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Class of Semihyperrings from Partitions of a Set

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Abstract

In this paper we show that a partition $\{P_\alpha : \alpha \in \Lambda\}$ of a non-empty set S, where Λ is an ordered set with the least element α_0 and P_{α_0} is a singleton set, induces a hyperaddition $+$ such that $(S, +)$ is a commutative hypermonoid. Also by using a collection of subsets of S, induced by the partition of the set S, we define hypermultiplication on S so that $(S, +, \cdot)$ is a semihyperring.

Key words: hypermonoid, semihyperring, $*$-collection.

MSC 2010: 20N20.

1 Introduction

The theory of hyperstructures has been introduced by the French Mathematician Marty [11] in 1934 at the age of 23 during the 8^{th}congress of Scandinavian Mathematicians held in Stockholm. Since then many researchers have worked on this new area and developed it.

The theory of hyperstructure has been subsequently developed by Corsini [4, 5, 6], Mittas [13], Stratigopoulos [16] and various authors. Basic definitions and results about the hyperstructures are found in [5, 6]. Some researchers, namely, Davvaz [7], Massouros [12], Vougiouklis [18] and others developed the theory of algebraic hyperstructures.

There are different notions of hyperrings $(R, +, \cdot)$. If the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a binary operation then we say the hyperring is an Krasner (additive) hyperring [10]. Rota [15] introduced a multiplicative hyperring, where $+$ is a binary operation and $\cdot$ is a hyperoperation. De Salvo [8] introduced a hyperring in which addition and multiplication are hyperoperations. These hyperrings are studied by Rahnamani Barghi [14] and by Asokkumar and Velrajan [1, 2, 17]. Chvalina [3] and Hoskova [3, 9], studied $h\nu$-groups, $H\nu$-rings.

In this paper, by using different partitions of a set, we construct different semihyperrings $(S, +, \cdot)$ where both $+$ and $\cdot$ are hyperoperations.

2 Preliminaries

This section explains some basic definitions that have been used in the sequel.

A *hyperoperation* $\circ$ on a non-empty set H is a mapping of $H \times H$ into the family of non-empty subsets of H (i.e., $x \circ y \subseteq H$, for every $x, y \in H$). A *hypergroupoid* is a non-empty set H equipped with a hyperoperation $\circ$. For any two subsets A, B of a hypergroupoid H, the set $A \circ B$ means $\bigcup_{\substack{a \in A \\ b \in B}} (a \circ b)$.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$(the associative axiom). A semihypergroup H is said to be regular (in the sense of Von Neumann) if $a \in a \circ H \circ a$ for every $a \in H$. A hypergroupoid $(H, \circ)$ is called a *quasihypergroup* if $x \circ H = H \circ x = H$ for every $x \in H$(the reproductive axiom). A reproductive semihypergroup is called a *hypergroup*(in the sense of Marty).A comprehensive review of the theory of hypergroups appears in [5].

Definition 2.1. A *semihyperring* is a non-empty set R with two hyperoperations $+$ and $\cdot$ satisfying the following axioms:
(1) $(R, +)$ *is a commutative hypermonoid*, that is,
(a) $(x + y) + z = x + (y + z)$ for all $x, y, z \in R$,
(b) there exists $0 \in R$, such that $x + 0 = 0 + x = \{x\}$ for all $x \in R$,
(c) $x + y = y + x$ for all $x, y \in R$.
(2) $(R, \cdot)$ is a semihypergroup, that is, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in R$.
(3) The hyperoperation $\cdot$ is distributive with respect to hyperoperation '+', that is, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$.
(4) There exists element $0 \in R$, such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$.

Definition 2.2. Let S be a semihyperring, An element $a \in S$ is said to be regular if there exists an element $y \in S$ such that $x \in xyx$. A semihyperring S is said to be regular if each element of S is regular.

3 Semihyperring constructed from a $*$-collection.

In this section, for a given commutative hypermonoid $(S,+)$, we define hyperoperation $\cdot$ on S suitably so that $(S,+,\cdot)$ is a regular semihyperring.

Definition 3.1. Let S be a commutative hypermonoid. A collection of non-empty subsets $\{S_a : a \in S\}$ of S satisfying the following conditions is called a $*$-collection if (i) $S_a = \{0\}$ if and only if $a = 0$, (ii) if $a \neq 0$ then $\{0,a\} \subseteq S_a$, (iii) $\bigcup_{x\in S_a} S_x = S_a$ for every $a \in S$, (iv) $S_a + S_a = S_a$ for every $a \in S$ and (v) $\bigcup_{x\in a+b} S_x = S_a + S_b$ for every $a, b \in S$.

Example 3.2. Consider the set $S = \{0,a,b\}$. If we define a hyperoperation $+$ on S as in the following table, then (S,+) is a commutative hypermonoid.

+	0	a	b
0	0	a	b
a	a	{a,b}	{a,b}
b	b	{a,b}	{a,b}

Now it is easy to see that $S_0 = \{0\}; S_a = S; S_b = S$ is a $*$-collection.

Example 3.3. Consider the set S={0,a,b}. If we define a hyperoperation + on S as in the following table, then (S,+) is a commutative hypermonoid.

+	0	a	b
0	0	a	b
a	a	{a}	{a,b}
b	b	{a,b}	{b}

Now it is easy to see that $S_0 = \{0\}; S_a = S; S_b = S$ is a $*$-collection. Now we show that $S_0 = \{0\}; S_a = \{a,0\}; S_b = \{b,0\}$ is another $*$-collection.
For each $a \in S$, $\bigcup_{x\in S_a} S_x = \bigcup_{x\in\{a,0\}} S_x = S_a \bigcup S_0 = \{a,0\} \bigcup \{0\} = \{a,0\} = S_a$. Also $S_0 + S_0 = \{0\} + \{0\} = \{0\} = S_0; S_a + S_a = \{0,a\} + \{0,a\} = \{0,a\} = S_a; S_b + S_b = \{0,b\} + \{0,b\} = \{0,b\} = S_b$. Further, for $a, b \in S$, we get $\bigcup_{x\in a+b} S_x = \bigcup_{x\in\{a,b\}} S_x = S_a \bigcup S_b = \{0,a,b\} = S_a + S_b$.

Example 3.4. Consider the set S={0,a,b}. If we define a hyperoperation + on S as in the following table, then (S,+) is a commutative hypermonoid.

+	0	a	b
0	0	a	b
a	a	{0,a}	S
b	b	S	{0,b}

It is easy to show that $S_0 = \{0\}$; $S_a = S$ for every $a \neq 0 \in S$, is a $*$-collection and $S_0 = \{0\}$; $S_a = \{a, 0\}$ for every $a \neq 0 \in S$ is another $*$-collection

Example 3.5. Consider the set S={0,a,b,c}. If we define a hyperoperation + on S as in the following table, then (S,+) is a commutative hypermonoid.

+	0	a	b	c
0	{0}	{a}	{b}	{c}
a	{a}	{a}	{a, b}	{a, c}
b	{b}	{a, b}	{b}	{b, c}
c	{c}	{a, c}	{b, c}	{c}

In this commutative hypermonoid, each one of the following is a $*$-collection.

$S_0 = \{0\}$; $S_a = \{a, 0\}$ for every $a \neq 0 \in S$,
$S_0 = \{0\}$; $S_a = S$ for every $a \neq 0 \in S$,
$S_0 = \{0\}; S_a = \{0, a\}; S_b = \{0, b, a\}; S_c = \{0, c, a\}$,
$S_0 = \{0\}; S_a = \{0, a, b\}; S_b = \{0, b\}; S_c = \{0, c, b\}$,
$S_0 = \{0\}; S_a = \{0, a, c\}; S_b = \{0, b, c\}; S_c = \{0, c\}$.

Theorem 3.6. *Let S be a commutative hypermonoid with the additive identity 0 with the condition that $x + y = \{0\}$ for $x, y \in S$ implies either $x = 0$ or $y = 0$. Let $\{S_a : a \in S\}$ be a $*$-collection on S. For $a, b \in S$, if we define a hypermultiplication on S as*

$$a \cdot b = \begin{cases} S_a & \text{if } a \neq 0, b \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

then $(S, +, .)$ is a regular semihyperring.

Proof. From the definition of the hypermultiplication, $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$. Let $a, b, c \in S$. If any one of a, b, c is 0, then $a \cdot (b \cdot c) = \{0\} = (a \cdot b) \cdot c$. If $a \neq 0, b \neq 0$ and $c \neq 0$, then $a \cdot (b \cdot c) = a \cdot S_b = S_a$. Also, $(a \cdot b) \cdot c = S_a \cdot c = \bigcup_{x \in S_a} (x \cdot c) = \bigcup_{x \in S_a} S_x = S_a$. Thus $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. Therefore, $(S, \cdot)$ is a semihypergroup.

Let $a, b, c \in S$. If $a = 0$ or $b = 0$ or $c = 0$, then it is obvious that $a \cdot (b + c) = a \cdot b + a \cdot c$. Suppose $a \neq 0, b \neq 0$ and $c \neq 0$. If $0 \in b + c$, then $a \cdot (b + c) = S_0 \cup S_a = S_a = S_a + S_a = a \cdot b + a \cdot c$. If $0 \notin b + c$, then $a \cdot (b + c) = S_a = S_a + S_a = a \cdot b + a \cdot c$. Thus $a \cdot (b + c) = a \cdot b + a \cdot c$.

Now we prove $(a + b) \cdot c = a \cdot c + b \cdot c$. For, $(a + b) \cdot c = \bigcup_{x \in a+b} x.c = \bigcup_{x \in a+b} S_x = S_a + S_b = a \cdot c + b \cdot c$. Therefore, $(a + b) \cdot c = a \cdot c + b \cdot c$. Thus $(S, +, \cdot)$ is a semihyperring.

Let $x \neq 0 \in S$. Now, for any $y \neq 0 \in S$, we have $x \in S_x = x \cdot y \subseteq x \cdot S_y = x \cdot (y \cdot x)$. Hence the semihyperring is regular. $\square$

Example 3.7. The semihyperring obtained by using the Theorem 3.1 in the Example 3.1 is as follows.

+	0	a	b
0	0	a	b
a	a	{a,b}	{a,b}
b	b	{a,b}	{a,b}

.	0	a	b
0	0	0	0
a	0	S	S
b	0	S	S

Example 3.8. The semihyperrings obtained by using the Theorem 3.1 in the Example 3.2 are as follows.

+	0	a	b
0	0	a	b
a	a	{a}	{a,b}
b	b	{a,b}	{b}

.	0	a	b
0	0	0	0
a	0	S	S
b	0	S	S

.	0	a	b
0	0	0	0
a	0	{0, a}	{0,a}
b	0	{0,b}	{0,b}

Example 3.9. The semihyperrings obtained by using the Theorem 3.1 in the Example 3.3 are as follows.

+	0	a	b
0	0	a	b
a	a	{0,a}	S
b	b	S	{0,b}

.	0	a	b
0	0	0	0
a	0	S	S
b	0	S	S

.	0	a	b
0	0	0	0
a	0	{0, a}	{0,a}
b	0	{0,b}	{0,b}

Theorem 3.10. *Let S be a commutative hypermonoid with the additive identity 0 with the condition that $x + y = 0$ for $x, y \in S$ implies either $x = 0$ or $y = 0$. Let $\{S_a : a \in S\}$ be a $*$-collection on S. For $a, b \in S$, if we define a hypermultiplication on S as*

$$a \cdot b = \begin{cases} S_b & \text{if } a \neq 0, b \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

then $(S, +, .)$ is a regular semihyperring.

Proof. The proof follows by the same lines as in the Theorem 3.1. Let $x \neq 0 \in S$. Now, for any $y \neq 0 \in S$, we have $x \in S_x = y \cdot x \subseteq S_y \cdot x = (x \cdot y) \cdot x$. Hence the semihyperring is regular. □

Theorem 3.11. *Let S be a commutative hypermonoid with the additive identity 0 such that $x + y = 0$ for $x, y \in S$ implies either $x = 0$ or $y = 0$. Let $\{S_a : a \in S\}$ be a $*$-collection on S such that $S_a \cap S_b = X$ for all $a \neq 0, b \neq 0 \in S$ where X is a subset of S such that $X + X = X$. For $a, b \in S$, if we define a hypermultiplication on S as*

$$a \cdot b = \begin{cases} S_a \cap S_b = X & \text{if } a \neq 0, b \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

then $(S, +, .)$ is a regular semihyperring.

Proof. Since $0 \in S_a$ and $0 \in S_b$, we get $0 \in S_a \cap S_b$. This implies that the set X is non-empty. From the definition of hypermultiplication, $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$. Let $a, b, c \in S$. If any one of a, b, c is 0, then $a \cdot (b \cdot c) = \{0\} = (a \cdot b) \cdot c$. If $a \neq 0, b \neq 0$ and $c \neq 0$, then $a \cdot (b \cdot c) = X = (a \cdot b) \cdot c$. Thus $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. Therefore, $(S, \cdot)$ is a semihypergroup.

If $a = 0$ or $b = 0$ or $c = 0$, then it is obvious that $a \cdot (b + c) = a \cdot b + a \cdot c$. Suppose $a \neq 0, b \neq 0$ and $c \neq 0$ then, $a \cdot (b + c) = X = X + X = a \cdot b + a \cdot c$. Similarly we have $(a + b) \cdot c = X = a \cdot c + b \cdot c$. Thus $(S, +, \cdot)$ is a semihyperring. Let $x \neq 0 \in S$. Since $x \in S_x$, we have $x \in S_x = x \cdot x \subseteq x \cdot S_x = x \cdot (x \cdot x)$. Hence the semihyperring is regular. □

Example 3.12. Using the Theorem 3.3 in the commutative hypermonoid given in the Example 3.4 and by using the following each $*$-collection

$S_0 = \{0\}; S_a = \{0, a\}; S_b = \{0, b, a\}; S_c = \{0, c, a\}$ with $X = \{0, a\}$,

$S_0 = \{0\}; S_a = \{0, a, b\}; S_b = \{0, b\}; S_c = \{0, c, b\}$ with $X = \{0, b\}$,

$S_0 = \{0\}; S_a = \{0, a, c\}; S_b = \{0, b, c\}; S_c = \{0, c\}$ with $X = \{0, c\}$, we get three hypermultiplications so that we get three semihyperrings.

4 Semihyperrings induced by a Partition.

In this section we show that a partition of a non-empty set S induces a hyperaddition $+$ such that, $(S, +)$ is a commutative hypermonoid and also the partition induces a $*$-collection. Using this $*$-collection,we define hypermultiplication $\cdot$ on the set S, so that $(S, +, .)$ a regular semihyperring.

Theorem 4.1. *Let S be any non-empty set and $\{P_\alpha\}_{\alpha \in \Lambda}$ be a partition of S, where Λ is an ordered set with the least element $\alpha_0 \in \Lambda$ and P_{α_0} be a singleton set, say, $\{0\}$. Define a hyperaddition "+" on S as follows: For all $a \in S$, $0 + a = a + 0 = \{a\}$. For $a \neq 0, b \neq 0 \in S$, suppose $a \in P_\alpha$ and $b \in P_\beta$ and $\gamma = max \{\alpha, \beta\}$,*

$$a + b = \begin{cases} P_\gamma & \text{if } \alpha \neq \beta, \\ P_\alpha = P_\beta & \text{if } \alpha = \beta \end{cases}$$

Then (i) $(S,+)$ is a commutative monoid and (ii) the partition $\{P_\alpha\}_{\alpha\in\Lambda}$ induces a $*$-collection.

Proof. It is clear that $a+b=b+a$ for all $a,b\in S$. Let $a,b,c\in S$. Suppose that $a\in P_\alpha,\ b\in P_\beta$ and $c\in P_\gamma$, where $\alpha,\beta,\gamma\in\Lambda$.

Case 1 : Suppose $\alpha<\beta<\gamma$.
Then $a+(b+c)=a+P_\gamma=P_\gamma$. Also, $(a+b)+c=P_\beta+c=P_\gamma$. Therefore, $a+(b+c)=(a+b)+c$.

Case 2 : Suppose $\beta<\alpha<\gamma$.
Then $a+(b+c)=a+P_\gamma=P_\gamma$. Also, $(a+b)+c=P_\alpha+c=P_\gamma$. Therefore, $a+(b+c)=(a+b)+c$.

Case 3 : Suppose $\alpha<\gamma<\beta$.
Then $a+(b+c)=a+P_\beta=P_\beta$. Also, $(a+b)+c=P_\beta+c=P_\beta$. Therefore, $a+(b+c)=(a+b)+c$.

Case 4 : Suppose $\gamma<\alpha<\beta$.
Then $a+(b+c)=a+P_\beta=P_\beta$. Also, $(a+b)+c=P_\beta+c=P_\beta$. Therefore, $a+(b+c)=(a+b)+c$.

Case 5 : Suppose $\gamma<\beta<\alpha$.
Then $a+(b+c)=a+P_\beta=P_\alpha$. Also, $(a+b)+c=P_\alpha+c=P_\alpha$. Therefore, $a+(b+c)=(a+b)+c$.

Case 6 : Suppose $\beta<\gamma<\alpha$.
Then $a+(b+c)=a+P_\gamma=P_\alpha$. Also, $(a+b)+c=P_\alpha+c=P_\alpha$. Therefore, $a+(b+c)=(a+b)+c$.

Case 7 : Suppose $\alpha=\beta<\gamma$.
Then $a+(b+c)=a+P_\gamma=P_\gamma$. Also, $(a+b)+c=P_\beta+c=P_\gamma$. Therefore, $a+(b+c)=(a+b)+c$.

Case 8 : Suppose $\gamma<\alpha=\beta$.
Then $a+(b+c)=a+P_\alpha=P_\alpha$. Also, $(a+b)+c=P_\alpha+c=P_\alpha$. Therefore, $a+(b+c)=(a+b)+c$.

Case 9 : Suppose $\alpha=\gamma<\beta$.
Then $a+(b+c)=a+P_\beta=P_\beta$. Also, $(a+b)+c=P_\beta+c=P_\beta$. Therefore, $a+(b+c)=(a+b)+c$.

Case 10 : Suppose $\beta<\alpha=\gamma$.
Then $a+(b+c)=a+P_\alpha=P_\alpha$ Also, $(a+b)+c=P_\alpha+c=P_\alpha$. Therefore, $a+(b+c)=(a+b)+c$.

Case 11 : Suppose $\beta=\gamma<\alpha$.
Then $a+(b+c)=a+P_\gamma=P_\alpha$. Also, $(a+b)+c=P_\alpha+c=P_\alpha$. Therefore, $a+(b+c)=(a+b)+c$.

Case 12 : Suppose $\alpha<\beta=\gamma$.
Then $a+(b+c)=a+P_\gamma=P_\gamma$. Also, $(a+b)+c=P_\beta+c=P_\gamma$. Therefore, $a+(b+c)=(a+b)+c$.

Case 13 : Suppose $\alpha = \beta = \gamma$.
Then $a+(b+c) = P_\alpha = P_\beta = P_\gamma = (a+b)+c$. Therefore, $a+(b+c) = (a+b)+c$.

Thus the hyperoperation $+$ is associative. So, $(S, +)$ is a commutative hypermonoid. Let $S_0 = P_{\alpha_0} = \{0\}$. For $a \neq 0 \in S$, then $S_a = \bigcup_{\alpha_0 \leq t \leq \alpha} P_t$ where $a \in P_\alpha$. It is clear that $S_a = \bigcup_{x \in S_a} S_x$. For $a \neq 0 \in S$, and $a \in P_\alpha$, then $S_a + S_a = \bigcup_{\alpha_0 \leq t \leq \alpha} P_t + \bigcup_{\alpha_0 \leq t \leq \alpha} P_t = \bigcup_{\alpha_0 \leq t \leq \alpha} P_t = S_a$. Also $S_0 + S_0 = \{0\} + \{0\} = \{0\} = S_0$. If either $a = 0$ or $b = 0$, then $\bigcup_{x \in a+b} S_x = S_a + S_b$. Let $a \neq 0, b \neq 0 \in S$. Then $a \in P_\alpha$ and $b \in P_\beta$ for some $\alpha, \beta \in \Lambda$.

Case 1 : Suppose $\alpha \neq \beta$, say $\alpha < \beta$, then $a + b = P_\beta$. Now $x \in a + b$ implies $x \in P_\beta$. Therefore, $S_x = \bigcup_{\alpha_0 \leq t \leq \beta} P_t$. Hence

$$\bigcup_{x \in a+b} S_x = \bigcup_{x \in a+b} \Big(\bigcup_{\alpha_0 \leq t \leq \beta} P_t \Big) = \bigcup_{\alpha_0 \leq t \leq \alpha} P_t = \bigcup_{\alpha_0 \leq t \leq \alpha} P_t + \bigcup_{\alpha_0 \leq t \leq \beta} P_t = S_a + S_b.$$

Case 2 : Suppose $\alpha = \beta$ then $a + b = P_\alpha$. Therefore, $\bigcup_{x \in a+b} S_x = \bigcup_{x \in P_\alpha} S_x = S_a + S_b$. Therefore, $\bigcup_{x \in a+b} S_x = S_a + S_b$. Thus $\{S_a : a \in S\}$ is a $*$-collection.

□

Remark 4.2. Let S be any non-empty set and $x_0 \in S$. Let $P_0 = \{x_0\}$ and $\{P_1, P_2, P_3, \cdots, P_n, \cdots\}$ be a partition of $S \setminus \{x_0\}$. Then the partition $\{P_0, P_1, P_2, \cdots, P_n, \cdots\}$ of S induces a hyperoperation $+$ on S so that $(S, +)$ is a commutative hypermonoid and $\{P_0, P_1, P_2, \cdots, P_n, \cdots\}$ induces a $*$-collection.

Theorem 4.3. *Let S be any non-empty set and $\{P_\alpha\}_{\alpha \in \Lambda}$ be a partition of S, where Λ is an ordered set with the least element α_0 and P_{α_0} is a singleton set. Then the partition induces a semihyperring.*

Proof. By the Theorem 4.1, the partition induces a hyperaddition $+$ such that $(S, +)$ is a commutative hypermonoid and it also induces a $*$-collection. Hence by the Theorem 3.1, we get a regular semihyperring. □

Example 4.4. We illustrate the construction of semihyperrings from the following examples. Let $S = \{0, a, b\}$. Consider a partition $P_1 = \{0\}, P_2 = \{a\}, P_3 = \{b\}$ of S. Here, the indexing set is $\Lambda = \{1, 2, 3\}$ which is an ordered set. The commutative hypermonoid induced by this partition is given by the following Caley table.

+	0	a	b
0	0	a	b
a	a	{a}	{b}
b	b	{b}	{b}

The $*$-collection induced by this partition is $S_0 = \{0\}, S_a = \{0, a\}, S_b = \{0, a, b\}$ and the hypermultiplication induced by the $*$-collection is given in the Caley table.

.	0	a	b
0	0	0	0
a	0	{0,a}	{0,a}
b	0	{0,a,b}	{0,a,b}

Example 4.5. Let $S = \{0, a, b\}$. Consider a partition $P_1 = \{0\}, P_2 = \{b\}, P_3 = \{a\}$ of S. Here, the indexing set is $\Lambda = \{1, 2, 3\}$ which is an ordered set. The commutative hypermonoid induced by this partition is given by the following Caley table.

+	0	a	b
0	0	a	b
a	a	{a}	{a}
b	b	{a}	{b}

The $*$-collection induced by this partition is $S_0 = \{0\}, S_a = \{0, a, b\}, S_b = \{0, b\}$ and the hypermultiplication induced by the $*$-collection is given in the Caley table.

.	0	a	b
0	0	0	0
a	0	{0,a,b}	{0,a,b}
b	0	{0,b}	{0,b}

Example 4.6. Let $S = \{0, a, b\}$. Consider a partition $P_1 = \{0\}, P_2 = \{a, b\}$ of S. Here, the indexing set is $\Lambda = \{1, 2\}$ which is an ordered set. The commutative hypermonoid induced by this partition is given by the following Caley table.

+	0	a	b
0	0	a	b
a	a	{a,b}	{a,b}
b	b	{a,b}	{a,b}

The $*$-collection induced by this partition is $S_0 = \{0\}, S_a = \{0, a, b\}, S_b = \{0, a, b\}$ and the hypermultiplication induced by the $*$-collection is given in the Caley table.

.	0	a	b
0	0	0	0
a	0	{0,a,b}	{0,a,b}
b	0	{0,a,b}	{0,a,b}

Thus we have a regular semihyperring.

Conclusion : In the section 3 of this paper, for the given commutative hypermonoid, given $*$-collection, we construct three semihyperrings. In the section 4, by the Theorem 4.1, a partition of a set S induces a hyperaddition $+$ such that $(S, +)$ is a commutative hypermonoid and it also induces a $*$-collection. Hence by the Theorem 3.1, we get a semihyperring. Thus we get semihyperrings depending on the partitions of the set satisfies the conditions of the Theorem 4.2. All the semihyperrings so constructed are regular.

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Classification of Hyper MV-algebras of Order 3

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Abstract

In this paper, we investigated the number of hyper MV-algebras of order 3. In fact, we prove that there are 33 hyper MV-algebras of order 3, up to isomorphism.

Key words: hyper MV-algebra

MSC 2010: 97U99.

