx\} \ni 0$, so the "weak" anticommutativity is valid, and

(b) $[x, [y, z]p]p + [y, [z, x]p]p + [z, [x, y]p]p =$
$(xP(yPz - zPy) - (yPz - zPy)Px) + (yP(zPx - xPz)-$
$-(zPx - xPz)Py) + (zP(xPy - yPx) - (xPy - yPx)Pz)$.

But this set contains the element

$$xpypz - xpzpy - ypzpx + zpypx + ypzpx - ypxpz-$$

$$-zpxpy + xpzpy + zpxpy - zpypx - xpypz + ypxpz = 0$$

So the "weak" Jacobi identity is valid.

Thus, zero belongs to the above results, as it has to be, but there are more elements because it is a multivalued operation.

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