1 Introduction

The concept of MV-algebras was introduced by Chang in [1] in order to show Lukasiewicz logic to be standard complete, i.e. complete with respect to evaluations of propositional variables in the real unit interval $[0,1]$. In [6], Mundici showed that any MV-algebra is an interval of an Abelian lattice ordered group with a strong unit. Also, he introduced the concept of state on MV-algebra. Georgescu and Iorgulescu [2] introduced a new non-commutative algebraic structures, which were called pseudo MV-algebras. It can be obtained by dropping commutative axioms in MV-algebras, which are a generalization of MV-algebras. The hyper structure theory was introduced by F. Marty [5] at the 8th congress of Scandinavian Mathematicians in 1934. Since then many researches have worked in these areas. Recently in [4], Sh. Ghorbani, A. Hasankhni and E. Eslami applied the hyper structure to MV-algebras and introduced the concept of a hyper MV-algebra which is a generalization of an MV-algebra and investigated some related results. Now, in this paper we find all hyper MV-algebras of order 3.

2 Preliminary

Definition 2.1. [1] An MV-algebra $(X, \oplus, ^*, 0)$ is a set X equipped with a binary operation $\oplus$, a unary operation $*$ and a constant 0 satisfying the following equations:

(MV_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(MV_2) $x \oplus y = y \oplus x$,

(MV_3) $x \oplus 0 = x$,

(MV_4) $(x^*)^* = x$,

(MV_5) $x \oplus 0^* = 0^*$,

(MV_6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,

for all $x, y, z \in X$.

Definition 2.2. [3]

A hyperalgebra $(M, \oplus, ^*, 0)$ with a hyperoperation $\oplus : M \times M \longrightarrow \mathcal{P}^*(M)$, a unary operation $^* : M \longrightarrow M$ and a constant 0, is said to be a hyper MV-algebra if and only if satisfies the following axioms, for all $x, y, z \in M$:

(HMV_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(HMV_2) $x \oplus y = y \oplus x$,
(HMV_3) $(x^*)^* = x$,
(HMV_4) $0^* \in x \oplus 0^*$,
(HMV_5) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(HMV_6) $0^* \in x \oplus x^*$,
(HMV_7) If $x \leqslant y$ and $y \leqslant x$, then $x = y$,

where $x \leqslant y$ is defined by $0^* \in x^* \oplus y$. For every $X, Y \subseteq M$, $X \leqslant Y$ if there exist $x \in X$ and $y \in Y$ such that $x \leqslant y$. We define $1 = 0^*$

Theorem 2.3. *[3] Let $(M, \oplus, ^*, 0)$ be a hyper-MV algebra. Then for all $x, y, z \in M$ and for all non-empty subsets A, B and C of M the following hold:*

(i) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
(ii) $0 \leqslant x \leqslant 1$, $x \leqslant x$ *and* $A \leqslant A$,
(iii) If $x \leqslant y$ *then* $y^* \leqslant x^*$ *and* $A \leqslant B$ *implies* $B^* \leqslant A^*$,
(iv) If $x \leqslant 0$ *or* $1 \leqslant x$, *then* $x = 0$ *or* $x = 1$, *respectively,*
(v) $0 \oplus 0 = \{0\}$,
(vi) $x \in x \oplus 0$,
(vii) If $x \oplus 0 = y \oplus 0$, *then* $x = y$.

3 Classification of hyper MV-algebras of order 3

In this section we try to find all hyper MV-algebras of order 3, up to isomorphism.

Theorem 3.1. *Let M be a hyper MV-algebra and x be an element of M such that $0 \oplus x = \{x\}$ and $x^* = x$. Then the following statements hold:*
(i) $(1 \oplus x)^* \oplus x = \{x\}$,
(ii) $(1 \oplus x)^* \oplus 1 = x \oplus x$,
(iii) $x \notin 1 \oplus x$ *and* $0 \notin 1 \oplus x$.

Proof. Since $0^* = 1$, then by hypothesis and $(HMV5)$;

$$(1\oplus x)^*\oplus x = (0^*\oplus x)^*\oplus x = (x^*\oplus 0)^*\oplus 0 = (x\oplus 0)^*\oplus 0 = x^*\oplus 0 = x\oplus 0 = \{x\}$$

$$\begin{aligned}(1 \oplus x)^* \oplus 1 = (x \oplus 1)^* \oplus 1 = ((x^*)^* \oplus 1)^* \oplus 1 = \\ = (1^* \oplus x^*)^* \oplus x^* = (0 \oplus x)^* \oplus x^* = x^* \oplus x^* = x \oplus x\end{aligned}$$

and so (i) and (ii) hold.

(iii) If $x \in 1\oplus x$, then $x = x^* \in (1\oplus x)^*$ and so $x\oplus x = x^*\oplus x \subseteq (1\oplus x)^*\oplus x$. By (i), $x \oplus x \subseteq \{x\}$. Hence $x \oplus x = \{x\}$. Now, since by (HMV_6), $1 = 0^* \in x \oplus x^* = x \oplus x = \{x\}$, then $x = 1$ and so $0 = 1^* = x^* = x = 1$, which is a contradiction. Hence $x \notin 1 \oplus x$. Now, let $0 \in 1 \oplus x$. Then $1 = 0^* \in (1 \oplus x)^*$ and so $1 \oplus x \subseteq (1 \oplus x)^* \oplus x$. By (i), $1 \oplus x \subseteq \{x\}$. Thus $1 \oplus x = \{x\}$, which is a contradiction. Hence $0 \notin 1 \oplus x$.

Note. From now one in this paper, we let $M = \{0, a, 1\}$ be a hyper MV-algebra of order 3.

Theorem 3.2. *(i)* $1 \leq 1$, $0 \leq 0$, $a \leq a$, $0 \leq 1$ *and* $0 \leq a$,
(ii) $a \nleq 0$,
(iii) $a^* = a$,
(iv) $1 \in 1 \oplus a$.

Proof. (i). By Theorem 2.3(ii), the proof is clear.

(ii). By Theorem 2.3(iv), the proof is clear.

(iii). By Definition 2.2, $0^* = 1$ and by (HMV_3), $0 = (0^*)^* = 1^*$. Now, if $a^* = 1$, then $0 = 1^* = (a^*)^* = a$, which is a contradiction. By similar way, if $a^* = 0$, then $1 = 0^* = (a^*)^* = a$, which is a contradiction. Hence, $a^* = a$.

(iv). By (HMV_4), $1 = 0^* \in 0^* \oplus a = 1 \oplus a$. $\square$

Theorem 3.3. *If $0 \oplus a = \{a\}$ or $1 \oplus a = \{1\}$, then M is an MV-algebra.*

Proof. Let $0 \oplus a = \{a\}$. Since $a^* = a$, then by Theorem 3.1(iii), $a \notin 1 \oplus a$ and $0 \notin 1 \oplus a$ and so $1 \oplus a = \{1\}$.

Moreover, By Theorem 3.1(iii) and (i), $0 \notin 1 \oplus 0$ and $(1 \oplus 0)^* \oplus 0 = \{0\}$. Since $0 \notin \{a\} = 0 \oplus a$ and $0 \notin 1 \oplus 0$, then $(1 \oplus 0)^* = \{0\}$ and so $1 \oplus 0 = \{1\}$. By Theorem 3.1(i) and (ii), $0 \oplus 1 = \{1\} = (1 \oplus a)^* \oplus 1 = a \oplus a$. Hence $a \oplus a = \{1\}$. Now, by (HMV_1),

$$1 \oplus 1 = (a \oplus a) \oplus 1 = a \oplus (1 \oplus a) = a \oplus 1 = \{1\}.$$

Therefore, $x \oplus y$ is singleton for all $x, y \in M$ and so M is an MV-algebra. □

Now, if $1 \oplus a = \{1\}$, then $\{0\} = \{1^*\} = (1 \oplus a)^*$ and so $0 \oplus a = (1 \oplus a)^* \oplus a$. By (HMV_5),

$$0 \oplus a = (1 \oplus a)^* \oplus a = 0 \oplus (0 \oplus a)^*.$$

By Theorem 3.2, $a \not< 0$, $1 \notin 0 \oplus a$. If $0 \in 0 \oplus a$, then $0 \oplus a = \{0, a\}$ and

$$\begin{aligned}\{0, a\} = 0 \oplus a = 0 \oplus (0 \oplus a)^* = 0 \oplus \{0, a\}^* = \\ = 0 \oplus \{1, a\} = (0 \oplus 1) \cup (0 \oplus a) = (0 \oplus 1) \cup \{0, a\}.\end{aligned}$$

Hence $0 \oplus 1 \subseteq \{0, a\}$. By $(HMV4)$, $1 \in 0 \oplus 1$. Thus $1 \in \{0, a\}$, which is a contradiction. Thus $0 \notin 0 \oplus a$ and so $0 \oplus a = \{a\}$. Therefore, M is a same MV-algebra, which is as follows:

$\oplus_1$	0	a	1
0	$\{0\}$	$\{a\}$	$\{1\}$
a	$\{a\}$	$\{1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

Definition 3.4. We call a hyper MV-algebra is proper, if it is not an MV-algebra.

Lemma 3.5. *Let $M = \{0, a, 1\}$ be a proper hyper MV-algebra of order 3. Then*

(i) $0 \oplus a = \{0, a\}$,
(ii) $0 \oplus 1 = \{1\}$, $\{0, 1\}$ *or* M,
(iii) $a \oplus a = \{1\}$, $\{0, 1\}$, $\{1, a\}$ *or* M,
(iv) $1 \oplus a = \{0, 1\}$, $\{1, a\}$ *or* M,
(v) $1 \oplus 1 = \{1\}$, $\{0, 1\}$ $\{1, a\}$ *or* M,
(vi) *If* $a \oplus a = \{1\}$, *then* $0 \oplus 1 = M$.

Proof. (i). Since $a \not< 0$, then $1 \notin 0 \oplus a$. By Theorem 2.3 (vi), $a \in 0 \oplus a$. Thus $0 \oplus a = \{a\}$ or $\{0, a\}$. If $0 \oplus a = \{a\}$, then by Theorem 3.3, M is not proper. Thus $0 \oplus a = \{0, a\}$

(ii). Since $0 \leq 0$, then $1 = 0^* \in 0^* \oplus 0 = 1 \oplus 0 = 0 \oplus 1$. Hence it is sufficient to show that $0 \oplus 1 \neq \{1, a\}$. Let $0 \oplus 1 = \{1, a\}$, by the contrary. Then by (HMV_1),

$$\{1, a\} = 0 \oplus 1 = (0 \oplus 0) \oplus 1 = (0 \oplus 1) \oplus 0 = \{1, a\} \oplus 0 = \{0, a, 1\},$$

which is impossible. Therefore, $0 \oplus 1 \neq \{1, a\}$ and so $0 \oplus 1 = \{1\}$, $\{0, 1\}$ or M.

(iii), (v). Since $a \leq a$ and $0 \leq 1$, then $1 \in a \oplus a$ and $1 \in 1 \oplus 1$ and so (v) and (iii) are hold.

(iv). Since $0 \leq a$, then $1 \in 1 \oplus a$. By Theorem 3.3, if $a \oplus 1 = \{1\}$, then M is an MV algebra which is impossible. Hence $1 \oplus a = \{0, 1\}$, $\{1, a\}$ or M.

(vi). Let $a \oplus a = \{1\}$. Then by (HMV_1),

$$0 \oplus 1 = 0 \oplus (a \oplus a) = (0 \oplus a) \oplus a = \{0, a\} \oplus a = (0 \oplus a) \cup (a \oplus a) = M.$$

By Lemma 3.5 (ii), we know that $0 \oplus 1 = \{1\}$, $\{0, 1\}$ or M. So, for the classification of all hyper MV-algebras of order 3, we consider the following three cases.

Case 1: $0 \oplus 1 = \{1\}$

Lemma 3.6. *Let* $M = \{0, a, 1\}$ *be a proper hyper* MV*-algebra of order* 3 *and* $0 \oplus 1 = \{1\}$. *Then*

(i) $a \oplus a = \{1, a\}$ *or* M,

(ii) $1 \oplus 1 = \{1\}$,

(iii) $1 \oplus a = M$.

Proof. (i). By Lemma 3.5 (i) and (iii), $0 \oplus a = \{0, a\}$ and $1 \in a \oplus a$. Hence

$$(0 \oplus a) \oplus a = \{0, a\} \oplus a = (0 \oplus a) \cup (a \oplus a) = \{0, a\} \cup (a \oplus a) = M.$$

Since by (HMV_1), $(0 \oplus a) \oplus a = 0 \oplus (a \oplus a)$, then $0 \oplus (a \oplus a) = M$. By Lemma 3.5$(iii)$, $a \oplus a = \{1\}$, $\{0, 1\}$, $\{1, a\}$ or M. If $a \oplus a = \{1\}$, then $0 \oplus (a \oplus a) = 0 \oplus 1 = \{1\}$, which is a contradiction.

If $a \oplus a = \{0, 1\}$, then by Theorem 2.3(v), $0 \oplus (a \oplus a) = 0 \oplus \{0, 1\} = (0 \oplus 0) \cup (0 \oplus 1) = \{0, 1\}$, which is a contradiction. Hence, $a \oplus a = \{1, a\}$ or M.

(ii). By (HMV_5), and Theorem 2.3(v),

$$(1 \oplus 1)^* \oplus 1 = (0^* \oplus 1)^* \oplus 1 = (1^* \oplus 0)^* \oplus 0 = (0 \oplus 0)^* \oplus 0 = 1 \oplus 0 = \{1\}.$$

If $0 \in 1 \oplus 1$, then $1 \oplus 1 \subseteq (1 \oplus 1)^* \oplus 1 = \{1\}$ and so $0 \notin 1 \oplus 1$, which is a contradiction. If $a \in 1 \oplus 1$, then $a \oplus 1 \subseteq (1 \oplus 1)^* \oplus 1 = \{1\}$. Thus $a \oplus 1 = \{1\}$ and so by Theorem 3.3, M is an MV-algebra, which is a contradiction. Hence, $1 \oplus 1 = \{1\}$.

(iii). By Lemma 3.5, $1 \oplus a = \{0, 1\}$, $\{1, a\}$ or M. If $1 \oplus a = \{0, 1\}$, since by (HMV_1), $1 \oplus (1 \oplus a) = (1 \oplus 1) \oplus a = 1 \oplus a$, then $1 \oplus (1 \oplus a) = \{1\}$, which is a contradiction. If $1 \oplus a = \{1, a\}$, since by (HMV_1), $0 \oplus (1 \oplus a) = (0 \oplus 1) \oplus a = 1 \oplus a$, then $0 \oplus (1 \oplus a) = (0 \oplus 1) \cup (0 \oplus a) = M$, which is a contradiction. Hence, $1 \oplus a = M$.

Theorem 3.7. *There are two non-isomorphic proper hyper MV-algebras of order 3 such that $0 \oplus 1 = \{1\}$.*

Proof. According Theorem 3.6, if M is a proper hyper MV-algebra of order 3 and $0 \oplus 1 = \{1\}$, then we must investigate two following tables, which both of them are non-isomorphic hyper MV-algebras.

$\oplus_2$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{1\}$
a	$\{0, a\}$	$\{1, a\}$	$\{0, a, 1\}$
1	$\{1\}$	$\{0, a, 1\}$	$\{1\}$

$\oplus_3$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{0, a, 1\}$
1	$\{1\}$	$\{0, a, 1\}$	$\{1\}$

Case 2: $0 \oplus 1 = \{0, 1\}$

Lemma 3.8. *Let $M = \{0, a, 1\}$ be a proper hyper MV-algebra of order 3 and $0 \oplus 1 = \{0, 1\}$. Then*
(i) $(a \oplus a) \cup (1 \oplus a) = M$,
(ii) $a \oplus 1 = \{a, 1\}$ *or* M,
(iii) $a \oplus a = \{a, 1\}$ *or* M,
(iv) $1 \oplus 1 = \{0, 1\}$ *or* $\{1\}$.

Proof. (i). Let $0 \oplus 1 = \{0, 1\}$. By Theorem 3.5(iv), since $1 \in 1 \oplus a$, by (HMV_1),

$$(0 \oplus a) \oplus 1 = (0 \oplus 1) \oplus a = \{0, 1\} \oplus a = (0 \oplus a) \cup (1 \oplus a) = \{0, a\} \cup (1 \oplus a) = M.$$

On the other hands

$$(0 \oplus a) \oplus 1 = \{0, a\} \oplus 1 = (0 \oplus 1) \cup (a \oplus 1) = \{0, 1\} \cup (a \oplus 1).$$

Thus $\{0,1\} \cup (a \oplus 1) = M$ and so $a \in a \oplus 1$. New, we consider two cases $0 \in a \oplus 1$ or $0 \neq a \oplus 1$. If $0 \in a \oplus 1$, since by Theorem 3.5, $1 \in a \oplus 1$, then $a \oplus 1 = M$ and so $(a \oplus a) \cup (1 \oplus a) = M$. Now, if $0 \neq a \oplus 1$, then by Theorem 3.5, $a \in a \oplus 1$. Hence by Theorem 3.2(iv), $\{1, a\} \subseteq a \oplus 1$. Thus

$$M = (0 \oplus 1) \cup (a \oplus 1) = \{0, a\} \oplus 1 = \{1, a\}^* \oplus 1 \subseteq (a \oplus 1)^* \oplus 1 \subseteq M$$

and so $(a \oplus 1)^* \oplus 1 = M$. On the other hands, by (HMV_5), $(a \oplus 1)^* \oplus 1 = (0 \oplus a)^* \oplus a$. Hence $(0 \oplus a)^* \oplus a = M$. Since $0 \oplus a = \{0, a\}$, then

$$M = (0 \oplus a)^* \oplus a = \{1, a\} \oplus a = (1 \oplus a) \cup (a \oplus a).$$

(ii). By Lemma 3.5(iv), it is enough to show that $1 \oplus a = \{0, 1\}$. Let $0 \in a \oplus 1$, by the contrary. Since by Lemma 3.5(iv) and (i), $0 \oplus a = \{0, a\}$ and $1 \in 1 \oplus a$, then

$$(0 \oplus 1) \oplus a = \{0, 1\} \oplus a = (0 \oplus a) \cup (1 \oplus a) = M.$$

Thus by (HMV_1),

$$M = (0 \oplus 1) \oplus a = (0 \oplus a) \oplus 1 = \{0, 1\} \cup (1 \oplus a).$$

and so $a \in 1 \oplus a$. Hence $a \oplus 1 \neq \{0, 1\}$ and so by lemma 3.5(iv), $a \oplus 1 = \{a, 1\}$ or M.

(iii). By Lemma 3.5(i), $0 \oplus a = \{0, a\}$. Now, since $1 \in a \oplus a$, then

$$(0 \oplus a) \oplus a = \{0, a\} \oplus a = (0 \oplus a) \cup (a \oplus a) = M.$$

Hence, by (HMV_1), $0 \oplus (a \oplus a) = (0 \oplus a) \oplus a = M$. Since $a \notin 0 \oplus 0$ and $a \notin 0 \oplus 1$, then $a \in a \oplus a$. Hence $a \oplus a = \{a, 1\}$ or M.

(iv). Let $a \in 1 \oplus 1$. By (HMV_5),

$$a \oplus 1 = a^* \oplus 1 \subseteq (1 \oplus 1)^* \oplus 1 = (0 \oplus 0)^* \oplus 0 = \{0, 1\}.$$

which is a contradiction by (i). Hence $a \notin 1 \oplus 1$ and so by Lemma 3.5(v), $1 \oplus 1 = \{0, 1\}$ or $\{1\}$.

Theorem 3.9. *There are 6 non-isomorphic proper hyper MV-algebras of order 3 such that $0 \oplus 1 = \{0, 1\}$.*

Proof. By Lemma 3.8 (iii), $a \oplus a = \{a, 1\}$ or M. If $a \oplus a = \{a, 1\}$, then by Lemma 3.8 (ii), $a \oplus 1 = \{a, 1\}$ or M. By Lemma 3.8 (i), if $a \oplus a = \{a, 1\}$, then $a \oplus 1 \neq \{a, 1\}$. Hence we must investigate 2 following tables which both of them are hyper MV-algebras.

$\oplus_4$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{0, 1\}$	$\{0, a, 1\}$	$\{1\}$

$\oplus_5$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{1\}$	$\{0, a, 1\}$	$\{0, 1\}$

Now, if $a \oplus a = M$, then by Lemma 3.8 (ii) and (iv), $a \oplus 1 = \{a, 1\}$ or M and $1 \oplus 1 = \{0, 1\}$ or $\{1\}$. Thus we must investigate 4 following tables, which all of them are hyper MV-algebras.

$\oplus_6$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{a, 1\}$
1	$\{0, 1\}$	$\{a, 1\}$	$\{1\}$

$\oplus_7$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{a, 1\}$
1	$\{1\}$	$\{a, 1\}$	$\{0, 1\}$

$\oplus_8$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{0, a, 1\}$
1	$\{0, 1\}$	$\{0, a, 1\}$	$\{1\}$

$\oplus_9$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{0, a, 1\}$
1	$\{0, 1\}$	$\{0, a, 1\}$	$\{0, 1\}$

Case 3: $0 \oplus 1 = M$

Lemma 3.10. *Let $M = \{0, a, 1\}$ be a proper hyper MV-algebra of order 3 such that $0 \oplus 1 = M$. Then*

(i) $(a \oplus a) \cup (1 \oplus a) = M$,

(ii) If $a \oplus a = \{1\}$, then $a \oplus 1 = 1 \oplus 1 = M$,

(iii) If $a \oplus a = \{0, 1\}$, then $a \oplus 1 = \{a, 1\}$ or M and if $a \oplus 1 = \{a, 1\}$, then $1 \oplus 1 = \{1\}, \{0, 1\}$ or M,

(iv) If $a \oplus a = \{a, 1\}$, then $a \oplus 1 = \{0, 1\}$ or M and if $a \oplus 1 = \{0, 1\}$, then $1 \oplus 1 = \{a, 1\}$ or M,

(v) If $a \oplus a = M$ and $a \oplus 1 = \{1, a\}$, then $1 \oplus 1 = \{1\}, \{0, 1\}$ or M,

(vi) If $a \oplus a = M$ and $a \oplus 1 = \{0, 1\}$, then $1 \oplus 1 = \{0, 1\}, \{a, 1\}$ or M.

Proof.

(i). Since by Lemma 3.5(iv), $1 \in 1 \oplus a$, then $M = 0 \oplus 1 = 1^* \oplus 1 \subseteq (a \oplus 1)^* \oplus 1$ and so $(a \oplus 1)^* \oplus 1 = M$. Hence by (HMV_5), $(0 \oplus a)^* \oplus a = (a \oplus 1)^* \oplus 1 = M$ and so by Lemma 3.5(i),

$$M = (0 \oplus a)^* \oplus a = \{0, a\}^* \oplus a = \{1, a\} \oplus a = (1 \oplus a) \cup a \oplus a.$$

(ii). Let $a \oplus a = \{1\}$. Since $1 \in 1 \oplus a$, then by (HMV_5) and Lemma 3.5(i),

$$\begin{aligned} 1 \oplus a &= (1 \oplus a) \cup (a \oplus a) = \{1, a\} \oplus a = \{0, a\}^* \oplus a = (0 \oplus a)^* \oplus a \\ &= (a \oplus 0)^* \oplus 0 = \{1, a\} \oplus 0 = (1 \oplus 0) \cup (a \oplus 0) \\ &= M \end{aligned}$$

Now, since $a \oplus a = \{1\}$ and $1 \oplus a = M$, then by (HMV_1),

$$\begin{aligned} 1 \oplus 1 &= (a \oplus a) \oplus (a \oplus a) = a \oplus (a \oplus (a \oplus a)) \\ &= a \oplus (a \oplus 1) = a \oplus M = (a \oplus 1) \cup (a \oplus a) \cup (a \oplus 0) = M. \end{aligned}$$

(iii). If $a \oplus a = \{0, 1\}$, then by (i) and Lemma 3.5(iv), $a \oplus 1 = \{a, 1\}$ or M. Let $a \oplus 1 = \{a, 1\}$. If $1 \oplus 1 = \{a, 1\}$, then by (HMV_1) and (i),

$$\begin{aligned} M &= (a \oplus a) \cup (1 \oplus a) = \{a, 1\} \oplus a = (1 \oplus 1) \oplus a \\ &= 1 \oplus (1 \oplus a) = 1 \oplus \{1, a\} = (1 \oplus 1) \cup (1 \oplus a) \\ &= (1 \oplus 1) \ \cup \ \{1, a\} \end{aligned}$$

Hence $0 \in 1 \oplus 1 = \{a, 1\}$, which is a contradiction. Thus $1 \oplus 1 \neq \{a, 1\}$ and so by Lemma 3.5(v), $1 \oplus 1 = \{1\}, \{0, 1\}$ or M.

(iv). By (i), if $a \oplus a = \{a, 1\}$, then $a \oplus 1 = \{0, 1\}$ or M.

If $a \oplus 1 = \{0, 1\}$, then by (HMV_1),

$$\begin{aligned} M &= \{0, a\} \cup (1 \oplus a) = \{0, 1\} \oplus a = (1 \oplus a) \oplus a \\ &= 1 \oplus (a \oplus a) = 1 \oplus \{a, 1\} = (1 \oplus a) \cup (1 \oplus 1) \\ &= \{0, 1\} \cup (1 \oplus 1) \end{aligned}$$

Hence $a \in 1 \oplus 1$. By Lemma 3.5(v), $1 \oplus 1 = \{1, a\}$ or M.

(v). Let $a \oplus a = M$ and $1 \oplus a = \{1, a\}$. If $1 \oplus 1 = \{a, 1\}$, then by (HMV_1),

$$\begin{aligned} M &= (a \oplus a) \cup (1 \oplus a) = \{1, a\} \oplus a = (1 \oplus 1) \oplus a = 1 \oplus (1 \oplus a) \\ &= 1 \oplus \{1, a\} = (1 \oplus 1) \cup (1 \oplus a) \\ &= (1 \oplus 1) \cup \{1, a\} \end{aligned}$$

Hence $0 \in 1 \oplus 1 = \{a, 1\}$, which is impossible. Thus $1 \oplus 1 \neq \{1, a\}$ and so by Lemma 3.5(v), $1 \oplus 1 = \{1\}$, $\{0, 1\}$ or M.

(vi). Let $a \oplus a = M$ and $1 \oplus a = \{0, 1\}$. Then by (HMV_1),

$$(1 \oplus 1) \oplus a = 1 \oplus (1 \oplus a) = 1 \oplus \{0, 1\} = (0 \oplus 1) \cup (1 \oplus 1) = M.$$

Now, if $1 \oplus 1 = \{1\}$, then $1 \oplus a = (1 \oplus 1) \oplus a = M$, which is a contradiction. Hence $1 \oplus 1 \neq \{1\}$ and so by Theorem 3.5(v), $1 \oplus 1 = \{0, 1\}, \{a, 1\}$ or M

Theorem 3.11. *There are 24 non-isomorphic proper hyper MV-algebras of order 3 such that $0 \oplus 1 = M$.*

Proof. By Lemma 3.5 (iii), $a \oplus a = \{1\}$, $\{0,1\}$, $\{1,a\}$ or M. If $a \oplus a = \{1\}$, then by Lemma 3.10 (ii), $a \oplus 1 = 1 \oplus 1 = M$ and so we must investigate the following table, which is a hyper MV-algebra.

$\oplus_{10}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{0,a,1\}$

If $a \oplus a = \{0,1\}$, then by Lemma 3.10 (iii), $a \oplus 1 = \{a,1\}$ or M and if $a \oplus 1 = \{a,1\}$, then $1 \oplus 1 = \{1\}$, $\{0,1\}$ or M. Thus we must investigate the following 3 cases which all of them are hyper MV-algebras.

$\oplus_{11}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,1\}$	$\{a,1\}$
1	$\{0,a,1\}$	$\{a,1\}$	$\{1\}$

$\oplus_{12}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,1\}$	$\{a,1\}$
1	$\{0,a,1\}$	$\{a,1\}$	$\{0,1\}$

$\oplus_{13}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,1\}$	$\{a,1\}$
1	$\{0,a,1\}$	$\{a,1\}$	$\{0,a,1\}$

If $a \oplus 1 = M$, then by Lemma 3.5 (v), $1 \oplus 1 = \{1\}$, $\{0,1\}$, $\{1,a\}$ or M. Hence we must investigate the following 4 cases which all of them are hyper MV-algebras.

$\oplus_{14}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{1\}$

$\oplus_{15}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{0,1\}$

$\oplus_{16}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{a,1\}$

$\oplus_{17}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{0,a,1\}$

Now, if $a \oplus a = \{a, 1\}$, then by Lemma 3.10 (iv), $a \oplus 1 = \{0, 1\}$ or M and if $a \oplus 1 = \{0, 1\}$, then $1 \oplus 1 = \{a, 1\}$ or M. Hence we must investigate the following 2 cases which both of them are hyper MV-algebras.

$\oplus_{18}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, 1\}$
1	$\{0, a, 1\}$	$\{0, 1\}$	$\{a, 1\}$

$\oplus_{19}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, 1\}$
1	$\{0, a, 1\}$	$\{0, 1\}$	$\{0, a, 1\}$

If $a \oplus 1 = M$, then by Lemma 3.5 (v), $1 \oplus 1 = \{1\}$, $\{0, 1\}$, $\{a, 1\}$ or M and so we must investigate the following 4 cases which all of them are hyper MV-algebras.

$\oplus_{20}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{1\}$

$\oplus_{21}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{0, 1\}$

$\oplus_{22}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{a, 1\}$

$\oplus_{23}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{0, a, 1\}$

Now, let $a \oplus a = M$. Then by Lemma 3.10 (v), $a \oplus 1 = \{1, a\}$, $\{0, 1\}$ or M. If $a \oplus 1 = \{1, a\}$, then $1 \oplus 1 = \{1\}, \{0, 1\}$ or M. Thus we must investigate the following 3 cases which all of them are hyper MV-algebras.

$\oplus_{24}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{a, 1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{1\}$

$\oplus_{25}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{a, 1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{0, 1\}$

$\oplus_{26}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{a, 1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{0, a, 1\}$

Also by Lemma 3.10 (v), if $a \oplus 1 = \{0, 1\}$, then $1 \oplus 1 = \{0, 1\}, \{a, 1\}$ or M. Hence we must investigate the following 3 cases which all of them are hyper

MV-algebras.

$\oplus_{27}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,1\}$
1	$\{0,a,1\}$	$\{0,1\}$	$\{0,1\}$

$\oplus_{28}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,1\}$
1	$\{0,a,1\}$	$\{0,1\}$	$\{a,1\}$

$\oplus_{29}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,1\}$
1	$\{0,a,1\}$	$\{0,1\}$	$\{0,a,1\}$

Finally, if $a\oplus 1 = M$, then by Lemma 3.5 (v), $1\oplus 1 = \{1\}, \{0,1\}, \{a,1\}$ or M. Hence we must investigate the following 4 cases which all of them are hyper MV-algebras.

$\oplus_{30}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{1\}$

$\oplus_{31}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{0,1\}$

$\oplus_{32}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{a,1\}$

$\oplus_{33}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{0,a,1\}$

Corolary 3.12. *There are* 33 *non-isomorphic hyper MV-algebras of order* 3.

Proof. By Theorems 3.3, 3.7, 3.9 and 3.11, we have 33 non-isomorphic hyper MV-algebras of order 3. □

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Optimal Control Policy of a Production and Inventory System for multi-product in Segmented Market

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Abstract

In this paper, we use market segmentation approach in multi-product inventory - production system with deteriorating items. The objective is to make use of optimal control theory to solve the production inventory problem and develop an optimal production policy that maximize the total profit associated with inventory and production rate in segmented market. First, we consider a single production and inventory problem with multi-destination demand that vary from segment to segment. Further, we described a single source production multi destination inventory and demand problem under the assumption that firm may choose independently the inventory directed to each segment. This problem has been discussed using numerical example.

Key words: Market Segmentation, Production-Inventory System, Optimal Control Problem

MSC2010: 97U99.

1 Introduction

Market segmentation is an essential element of marketing in industrialized countries. Goods can no longer be produced and sold without considering customer needs and recognizing the heterogeneity of these needs [1]. Earlier in this century, industrial development in various sectors of economy induced strategies of mass production and marketing. Those strategies were manufacturing oriented, focusing on reduction of production costs rather than satisfaction of customers. But as production processes become more flexible, and customer's affluence led to the diversification of demand, firms that identified the specific needs of groups of customers were able to develop the right offer for one or more submarkets and thus obtained a competitive advantage. Segmentation has emerged as a key planning tool and the foundation for effective strategy formulation. Nevertheless, market segmentation is not well known in mathematical inventory-production models. Only a few papers on inventory-production models deal with market segmentation [2, 3]. Optimal control theory, a modern extension of the calculus of variations, is a mathematical optimization tool for deriving control policies. It has been used in inventory-production [4, 6] to derive the theoretical structure of optimal policies. Apart from inventory-production, it has been successfully applied to many areas of operational research such as Finance [7, 8], Economics [9, 10, 11], Marketing [12, 13, 14, 15], Maintenance [16] and the Consumption of Natural Resources [17, 18, 19] etc. The application of optimal control theory in inventory-production control analysis is possible due to its dynamic behaviour. Continuous optimal control models provide a powerful tool for understanding the behaviour of production-inventory system where dynamic aspect plays an important role. Several papers have been written on the application of optimal control theory in Production-Inventory system with deteriorating items [20, 21, 22, 23].

In this paper, we assume that firm has defined its target market in a segmented consumer population and that it develop a production-inventory plan to attack each segment with the objective of maximizing profit. In addition, we shed some light on the problem in the control of a single firm with a finite production capacity (producing a multi-product at a time) that serves as a supplier of a multi product to multiple market segments. Segmented customers place demand continuously over time with rates that vary from segment to segment. In response to segmented customer demand, the firm must decide on how much inventory to stock and when to replenish this stock by producing. We apply optimal control theory to solve the problem and find the optimal production and inventory policies. The rest of the paper is organized as follows. Following this introduction, all the notations and assumptions needed in the sequel is stated in Section 2. In section 3, we described the single source production-inventory problem with multi-destination demand that vary from segment to segment and developed the optimal control theory problem and its solution. In section 4 of this paper we introduce optimal control formulation of a single source production- multi destination demand and inventory problem and discussion of solution. Numerical illustration is presented in the section 5 and finally conclusions are drawn in section 6 with some future research directions.

2 Notations and Assumptions

Here we begin the analysis by stating the model with as few notations as possible. Let us consider a manufacturing firm producing m product in segmented market environment. We introduce the notation that is used in the development of the model:

Notations:

T	:	Length of planning period,
$P_j(t)$	:	Production rate for j^{th} product,
$I_j(t)$	:	Inventory level for j^{th} product,
$I_{ij}(t)$	:	Inventory level for j^{th} product in i^{th} segment,
$D_{ij}(t)$	:	Demand rate for j^{th} product in i^{th} segment,
$h_j(I_j(t))$	:	Holding cost rate for j^{th} product, (single source inventory)
$h_{ij}(I_{ij}(t))$	:	Holding cost rate for j^{th} product in i^{th} segment, (multi destination)
c_j	:	The unit production cost rate for j^{th} product,
$\theta_j(t, I_j(t))$	:	Deterioration rate for j^{th} product, (single source inventory)
$\theta_{ij}(t, I_{ij}(t))$	:	The deterioration rate for j^{th} product in i^{th} segment, (multi destination)
$K_j(P_j(t))$	:	cost rate corresponding to the production rate for j^{th} product,
r_{ij}	:	The revenue rate per unit sale for j^{th} product in i^{th} segment,
ρ	:	Constant non-negative discount rate.

The model is based on the following assumptions: We assume that the time horizon is finite. The model is developed for multi-product in segmented market. The production, demand, and deterioration rates are function of time. The holding cost rate is function of inventory level & production cost rate depends on the production rate. The functions $h_{ij}(I_{ij}(t))$ (in case of single source $h_j(I_j(t))$ and $\theta_{ij}(t, I_{ij}(t))$ (in case of single source $\theta_j(t, I_j(t))$) are convex. All functions are assumed to be non negative, continuous and differentiable functions. This allows us to derive the most general and robust conclusions. Further, we will consider more specific cases for which we obtain some important results.

3 Single Source Production and Inventory-Multi-Destination Demand Problem

Many manufacturing enterprises use a production-inventory system to manage fluctuations in consumers demand for the product. Such a system consists of a manufacturing plant and a finished goods warehouse to store those products which manufactured but not immediately sold. Here, we assume that once a product is made and put inventory into single warehouse, and demand for all products comes from each segment. Let there be m products and n segments. (i.e., $j = 1, \ldots, m$ and $i = 1, \ldots, n$).

Therefore, the inventory evolution in segmented market is described by the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} I_j(t) = P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)), \ \forall j = 1, \ldots, m. \tag{1}$$

So far, firm want to maximize the total Profit during planning period in segmented market. Therefore, the objective functional for all segments is defined as

$$\max_{P_j(t) \geq \sum_{i=1}^{m} D_{ij}(t) + \theta_j(t, I_j(t))} J = \int_0^T \mathrm{e}^{-\rho t} \sum_{j=1}^{m} \left[\sum_{i=1}^{n} r_{ij} D_{ij}(t) - K_j(P_j(t)) - h_j(I_j(t)) \right] \mathrm{d}t + \int_0^T \mathrm{e}^{-\rho t} \sum_{j=1}^{m} \left[c_j \left(\sum_{i=1}^{n} D_{ij}(t) - P_j(t) \right) \right] \mathrm{d}t \tag{2}$$

Subject to the equation (1).This is the optimal control problem with m-control variable (rate of production) with m-state variable (inventory states). Since total demand occurs at rate $\sum_{i=1}^{n} D_{ij}(t)$ and production occurs at controllable rate $P_j(t)$ for j^{th}, it follows that $I_j(t)$ evolves according to the above state equation (1). The constraints $P_j(t) \geq \sum_{i=1}^{m} D_{ij}(t) - \theta_j(t, I_j(t))$ and $I_j(0) = I_{j0} \geq 0$ ensure that shortage are not allowed.
Using the maximum principle [10], the necessary conditions for (P_j^*, I_j^*) to be an optimal solution of above problem are that there should exist a piecewise continuously differentiable function λ and piecewise continuous function μ ,

called the adjoint and Lagrange multiplier function, respectively such that

$$H(t, I^*, P^*, \lambda) \geq H(t, I^*, P, \lambda),\ \text{for all } P_j(t) \geq \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t) \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_j(t) = -\frac{\partial}{\partial I_j} L(t, I_j, P_j, \lambda_j, \mu_j) \tag{4}$$

$$I_j(0) = I_{j0},\ \lambda_j(T) = 0 \tag{5}$$

$$\frac{\partial}{\partial P_j} L(t, I_j, P_j, \lambda_j, \mu_j) = 0 \tag{6}$$

$$\begin{aligned} &P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) \geq 0,\ \mu_j(t) \geq 0, \\ &\mu_j(t)\left[P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t))\right] = 0 \end{aligned} \tag{7}$$

Where, $H(t, I, P, \lambda)$ and $L(t, I, P, \lambda, \mu)$ are Hamiltonian function and Lagrangian function respectively. In the present problem Hamiltonian function and Lagrangian function are defined as

$$\begin{aligned} H = &\sum_{j=1}^{m}\left[\sum_{i=1}^{n} r_{ij} D_{ij}(t) + c_j\left(\sum_{i=1}^{n} D_{ij}(t) - P_j(t)\right) - K_j(P_j(t)) - h_j(I_j(t))\right] \\ &+ \sum_{j=1}^{m}\left[\lambda_j(t)\left(P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t))\right)\right] \end{aligned} \tag{8}$$

$$\begin{aligned} L = &\sum_{j=1}^{m}\left[\sum_{i=1}^{n} r_{ij} D_{ij}(t) + c_j\left(\sum_{i=1}^{n} D_{ij}(t) - P_j(t)\right) - K_j(P_j(t)) - h_j(I_j(t))\right] \\ &+ \sum_{j=1}^{m}\left[(\lambda_j(t) + \mu_j(t))\left(P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t))\right)\right] \end{aligned} \tag{9}$$

A simple interpretation of the Hamiltonian is that it represents the overall profit of the various policy decisions with both the immediate and the future effects taken into account and the value of $\lambda_j(t)$ at time t describes the future effect on profits upon making a small change in $I_j(t)$. Let the Hamiltonian H for all segments is strictly concave in $P_j(t)$ and according to the Mangasarian sufficiency theorem [4, 10]; there exists a unique Production rate.

From equation (4) and (6), we have following equations respectively

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_j(t) = \rho\lambda_j(t) - \left\{-\frac{\partial h_j(I_j(t))}{\partial I_j} - (\lambda_j(t) + \mu_j(t))\frac{\partial \theta_j(t, I_j(t))}{\partial I_j}\right\}, \tag{10}$$

for all $j = 1, \cdots, m$

$$\lambda_j(t) + \mu_j(t) = c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(t)). \tag{11}$$

Now, consider equation (7). Then for any t, we have either

$$P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) = 0 \text{ or}$$

$$P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) > 0 \ \forall\ j = 1, \cdots, m.$$

3.1 Case 1:

Let S is a subset of planning period $[0, T]$, when $P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) = 0$. Then $\frac{\mathrm{d}}{\mathrm{d}t}I_j(t) = 0$ on S, in this case $I^*(t)$ is obviously constant on S and the optimal production rate is given by the following equation

$$P_j^*(t) = \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j^*(t)), \text{ for all } t \in S \tag{12}$$

By equation (10) and (11), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_j(t) = \rho\lambda_j(t) - \left\{-\frac{\partial h_j(I_j(t))}{\partial I_j} - \left(c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(t))\right)\frac{\partial \theta_j(t, I_j(t))}{\partial I_j}\right\} \tag{13}$$

After solving the above equation, we get a explicit from of the adjoint function $\lambda_j(t)$. From the equation (10)), we can obtain the value 0f Lagrange multiplier $\mu_j(t)$.

3.2 Case 2:

$P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) > 0$, for $t \in [0, T]\backslash S$. Then $\mu_j(t) = 0$ on $t \in [0, T]\backslash S$. In this case the equation (10) and (11) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_j(t) = \rho\lambda_j(t) - \left\{-\frac{\partial h_j(I_j(t))}{\partial I_j} - \lambda_j(t)\frac{\partial \theta_j(t, I_j(t))}{\partial I_j}\right\}, \ \forall\ j = 1, \cdots, m \tag{14}$$

$$\lambda_j(t) = c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(t)) \tag{15}$$

Cobining these equation with the state equation, we have the following second order differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_j(t)\frac{d^2}{dP_j^2}K_j(P_j) - \left[\rho + \frac{\partial\theta_j(t, I_j(t))}{\partial I_j}\right]\frac{\mathrm{d}}{dP_j}K_j(P_j) = c_j\left(\rho + \frac{\partial\theta_j(t, I_j(t))}{\partial I_j}\right) + \frac{\partial h_j(t, I_j(t))}{\partial I_j} \quad (16)$$

and $I_j(0) = I_{j0}$, $c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(T)) = 0$. For illustration purpose, let us assume the following forms the exogenous functions $K_j(P_j) = k_jP_j^2/2$, $h_j(t, I_j(t)) = h_jI_j(t)$ and $\theta_j(t, I_j(t)) = \theta_jI_j(t)$, where k_j h_j θ_j are positive constants for all $j = 1, \cdots, m$.
For these functions the necessary conditions for (P_j^*, I_j^*) to be optimal solution of problem (2) with equation (1) becomes

$$\frac{d^2}{\mathrm{d}t^2}I_j(t) - \rho\frac{\mathrm{d}}{\mathrm{d}t}I_j(t) - (\rho + \theta_j)\theta_{1j}I_j(t) = \eta_j(t) \quad (17)$$

with $I_j(0) = I_{j0}$, $c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(T)) = 0$.
Where, $\eta_j(t) = -\sum_{i=1}^n\left(\frac{\mathrm{d}}{\mathrm{d}t}D_{ij}(t)\right) + (\rho + \theta_{1j})\left(\sum_{i=1}^n D_{ij}(t)\right) + \frac{(c_j(\rho+\theta_{1j})+h_j)}{k_j}$.
This problem is a two point boundary value problem.

Proposition 3.1. *The optimal solution of (P_j^*, I_j^*) to the problem is given by*

$$I_j^*(t) = a_{1j}\mathrm{e}^{m_{1j}t} + a_{2j}\mathrm{e}^{m_{2j}t} + Q_j(t), \quad (18)$$

and the corresponding P_j^* is given by

$$P_j^*(t) = a_{1j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{1j}t} + a_{2j}(m_{2j} + \theta_j)\mathrm{e}^{m_{2j}t} + \frac{\mathrm{d}}{\mathrm{d}t}Q_j(t) + \theta_{1j}Q_j(t) + \sum_{i=1}^n D_{ij}. \quad (19)$$

The values of the constant a_{1j}, a_{2j}, m_{1j}, m_{2j} are given in the proof, and $Q_j(t)$ is a particular solution of the equation (17).

Proof. The solution of the two point boundary value problem (17) is given by standard method. Its characteristic equation $m_j^2 - \rho m_j - (\rho + \theta_j)\theta_{1j} = 0$, has two real roots of opposite sign, given by

$$m_{1j} = \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4(\rho + \theta_{1j})\theta_{1j}}\right) < 0,$$

$$m_{2j} = \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4(\rho + \theta_{1j})\theta_{1j}}\right) > 0,$$

and therefore $I_j^*(t)$ is given by (18), where $Q_j(t)$ is the particular solution. Then initial and terminal condition used to determineed the values of constant a_{1j} and a_{2j} as follows

$$\begin{aligned}
&a_{1j} + a_{2j} + Q_j(0) = I_{j0},\\
&a_{1j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{1j}T} + a_{2j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{2j}T}\\
&+\left(\frac{c_j}{k_j} + \frac{\mathrm{d}}{\mathrm{d}t}Q_j(T) + \theta_{1j}Q_j(T) + \sum_{i=1}^{n} D_{ij}(T)\right) = 0.
\end{aligned}$$

By putting $b_{1j} = I_{j0} - Q_j(0)$ and $b_{2j} = -(\frac{c_j}{k_j} + \frac{\mathrm{d}}{\mathrm{d}t}Q_j(T) + \theta_{1j}Q_j(T) + \sum_{i=1}^{n} D_{ij}(T))$, we obtain the following system of two linear equation with two unknowns

$$\begin{aligned}
&a_{1j} + a_{2j} = b_{1j}\\
&a_{1j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{1j}T} + a_{2j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{2j}T} = b_{2j}
\end{aligned} \tag{20}$$

The value of P_j^* is deduced using the values of I_j^* and the state equation. $\square$

4 Single Source Production- Multi Destination Demand and Inventory Problem

We assume the single source production and multi destination demand-inventory system. Hence, the inventory evolution in each segmented is described by the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}I_{ij}(t) = \gamma_{ij}P_j(t) - D_{ij}(t) - \theta_{ij}(t, I_{ij}(t)),\ \forall\ j = 1, \cdots, m;\ i = 1, \cdots, n. \tag{21}$$

Here, $\gamma_{ij} > 0$, $\sum_{i=1}^{n} \gamma_{ij} = 1$, $\forall\ j = 1, \cdots m$ with the conditions $I_{ij}(0) = I_{ij}^0$ and $\gamma_{ij}P_j(t) \geq D_{ij}(t) - \theta_{ij}(t, I_{ij}(t))$. We called $\gamma_{ij} > 0$ the segment production spectrum and $\gamma_{ij}P_j(t)$ define the relative segment production rate of j^{th} product towards i^{th} segment. We develop a marketing-production model in which firm seeks to maximize its all profit by properly choosing production and market segmentation. Therefore, we defined the profit maximization objective function as follows:

$$\begin{aligned}
&\max_{\gamma_{ij}P_j(t)\geq D_{ij}(t)-\theta_{ij}(t,I_{ij}(t))} J = \\
&= \int_0^T \mathrm{e}^{-\rho t}\sum_{j=1}^{m}\left[\sum_{i=1}^{n} r_{ij}D_{ij}(t) + c_j\left(\sum_{i=1}^{n}(D_{ij}(t)-\gamma_{ij}P_j(t))\right)\right]\mathrm{d}t \\
&- \int_0^T \mathrm{e}^{-\rho t}\sum_{j=1}^{m}\left[\sum_{i=1}^{n} h_{ij}(I_{ij}(t)) - K_j(P_j(t))\right]\mathrm{d}t \qquad (22)
\end{aligned}$$

subject to the equation (21). This is the optimal control problem (production rate) with m control variable with mn state variable (stock of inventory). To solve the optimal control problem expressed in equation (21) and (22), the following Hamiltonian and Lagrangian are defined as

$$\begin{aligned}
H =& \sum_{j=1}^{m}\left[\sum_{i=1}^{n} r_{ij}D_{ij}(t) + c_j\left(\sum_{i=1}^{n}(D_{ij}(t)-\gamma_{ij}P_j(t))\right)\right] \\
&- \sum_{j=1}^{m}\left[\sum_{i=1}^{n} h_{ij}(I_{ij}(t)) + K_j(P_j(t))\right] \\
&+ \sum_{j=1}^{m}\sum_{i=1}^{n}\lambda_{ij}(t)[\gamma_i P_j(t) - D_{ij}(t) - \theta_{ij}(t, I_{ij}(t))] \qquad (23)
\end{aligned}$$

$$\begin{aligned}
L =& \sum_{j=1}^{m}\left[\sum_{i=1}^{n} r_{ij}D_{ij}(t) + c_j\left(\sum_{i=1}^{n}(D_{ij}(t)-\gamma_{ij}P_j(t))\right)\right] \\
&- \sum_{j=1}^{m}\left[\sum_{i=1}^{n} h_{ij}(I_{ij}(t)) + K_j(P_j(t))\right] \\
&+ \sum_{j=1}^{m}\sum_{i=1}^{n}(\lambda_{ij} + \mu_{ij}(t))[\gamma_i P_j(t) - D_{ij}(t) - \theta_{ij}(t, I_{ij}(t))] \qquad (24)
\end{aligned}$$

Equation (4), (6) and (21) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}(t) = \rho\lambda_{ij}(t) - \left\{-\frac{\partial h_{ij}(I_{ij}(t))}{\partial I_{ij}} - \lambda_{ij}(t)\frac{\partial\theta_{ij}(t, I_{ij}(t))}{\partial I_{ij}}\right\}, \qquad (25)$$

for all $i = 1, \cdots, n,\ j = 1, \cdots, m$

$$\sum_{i=1}^{n}(\lambda_{ij}(t) + \mu_{ij}(t))\gamma_i = c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(t)) \qquad (26)$$

In the next section of the paper, we consider only case when

$$\gamma_{ij}P_j(t) - D_{ij}(t) - \theta_{ij}(t, I_{ij}(t)) > 0,\ \forall\, i,\ j.$$

4.1 Case 2:

$\gamma_{ij}P_j(t)-D_{ij}(t)-\theta_{ij}(t,I_{ij}(t))>0\;\forall\; i,\; j$, for $t\in[0,T]\backslash S$. Then $\mu_{ij}(t)=0$ on $t\in[0,T]\backslash S$. In this case, the equation (25) and (26) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}(t)=\rho\lambda_{ij}(t)-\left\{-\frac{\partial h_{ij}(I_{ij}(t))}{\partial I_{ij}}-\lambda_{ij}(t)\frac{\partial\theta_{ij}(t,I_{ij}(t))}{\partial I_{ij}}\right\} \tag{27}$$

$$\sum_{i=1}^{n}\gamma_{ij}\lambda_{ij}(t)=c_j+\frac{\mathrm{d}}{dP_j}K_j(P_j(t)) \tag{28}$$

Cobining these equation with the state equation, we have the following second order differential equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}P_j(t)\frac{d^2}{dP_j^2}K_j(P_j)-\frac{1}{n}\sum_{i=1}^{n}\left(\rho+\frac{\partial\theta_i(t,I_{ij}(t))}{\partial I_{ij}}\right)\frac{\mathrm{d}}{dP_j}K_j(P_j)\\ =\sum_{i=1}^{n}c_j\gamma_i\left(\rho+\frac{\partial\theta_{ij}(t,I_{ij}(t))}{\partial I_{ij}}\right)+\sum_{i=1}^{n}\gamma_i\frac{\partial h_{ij}(t,I_{ij}(t))}{\partial I_{ij}}\end{aligned} \tag{29}$$

with $I_j(0)=I_{ij}^0$, $\sum_{i=1}^{n}\gamma_{ij}\lambda_{ij}(T)=0\quad\rightarrow\lambda_{ij}(T)=0\;\forall\; i$ and j, $c_j+\frac{\mathrm{d}}{dP_j}K_j(P_j(T))=0$. For illustration, let us assume the following forms the exogenous functions $K_j(P_j)=k_jP_j^2/2$, $h_{ij}(t,I_{ij}(t))=h_{ij}I_{ij}(t)$ and $\theta_{ij}(t,I_{ij}(t))=\theta_{ij}I_{ij}(t)$, where $k_j\; h_{ij}\;\theta_{ij}$ are positive constants.
For these functions the necessary conditions for (P_j^*,I_{ij}^*) to be optimal solution of problem (19) with equation (18) becomes

$$\frac{d^2I_{ij}(t)}{\mathrm{d}t^2}+(\theta_{ij}-A)\frac{dI_{ij}(t)}{\mathrm{d}t}-A\theta_iI_{ij}(t)=\eta_{ij}(t) \tag{30}$$

with $I_{ij}(0)=I_{ij}^0$, $\lambda_{ij}(T)=0\;\forall\; i$, $c_j+\frac{\mathrm{d}}{dP_j}K_j(P_j(T))=0$.
Where, $\eta_{ij}(t)=-D_{ij}(t)A+\frac{\gamma_j}{k_j}\left[\sum_{i=1}^{n}\gamma_i(h_{ij}+c_j(\rho+\theta_{ij}))\right]+\frac{dD_{ij}(t)}{\mathrm{d}t}$, $A=\sum_{i=1}^{n}\frac{(\rho+\theta_{ij})}{n}$. This problem is a two point boundary value problem.
The above system of two point boundary value problem (29) is solved by same method that we used in to solve (17).

5 Numerical Illustration

In order to demonstrate the numerical results of the above problem, the discounted continuous optimal problem (2) is transferred into equivalent discrete problem [24] that is solved to present numerical solution. The discrete optimal control can be written as follows:

$$J = \sum_{k=1}^{T}\left(\sum_{j=1}^{m}\left[\sum_{i=1}^{n}(r_{ij}(k-1)D_{ij}(k-1))\right]\right)\left(\frac{1}{(1+\rho)^{k-2}}\right)$$
$$+\sum_{k=1}^{T}\left(\sum_{j=1}^{m}c_j\left(\sum_{i=1}^{n}D_{ij}(k-1)-P_j(k-1)\right)\right)\left(\frac{1}{(1+\rho)^{k-2}}\right)$$
$$-\sum_{k=1}^{T}\left(\sum_{j=1}^{m}[K_j(P_j(k-1)+h_j(I_j(k-1)))]\right)\left(\frac{1}{(1+\rho)^{k-2}}\right)$$

such that

$$I_j(k) = I_j(k-1) + p_j(k-1) - \sum_{i=1}^{n} D_{ij}(k-1) - \theta_j(k-1, I_j(k-1))$$

for all $j = 1, \cdots, m$.

Similar discrete optimal control problem can be written for single source production multi destination and inventory control problem. These discrete optimal control problems are solved by using Lingo11. We assume that the duration of all the time periods are equal and demand are equal from segment for each product. The number of market segmentsis 4 and the number of products is 3. The value of parameters are r_{i1} = 2.55, 2.53, 2.53, 2.54;

Table 1: The Optimal production and inventory rate in segment market

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10
P_1	100	86	80	73	64	53	39	21	5	0
P_2	110	81	76	70	62	52	38	21	5	0
P_3	140	79	75	69	61	51	38	21	5	0
I_1	20	98	154	199	232	254	262	255	231	193
I_2	20	107	156	194	222	238	241	231	205	166
I_3	20	137	179	211	233	244	244	231	203	161

r_{i2} = 2.52, 2.53, 2.54, 2.53; r_{i3} = 2.51, 2.54, 2.54, 2.52 for segments $i = 1$ to 4; $c_j = 1$; $k_j = 2$; $\theta_j = 0.10,\ 0.12,\ 0.13$; $h_j = 1$; for all the three products. The optimal production rate and inventory for every product for each segment is shown in Table 1 and their corresponding total profit is \$177402.70.

The optimal trajectories of production and inventory rate for every product for each segment are shown in Fig1, Fig2 and Fig3 respectively (Appendix). In case of single source production-multi destination demand and inventory, the number of market segments M is 4 and the number of products is 3. The values of additional parameters are each segment is shown in Table 2.

Table 2: The values of parameter of deteriorating rate and holding cost rate constant

Segment	θ_{i1}	θ_{i2}	θ_{i3}	h_{i1}	h_{i2}	h_{i3}
$M1$	0.10	0.11	0.11	1.0	1.1	1.0
$M2$	0.11	0.12	0.12	1.1	1.2	1.1
$M3$	0.13	0.11	0.11	1.2	1.1	1.2
$M4$	0.11	0.13	0.11	1.1	1.0	1.3

Table 3: Values of the parameter for single source production-multi destination demand and inventory problem in each segment

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10
P_1	100	85	79	73	65	54	41	23	5	0
P_2	110	82	77	70	62	52	38	21	5	0
P_3	140	83	77	71	63	52	38	21	5	0
I_{11}	20	98	153	197	231	254	263	258	236	197
I_{12}	20	97	152	195	227	247	255	248	225	185
I_{13}	20	97	150	192	223	242	247	239	214	173
I_{14}	20	97	149	190	218	236	240	230	204	162
I_{21}	20	108	158	198	227	245	250	242	217	178
I_{22}	20	107	157	195	222	238	242	232	206	167
I_{23}	20	108	158	198	227	245	250	242	217	178
I_{24}	20	107	156	192	218	232	235	223	196	156
I_{31}	20	138	186	223	250	265	268	258	231	191
I_{32}	20	138	184	220	244	258	260	248	219	178
I_{33}	20	138	185	223	250	265	268	258	231	191
I_{34}	20	138	186	223	250	265	268	258	231	191

The optimal production rate and inventory for every product for each segment is shown in Table 3 with production spectrum $\gamma_{11} = 0.10,\ \gamma_{12} = 0.10,\ \gamma_{13} = 0.77,\ \gamma_{14} = 0.03;\ \gamma_{21} = 0.12,\ \gamma_{22} = 0.12,\ \gamma_{23} = 0.75,\ \gamma_{24} = 0.01;\ \gamma_{31} = 0.14,\ \gamma_{32} = 0.14,\ \gamma_{33} = 0.72,\ \gamma_{34} = 0.04$. The optimal value of total profit for all products is $185876.90. In case of single source production-multi destination demand and inventory, The optimal trajectories of production and inventory rate for every product for each segment are shown in Fig4, Fig5, Fig6 and Fig7 respectively (Appendix).

6 Conclusion

In this paper, we have introduced market segmentation concept in the production inventory system for multi product and its optimal control formulation. We have used maximum principle to determine the optimal production rate policy that maximizes the total profit associated with inventory and production rate. The resulting analytical solution yield good insight on how production planning task can be carried out in segmented market environment. In order to show the numerical results of the above problem, the discounted continuous optimal problem is transferred into equivalent discrete problem [24] that is solved using Lingo 11 to present numerical solution. In the present paper, we have assumption that the segmented demand for each product is a function of time only. A natural extension to the analysis developed here is the consideration of segmented demand that is a general functional of time and amount of onhand stock (inventory).

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Appendix

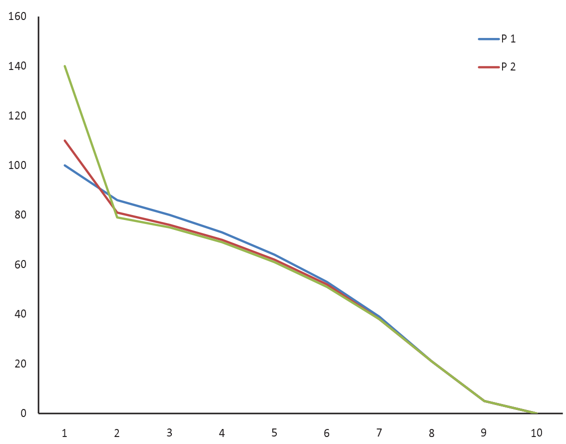

Fig-1

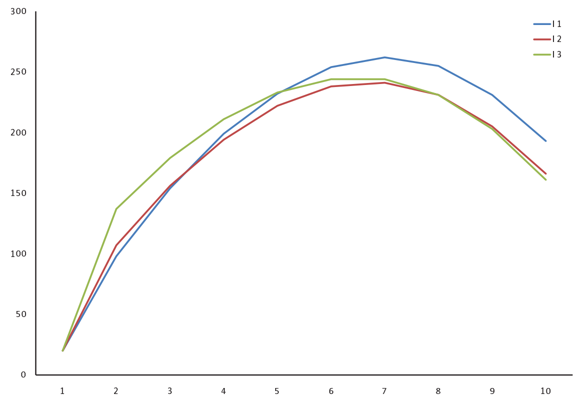

Fig-2

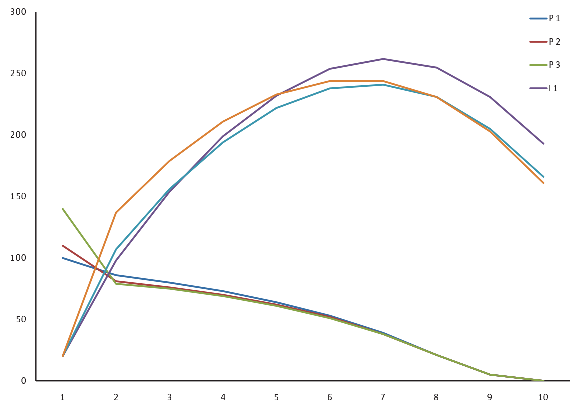

Fig-3

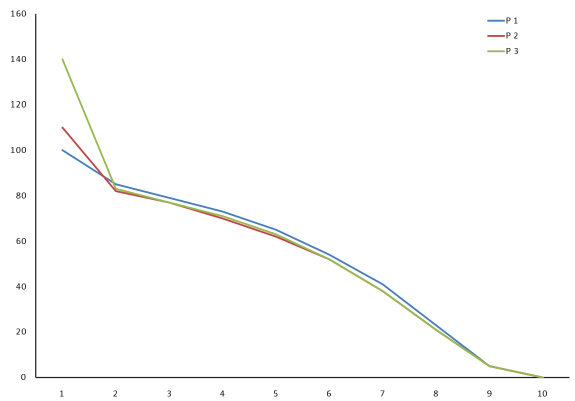

Fig-4

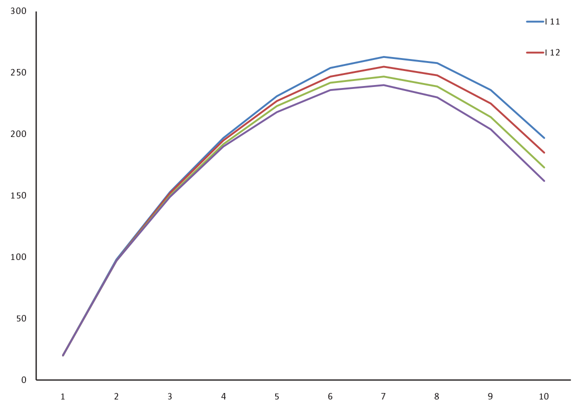

Fig-5

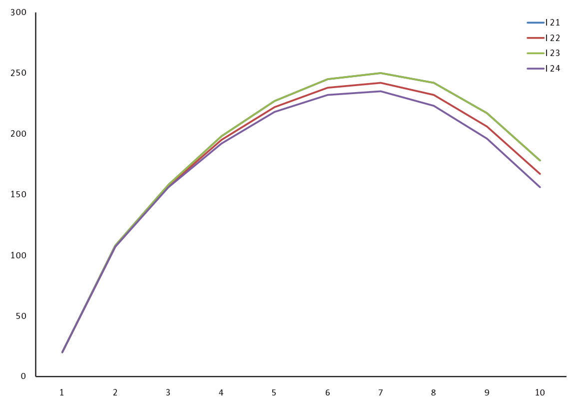

Fig-6

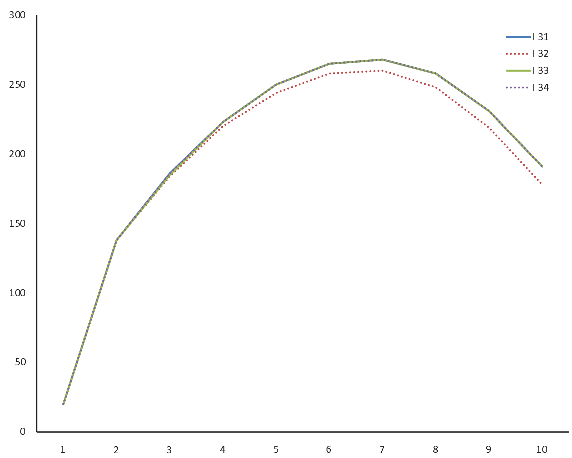

Fig-7

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The Divisors' Hyperoperations

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Abstract

In the set $\mathbb{N}$ of the Natural Numbers we define two hyperoperations based on the divisors of the addition and multiplication of two numbers. Then, the properties of these two hyperoperations are studied together with the resulting hyperstructures. Furthermore, from the coexistence of these two hyperoperations in $\mathbb{N}^*$, an H_v-ring is resulting which is dual.

Key words: Hyperstructures, H_v-structures

MSC2010: 20N20, 16Y99.

1 Introduction

In 1934, F. Marty introduced the definitions of the *hyperoperation* and of the *hypergroup* as a generalization of the operation and the group respectively.

Let H be a set and $\circ : H \times H \to P'(H)$ be a hyperoperation, [2], [3], [5], [6], [8]:
The hyperoperation $(\circ)$ in H is called ***associative***, if

$$(x \circ y) \circ z = x \circ (y \circ z), \forall x, y, z \in H.$$

The hyperoperation $(\circ)$ in H is called ***commutative***, if

$$x \circ y = y \circ x, \forall x, y \in H.$$

An algebraic hyperstructure $(H, \circ)$, i.e. a set H equipped with the hyperoperation $(\circ)$, is called ***hypergroupoid***. If this hyperoperation is associative, then the hyperstructure is called ***semihypergroup***. The semihypergroup $(H, \circ)$, is called ***hypergroup*** if it satisfies the ***reproduction*** axiom:

$$x \circ H = H \circ x, \forall x \in H.$$

One of the topics of great interest, in the last years, is the H_V*-structures*, which was introduced by T. Vougiouklis in 1990 [7]. The class of H_V-structures is the largest class of algebraic hyperstructures. These structures satisfy weak axioms, where the non-empty intersection replaces the equality, as bellow [8]:

i) The $(\circ)$ in H is called ***weak associative***, we write ***WASS***, if

$$(x \circ y) \circ z \cap x \circ (y \circ z) \neq \emptyset, \forall x, y, z \in H.$$

ii) The $(\circ)$ is called ***weak commutative***, we write ***COW***, if

$$(x \circ y) \cap (y \circ x) \neq \emptyset, \forall x, y \in H.$$

iii) If H is equipped with two hyperoperations $(\circ)$ and $(*)$, then $(*)$ is called ***weak distributive*** with respect to $(\circ)$, if

$$[x * (y \circ z)] \cap [(x * y) \circ (x * z)] \neq \emptyset, \forall x, y, z \in H.$$

The hyperstructure $(H, \circ)$ is called ***H_v-semigroup*** if it is *WASS* and it is called ***H_v-group*** if it is a reproductive (i.e. $x \circ H = H \circ x = H, \forall x \in H$) H_v-semigroup. It is called ***commutative H_v-group*** if $(\circ)$ is commutative and it is called ***H_v- commutative group*** if $(\circ)$ is weak commutative. The hyperstructure $(H, \circ, *)$ is called ***H_v-ring*** if both hyperstructures $(\circ)$ and $(*)$ are *WASS*, the reproduction axiom is valid for $(\circ)$, and $(*)$ is weak distributive with respect to $(\circ)$.

It is denoted [4] by E_* the set of the unit elements with respect to $(*)$ and by $I_*(x, e)$ the set of the inverse elements of x associated with the unit e, with respect to $(*)$.

An H_v-ring $(R, +, \cdot)$ is called ***Dual H_v*-ring**, if $(R, \cdot, +)$ is an H_v-ring, too [4].

Let $(H, \cdot)$ be a hypergroupoid. An element $e \in H$ is called *right unit* element if $a \in a \cdot e, \forall a \in H$ and is called *left unit element* if $a \in e \cdot a, \forall a \in H$. The element $e \in H$ is called *unit* element if $a \in a \cdot e \cap e \cdot a, \forall a \in H$.

Let $(H, \cdot)$ be a hypergroupoid endowed with at least one unit element. An element $a' \in H$ is called an *inverse* element of the element $a \in$H, if there exists a unit element $e \in H$ such that $e \in a' \cdot a \cap a \cdot a'$.

Moreover, let us define here: If $x \in x \cdot y (resp. x \in y \cdot x) \forall y \in H$, then, x is called *left absorbing-like* element (resp. *right absorbing-like* element). An element $a \in H$ is called *idempotent* element if $a^2 = a$. The n^{th} power of an element h, denoted h^s, is defined to be the union of all expressions of n times of h, in which the parentheses are put in all possible ways. An H_v-group (H,$\cdot$) is called *cyclic* with finite period with respect to $h \in H$, if there exists a positive integer s, such that $H = h^1 \cup h^2 \quad \ldots \cup h^s$. The minimum such s is called *period of the generator h.* If all generators have the same period, then H is *cyclic with period.* If there exists $h \in H$ and s positive integer, the minimum one, such that $H = h^s$, then H is called *single-power cyclic* and h is a generator with *single-power period s*. The cyclicity in the infinite case is defined similarly. Thus, for example, the H_v-group $(H, \cdot)$ is called *single-power cyclic with infinite period* with generator h if every element of H belongs to a power of h and there exists $s_0 \geq 1$, such that $\forall s \geq s_0$ we have:

$$h^1 \cup h^2 \cup \cdots \cup h^{s-1} \subset h^s.$$

2 The divisors' hyperoperation due to addition in N

Let $\mathbb{N}$ be the set of the Natural Numbers. Let us define the hyperoperation $(\oslash)$ in $\mathbb{N}$ as follows:

Definition 2.1. For every $x, y \in \mathbb{N}$

$$\oslash \colon \mathbb{N} \times \mathbb{N} \to \mathcal{P}(\mathbb{N}) - \{\emptyset\} : (x, y) \mapsto x \oslash y \subset \mathbb{N}$$

such that

$$x \oslash y = \{z \in \mathbb{N} : x + y = z \cdot \lambda, \lambda \in \mathbb{N}\}$$

where $(+)$ and $(\cdot)$ are the usual operations of the addition and multiplication in $\mathbb{N}$, respectively.
We call the above hyperoperation, *divisors' hyperoperation due to addition.*

Some properties of the divisors' hyperoperation due to addition

1. $x \oslash y = y \oslash x, \forall x, y \in \mathbb{N}$
2. $0 \oslash 0 = \mathbb{N}$
3. $0 \oslash 1 = 1 \oslash 0 = 1$
4. $\{1, x + y\} \subset x \oslash y, \forall x, y \in \mathbb{N}$
5. If $x + y = \kappa \cdot \nu \Rightarrow \{1, \kappa, \nu, \kappa \cdot \nu\} \subset x \oslash y, x, y, \kappa, \nu \in \mathbb{N}$.

Remark 2.2. If $x + y = p$, where $p \in \mathbb{N}$ is a prime number, then $x \oslash y = \{1, p\}, x, y \in \mathbb{N}$.

Proposition 2.3. The number 0 is a unit element of the divisors' hyperoperation due to addition.

Proof. Indeed, for $x \in \mathbb{N}$

$$x \oslash 0 = \{z \in \mathbb{N} : x + 0 = z \cdot \lambda, \lambda \in \mathbb{N}\} = \{z \in \mathbb{N} : x = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni x.$$

Then,

$$x \in (x \oslash 0) \cap (0 \oslash x), \forall x \in \mathbb{N}. \quad \square$$

Remark 2.4. Since, there is no $x' \in \mathbb{N}$ such that $0 \in (x \oslash x') \cap (x' \oslash x)$ when $x \neq 0$, the number 0 is the only one in $\mathbb{N}$ having an inverse element (and that is 0) associated with the unique unit element 0 of the divisors' hyperoperation due to addition, i.e. $0 \in 0 \oslash 0$.

Proposition 2.5. The number 1 is an absorbing-like element of the divisors' hyperoperation due to addition.

Proof. Indeed,

$$1 \in x \oslash y, \forall x, y \in \mathbb{N} \Rightarrow 1 \in 1 \oslash y, \forall y \in \mathbb{N} \Rightarrow 1 \in (1 \oslash y) \cap (y \oslash 1), \forall y \in \mathbb{N}. \quad \square$$

Proposition 2.6. If $y = n \cdot x$, $x, n \in \mathbb{N}$ then $\{1, x, 1 + n, x(1 + n)\} \subset x \oslash y$.

Proof. Let $y = n \cdot x$, $x, n \in \mathbb{N}$ then

$$\begin{aligned} x \oslash y &= \{z \in \mathbb{N} : x + y = z \cdot \lambda, \lambda \in \mathbb{N}\} = \{z \in \mathbb{N} : x + nx = z \cdot \lambda, \lambda \in \mathbb{N}\} \\ &= \{z \in \mathbb{N} : x(1 + n) = z \cdot \lambda, \lambda \in \mathbb{N}\} \supset \{1, x, 1 + n, x(1 + n)\}. \end{aligned} \quad \square$$

Proposition 2.7. If $x \in \mathbb{N}$ is a prime number then $x^2 = \{1, 2, x, 2x\}$.

Proof. Let $x \in \mathbb{N}$, be a prime number then

$$x^2 = x \oslash x = \{z \in \mathbb{N} : x + x = z \cdot \lambda, \lambda \in \mathbb{N}\} = \{z \in \mathbb{N} : 2x = z \cdot \lambda, \lambda \in \mathbb{N}\}.$$

According to property 5, $\{1, 2, x, 2x\} \subset x^2$, but since x is prime, $x^2 = \{1, 2, x, 2x\}$. $\square$

Proposition 2.8. $x \oslash \mathbb{N}^* = \mathbb{N}^* \oslash x = \mathbb{N}^*, \forall x \in \mathbb{N}^*$.

Proof. Let $x \in \mathbb{N}^*$, then

$$x \oslash \mathbb{N}^* \supset x \oslash (nx) \ni n + 1,\ n \in \mathbb{N}^*, \text{ according to Proposition 2.6.}$$

So, we proved that $n+1 \in x \oslash \mathbb{N}^*, \forall x, n \in \mathbb{N}^*$ and since $1 \in x \oslash \mathbb{N}^*, \forall x \in \mathbb{N}^*$, we get

$$x \oslash \mathbb{N}^* = \mathbb{N}^* \oslash x = \mathbb{N}^*, \forall x \in \mathbb{N}^*. \quad \square$$

Remark 2.9. Notice that, for $x \in \mathbb{N}^*$

$$x \oslash \mathbb{N} = \bigcup_{n \in IN} (x \oslash n) = \bigcup_{n \in I\mathbb{N}} \{z \in \mathbb{N} : x + n = z \cdot \lambda, \lambda \in \mathbb{N}^*\} \supseteq$$
$$\supseteq \bigcup_{n \in I\mathbb{N}} \{z \in \mathbb{N} : x + nx = z \cdot \lambda, \lambda \in \mathbb{N}^*\} = \bigcup_{n \in IN} (x \oslash nx).$$

But from Proposition 2.6,

$$\bigcup_{n \in I\mathbb{N}} (x \oslash nx) \supset \bigcup_{n \in IN} \{1, x, n+1, x(n+1)\} \supset \bigcup_{n \in IN} \{n+1\} = \mathbb{N}^*.$$

So,

$$x \oslash N = N \oslash x = N^*, \forall x \in N^*.$$

Proposition 2.10. The divisors' hyperoperation due to addition is a weak associative one in $\mathbb{N}^*$.

Proof. For $x, y, z \in \mathbb{N}^*$

$$(x \oslash y) \oslash z = \{w \in \mathbb{N}^* : x + y = w \cdot \lambda, \lambda \in \mathbb{N}^*\} \oslash z = \bigcup_{w \in IN*} (w \oslash z) =$$
$$= \bigcup_{w \in IN*} \{w' \in \mathbb{N}^* : w + z = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\}$$
$$\supset \{w' \in \mathbb{N}^* : x + y + z = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\} \qquad (I)$$

On the other hand

$$x \oslash (y \oslash z) = x \oslash \{v \in \mathbb{N}^* : y + z = v \cdot \rho, \rho \in \mathbb{N}^*\} = \bigcup_{v \in IN*} (x \oslash v) =$$
$$= \bigcup_{v \in IN*} \{v' \in \mathbb{N}^* : x + v = v' \cdot \rho', \rho' \in \mathbb{N}^*\}$$
$$\supset \{v' \in \mathbb{N}^* : x + y + z = v' \cdot \rho', \rho' \in \mathbb{N}^*\} \qquad (II)$$

From (I) and (II) we get:

$$(x \oslash y) \oslash z \cap x \oslash (y \oslash z) = \{n \in N^* : x+y+z = n \cdot \mu, \mu \in N^*\} \neq \emptyset, \forall x, y, z \in \mathbb{N}^*. \quad \square$$

Since the divisors' hyperoperation due to addition is commutative, according to Propositions 2.8 and 2.10, we get the following:

Proposition 2.11. The hyperstructure $(\mathbb{N}^*, \oslash)$ is a commutative H_v-group.

Proposition 2.12. For $(x, y, z) \in N^* \times N^* \times N^*$, if $x = z$, then the divisors' hyperoperation due to addition is strong associative.

Proof. Let $(x, y, z) \in \mathbb{N}^* \times \mathbb{N}^* \times N^*$ such that $x = z$, then due to commutativity we get:

$$(x \oslash y) \oslash z = (x \oslash y) \oslash x = x \oslash (x \oslash y) = x \oslash (y \oslash x) = x \oslash (y \oslash z). \quad \square$$

Proposition 2.13. The $(\mathbb{N}^*, \oslash)$, is a single-power cyclic H_v-group with infinite period where every $x \in \mathbb{N}^*$ is a generator.

Proof. For $x \in \mathbb{N}^*$, notice that

$$\begin{aligned}
&x^1 = \{x\} \\
&x^2 = x \oslash x = \{z \in \mathbb{N}^* : 2x = z \cdot \lambda, \lambda \in \mathbb{N}^*\} \supset \{1, 2\} \\
&x^3 = x^2 \oslash x = \{z \in \mathbb{N}^* : 2x = z \cdot \lambda, \lambda \in \mathbb{N}^*\} \oslash x = \bigcup_{z \in IN*} (z \oslash x) \\
&\quad = \bigcup_{z \in IN*} \{w \in \mathbb{N}^* : z + x = w \cdot \rho, \rho \in \mathbb{N}^*\} \supset \\
&\quad \supset \{w' \in \mathbb{N}^* : 2x + x = w' \cdot \rho', \rho' \in \mathbb{N}^*\} \cup \\
&\quad \cup \{w'' \in \mathbb{N}^* : x + x = w'' \cdot \rho'', \rho'' \in \mathbb{N}^*\} \supset \\
&\quad \supset \{1, 3\} \cup \{1, 2\} = \{1, 2, 3\}.
\end{aligned}$$

We shall prove that $x^n \supset \{1, 2, 3, \dots, n\}$, $\forall x \in \mathbb{N}^*$, $n \in \mathbb{N}^*, n \geq 2$, by induction.

Suppose that for $n = k, k \in \mathbb{N}^*, k \geq 2$:

$$x^k \supset \{1, 2, 3, \dots, k\}$$

We shall prove that the above is valid for $n = k + 1$, i.e.

$$x^{k+1} \supset \{1, 2, 3, \dots, k, k + 1\}.$$

Indeed,

$$x^{k+1} = (x^k \oslash x) \cup (x^{k-1} \oslash x^2) \cup \dots \cup (x \oslash x^k).$$

Then

$$\begin{aligned}
x^{k+1} &\supset (x^{k-1} \oslash x^2) \supset \{1, 2, 3, \dots, k - 1\} \oslash \{1, 2\} \supset \\
&\subset \{1, 2, 3, \dots, k\} \cup \{k + 1\} = \{1, 2, 3, \dots, k, k + 1\}.
\end{aligned}$$

Therefore every element of $\mathbb{N}^*$ belongs to a special power of x, thus, is a generator of the single-power cyclic H_v-group. $\square$

3 The divisors' hyperoperation due to multiplication in N

Now, let us define the hyperoperation $(\otimes)$ in $\mathbb{N}$ as follows:

Definition 3.1. For every $x, y \in IN$

$$\otimes \quad : \mathbb{N} \times \mathbb{N} \to \quad P(\mathbb{N}) - \{\emptyset\} : (x, y) \mapsto x \otimes y \subset \mathbb{N}$$

such that

$$x \odot y = \{z \in \mathbb{N} : x \cdot y = z \cdot \lambda, \lambda \in \mathbb{N}\}$$

where $(\cdot)$ is the usual operation of the multiplication in $\mathbb{N}$.

We call the above hyperoperation, *divisors' hyperoperation due to multiplication.*

Some properties of the divisors' hyperoperation due to multiplication

1. $x \otimes y = y \otimes x, \forall x, y \in \mathbb{N}$
2. $0 \otimes x = x \otimes 0 = \mathbb{N}, \forall x \in \mathbb{N}$
3. $1 \otimes 1 = 1$, i.e 1 is an idempotent element
4. $\{1, x, y, xy\} \subset x \otimes y, \forall x, y \in \mathbb{N}$

Remark 3.2. If x is a prime number, then $1 \otimes x = x \otimes 1 = \{1, x\}$.

Proposition 3.3. $E_{\otimes} = \mathbb{N}$.

Proof. For $x, e \in \mathbb{N}, x \otimes e = \{z \in \mathbb{N} : x \cdot e = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni x$. So, according to property 1, we get

$$x \in (x \otimes e) \cap (e \otimes x), \forall x, e \in \mathbb{N}$$

That means that the set of the unit elements with respect to $(\otimes)$ is the set $\mathbb{N}$, i.e. $E_{\otimes} = \mathbb{N}$. □

Proposition 3.4. i) $I_{\otimes}(x, 0) = \{0\}, y \in \mathbb{N}$ ii) $I_{\otimes}(x, 1) = \mathbb{N}$.

Proof. i) Straightforward from property 2.

ii) Take the unit element 1, then from property 4, we get

$$1 \in (x \otimes y) \cap (y \otimes x), \forall x, y \in \mathbb{N}$$

which means that $I_{\otimes}(x, 1) = \mathbb{N}$. □

Proposition 3.5. If a unit element p is a prime number, then

$$I_{\otimes}(x,p) = \begin{cases} \mathbb{N}, & x = np, n \in \mathbb{N} \\ pN, & x \neq np, n \in \mathbb{N}. \end{cases}$$

Proof. Let $p \in \mathbb{N}$ be a unit element and $p =$ prime number. Then p has no other divisors than 1 and itself. So, let $x = np$, $n \in \mathbb{N}$, then for $x' \in \mathbb{N}$

$$\begin{aligned} x \otimes x' &= \{z \in \mathbb{N} : x \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} = \\ &= \{z \in \mathbb{N} : (np) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni p, \forall x' \in \mathbb{N}. \end{aligned}$$

That means that $I_{\otimes}(x,p) = \mathbb{N}$. Let $x \neq np, n \in \mathbb{N}$, then $p \in \{z \in \mathbb{N} : x \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \Leftrightarrow x' = pn, n \in \mathbb{N} \Leftrightarrow I_{\otimes}(x,p) = pN$. □

Seems to be particularly interesting, one to study cases where the unit element is not a prime number. The following two examples study the cases where the unit element is 6 and 9.

Example 3.6. Let 6 be the unit element. Assume that $x = 6n$, $n \in \mathbb{N}$, then

$$x \otimes x' = \{z \in \mathbb{N} : (6n) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni 6, \forall x' \in \mathbb{N}.$$

Then, $I_{\otimes}(x,6) = \mathbb{N}$.
Assume that $x = 3m \neq 6n$, $n,m \in \mathbb{N}$, then

$$\begin{aligned} 6 \in x \otimes x' = \{z \in \mathbb{N} : (3m) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \Leftrightarrow x' = 2n, n \in \mathbb{N} \\ \Leftrightarrow I_{\otimes}(x,6) = 2\mathbb{N}. \end{aligned}$$

Assume that $x = 2m \neq 6n$, $n,m \in \mathbb{N}$, then

$$\begin{aligned} 6 \in x \otimes x' = \{z \in \mathbb{N} : (2m) \cdot x' = z \cdot \lambda, \lambda \in N\} \Leftrightarrow x' = 3n, n \in \mathbb{N} \\ \Leftrightarrow I_{\otimes}(x,6) = 3N. \end{aligned}$$

Assume that $x = 2m+1 \neq 3n$, $n,m \in \mathbb{N}$, then

$$\begin{aligned} 6 \in x \otimes x' = \{z \in \mathbb{N} : (2m+1) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \\ \Leftrightarrow x' = 6n, n \in \mathbb{N} \Leftrightarrow I_{\otimes}(x,6) = 6\mathbb{N}. \end{aligned}$$

Example 3.7. Let 9 be the unit element. Assume that $x = 9n$, $n \in \mathbb{N}$, then

$$x \otimes x' = \{z \in \mathbb{N} : (9n) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni 9, \forall x' \in \mathbb{N}.$$

Then, $I_{\otimes}(x,9) = \mathbb{N}$. Assume that $x = 3m \neq 9n$, $n,m \in \mathbb{N}$, then

$$\begin{aligned} 9 \in x \otimes x' = \{z \in \mathbb{N} : (3m) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \\ \Leftrightarrow x' = 3n, n \in \mathbb{N} \Leftrightarrow I_{\otimes}(x,9) = 3N. \end{aligned}$$

Assume that $x \neq 3m$, $m \in \mathbb{N}$, then

$$9 \in x \otimes x' = \{z \in \mathbb{N} : x \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \Leftrightarrow x' = 9n, n \in \mathbb{N} \Leftrightarrow I_{\otimes}(x,9) = 9N.$$

Proposition 3.8. Every element $x \in \mathbb{N}$ is an absorbing-like element of the divisors' hyperoperation due to multiplication.

Proof. According to property 4, $x \in x \otimes y$, $\forall x, y \in \mathbb{N}$, which means, that for every x$\in$ $\mathbb{N}$, $x \in x \otimes y$, $\forall y \in \mathbb{N}$ and due to property 1, $\forall x \in \mathbb{N}, x \in (x \otimes y) \cap (y \otimes x), \forall y \in \mathbb{N}$. Then, every natural number is an absorbing-like element of the divisors' hyperoperation due to multiplication. □

Proposition 3.9. The divisors' hyperoperation due to multiplication is a strong associative one in $\mathbb{N}$.

Proof. For $x, y, z \in \mathbb{N}$

$$
\begin{aligned}
(x \otimes y) \otimes z = \{w \in \mathbb{N} : x \cdot y = w \cdot \lambda, \lambda \in \mathbb{N}\} \otimes z = \bigcup_{w \in IN} (w \otimes z) = \\
= \bigcup_{w \in IN} \{w' \in \mathbb{N} : w \cdot z = w' \cdot \lambda', \lambda' \in \mathbb{N}\} = \\
= \bigcup_{\lambda \in IN} \{w' \in \mathbb{N} : \big[\frac{1}{\lambda}(xy)\big] z = w' \cdot \lambda', \lambda' \in \mathbb{N}\} = \\
= \bigcup_{\lambda \in IN} \{w' \in \mathbb{N} : x\big[\frac{1}{\lambda}(yz)\big] = w' \cdot \lambda', \lambda' \in \mathbb{N}\} = \\
= \bigcup_{v \in IN} \{w' \in \mathbb{N} : x \cdot v = w' \cdot \lambda', \lambda' \in N\} = \\
= \bigcup_{v \in IN} (x \otimes v) = x \otimes \{v \in \mathbb{N} : y \cdot z = v \cdot \lambda, \lambda \in \mathbb{N}\} = x \otimes (y \otimes z).
\end{aligned}
$$

So, $(x \otimes y) \otimes z = x \otimes (y \otimes z), \forall x, y, z \in \mathbb{N}$. □

Proposition 3.10. The hyperstructure $(\mathbb{N}, \otimes)$ is a commutative hypergroup.

Proof. Indeed, for $x \in \mathbb{N}$,

$$x \otimes \mathbb{N} = (x \otimes 0) \cup [\bigcup_{n \in IN*} (x \otimes n)] = \mathbb{N} \cup \Big[\bigcup_{n \in I\mathbb{N}*} (x \otimes n)\Big] = \mathbb{N}.$$

So, $x \otimes \mathbb{N} = \mathbb{N} \otimes x = \mathbb{N}$, $\forall x \in \mathbb{N}$.

Also, according to property 1 and Proposition 3.9 we get that $(\mathbb{N}, \otimes)$ is a commutative hypergroup. □

Remark 3.11. For $x \in \mathbb{N}^*$,

$$x \otimes \mathbb{N}^* = \bigcup_{n \in IN*} (x \otimes n) = \bigcup_{n \in IN*} \{z \in \mathbb{N} : x \cdot n = z \cdot \lambda, \lambda \in \mathbb{N}^*\} \supset \bigcup_{n \in IN*} \{n\} = \mathbb{N}^*.$$

So, $x \otimes \mathbb{N}^* = \mathbb{N}^* \otimes x = \mathbb{N}^*, \forall x \in \mathbb{N}^*$.

Proposition 3.12. For every $x \in \mathbb{N}, x^{n-1} \subseteq x^n, n \in \mathbb{N}, n \geq 2$.

Proof. For $x \in \mathbb{N}$ and $n \in \mathbb{N}, n \geq 2$

$$x^n = (x^{n-1} \otimes x) \cup (x^{n-2} \otimes x^2) \cup \ldots \cup (x^{n-p} \otimes x^p)$$

where $p = \left[\frac{n}{2}\right]$ the integer part of $\frac{n}{2}$, [1]. Then,

$$x^n \supseteq x^{n-1} \otimes x \supseteq x^{n-1} \otimes 1 \supseteq x^{n-1}. \qquad \square$$

4 On a dual H_v-ring in $\mathbb{N}^*$

Proposition 4.1. $(x \otimes y) \oslash (x \otimes z) \supset x \otimes (y \oslash z), \forall x, y, z \in \mathbb{N}^*$.

Proof. For $x, y, z \in \mathbb{N}^*$, we get

$$\begin{aligned} x \otimes (y \oslash z) &= x \otimes \{w \in \mathbb{N}^* : y + z = w \cdot \lambda, \lambda \in \mathbb{N}^*\} = \bigcup_{w \in IN*} (x \otimes w) = \\ &= \bigcup_{w \in IN*} \{w' \in \mathbb{N}^* : x \cdot w = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\} = \\ &= \bigcup_{\lambda \in IN*} \{w' \in \mathbb{N}^* : x \cdot \frac{y+z}{\lambda} = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(x \otimes y) \oslash (x \otimes z) = \\ &\quad = \{v \in \mathbb{N}^* \colon x \cdot y = v \cdot \rho, \rho \in \mathbb{N}^*\} \oslash \{v' \in \mathbb{N}^* \colon x \cdot z = v' \cdot \rho', \rho' \in \mathbb{N}^*\} \\ &\quad = \bigcup_{v, v' \in IN*} (v \oslash v') = \bigcup_{v, v' \in IN*} \{k \in \mathbb{N}^* \colon v + v' = k \cdot \mu, \mu \in \mathbb{N}^*\} = \\ &\quad = \bigcup_{\rho, \rho' \in IN*} \{\kappa \in \mathbb{N}^* \colon \frac{xy}{\rho} + \frac{xz}{\rho'} = \kappa \cdot \mu, \mu \in \mathbb{N}^*\} \supset \\ &\quad \supset \bigcup_{\mu' \in IN*} \{\kappa' \in \mathbb{N}^* \colon x \cdot \frac{y+z}{\mu'} = \kappa' \cdot \tau', \tau' \in \mathbb{N}^*\} = x \otimes (y \oslash z). \end{aligned}$$

So, $(x \otimes y) \oslash (x \otimes z) \supset x \otimes (y \oslash z)$ and then,

$$x \otimes (y \oslash z) \cap (x \otimes y) \oslash (x \otimes z) \neq \emptyset, \forall x, y, z \in \mathbb{N}^*. \qquad \square$$

Proposition 4.2. The divisors' hyperoperation due to addition is weak distributive with respect to the divisors' hyperoperation due to multiplication in $\mathbb{N}^*$.

Proof. For $x, y, z \in \mathbb{N}^*$, we get

$$\begin{aligned} x \oslash (y \otimes z) &= x \oslash \{w \in \mathbb{N}^* : y \cdot z = w \cdot \lambda, \lambda \in \mathbb{N}^*\} = \bigcup_{w \in IN*} (x \oslash w) = \\ &= \bigcup_{w \in IN*} \{w' \in \mathbb{N}^* : x + w = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\} \supset \\ &\supset \{w'' \in \mathbb{N}^* : x + y = w'' \cdot \lambda'', \lambda'' \in \mathbb{N}^*\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(x \oslash y) \otimes (x \oslash z) = \\ &\quad = \{v \in \mathbb{N}^* : x + y = v \cdot \rho, \rho \in \mathbb{N}^*\} \otimes \{v' \in \mathbb{N}^* : x + z = v' \cdot \rho', \rho' \in \mathbb{N}^*\} = \\ &\quad = \bigcup_{v, v' \in IN*} (v \otimes v') = \bigcup_{v, v' \in IN*} \{v'' \in \mathbb{N}^* : v \cdot v' = v'' \cdot \rho'', \rho'' \in \mathbb{N}^*\} \supset \\ &\quad \supset \{\kappa \in N^* : (x + y) \cdot (x + z) = \kappa \cdot \mu, \mu \in \mathbb{N}^*\} \supset \\ &\quad \supset \{\kappa' \in \mathbb{N}^* : x + y = \kappa' \cdot \mu', \mu' \in \mathbb{N}^*\}. \end{aligned}$$

So, $x \oslash (y \otimes z) \cap (x \oslash y) \otimes (x \oslash z) \supset \{\tau \in \mathbb{N}^* : x + y = \tau \cdot \sigma, \sigma \in \mathbb{N}^*\}$ and then

$$x \oslash (y \otimes z) \cap (x \oslash y) \otimes (x \oslash z) \neq \emptyset, \forall x, y, z \in \mathbb{N}^*.$$ □

Proposition 4.3. The hyperstructure $(\mathbb{N}^*, \oslash, \otimes)$ is a commutative dual H_v-ring.

Proof. Indeed, according to Propositions 2.11 and 3.10 the hyperstructures $(\mathbb{N}^*, \oslash)$ and $(\mathbb{N}^*, \otimes)$ are commutative H_v-group and commutative hypergroup respectively. From Propositions 4.1 and 4.2 we get that $(\otimes)$ is weak distributive with respect to $(\oslash)$ and $(\oslash)$ is weak distributive with respect to $(\otimes)$, respectively. □

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The LV-hyperstructures

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Abstract

The largest class of hyperstructures is the one which satisfy the weak properties and they are called H_v-structures introduced in 1990. The H_v-structures have a partial order (poset) on which gradations can be defined. We introduce the LV-construction based on the Levels Variable.

Key words: hyperstructures, H_v-structures, hopes, weak hopes.

MSC2010: 20N20.

1 Fundamental Definitions

In a set H is called *hyperoperation* (abbreviation *hyperoperation=hope*) in a set H, is called any map $\cdot : H \times H \to \mathcal{P}(H) - \{\emptyset\}$.

Definition 1.1 (Marty 1934)**.** A hyperstructure $(H, \cdot)$ is a *hypergroup* if $(\cdot)$ is an associative hyperoperation for which the reproduction axiom: $hH = Hh = H, \forall x \in H$, is valid.

Definition 1.2 (Vougiouklis 1990)**.** In a set H with a hope we abbreviate by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \emptyset, \forall x, y \in H$. The hyperstructure $(H, \cdot)$ is called H_v*-semigroup* if it is *WASS,* it is called H_v*-group* if it is reproductive H_v-semigroup, i.e. $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called H_v*-ring* if both $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to

$$(+) : x(y + z) \cap (xy + xz) \neq \emptyset, (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R$$

.

Definition 1.3 (Santilly-Vougiouklis)**.** A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hyperoperation, is called *e-hyperfield* if the following axioms are valid:

1. $(F, +)$ is an abelian group with the additive unit 0,
2. $(\cdot)$ is WASS,
3. $(\cdot)$ is weak distributive with respect to $(+)$,
4. 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$,
5. there exists a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$,
6. for every $x \in F$ there exists a unique inverse x^{-1}, such that
$$1 \in x \cdot x^{-1} \cap x^{-1} \cdot x.$$

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Construction 1.4. *The Main e-Construction.* Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hyperoperations $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, \ldots\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, \ldots \in G - \{e\}$$

$g_1, g_2, \ldots$ are not necessarily the same for each pair (x, y). Then $(G, \otimes)$ becomes an H_v-group, in fact is H_b-group which contains the $(G, \cdot)$. The H_v-group $(G, \otimes)$ is an e-hypergroup. Moreover, if for each x, y such that $xy = e$, so we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong e-hypergroup.

For more definitions and applications on H_v-structures, see the books and papers [1-20].

The main tool to study hyperstructures are the *fundamental relations* β^*, γ^* and ε^*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space, resp. Fundamental relations are used for general definitions. Thus, an H_v-ring $(R, +, \cdot)$ is called *H_v-field* if R/γ^* is a field.

Definition 1.5. Let $(H, \cdot)$, $(H, *)$ be H_v-semigroups defined on the same set H. Then $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an $f \in Aut(H, *)$ such that $xy \subset f(x * y), \forall x, y \in H$. Then we write $\cdot \leq *$ and we say that $(H, *)$ *contains* $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, *)$ is called *H_b-structure.*

Theorem 1.6 (The Little Theorem)**.** *Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

This Theorem leads to a partial order on H_v-structures, thus we have posets. The determination of all H_v-groups and H_v-rings is very interesting. To compare classes we can see the small sets. The problem of enumeration of classes of H_v-structures was started very early but recently we have results by using computers. The partial order in H_v-structures restricts the problem in finding the minimals.

2 Enumeration Theorems

Theorem 2.1 (Chung-Choi)**.** *There exists up to isomorphism,* 13 *minimal H_v-groups of order* 3 *with scalar unit, i.e. minimal e-hyperstructures of order* 3*.*

Theorem 2.2 (Bayon-Lygeros)**.**

- *There exist, up to isomorphism,* 20 *H_v-groups of order* 2*.*
- *There exist, up to isomorphism,* 292 *H_v-groups of order* 3 *with scalar unit, i.e. e-hyperstructures of order* 3*.*
- *There exist, up to isomorphism,* 6494 *minimal H_v-groups of order* 3*.*
- *There exist, up to isomorphism,* 1026462 *H_v-groups of order* 3*.*

Theorem 2.3 (Bayon-Lygeros)**.**

- *There exist, up to isomorphism,* 631609 *H_v-groups of order* 4 *with scalar unit, i.e. e-hyperstructures of order* 4*.*
- *There exist, up to isomorphism,* 8.028.299.905 *abelian Hv-groups of order* 4*.*

Theorem 2.4 (Bayon-Lygeros)**.**

- *The number of abelian H_v-groups of order* 4 *with scalar unit (i.e. abelian e-hyperstructures) in respect with their automorphism group are the following*

$\lvert\mathrm{Aut}\,(H_v)\rvert$	1	2	3	4	6	8	12	24
	—	—	—	32	—	46	5510	626021

- *There are* 63 *isomorphism classes of hyperrings of order* 2.
- *There are* 875 *isomorphism classes of* H_v*-rings of order* 2.
- *There are* 33277642 *isomorphism classes of hyperrings of order* 3.

In all the above results we construct the poset of hyperstructures of order 2 and 3 in the sense of inclusion for hyperproducts. We compute the Betti numbers of the poset of Hv-groups of order 2 and we have the following results: $(1,5),(2,4),(3,6),(4,4),(5,1)$. We also compute the Betti numbers of the poset of hypergroups of order 3 and we have the following results: $(1,59),(2,168),(3,294),(4,438),(5,568),(6,585),$ $(7,536),(8,480),$ $(9,358),$ $(10,245),(11,160),(12,66),(13,29),(14,10),$ $(15,2),(16,1)$.

We explicitly compute the Cayley subtables of the minimal e-hyperstructures with $H=\{e,a,b\}$ and we have for the products (aa, ab, ba, bb) the following results: (b; e; e; a), (eb; a; a; e), (e; ab; ab; e), (a; eb; eb; a), (ab; ea; ea; e), (H; eb; a; ea), (H; a; eb; ea), (a; H; H; e), (b; H; H; e), (a; H; H; b), (H; b; a; H), (H; a; b; H), (H; e; ab; H).

3 Construction Theorems

There are several ways to organize such posets using hyperstructure theory. We present now a new construction on posets and we name this LV-construction since it is based on gradations where the Levels are used as Variable. Thus LV means Level Variable.

Theorem 3.1. *The LV-Construction I*

Consider the set $\mathbf{P}_n$ *of all* H_v*-groups defined on a set of* n *elements. Take the following gradation on* $\mathbf{P}_n$ *based on posets:*

Level 0 *(or grade* 0*), denoted by* $\mathbf{g}_0$*, is the set of all minimals of* $\mathbf{P}_n$*. Level (grade)* 1*, denoted by* $\mathbf{g}_1$*, is the set of all* H_v*-groups obtained from minimals by adding one only element to anyone of the results of the products of two elements on the minimals of* $\mathbf{P}_n$*, i.e. of* $\mathbf{g}_0$*. Level* 2 *(or grade* 2*), denoted by* $\mathbf{g}_2$*, is the set of all* H_v*-groups obtained from minimals by adding only two elements to anyone of the results of the products of two elements of the minimals* $\mathbf{g}_0$*. Then inductively the Level* k *is defined, denoted by* $\mathbf{g}_k$*. In the* *case that an H_v-group is obtained by adding k_1 elements of one minimal and by adding k_2 elements of another minimal then we consider that it belongs to the Level* $\min(k_1, k_2)$.

Denote by r the cardinality of the minimals, $|\mathbf{g}_0| = r$, and by s the number of levels. Take any H_v-group with r elements corresponding to the r elements of $\mathbf{g}_0$, so we have an H_v-group $(\mathbf{g}_0, *)$. Then we define a hope on

$$\mathbf{P}_n = \mathbf{g}_0 \cup \mathbf{g}_1 \cup, \ldots, \cup \mathbf{g}_{s-1},$$

as follows

$$x \otimes y = \begin{cases} x * y, & \forall x, y \in \mathbf{g}_0 \\ \mathbf{g}_{\kappa+\lambda}, & \forall x \in \mathbf{g}_\kappa, y \in \mathbf{g}_\lambda, \text{ where } (\kappa, \lambda) \neq (0,0) \end{cases}$$

Then the hyperstructure $(\mathbf{P}_n, \otimes)$ is an H_v-group where its fundamental group is isomorphic to $\mathbf{Z}_s$, thus we have

$$\mathbf{P}_n/\beta^* \approx \mathbf{Z}_s.$$

Proof. Let us correspond, numbered, the levels with the elements of $\mathbf{Z}_s$: $\mathbf{g}_i \rightarrow \underline{i}, i = 0, \ldots, s-1$.

From the definition of $(\otimes)$ any hyperproduct of elements from several levels, apart of $\mathbf{g}_0$, equals to only one special set of H_v-groups that constitute one level. Moreover we have

$$x \otimes y = \mathbf{g}_0, \forall x \in \mathbf{g}_\kappa, y \in \mathbf{g}_{-\kappa}, \text{ for any } \kappa \neq 0.$$

That means that the elements of $\mathbf{g}_0$are β^*-equivalent. Therefore all elements of each level are β^*-equivalent and there are no β^*-equivalent elements from different levels. That proves that

$$\mathbf{P}_n/\beta^* \approx \mathbf{Z}_s. \qquad \square$$

The above is a construction similar to the one from the book [15, p.27]

A generalization of the above construction is the following:

Theorem 3.2. *The LV-Construction II*

Consider a graded finite poset with n elements: $\mathbf{P}_n = \mathbf{g}_0 \cup \mathbf{g}_1 \cup, \ldots, \cup \mathbf{g}_{s-1}$, with s levels (grades) $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_{s-1}$, such that

$$\sum_{i=0}^{s-1} |\mathbf{g_i}| = n.$$

Denoting $|\mathbf{g}_0| = r$, *we consider two* H_v*-groups* $(\mathbf{E}, \cdot)$ *and* $(\mathbf{S}, *)$ *such that* $|\mathbf{E}| = r$, $|\mathbf{S}| = s$ *and moreover* $\mathbf{S}$ *has a unit single element* e. *Then we take 1:1 maps from* $\mathbf{E}$ *onto* $\mathbf{g}_0$ *and from* $\mathbf{S}$ *onto* $\{\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_{s-1}\}$, *so we obtain two* H_v*-groups:* $(\mathbf{g}_0, \cdot)$ *and* $\big(\mathbf{G} = \{\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_{s-1}\}, *\big)$ *where* $\mathbf{E} = \mathbf{g}_0$ *corresponds to the single element* e. *We define a hope on* $\mathbf{P}_n$ *as follows:*

$$x \otimes y = \begin{cases} x \cdot y, & \forall x, y \in \mathbf{g}_0 \\ \mathbf{g}_\kappa * \mathbf{g}_\lambda, & \forall \mathbf{g}_\kappa, \mathbf{g}_\lambda \in \mathbf{G}, \text{ where } (\kappa, \lambda) \neq (0,0) \end{cases}$$

Then the hyperstructure $(\mathbf{P}_n, \otimes)$ *is an* H_v*-group where its fundamental group is isomorphic to the fundamental group of* $(\mathbf{S}, *)$, *therefore we have*

$$(\mathbf{P}_n, \otimes)/\beta^* \approx (\mathbf{S}, *)/\beta^*.$$

Proof. From the reproductivity of $(\mathbf{G}, *)$, for each $\mathbf{g}_\kappa, \kappa \neq 0$, there exists a $\mathbf{g}_\lambda$ such that $\mathbf{g}_0 \in \mathbf{g}_\kappa * \mathbf{g}_\lambda$. But $\mathbf{g}_0$ is a single element of $(\mathbf{S}, *)$, therefore we have $\mathbf{g}_0 = \mathbf{g}_\kappa * \mathbf{g}_\lambda$. Then, by the definition, for any $x \in \mathbf{g}_\kappa$, $y \in \mathbf{g}_\lambda$ we have, $x \otimes y = \mathbf{g}_0$. Therefore, all the elements of $\mathbf{g}_0$ are β^*-equivalent. On the other side, from the definition, all elements of each level are β^*-equivalent and they are β^*-equivalent elements with different levels if and only if they are β^*-equivalent in $(\mathbf{G}, *)$. In other wards they follow exactly the β^*-equivalence of $(\mathbf{G}, *)$.

That proves that

$$(\mathbf{P}_n, \otimes)/\beta^* \approx (\mathbf{S}, *)/\beta^*. \qquad \square$$

With this LV-construction we can define the poset for H_v-groups of order 2. So we get a non-connected poset with Betti numbers for the two subposets (1,4), (2,4), (3,1) and (1,1), (2, 4), (3,6).

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Some properties of certain Subhypergroups

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Abstract

The structure of the hypergroup is much more complicated than that of the group. Thus there exist various kinds of subhypergroups. This paper deals with some of these subhypergroups and presents certain properties of the closed, invertible and ultra-closed subhypergroups.

Key words: Hypergroup, Subhypergroup.

MSC2010: 20N20.

1 Introduction

In 1934 F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the *hypergroup* [11]. An *operation* or *composition* in a non-void set H is a function from $H \times H$ to H, while a *hyperoperation* or *hypercomposition* is a function from $H \times H$ to the powerset $P(H)$ of H. An algebraic structure that satisfies the axioms:

i. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in H$ (associative axiom) and

ii. $a \cdot H = H \cdot a = H$ for every $a \in H$ (reproductive axiom).

is called *group* if $\cdot$ is a composition [16] and *hypergroup* if $\cdot$ is a hypercomposition [11]. When there is no likelihood of confusion $\cdot$ can be omitted. If A and B are subsets of H, then AB signifies the union$\bigcup_{(a,b)\in A\times B} ab$, in particular if A=$\emptyset$ or B=$\emptyset$ then AB=$\emptyset$. Ab and aB have the same meaning as $A\{b\}$ and $\{a\}B$. In general, the singleton $\{a\}$ is identified with its member a.

Proposition 1.1. *If a non-void set H is endowed with a composition which satisfies the associative and the reproductive axioms, then H has a bilateral neutral element and any element in H has a bilateral symmetric.*

Proof. Let $x \in H$. Per reproductive axiom $x \in xH$. Therefore there exists $e \in H$ such that $xe = x$. Next, let y be an arbitrary element in H. Per reproductive axiom there exists $z \in H$ such that $y = zx$. Consequently $ye = (zx)\,e = z\,(xe) = zx = y$. Hence e is a right neutral element. In an analogous way there exists a left neutral element e'. Then the equality $e = e'e = e'$ is valid. Therefore e is the bilateral neutral element of H. Now, per reproductive axiom $e \in xH$. Thus there exists $x' \in H$, such that $e = xx'$. Hence any element in H has a right symmetric. Similarly any element in H has a left symmetric and it is easy to prove that these two symmetric elements coincide. □

Remark 1.2. An analogous Proposition to Proposition 1.1 is not valid when H is endowed with a hypercomposition. In hypergroups there exist different types of neutral elements [15] (e.g. scalar [4], strong [8,17] ect). There also exist special types of hypergroups which have a neutral element and each one of their elements has one or more symmetric. Such hypergroups are for example the canonical hypergroups [21], the quasicanonical hypergroups [12], the fortified join hypergroups [17], the fortified transposition hypergroups [8], the transposition polysymmetrical hypergroups [19], the canonical polysymmetrical hypergroups [14], etc.

Proposition 1.3. *If H is a hypergroup, then $ab \neq \emptyset$ is valid for all the elements a, b of H.*

Proof. Suppose that $ab = \emptyset$ for some $a, b \in H$. Per reproductive axiom, $aH = H$ and $bH = H$. Hence, $H = aH = a\,(bH) = (ab)\,H = \emptyset H = \emptyset$, which is absurd. □

In [11], F. Marty also defined the two induced hypercompositions (right and left division) that result from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|b} = \{x \in H | a \in xb\} \text{ and } \frac{a}{b|} = \{x \in H | a \in bx\}.$$

It is obvious that the two induced hypercompositions coincide, if the hypergroup is commutative. For the sake of notational simplicity, a/b or $a : b$ is used for right division and $b\backslash a$ or $a..b$ for left division [7, 13].

Proposition 1.4. *If H is a hypergroup, then $a/b \neq \emptyset$ and $b\backslash a \neq \emptyset$ for all the elements a, b of H.*

Proof. Per reproductive axiom, $Hb = H$ for all $b \in H$. Hence, for every $a \in H$there exists $x \in H$, such that $a \in xb$. Thus, $x \in a/b$ and, therefore, $a/b \neq \emptyset$. Dually, $b\backslash a \neq \emptyset$. □

In Proposition 2.3 of [13] the following properties were proved for any hypergroup H (see also Proposition 1 in [7])

Proposition 1.5. i) $(a/b)/c = a/(cb)$ and $c\backslash(b\backslash a) = (bc)\backslash a$, for all $a, b, c \in H$.

ii) $b \in (a/b)\backslash a$ and $b \in a/(b\backslash a)$, for all $a, b \in H$.

In [7] and then in [8] a principle of duality is established in the theory of hypergroups and in the theory of transposition hypergroups as follows:
Given a theorem, the dual statement which results from the interchanging of the order of the hypercomposition . (and necessarily interchanging of the left and the right division), is also a theorem.
This principle is used throughout this paper.

2 Closed, invertible and ultra-closed subhypergoups

The structure of the hypergroup is much more complicated than that of the group. There are various kinds of subhypergroups. In particular a non-empty subset K of H is called semi-subhypergroup when it is stable under the hypercomposition, i.e. it has the property $xy \subseteq K$ for all $x, y \in K$. K is a subhypergroup of H if it satisfies the reproductive axiom, i.e. if the equality $xK = Kx = K$ is valid for all $x \in K$(for the fuzzy case see e.g [3]). This means that when K is a subhypergroup and $a, b \in K$, the relations $a \in bx$ and $a \in yb$ always have solutions in K. Although the non-void intersection of two subhypergroups is stable under the hypercomposition, it usually is not a subhypergroup since the reproductive axiom fails to be valid for it. This led, from the very early steps of hypergroup theory, to the consideration of more special types of subhypergroups. One of them is the *closed subhypergroup* (e.g. see [5], [9]). A subhypergroup K of H is called *left closed* with respect to H if for any two elements a and b in K, all the solutions of the relation $a \in yb$ lie in K. This means that K is left closed if and only if $a/b \subseteq K$, for all $a, b \in K$ (see [13]). Similarly K is *right closed* when all the solutions of the relation $a \in bx$ lie in K or equivalently if $b\backslash a \subseteq K$, for all $a, b \in K$ [13]. Finally K is *closed* when it is both right and left closed. In the case of the closed subhypergroups, the non-void intersection of any family of closed subhypergroups is a closed subhypergroup. It must be mentioned though that a hypergroup may have subhypergroups, but no proper closed ones. For example if Q is a quasi-order hypergroup [6], a^2 is a subhypergroup of Q, for each $a \in Q$, but $a/a = a\backslash a = Q$ for all $a \in Q$. Also fortified transposition hypergroups [8, 17] consisting only of attractive elements have no proper closed subhypergroups [18].

Proposition 2.1. *If K is a subset of a hypergroup H such that $a/b \subseteq K$* and $b\backslash a \subseteq K$, *for all $a, b \in K$, then K is a subhypergroup of H.*

Proof. Let a be an element of K. It must be shown that $aK = Ka = K$. Suppose that $x \in K$. Then $a\backslash x \subseteq K$, therefore $x \in aK$, hence $K \subseteq aK$. For the reverse inclusion now suppose that $y \in aK$. Then $K/y \subseteq K/aK$. So $K \cap (K/aK)\, y \neq \emptyset$. Thus, $y \in (K/aK)\backslash K$. Per Proposition 1.4 (i) the equality $K/aK = (K/K)/a$ is valid. Thus $(K/aK)\backslash K = ((K/K)/a)\backslash K \subseteq (K/a)\backslash K \subseteq (K/K)\backslash K \subseteq K\backslash K \subseteq K$. Hence $y \in K$ and so $aK \subseteq K$. Therefore $aK = K$. The equality $Ka = K$ follows by duality. □

In [13] it is also proved that the equalities

$$K = K/a = a/K = a\backslash K = K\backslash a$$

are valid for every element a of a closed subhypergroup K.

Next some properties of these subhypergroups will be presented.

Proposition 2.2. *If K is a subhypergroup of H, then $H - K \subseteq (H - K)\, s$ and $H - K \subseteq s\,(H - K)$, for all $s \in K$.*

Proof. Let r be an element in $H - K$ which does not belong to $(H - K)\, s$. Because of the reproductive axiom, $r \in Hs$ and since $r \notin (H - K)\, s$, r must be a member of Ks. Thus, $r \in Ks \subseteq KK = K$. This contradicts the assumption and so $H - K \subseteq (H - K)\, s$. The second inclusion follows by duality. □

Proposition 2.3. (i) *A subhypergroup K of H is left closed in H, if and only if $(H - K)\, s = H - K$ for all $s \in K$.*
(ii) *A subhypergroup K of H is right closed in H, if and only if*
$s\,(H - K) = H - K$ for all $s \in K$.
(iii) *A subhypergroup K of H is closed in H, if and only if*
$s\,(H - K) = (H - K)\, s = H - K$ for all $s \in K$.

Proof. (i) Let K be left closed in H. Suppose that z lies in $H-K$ and assume that $zs \cap K \neq \emptyset$. Then, there exists an element y in K such that $y \in zs$, or equivalently, $z \in y/s$. Therefore $z \in K$, which is absurd. Hence $(H-K)\,s \subseteq H-K$. Next, because of Proposition 1, $H-K \subseteq (H-K)\,s$ and therefore $H-K = (H-K)\,s$. Conversely now. Suppose that $(H-K)\,s = H-K$ for all $s \in K$. Then $(H-K)\,s \cap K = \emptyset$ for all $s \in K$. Hence $x \notin rs$ and so $r \notin x/s$ for all $x, s \in K$ and $r \in H-K$. Therefore $x/s \cap (H-K) = \emptyset$ which implies that $x/s \subseteq K$. Thus K is closed in H. (ii) follows by duality and (iii) is an obvious consequence of (i) and (ii). $\square$

Corolary 2.4. *(i) If K is a left closed subhypergroup in H, then $xK \cap K = \emptyset$, for all $x \in H-K$.*
(ii) If K is a right closed subhypergroup in H, then $Kx \cap K = \emptyset$, for all $x \in H-K$.
(iii) If K is a closed subhypergroup in H, then $xK \cap K = \emptyset$ and $Kx \cap K = \emptyset$, for all $x \in H-K$.

Proposition 2.5. *If K is a subhypergroup of H, $A \subseteq K$ and $B \subseteq H$, then* (i) $A\,(B \cap K) \subseteq AB \cap K$ *and* (ii) $(B \cap K)\,A \subseteq BA \cap K$.

Proof. Let $t \in A\,(B \cap K)$. Then $t \in ax$, with $a \in A$ and $x \in B \cap K$. Since x lies in $B \cap K$, it derives that $x \in B$ and $x \in K$. Hence $ax \subseteq aB$ and $ax \subseteq aK = K$. Thus $ax \subseteq AB \cap K$ and therefore $t \in AB \cap K$. Duality gives (ii) and so the Proposition. $\square$

Proposition 2.6. (i) *If K is a left closed subhypergroup in H, $A \subseteq K$ and $B \subseteq H$, then $(B \cap K)\,A = BA \cap K$.*

(ii) *If K is a right closed subhypergroup in H, $A \subseteq K$ and $B \subseteq H$, then $A\,(B \cap K) = AB \cap K$.*

Proof. (i) Let $t \in BA \cap K$. Since K is right closed, for any element y in $B-K$, it is valid that $yA \cap K \subseteq yK \cap K = \emptyset$. Hence $t \in (B \cap K)\,A \cap K$. But $(B \cap K)\,A \subseteq KK = K$. Thus $t \in (B \cap K)\,A$. Therefore $BA \cap K \subseteq (B \cap K)\,A$. Next the inclusion becomes equality because of Proposition 2.5. (ii) derives from the duality. $\square$

Proposition 2.7. (i) *If K is a left closed subhypergroup in H, $A \subseteq K$ and $B \subseteq H$, then $(B \cap K)\,/A = (B/A) \cap K$.*

(ii) *If K is a right closed subhypergroup in H, $A \subseteq K$ and $B \subseteq H$, then $(B \cap K)\,\backslash A = B\backslash A \cap K$.*

Proof. (i) Since $B \cap K \subseteq B$, it derives that $(B \cap K)\,/A \subseteq B/A$. Moreover $A \subseteq K$ and $B \cap K \subseteq K$, thus $(B \cap K)\,/A \subseteq K$. Hence $(B \cap K)\,/A \subseteq$

$(B/A) \cap K$. For the reverse inclusion now suppose that $x \in (B/A) \cap K$. Then, there exist $a \in A$, $b \in B$ such that $x \in b/a$ or equivalently $b \in ax$. Since $ax \subseteq K$ it derives that $b \in K$ and so $b \in B \cap K$. Therefore $b/a \subseteq (B \cap K)/A$. Thus $x \in (B \cap K)/A$. Hence $(B/A) \cap K \subseteq (B \cap K)/A$, QED. Duality gives (ii) and so the Proposition. □

Krasner generalized the notion of the closed subhypergroups, considering closed subhypergroups in other subhypergroups [9]. Let us define the restriction of the right and left division in subset A of a hypergroup H as follows:

$$a/_A b = \{x \in A | a \in xb\} \text{ and } b\backslash_A a = \{x \in A | a \in bx\}$$

Thus, if K is a subhypergroup of H and $K \subseteq A$, then K is right closed in A, if $b\backslash_A a \subseteq K$ for all $a, b \in K$ and K is left closed in A, if $a/_A b \subseteq K$ for all $a, b \in K$.

Proposition 2.8. *Let K, M be two subhypergroups of a hypergroup H, such that $K \subseteq M$. If K is left (or right) closed in M and M is left (or right) closed in H, then K is left (or right) closed in H.*

Proof. Since K is left closed in M, the inclusion $a/_M b \subseteq K$ is valid, for all $a, b \in K$. This means that if x is an element of M such that $a \in xb$, then $x \in K$. Next if there exists $y \in H - M$ such that $a \in yb$, then a/b will not be a subset of M. Hence M will not be left closed in H. This contradicts the assumption, and so the Proposition. □

Corolary 2.9. *Let K, M be two subhypergroups of a hypergroup H, such that $K \subseteq M$. If K is closed in M and M is closed in H, then K is closed in H.*

Proposition 2.10. *Let K, M be two subhypergroups of a hypergroup $H$and suppose that K is left (or right) closed in H. Then $K \cap M$ is left (or right) closed in M.*

Proof. Let $a, b \in K \cap M$. Then $a/b = \{x \in H | a \in xb\} \subseteq K$. Hence $\{x \in M | a \in xb\} \subseteq K \cap M$. Therefore $a/_M b \subseteq K \cap M$. Thus $K \cap M$ is left closed in M. □

Corolary 2.11. *Let K, M be two subhypergroups of a hypergroup $H$and suppose that K is closed in H. Then $K \cap M$ is closed in M.*

Proposition 2.12. *If two subhypergroups K, $M$of a hypergroup $H$are left (or right) closed in $H$and their intersection is not void, then $K \cap M$ is left (or right) closed in M.*

Proof. Let $a, b \in K \cap M$. Since K, M are left closed in H, $a/b = \{x \in H | a \in xb\}$ is a subset of both K and M. Hence $a/b \subseteq K \cap M$ and so the Proposition. □

Corolary 2.13. *The non-void intersection of two closed subhypergroups is a closed subhypergroup.*

The next type of hypergroups was introduced by Dresher and Ore in [5] and immediately after that, M. Krasner used them in [9]. In both [5] and [9] they are named reversible subhypergoups. In our days these subhypergroups are called invertible. The Definition that follows was given by Jantosciak in [7].

Definition 2.14. A subhypergroup K of a hypergroup H is **right invertible** if $a/b \cap K \neq \emptyset$, implies that $b/a \cap K \neq \emptyset$, $a, b \in H$. K is **left invertible** if $b\backslash a \cap K \neq \emptyset$, implies that $a\backslash b \cap K \neq \emptyset$, $a, b \in H$. If K is both right and left invertible, then it is called **invertible**.

Theorem 4 in [1] gives an interesting example of an invertible subhypergroup in a join hypergroup of partial differential operators. Moreover the closed subhypergroups of the quasicanonical or of the canonical hypergroups are invertible [21].

Direct consequences of the above definition are the following propositions:

Proposition 2.15. (i) *K is right invertible in H, if and only if the following implication is valid: $b \in Ka \Rightarrow a \in Kb$, $a, b \in H$.*

(ii) *K is left invertible in H, if and only if the following implication is valid: $b \in aK \Rightarrow a \in bK$, $a, b \in H$.*

Proposition 2.16. (i) K is *right invertible in H, if and only if the following implication is valid: $Ka \neq Kb \Rightarrow Ka \cap Kb = \emptyset$, $a, b \in H$.*

(ii) K is *left invertible in H, if and only if the following implication is valid: $aK \neq bK \Rightarrow aK \cap bK = \emptyset$, $a, b \in H$.*

Proposition 2.17. *If K is right (left) invertible in H, then K is right (left) closed in H.*

In [2] one can find examples of closed hypegroups that are not invertible.

Definition 2.18. A subhypergroup K of a hypergroup H is **right ultra-closed** if it is right closed and $a/a \subseteq K$ for each $a \in H$. K is **left ultra-closed** if it is left closed and $a\backslash a \subseteq K$ for each $a \in H$. If K is both right and left ultra-closed, then it is called **ultra-closed**.

Proposition 2.19. (i) *If K is right ultra-closed in H, then either $a/b \subseteq K$ or $a/b \cap K = \emptyset$, for all $a, b \in H$. Moreover if $a/b \subseteq K$, then $b/a \subseteq K$.*

(ii) *If K is left ultra-closed in H, then either $b\backslash a \subseteq K$ or $b\backslash a \cap K = \emptyset$, for all $a, b \in H$. Moreover if $b\backslash a \subseteq K$, then $a\backslash b \subseteq K$.*

Proof. Suppose that $a/b \cap K \neq \emptyset$, $a, b \in H$. Then $a \in kb$, for some $k \in K$. Next assume that $b/a \cap (H-K) \neq \emptyset$. Then $b \in ra$, $r \in H-K$. Thus $a \in k\,(ra) = (kr)\,a$. Since K is right closed, per Proposition 2.3, $kr \subseteq H-K$. So $a \in va$, for some $v \in H-K$. Therefore $a/a \cap (H-K) \neq \emptyset$, which is absurd. Hence $b/a \subseteq K$. Now let there be x in K such that $b \in xa$. If $a/b \cap (H-K) \neq \emptyset$, there exists $y \in H-K$ such that $a \in yb$. Therefore $b \in x\,(yb) = (xy)\,b$. Since K is right closed, per Proposition 2.3, $xy \subseteq H-K$. So $b \in zb$, for some $z \in H-K$. Therefore $b/b \cap (H-K) \neq \emptyset$, which is absurd. Hence $a/b \subseteq K$. Duality gives (ii). □

Corolary 2.20. *If K is right (left) ultra-closed in H, then K is right (left) invertible in H.*

Ultra-closed subhypergroups were introduced by Y. Sureau [22] (see also [2, 20]). The following Proposition proves that the above given definition is equivalent to the definition used by Sureau:

Proposition 2.21. (i) *K is right ultra-closed in H, if and only if $Ka \cap (H-K)\,a = \emptyset$ for all $a \in H$.*

(ii) *K is left ultra-closed in H, if and only if $aK \cap a\,(H-K) = \emptyset$ for all $a \in H$.*

Proof. Suppose that K is right ultra-closed in H. Then $a/a \subseteq K$ for all $a \in H$. Since K is right closed, $(a/a)\,/k \subseteq K$ is valid, or equivalently $a/\,(ak) \subseteq K$ for all $k \in K$. Proposition 2.19 yields $(ak)\,/a \subseteq K$ for all $k \in K$. If $Ka \cap (H-K)\,a \neq \emptyset$, then there exist $k \in K$ and $v \in H-K$, such that $ka \cap va \neq \emptyset$, which implies that $v \in ak/a$. But $(ak)\,/a \subseteq K$, hence $v \in K$ which is absurd. Conversely now: Let $Ka \cap (H-K)\,a = \emptyset$ for all $a \in H$. If $a \in K$, then $K \cap (H-K)\,a = \emptyset$. Therefore $k \notin ra$, for each $k \in K$ and $r \in H-K$. Equivalently $k/a \cap (H-K) = \emptyset$, for all $k \in K$. Hence $k/a \subseteq K$ for all $k \in K$ and $a \in K$. So K is right closed. Next suppose that $a/a \cap (H-K) \neq \emptyset$ for some $a \in H$. Then $a \in (H-K)\,a$, or $Ka \subseteq K\,(H-K)\,a$. Since K is closed, per Proposition 2.3, $K\,(H-K) \subseteq H-K$ is valid. Thus $Ka \subseteq (H-K)\,a$, which contradicts the assumption. Duality gives (ii). □

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Hypermatrix Based on Krasner Hypervector Spaces

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Abstract

In this paper we extend a very specific class of hypervector spaces called Krasner hypervector spaces in order to obtain a hypermatrix. For reaching to this goal, we will define dependent and independent vectors in this kind of hypervector space and define basis and dimension for it. Also, by using multivalued linear transformations, we examine the possibility of existing a free object here. Finally, we study the fundamental relation on Krasner hypervector spaces and we define a functor.

Key words: Hypermatrix, Hypervector spaces, Basis of a hypervector space, Multivalued linear transformations.

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1 Introduction

The notion of a hypergroup was introduced by F. Marty in 1934 [5]. Since then many researchers have worked on hyperalgebraic structures and developed this theory (for more details see [2],[3]). Using hyperstructures

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theory, mathematicians have defined and studied variety of algebraic structures. Among them the notion of hypervector spaces has been studied mainly by Vougiuklis [8, 9], Tallini [6, 7] and Krasner [3].(see also [1]). There are differences mainley about operation or a hyperoperation in these three type of hypervector spaces. Vougiuklis has studied H_V vector spaces which deals with a very weak condition regarding intersections. Tallini defined a hypervector spaces considering a crisp sum and using a hyperexternal operation which assigns to the production every element of a field and every element of the abelian geroup $(V,+)$, a non empty subset of V, while Krasner in the definition of a hypervector space used a hypersum to make a canonical hypergroup and by using a singlevalued operation he defined the Krasner hypervector space with some definitions.
In this paper we have chosen the definition of Krasner and we defined the generalized subset of it. Also, to make a correct logical relation between definitions we had to define the notion of a multivalued linear transformation and by using this notion we could talk about basis and dimension of a Krasner hypervector space. In the sequel, considering the multivalued functions, we have constructed a kind of matrix whith hyperarrays with coefficients taken from the hyperfield of Krasner and elements of the basis. Also, we studied the notion of singular and nonsingular transformations. Finally, we studied the category of Krasner hypervector spaces and defines the fundamental relation on it. In the last part we have defines a functor.

2 Preliminaries

In this section we present definitions and properties of hypervector spaces and subsets, that we need for developing our paper.

A mapping $\circ : H \times H \longrightarrow P^*(H)$ is called a *hyperoperation* (or a join operation), where $P^*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B\}$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$, respectively. Generally, the singleton $\{a\}$ is identified by its element a.

A hypergroupoid $(H, \circ)$, which is associative, i. e, $x \circ (y \circ z) = (x \circ y) \circ z$, $\forall x, y, z \in H$ is called a semihypergroup. A hypergroup is a semihypergroup such that for all $x \in H$, we have $x \circ H = H = H \circ x$, which is called reproduction axiom.

Definition 2.1. [3] A semihypergroup $(H, +)$ is called a canonical hypergroup if the following conditions are satisfied:
$(i) x + y = y + x, \forall x, y \in R$;
$(ii) \exists 0 \in R$(unique) such that for every $x \in R, x \in 0 + x = x$;
(iii) for every $x \in R$, there exists a unique element, say $\acute{x}$ such that $0 \in x + \acute{x}$.(we denote $\acute{x}$ by -x);
(iv) for every $x, y, z \in R$, $z \in x + y \Longleftrightarrow x \in z - y \Longleftrightarrow y \in z - x$.
from the definition it can be easily verified that $-(-x) = x$ and $-(x + y) = -x - y$.

Definition 2.2. [3] A Krasner hyperring is a hyperstructure $(R, \oplus, \star)$ where
(i) $(A, \oplus)$ is a canonical hypergroup;
(ii) $(A, \star)$ is a semigroup endowed with a two-sided absorbing element 0;
(iii) the product distributes from both sides over the sum.

A hyperfield is a Krasner hyperring $(K, \oplus, \star)$, such that $(K - \{0\}, \star)$ is a group.

Definition 2.3. [3] Let $(K, \oplus, \star)$ be a hyperskewfield and $(V, \oplus)$ be a canonical hypergroup. We define a *Krasner hypervector* space over K to be the quadrupled $(V, \oplus, \cdot, K)$, where " $\cdot$ " is a single-valued operation

$$\cdot : K \times V \quad \longrightarrow \quad V,$$

such that for all $a \in K$ and $x \in V$ we have $a \cdot x \in V$, and for all $a, b \in K$ and $x, y \in V$ the following conditions hold:
(H_1) $a \cdot (x \oplus y) = a \cdot x \oplus a \cdot y$;
(H_2) $(a \oplus b) \cdot x = a \cdot x \oplus b \cdot x$;
(H_3) $a \cdot (b \cdot x) = (a \star b) \cdot x$;
(H_4) $0 \cdot x = 0$;
(H_5) $1 \cdot x = x$.
We say that $(V, \oplus, \cdot, K)$ is *anti-left distributive* if for all $a, b \in K$, $x \in V$, $(a + b) \cdot x \supseteq a \cdot x + b \cdot x$, and *strongly left distributive*, if for all $a, b \in K$, $x \in V$, $(a \oplus b) \cdot x = a \cdot x \oplus b \cdot x$,
In a similar way we define the *anti-right distributive and* strongly right distributive hypervector spaces, respectively. V is called strongly distributive if it is both strongly left and strongly right distributive.

In the sequel by a hypervector space we mean a Krasner hypervector space.

3 Krasner Subhypervector Space

Here we study some basic results of Krasner hypervector spaces and after defining the category of Krasner hypervector spaces, we continue to find a free object in the category of Krasner hypervector spaces.

Definition 3.1. A nonempty subset S of V is a *subhyperspace* if $(S, \oplus)$ is a canonical subhypergroup of V and for all $a \in K$, $x \in S$, we have $a \cdot x \in S$.

Here we present example of a Krasner hypervector spaces.

Example 3.2. Let F be a field , V be a vector space and F^* be a multiplicative subgroup of F. For all $x, y \in V$ we define the equivalence relation $\sim$ on V as follows:

$$x \sim y \Longleftrightarrow x = ty \qquad t \in F^*$$

Now, let $\bar{V}$ be the set of all classes of V modulo $\sim$. $\bar{V}$ together with the hypersum $\oplus$, construct a canonical hypergroup:

$$\bar{x} \oplus \bar{y} = \{\bar{v} \in \bar{V} |\ \bar{v} \subseteq \bar{x} \oplus \bar{y}\}$$

Here we consider the external composition

$$\cdot : \bar{f} \times \bar{V} \longrightarrow \bar{V}$$
$$\bar{a} \cdot \bar{v} \longmapsto \bar{a}\bar{v}$$

Now, $(\bar{V}, \oplus, \cdot, F)$ is a hypervector space.

Lemma 3.3. *Let V_i be a hypervector space, for all $i \in I$, then $\bigcap V_i$ is also a hypervector space.*

Definition 3.4. Let V be a hypervector spaces and S a nonempty subset of it, then the smallest subhypervector space of V containing S is called linear space generated by S and is denoted by $< S >$. Moreover, $< S >= \bigcap_{S \subseteq W \leq V} W$.

Theorem 3.5. *Let V be a hypervector space and S a nonempty subset of it, then*

$$< S > = \{t \in V | t \in \sum_{i=1}^{n} a_i \cdot s_i, a_i \in K, s_i \in S, n \in N\} =$$
$$= \{t_1 \oplus ... \oplus t_n | t_i = a_i \cdot s_i\}.$$

Proof. Let $A = \{t \in V | t \in \sum_{i=1}^{n} a_i \cdot s_i, a_i \in K, s_i \in S, n \in N\}$. We claim that $(A, \oplus, \cdot, K)$ is the smallest hypervector space generated by S.
First we show that $(A, \oplus)$ is a canonical hypergroup. Commutativity is obvious.
For all $x \in A$, we have $x \in \sum_{i=1}^{n} a_i \cdot s_i$. Suppose there exists a scalar identity

$0_A \in A$ such that $0_A \in \sum_{i=1}^{n} b_i \cdot r_i$, for $b_i \in K$ and $r_i \in S$, we should have
$$x \oplus 0_A = \sum_{i=1}^{n} a_i \cdot s_i \oplus \sum_{i=1}^{n} b_i \cdot r_i = \sum_{i=1}^{n} a_i \cdot s_i \ni x.$$
Since for all $s_i \in A$, we have $s_i \in S \subseteq V$, and $(V, \oplus)$ is a canonical hypergroup, then there exists a scalar identity in V called 0_V such that $s_i \oplus 0_V = s_i$. Hence in the above equation it is enough to choose $b_i = a_i$ and $r_i = 0_V$, we obtain
$$x \oplus 0_A = \sum_{i=1}^{n} a_i \cdot s_i \oplus \sum_{i=1}^{n} a_i \cdot 0_V = \sum_{i=1}^{n} a_i \cdot (s_i \oplus 0_V) = \sum_{i=1}^{n} a_i \cdot s_i \ni x.$$
Now for all $x \in A$ we define $-x = \sum_{i=1}^{n} a_i \cdot (-s_i)$, then we have
$$0_S = \sum_{i=1}^{n} a_i \cdot 0_V \in \sum_{i=1}^{n} a_i \cdot s_i \oplus \sum_{i=1}^{n} a_i \cdot (-s)_i = \sum_{i=1}^{n} a_i \cdot (s_i \oplus (-s_i))$$
Hence every element in $(A, \oplus)$ has a unique identity. Moreover, every element in $(A, \oplus)$ is reversible, because suppose for all $x, y, z \in A$, we have $x = \sum_{i=1}^{n} a_i \cdot s_i,\ y = \sum_{i=1}^{n} \acute{a}_i \cdot \acute{s}_i,\ z = \sum_{i=1}^{n} \acute{\acute{a}}_i \cdot \acute{\acute{s}}_i$. Since for $s_i, \acute{s}_i, \acute{\acute{s}}_i \in S \subseteq V$, if $\acute{\acute{s}}_i \in s_i \oplus \acute{s}_i$ we have $\acute{s}_i \in \acute{\acute{s}}_i \oplus (-s_i)$, then it is sufficient to choose $\acute{\acute{a}}_i = \acute{a}_i = a_i$. Therefore $(A, \oplus)$ is a canonical subhypergroup.
Now for all $t \in A$, $k \in K$, we have
$$k \cdot t \subseteq k \cdot \sum_{i=1}^{n} a_i \cdot s_i = \sum_{i=1}^{n} (k \star a_i) \cdot s_i \subseteq A.$$
Then $(A, \oplus, \cdot, K)$ is a subhypervector space of V.
Let W be another subhypervector space of V containg S. let $t \in A$, then $t \in \sum_{i=1}^{n} a_i \cdot s_i$, for $a_i \in K, s_i \in S, n \in N$. Since W is a subhypervector space of V containing S, then $\sum_{i=1}^{n} a_i \cdot s_i \subseteq W$ and $A \subseteq W$. So, A is the smallest subhypervector space of V. Also, for all $s \in S$, we have $s = 1 \cdot s$, then $s \in A$, therefore $S \subseteq A$. □

Definition 3.6. Let $(V, \oplus, \cdot), (W, \oplus, \cdot)$ be two hypervector spaces over a hyperskewfield K, then the mapping $T : V \longrightarrow P^*(W)$ is called

(i) *multivalued linear transformation* if
$$T(x \oplus y) \subseteq T(x) \oplus T(y) \qquad \text{and} \qquad T(a \cdot x) = a \cdot T(x).$$
(ii) *multivalued good linear transformation* if
$$T(x \oplus y) = T(x) \oplus T(y) \qquad \text{and} \qquad T(a \cdot x) = a \cdot T(x).$$
where, $P^*(W)$ is the nonempty power set of W.

From now on, by *mv- linear transformation* we mean a multivalued linear transformation.

Remark 3.7. We define $T(0_V) = 0_W$.

Definition 3.8. Let V, W be two hypervector spaces over a hyperskewfield K, and $T : V \longrightarrow P(W)$ be a mv-linear transformation. Then the kernel of T is denoted by *kerT* and defined by
$$KerT = \{x \in V \mid 0_W \in T(x)\}$$

Theorem 3.9. *Let V, W be two hypervector spaces on a hyperskewfield K and $T : V \longrightarrow W$ be a linear transformation. Then $KerT$ is a subhypervector space of V.*

Proof. By Remark 4.18, we have $T(0_V) = 0_W$ which means that $0_V \in KerT$ and $KerT \neq \varnothing$, then we have $x \in x \oplus 0_V = x$, for all $x \in KerT$. The other properties of a canonical subhypervector space will inherit from V. $\square$

Theorem 3.10. *Let V, U be two hypervector spaces and $T : V \longrightarrow P^*(U)$ be a mv-linear transformation :*

(i) if W is a subhypervector space of V, then $T(W)$ is also a subhypervector space of U.

(ii) if L is a subhypervector space of U, then $T^{-1}(L)$ is also a subhypervector space of V containing $kerT$.

Proof. (i) Let $a \in K$ and $\acute{x}, \acute{y} \in T(W)$, such that $\acute{x} = T(x), \acute{y} = T(y)$ for some $x, y \in W$. Then $\acute{x} \oplus \acute{y} = T(x) \oplus T(y) = T(y) \oplus T(x) = \acute{y} \oplus \acute{x}$, hence commutativity holds.
For all $x \in V$ we have $x = x \oplus 0_V$, then we obtain

$$T(x) = T(x \oplus 0_V) \subseteq T(x) \oplus T(0_V) = T(x) \oplus 0_U.$$

Also, for all $x \in V$, there exists $\acute{x} = -x \in V$ such that $0_V \in x \oplus (-x)$. By Remark 4.18 we have

$$0_U = T(0_V) \in T(x \oplus (-x)) \subseteq T(x) \oplus T(-x) = \acute{x} \oplus \overline{\acute{x}}.$$

where $\overline{\acute{x}} = T(-x)$ is the unique inverse of $\acute{x}$.
Now suppose for all $x, y, z \in V$ we have

$$x \in y \oplus z \Longrightarrow y \in x \oplus (-z)$$

This is equivalent to

$$T(x) \in T(y \oplus z) \subseteq T(y) \oplus T(z) \Longrightarrow T(y) \in T(x) \oplus T(-z).$$

So, $(T(W), \oplus)$ is a canonical hypergroup. Now for $a \in K$ and $\acute{x} \in T(W)$, we have

$$a \cdot \acute{x} = a \cdot T(x) = T(a \cdot x) \subseteq T(W).$$

Hence, $(T(W), \oplus, \cdot)$, is a subhypervector space of V.

(ii) let $a \in K$ and $x, y \in T^{-1}(L)$. Suppose $\acute{x} = T(x), \acute{y} = T(y)$, for $\acute{x}, \acute{y} \in L$. Since $(U, \oplus)$ is a canonical hypergroup, then we have

$$x \oplus y = T^{-1}(\acute{x}) \oplus T^{-1}(\acute{y}) = T^{-1}(\acute{y}) \oplus T^{-1}(\acute{x}) = y \oplus x.$$

Also, we have

$$x \oplus 0_V = T^{-1}(\acute{x}) \oplus T^{-1}(0_U) \supseteq T^{-1}(\acute{x} \oplus 0_U) \supseteq T^{-1}(\acute{x}) = x.$$

for all $\acute{x} \in V$, there exists $\acute{-x}$ such that $0_U \in \acute{x} \oplus (\acute{-x})$, hence for $x \in T^{-1}(\acute{x})$, there exists $T^{-1}(-\acute{x}) \in T^{-1}(L)$ such that

$$x \oplus (-x) = T^{-1}(\acute{x}) \oplus T^{-1}(-\acute{x}) = T^{-1}(x \oplus (-\acute{x})) = T^{-1}(0_U) = 0_V.$$

Now for all $\acute{x}, \acute{y}, \acute{z} \in L$, we have

$$\acute{x} \in \acute{y} \oplus \acute{z} \Longrightarrow \acute{y} \in \acute{x} \oplus (-\acute{z})$$

Suppose $x, y, z \in T^{-1}(L)$. The above relation is equivalent to

$$y \oplus z = T^{-1}(\acute{y}) \oplus T^{-1}(\acute{z}) \supseteq T^{-1}(\acute{y} \oplus \acute{z}) \supseteq T^{-1}(\acute{x}) = x$$
$$\Longrightarrow x \oplus (-z) = T(\acute{x}) \oplus T(-\acute{z}) \supseteq T(\acute{x} \oplus (-\acute{z})) \supseteq T(\acute{y}) = y.$$

which means that $x \in y \oplus z \Longrightarrow y \in x \oplus (-z)$.
Moreover, $a \cdot x = a \cdot T^{-1}(\acute{x}) = T^{-1}(a \cdot \acute{x}) \subseteq T^{-1}(L)$. Hence $(T^{-1}(L), \oplus, \cdot)$ is a subhypervector space of V.
Now for $x \in KerT$ we have $T(x) = 0_U \in L$, then we obtain $x \in T^{-1}(L)$, hence $KerT \subseteq T^{-1}(L)$. □

Theorem 3.11. *Let U, V be two hypervector spaces on a hyperskewfield K and $T : V \longrightarrow P^*(U)$ be a good linear transformation. Then there is a one to one correspondence between sunhypervector spaces of V containing $KerT$ and subhypervector spaces of U.*

Proof. Suppose $A = \{W|W \leq V,\ W \supseteq KerT\}$ and $B = \{L|L \leq U\}$. We show that the following map is one to one and onto:

$$\phi : A \longrightarrow B$$
$$W \longrightarrow T(W)$$

By Theorem 3.10, $T(W)$ belongs to B, for all $W \in A$. Now let W_1, W_2 be two elements of A such that $W_1 \neq W_2$, then there exists $w_1 \in W_1 - W_2$ or $w_2 \in W_2 - W_1$. Let $w_1 \in W_1 - W_2$, then $T(w_1) \in T(W_1) - T(W_2)$ and hence $T(W_1) \neq T(W_2)$. If $w_2 \in W_2 - W_1$, then $T(W_1) \neq T(W_2)$, too. So, ϕ is well defined and one to one. Now for $L \in B$, put $W = T^{-1}(L)$, then by Theorem 3.9 we have $W \in A$ and $T(W) = L$. Therefore, ϕ is onto, hence the result. □

4 Construction of a hypermatrix

Now, we will talk about the basis of a hypervector space and verify that considering a multivalued linear transformation will imply some conditions to this definition. Finally, with the elements of hyperfield and basis we will construct a hypermatrix.

Definition 4.1. A subset S of V is called linearly independent if for every vectors $v_1, ..., v_n \in S$, and $c_1, ..., c_n \in K$, if we have $0_V \in c_1 \cdot v_1 \oplus ... \oplus c_n \cdot v_n$, implies that $c_1 = ... = c_n = 0_K$. Otherwise S is called linearly dependent.

Theorem 4.2. *Let V be a hypervector space and $v_1, ..., v_n$ be independent in V. Then every element in the linear space $< v_1, ..., v_n >$ belongs to a unique sum of the form $\sum_{i=1}^{n} a_i \cdot v_i$ where $a_i \in K$.*

Proof. Every element of $< v_1, ..., v_n >$ belongs to a set of the form $\sum_{i=1}^{n} a_i \cdot v_i$ where $a_i \in K$. We will show that this form is unique. Let $u \in V$ such that $u \subseteq \sum_{i=1}^{n} a_i \cdot v_i$ and $u \subseteq \sum_{i=1}^{n} b_i \cdot v_i$, where $a_i, b_i \in K$. Since V is a hypervector space we have :

$$0_V \in u - u \subseteq \sum_{i=1}^{n} a_i \cdot v_i - \sum_{i=1}^{n} b_i \cdot v_i = \sum_{i=1}^{n} a_i \cdot v_i \oplus \sum_{i=1}^{n} (-b)_i \cdot v_i.$$

Therefore, $0_V \subseteq \sum_{i=1}^{n} (a_i \oplus (-b_i)) \cdot v_i$. And since $v_1, ...v_n$ are independent we have $a_i \oplus (-b_i) = 0, \ \forall i$, then $a_i = -(-b_i) = b_i$. □

Theorem 4.3. *Let V be a hypervector space. Then vectors $v_1, ...v_n \in V$ are independent or v_j for some $1 \leq j \leq r$, belongs to the linear combination of the other vectors.*

Proof. Let $v_1, ..., v_n$be dependent and let $0_V \subseteq \sum_{i=1}^{n} a_i \cdot v_i$ such that at least one of the scalars such as a_j is not zero. Then there exists $t_i, (i = 1, ..., n)$ such that

$$0_V \in t_1 \oplus t_2 \oplus ... \oplus t_n,$$

where $t_i = a_i \cdot v_i$, which means that

$$\begin{aligned} & t_j \in 0 \oplus (-(t_1 \oplus ... \oplus t_{j-1} \oplus t_{j+1} \oplus ... \oplus t_n)) \\ \Longrightarrow\ & t_j \in 0 \oplus ((-t_1) \oplus ... \oplus (-t_{j-1}) \oplus (-t_{j+1}) \oplus ... \oplus (-t_n)) \end{aligned}$$

Moreover, for at least one v_j we have $v_j = (a_j^{-1}) \cdot t_j$. which means

$$\begin{aligned} v_j & \in (a_j^{-1}) \cdot (-t_1 \oplus ... \oplus (-t_{j-1}) \oplus (-t_{j+1}) \oplus ... \oplus (-t_n)) \in \\ & \in ((a_j^{-1}) \cdot (-t_1)) \oplus ((a_j^{-1}) \cdot (-t_{j-1})) \oplus ((a_j^{-1}) \cdot (-t_{j+1})) \oplus ... \oplus ((a_j^{-1}) \cdot (-t_n)) \\ & \in ((a_j^{-1}) \cdot (-a_1 \cdot v_1)) \oplus ... \oplus ((a_j^{-1}) \cdot (-a_1 \cdot v_{j-1})) \oplus ((a_j^{-1}) \cdot (-a_{j+1} \cdot v_{j+1})) \\ & \qquad \oplus ... \oplus ((a_j^{-1}) \cdot (-a_n \cdot t_n)) \\ & \in ((a_j^{-1} \star (-a_1)) \cdot v_1) \oplus ... \oplus ((a_j^{-1} \star (-a_1)) \cdot v_{j-1}) \oplus ((a_j^{-1} \star (-a_{j+1})) \cdot v_{j+1}) \\ & \qquad \oplus ... \oplus ((a_j^{-1} \star (-a_n)) \cdot t_n) \\ & \in (c_1 \cdot v_1) \oplus ... \oplus (c_j \cdot v_{j-i}) \oplus (c_j \cdot v_{j+1}) \oplus ... \oplus (c_j \cdot v_n) \end{aligned}$$

where $c_j = (a_j^{-1} \star (-a_n))$. Therefore v_j belongs to a linear combination of $v_1, ..., v_{j-1}, v_{j+1}, ..., v_n$ as desired. □

Definition 4.4. We call β a basis for V if it is a linearly independent subset of V and it spans V. We say that V has finite dimensional if it has a finite basis.

The following results are the generalization of the same results for vector spaces, also the methods here are adopted from those in the ordinary vector spaces.

Theorem 4.5. *Let V be a hypervector space. If W is a subhypervector space of V generated by $\beta = \{v_1, ..., v_n\}$, then W has a basis contained in β.*

Corolary 4.6. *If V is a hypervector space, then every generating subset of V, contains a basis of V, which means every independent subset of V is included in a finite basis.*

Theorem 4.7. *Let V be a hypervector space. If V has a finite basis with n elements, then the number of elements of every independent subset of V is smaller or equal to n.*

Corolary 4.8. *Let V be strongly left distributive and hypervector space. If V is finite dimensional then every two basis of V have the same elements.*

Lemma 4.9. *Let V be a hypervector space. If V is finite dimensional, then every linearly independent subset of V is contained in a finite basis.*

Now, we want to determine that what is a free object in the category of hypervector spaces. First, notice that if we denote the category of hypervector spaces by **KrH-vect**, we define the category as follows:
(i) the objects in this category are hypervector spaces over a hyperskew field K;
(ii) for the objects V, W of **KrH-vect**, the set of morphisms from V to $P^*(W)$ is the multivalued linear transformations which we show by $Home(V, W)$.
(iii) combination of morphism is defined as usual;
(iv) for all objects V in the category, the morphism $1_V : V \longrightarrow V$ is the identity.

According to the definition of a free object in the category of hypersets [2], and considering the category of hypervector spaces, if X is a basis for the hypervector space V, then we say that F is a free object in **KrH-vect** then for every function $f : X \longrightarrow V$, there exists a homomorphism $\bar{f} : F \longrightarrow V$, such that $\bar{f} \circ i = f$, where i is the inclusion function. Now, we have

$$(\bar{f} \circ i)(x) = \bar{f}(i(x)) = \bar{f}(x) \qquad (\star)$$

Since the homomorphism $\bar{f}$ is defined in **H-vect**, it is a multivalued transformation, then we define $\bar{f}(x) = \{f(x)\}$ we obtain $\bar{f} \circ i = f$.

Let $g : F \longrightarrow V$ be another homomorphism such that $g(x_i) = f(x_i)$, then for $t \in \sum_{i=1}^{n} a_i \cdot x_i$, let $\bar{f}$ be defined by $\bar{f}(t) = \sum_{i=1}^{n} a_i \cdot f(x_i)$, we have

$$g(t) \subseteq g(\sum_{i=1}^{n} a_i \cdot x_i) = \sum_{i=1}^{n} a_i \cdot g(x_i) = \bar{f}(t).$$

hence $\bar{f}$ defined above is the maximum homomorphism such that $(\star)$ is satisfied.

Suppose $t \in \sum_{i=1}^{n} a_i \cdot x_i$ and $t \in \sum_{i=1}^{n} b_i \cdot x_i$, for $a_i, b_i \in K$, we have $\bar{f}(t) = \sum_{i=1}^{n} a_i \cdot f(x_i)$, and also $\bar{f}(t) = \sum_{i=1}^{n} b_i \cdot f(x_i)$, then $\sum_{i=1}^{n} a_i \cdot f(x_i) = \sum_{i=1}^{n} b_i \cdot f(x_i)$, we obtain

$$0 \in \sum_{i=1}^{n} a_i \cdot f(x_i) - b_i \cdot f(x_i) = \sum_{i=1}^{n} (a_i - b_i) \cdot f(x_i)$$

So $a_i = b_i$. Therefore, $\bar{f}$ is a unique mv-transformation.
Hence we have the following corollary:

Corolary 4.10. *Every hypervector space with a basis is a free object in the category of hypervector spaces.*

Theorem 4.11. *Let $(V, \oplus, \cdot), (W, \oplus, \cdot)$ be two hypervector spaces on a hyperskewfiled K . If $T : V \longrightarrow P^*(W)$ and $U : V \longrightarrow P^*(W)$ be two mv-transformations. We define $L(V, W) = \{T | T : V \longrightarrow P^*(W)\}$ and the hyperoperation " $\boxplus$ " as follows:*

$$(T \boxplus U)(\alpha) = T(\alpha) \boxplus U(\alpha)$$

Also, we define the external composition as

$$(c \boxdot T)(\alpha) = c \boxdot T(\alpha)$$

Then $(L(V, W), \boxplus, \boxdot))$ as defined above is a hypervectorspace over a hyperskewfield K.

Proof. The external composition " $\boxdot$ " is defined as follows:

$$\boxdot : K \times L(V, W) \longrightarrow P^*(L(V, W))$$
$$(\alpha, T) \longmapsto \alpha \boxdot T$$

First we show that $(L(V, W), \boxplus)$ is a canonical hypergroup.
Communativity and associativity is obvious. We consider the transformation $0 : V \longrightarrow 0$ as a "0" for the group and $1 : V \longrightarrow P^*(V)$ as the identity. Then there exists a unique inverse $(-T)$ such that $0 \in (T \boxplus (-T))(\alpha)$.
Now, let T, U, Z be three linear transformations that belong to $L(V, W)$ then if $Z \in T \boxplus U$ then we have $Z(\alpha) \in (T \boxplus U)(\alpha)$, which means $Z(\alpha) \in T(\alpha) \boxplus U(\alpha)$. Now since W is hypervector space then we obtain $T(\alpha) \in Z(\alpha) \boxplus (-U)(\alpha)$, hence $T \in Z \boxplus (-U), \forall \alpha \in K$. Therefore, $(L(V, W), \boxplus)$ is a canonical hypergroup.
Now, we check that $L(V, W)$ is a hypervector space. Let $x, y \in K$ and $T, U \in L(V, W)$ then we have
(1) $(x \boxdot (T \boxplus U))(\alpha) = x \boxdot (T \boxplus U)(\alpha) = (x \boxdot T(\alpha)) \boxplus (x \boxdot U(\alpha))$
(2) $((x \boxplus y) \boxdot T)(\alpha) = \bigcup_{z \in x \boxplus y} z \boxdot T(\alpha) = (x \boxdot T(\alpha)) \boxplus (y \boxdot T(\alpha))$.

The other conditions will be obtained immediately. Therefore, $(L(V,W), \boxplus, \boxdot)$ is a hypervector space. $\square$

Theorem 4.12. *Let $(V, \oplus, \cdot)$, $(W, \oplus, \cdot)$ be two hypervector spaces on a hyperskewfield K, if $A = \{\alpha_1, ..., \alpha_n\}$ be a basis for V and $\beta_1, ..., \beta_n$ be any vectors in W, then there is a unique linear transformation $T : V \longrightarrow P^*(W)$ such that $T(\alpha_i) = \beta_i,\ 1 \leq i \leq n$.*
In Other words, every linear transformation can be characterized by its operation on the basis of V.

Proof. Since for every $v \in V$, there exists scalars $c_1, ..., c_n \in K$ such that

$$(*) \qquad v \in \sum_{i=1}^{n} c_i \cdot \alpha_i$$

then we define a map $T : V \longrightarrow P^*(W)$ as follows:
$$T(v) = \sum_{i=1}^{n} c_i \cdot T(\alpha_i) = \sum_{i=1}^{n} c_i \cdot \beta_i$$
Since $(*)$ is unique then T is well-defined. Now, we check that T is a linear transformation. Let $v, w \in V$ and scalars $d_1, ..., d_n \in K$ then $v \in \sum_{i=1}^{n} c_i \cdot \alpha_i$ and $w \in \sum_{i=1}^{n} d_i \cdot \alpha_i$, then we have $T(v) = \sum_{i=1}^{n} c_i \cdot T(\alpha_i)$ and $T(w) = \sum_{i=1}^{n} d_i \cdot T(\alpha_i)$. Now since $v \oplus w \in \sum_{i=1}^{n} (c_i \oplus d_i) \cdot \alpha_i$, then we obtain
$$T(v \oplus w) \subseteq T(\sum_{i=1}^{n} (c_i \oplus d_i) \cdot \alpha_i) = \sum_{i=1}^{n} (c_i \oplus d_i) \cdot T(\alpha_i)$$
$$= \sum_{i=1}^{n} c_i \cdot T(\alpha_i) \oplus \sum_{i=1}^{n} d_i \cdot T(\alpha_i) = T(v) \oplus T(w).$$
Also, it is clear that $(c \circ T)(\alpha) = c \circ T(\alpha)$. Hence, T is a linear transformation.
Now, we shall check that T is unique. Let $S : V \longrightarrow P^*(W)$ be another linear transformation that satisfies $S(\alpha_i) = \beta_i$. We will show that $S = T$. We have

$$S(\alpha) = \sum_{i=1}^{n} c_i \cdot S(\alpha_i) = \sum_{i=1}^{n} c_i \cdot \beta_i = \sum_{i=1}^{n} c_i \cdot T(\alpha_i) = T(\alpha)$$

So, $S = T$ as desired. $\square$

Remark 4.13. Let $T : V \longrightarrow P^*(W)$ be a linear transformation. We denote $KerT = \{\alpha \in V |\ 0 \in T(\alpha)\}$ by N_T and by ImT we mean $R_T = \{T(\alpha) | \alpha \in V\}$.
We call dimension of R_T, rank of T and it is denoted by $R(T)$. Notice that N_T is a subhypervector space of V and R_T is a subhypervector space of W.

Theorem 4.14. *Let V, W be two hypervector spaces over a field K. Let $T : V \longrightarrow P^*(W)$ be a linear transformation and $dimV = n < \infty$. Then*

$$dimR_T + dimKerT = dimV$$

Proof. Let $W = N_T$ and let $\beta_1 = \{\alpha_1, ..., \alpha_k\}$ be a basis for W. We extend β_1 to $\beta_2 = \{\alpha_1, ..., \alpha_k, \alpha_{k+1}, ..., \alpha_n\}$. We will show that $\beta = \{T(\alpha_{k+1}), ..., T(\alpha_n)\}$ is a basis for R_T. Let $c_1, ..., c_n$ be scalars in K such that

$$0 \in \sum_{i=k+1}^{n} c_i \cdot T(\alpha_i)$$

then there exists $\gamma \in \sum_{i=k+1}^{n}(c_i \cdot \alpha_i)$ such that $0 \in T(\gamma)$, this implies that $\gamma \in KerT = N_T$, hence $\gamma \in \sum_{i=1}^{k}(c_i \cdot \alpha_i)$. Therefore

$$0 = \gamma - \gamma \in \sum_{i=1}^{k}(c_i \cdot \alpha_i) \oplus \sum_{i=k+1}^{n}((-c_i) \cdot \alpha_i) \Longrightarrow c_i = 0$$

Now, we claim that β generates R_T because if for all $\alpha \in V$ we have $T(\alpha) = \beta$, and since $0 \in \sum_{i=1}^{k} c_i \cdot T(\alpha_i)$, hence
$\beta = T(\alpha) \subseteq T(\sum_{i=1}^{n} c_i \cdot \alpha_i) = \sum_{i=1}^{n} c_i \cdot T(\alpha_i) = \sum_{i=1}^{k} c_i \cdot T(\alpha_i) + \sum_{i=k+1}^{n} c_i \cdot T(\alpha_i) = \sum_{i=k+1}^{n} c_i \cdot T(\alpha_i)$
Therefore, $dimR_T + dimN_T = (n-k) + k = n = dimV$. □

For all $1 \leqslant j \leqslant n$ and $1 \leqslant p \leqslant m$, we define C_{pj} as the coordinator of $T(\alpha_j)$ on the ordered basis $B = \{\beta_1, ..., \beta_p\}$ which means

$$T(\alpha_j) = \sum_{p=1}^{m} C_{pj} \cdot \beta_p$$

where for $C_{pj} = (c_{pj}), \beta_p = (\beta_{p1})$. Now, if we notice the following matrix with a crisp product and hypersum, we will have a hypermatrix as the following:

$$\underbrace{\begin{pmatrix} c_{11} & ... & c_{1p} \\ ... & ... & ... \\ c_{j1} & ... & c_{jp} \end{pmatrix}}_{C_{pj}} \underbrace{\begin{pmatrix} \beta_{11} \\ ... \\ \beta_{p1} \end{pmatrix}}_{\beta_p} = \begin{pmatrix} c_{11} \cdot \beta_{11} \oplus ... \oplus c_{1p} \cdot \beta_{p1} \\ ... \\ c_{j1} \cdot \beta_{11} \oplus ... \oplus c_{jp} \cdot \beta_{p1} \end{pmatrix} = \underbrace{\begin{pmatrix} T(\alpha_1) \\ ... \\ T(\alpha_j) \end{pmatrix}}$$

Theorem 4.15. *Let V, W be two hypervector spaces. If $dimV = n$ and $dimW = m$, then $dimL(V, W) = mn$.*

Proof. Let $A = \{\alpha_1, ..., \alpha_n\}$ and $B = \{\beta_1, ..., \beta_m\}$ be the basis of V, W respectively. For all (p, q), where $p, q \in \mathbb{Z}$, and $1 \leqslant q \leqslant n$, $1 \leqslant p \leqslant m$ by Theorem 4.12 we have a unique linear transformation $T_{pq} : V \longrightarrow P^*(W)$ which we define by $T_{pq}(\alpha_i) = \beta_p$, when $i = q$ and otherwise it is defined 0. Since we have mn linear transformation from V to $P^*(W)$, it is sufficient to

show that $\beta'' = \{T_{pq} | 1 \leqslant p \leqslant m,\ 1 \leqslant q \leqslant n\}$ is a basis for $L(V, W)$.
Let $T : V \longrightarrow P^*(W)$ be a linear transformation. For all $1 \leqslant j \leqslant n$, let $C_{1j}, ..., C_{mj}$ be the coordinate of $T(\alpha_j)$ in the ordered basis $\acute{\beta}$, i.e, $T(\alpha_j) = \sum_{p=1}^{m} C_{pj} \cdot \beta_p$. We will show that $T = \sum_{p=1}^{m} \sum_{q=1}^{n} C_{pq} \cdot T_{pq}$ generates $L(V, W)$. Because if we suppose $U = \sum_{p=1}^{m} \sum_{q=1}^{n} C_{pq} \cdot T_{pq}$, then if suppose $i = q$ we obtain

$U(\alpha_j) = \sum_{p=1}^{m} \sum_{q=1}^{n} C_{pq} \cdot T_{pq}(\alpha_j) = \sum_{p=1}^{m} \sum_{q=1}^{n} C_{pq} \cdot \beta_p = \sum_{p=1}^{m} A_{pj} \cdot \beta_p = T(\alpha_j)$.

Otherwise it will be 0. Also, it is obvious that β'' is independent. Hence the result. □

Remark 4.16. Let $T : V \longrightarrow P^*(W)$ and $S : W \longrightarrow P^*(Z)$ be two linear transformations and $\alpha \in V$, we define $(S \circ T)(\alpha) = S(T(\alpha)) = \bigcup_{\beta \in T(\alpha)} S(\beta)$ then $S \circ T$ is also a linear transformation.

Definition 4.17. Let $T : V \longrightarrow P^*(V)$ be a linear transformation, we call T a *linear operator*(or shortly an operator) on V, and If we have $T \circ T$, we denote it by T^2.

Lemma 4.18. *let V be a hypervector space on a field K. If U, T, S be three operators on V and $k \in K$, then the following results are immediate:*
(i) $I \circ U = U \circ I = U$;
(ii) $(S \oplus T) \circ U = S \circ U \oplus T \circ U$, $U \circ (S \oplus T) = U \circ S \oplus U \circ T$;
(iii) $k \oplus (U \circ T) = (kU) \circ T = U \circ (kT)$.□

Example 4.19. Let $\beta = \{\alpha_1, ..., \alpha_n\}$ be an ordered basis for the hypervector space V. Consider the operators $T_{(p,q)}$ regarding the proof of Theorem 4.15. These n^2 operators construct a basis for the space of operators of V. let S, U be two operators on V then we have

$$S = \sum_p \sum_q C_{pq} \cdot S_{pq}, \qquad U = \sum_r \sum_s B_{rs} \cdot S_{rs}.$$

Now by lemma 4.18, we have

$$(S \circ U)(\alpha_i) = S(U(\alpha_i)) = \bigcup_{\beta \in U(\alpha_i)} S(\beta) = \bigcup_{\beta \in \Sigma_r \Sigma_s B_{rs} \cdot T_{rs}(\alpha_i)} S(\beta)$$
$$= S(\sum_r \sum_s B_{rs} \cdot T_{(r,s)}(\alpha_i)) = S(\sum_r \sum_s B_{rs} \cdot \alpha_r)$$

when $i = s$ we have

$$\sum_r \sum_s B_{ri} \cdot S(\alpha_r) = \sum_r \sum_s B_{rs} \cdot (\sum_p \sum_q C_{pq} \cdot T_{pq}(\alpha_r))$$
$$= \sum_r \sum_s \sum_p \sum_q (B_{ri} A_{pq}) \circ \alpha_p$$

and when $r = q$ we have

$$= \sum_r \sum_s \sum_p \sum_q (B_{ri} C_{pr}) \cdot \alpha_p = \sum_r \sum_s \sum_p \sum_q (C_{pr} B_{ri}) \cdot \alpha_p$$

and since $1 \leqslant i \leqslant P$ then we have $\sum_r \sum_s \sum_p \sum_q (BC)_{n^2} \cdot \alpha_i$
Hence when we compose two operators S and U, the result is obtained by multiplying two matrices of them.□

Now, it is time to talk about the inverse of a transformation. As it is usual for defining an inverse we have:

Definition 4.20. let $T : V \longrightarrow P^*(W)$ be one to one and onto. T is said to have an inverse when there exists $U : W \longrightarrow P^*(V)$ such that $T \circ U = I_V$ and $U \circ T = I_W$. Also, the inverse of T is denoted by T^{-1} and obviously is not unique. We have $(U \circ T)^{-1} = T^{-1} \circ U^{-1}$.
We say that a linear transformation T is called *nonsingular* if $0 \in T(\alpha)$ implies that $\alpha = \{0\}$, which means that the null space of T is equal to $\{0\}$.

Lemma 4.21. *Let $T : V \longrightarrow P^*(W)$ be a linear transformation then T is one to one if and only if T is nonsingular if and only if $KerT = 0$*

Proof. Let T be one to one and suppose $0 \in T(\alpha)$, then since $T(0) = 0$, we have $T(0) \in T(\alpha)$ then

$$T(0) \in T(\alpha + 0) \subseteq T(\alpha) + T(0) \Longrightarrow T(\alpha) \in T(0) + (-T(0)) = 0$$

hence $\alpha = 0$. Conversely, let T is nonsingular and suppose for $x, y \in V$, we have $T(x) = T(y)$ then, $0 \in T(x) - T(y) = T(x - y)$ and since T is nonsingular we obtain $x - y = 0$, which means $x = y$.
Now let for all $\alpha \in KerT$ we have $0 \in T(\alpha)$, then since T is nonsingular we obtain $\alpha = 0$ which means $KerT = 0$. Conversely, if $KerT = 0$, then suppose $0 \in T(\alpha)$ implies that $\alpha \in KerT = 0$, hence $\alpha = 0$. □

Theorem 4.22. *Let V, W be two hypervector spaces on a hyperfiled K and let $T : V \longrightarrow P^*(W)$ be a linear transformation. If T is good reversible linear transformation, then the reverse of T is also a good linear transformation.*

Proof. Let $w_1, w_2 \in W$ and $k \in K$, then there exists $v_1, v_2 \in V$ such that $T^{-1}(w_1) = v_1, T^{-1}(w_2) = v_2$, where $T(v_1) = w_1$, and $T(w_2) = v_2$. We have

$$T^{-1}(w_1 \oplus w_2) = T^{-1}(T(v_1) \oplus T(v_2)) \supseteq T^{-1}(T(v_1 \oplus v_2)) = v_1 \oplus v_2 = T^{-1}(w_1) \oplus T^{-1}(w_2)$$

and when T is a good linear transformation, T^{-1} is also a good linear transformation. □

Theorem 4.23. *Let $T : V \longrightarrow P^*(W)$ be a linear transformation. T is nonsingular if and only if T corresponds every linearly independent subset of V onto a linearly independent subset of W.*

Proof. Let T be nonsingular and S be a linearly independent subset of V. We show that $T(S)$ is independent. Let $\acute{s}_i \in T(S)$ and for all i there exists $s_i \in S$ such that $T(s_i) = \acute{s}_i$. We assume

$$\sum_{i=1}^{n} c_i \cdot \acute{s}_i = 0 \Longrightarrow \sum_{i=1}^{n} c_i \cdot T(s_i) = 0 \Longrightarrow T(\sum_{i=1}^{n} c_i \cdot s_i) = 0$$

because T is nonsingular we have $\sum_{i=1}^{n} c_i \cdot s_i = 0$, and since s_i, for all i are

linearly independent then $c_i = 0$, hence $T(S)$ is linearly independent.
Conversely, let $0 \neq \alpha \in V$, then $\{0\}$ is an independent set. Hence by hypothesis T corresponds this independent set to a linearly in dependent set such as $T(\alpha) \in P^*(W)$, then we have $T(\alpha) \neq 0$. Therefore, T is nonsingular. $\square$

Theorem 4.24. *Let V, W be two hypervector spaces with finite dimension on a hyperskewfield K and $dimV = dimW$. If $T : V \longrightarrow P^*(W)$ is a linear transformation, then the followings are equivalent:*
(i) T is reversible;
(ii) T is nonsingular;
(iii) T is onto.
(iv) If $\{\alpha_1, ..., \alpha_n\}$ is a basis for V, then $\{T(\alpha_1), ..., T(\alpha_n)\}$ is a basis for W.
(v) There exists a basis like $\{\alpha_1, ..., \alpha_n\}$ for V such that $\{T(\alpha_1), ..., T(\alpha_n)\}$ is a basis for W.

Lemma 4.25. *let V be a hypervector space with finite dimension on a hyperfield K, then $V \cong K^n$.*

5 Fundamental Relations

Let $(V, \oplus, \cdot)$ be a hypervector space, we define the relation ε^* as the smallest equivalence relation on V such that the set of all equivalence classes,V/ε^*, is an ordinary vector space. ε^* is called fundamental equivalence relation on V and V/ε^* is the fundamental ring.
Let $\varepsilon^*(v)$ is the equivalence class containing $v \in V$, then we define $\boxplus$ and $\boxdot$ on V/ε^* as follows:

$$\varepsilon^*(v) \boxplus \varepsilon^*(w) = \varepsilon^*(z), \text{ for all } z \in \varepsilon^*(v) \oplus \varepsilon^*(w)$$
$$a \boxdot \varepsilon^*(v) = \varepsilon^*(z), \text{ for all } z \in a \cdot \varepsilon^*(v), \quad a \in K$$

Let $\mathbf{U}$ be the set of all finite linear combinations of elements of V with coefficients in K, which means

$$\mathbf{U} = \{\textstyle\sum_{i=1}^n a_i \cdot v_i;\ a_i \in K,\ v_i \in V,\ n \in \mathbf{N}\}$$

we define the relation ε as follows:

$$v \varepsilon w \Longleftrightarrow \exists u \in \mathbf{U}; \{v, w\} \subseteq u$$

Koskas [4] introduced the relation β^* on hypergroups as the smallest equivalence relation such that the quotient R/β^*_+ is a group. We will denote β_+ the relation in R as follows:

$$v \beta_+ w \Longleftrightarrow \exists (c_1, ..., c_n) \in V^n \text{ such that } \{v, w\} \subseteq c_1 \oplus ... \oplus c_n$$

Freni proved that for hyperrings we have $\beta_+^* = \beta_+$. Since in here $(V, \oplus)$ is a canonical hypergroup the we will have:

Theorem 5.1. *In the hypervector space $(V, \oplus, \cdot)$, we have $\varepsilon^* = \beta_+^*$.*

Vougiouklis [9] has proved that the sets $\{\varepsilon^*(z) : z \in \varepsilon^*(v) \oplus \varepsilon^*(w)\}$ and $\{\varepsilon^*(z) : z \in a \cdot \varepsilon^*(v)\}$ are singletons. With a similar method we can prove the following theorem:

Theorem 5.2. *Let $(V, \oplus, \cdot)$ be a hypervector space, then for all $a \in K$, $v, w \in V$, we have the followings:*
(i) $\varepsilon^*(v) \boxplus \varepsilon^*(w) = \varepsilon^*(z)$, $\forall z \in \varepsilon^*(v) \oplus \varepsilon^*(w)$
$a \boxdot \varepsilon^*(v) = \varepsilon^*(z)$, $\forall z \in a \cdot \varepsilon^*(w)$
$(ii)\varepsilon^*(0_V)$ is the zero element of $(V/\epsilon^*, \boxplus)$.
(iii) $(V/\varepsilon^*, \boxplus, \boxdot)$ is a hypervector space and is called the fundamental hypervector space of V.

Proof. (i) The proof is the same as [9], and we omit it.
(ii) since from (i) we obtain $\varepsilon^*(v) \boxplus \varepsilon^*(w) = \varepsilon^*(v \oplus w)$ and $a \boxdot \varepsilon^*(v) = \varepsilon^*(a \cdot v)$ we have

$$\varepsilon^*(v) \boxplus \epsilon^*(0) = \varepsilon(v \oplus 0) = \varepsilon^*(v)$$

(iii) The conditions for the vector space $(V/\epsilon^*, \boxplus, \boxdot)$ will be obtained from the hypervector space $(V, \oplus, \cdot)$. $\square$

Theorem 5.3. *Let $(V, \oplus, \cdot, K)$ be a hypervector space and $(V/\epsilon^*, \boxplus, \boxdot)$ be the fundamental relation of it then $dimV = dimV/\epsilon^*$.*

Proof. Let $B = \{v_1, ..., v_n\}$ be a basis for V. We show that the set $B^* = \{\varepsilon^*(v_1), ..., \varepsilon^*(v_n)\}$ is a basis for V/ϵ^*. For this let $\varepsilon^*(v) \in V/\varepsilon^*$, then for every $v \in V$ there exists $a_1, ..., a_n \in K$ such that $x \in \sum_{i=1}^n a_i \cdot v_i$, then $v = t_1 \oplus ... \oplus t_n$, where $t_i = a_i \cdot v_i, i \in \{1, ..., n\}$. Now by Theorem 5.2 we have $\varepsilon^*(t_i) = a_i \cdot \varepsilon^*(v_i)$ then

$$\varepsilon^*(v) = \varepsilon^*(t_1 \oplus .... \oplus t_n) = \varepsilon^*(t_1) \boxplus .... \boxplus \varepsilon^*(t_n) = (a_1 \boxdot \varepsilon^*(v_1)) \boxplus (a_n \boxdot \varepsilon^*(v_n)).$$

hence, V/ε^* is spanned by B^*.
Now we show that B^* is linearly independent. For this let

$$\begin{aligned}
&(a_1 \boxdot \varepsilon^*(v_1)) \boxplus ... \boxplus (a_n \boxdot \varepsilon^*(v_n)) = \varepsilon^*(0) \\
&\Longrightarrow \varepsilon^*(a_1 \cdot v_1) \boxplus ... \boxplus \varepsilon^*(a_n \cdot v_n) = \varepsilon^*(0) \\
&\Longrightarrow \varepsilon^*(a_1 \cdot v_1 \oplus ... \oplus a_n \cdot v_n) = \varepsilon^*(0) \\
&\Longrightarrow 0 \in a_1 \cdot v_1 \oplus ... \oplus a_n \cdot v_n
\end{aligned}$$

since B in linearly independent in V, then $a_1 = ... = a_n = 0$. Therefore, B^* is also linearly independent. $\square$

Lemma 5.4. *Let V, W be two hypervector spaces and $T : V \longrightarrow P^*(W)$ be a linear transformation, then*
(i) $T(\varepsilon^*(v)) \subseteq \varepsilon^*(T(v))$, for all $v \in V$;
(ii) The map $T^* : V/\varepsilon^* \longrightarrow W/\varepsilon^*$ defined as $T^*(\varepsilon^*(v)) = \varepsilon^*(T(v))$ is a linear transformation.

Proof. (i) straightforward.
(ii) It is obvious that T^* is well defined. Now we show that T^* is a linear transformation. Let $a \in K$, $x, y \in V$, then by Theorem 5.2 we have
$T^*(\varepsilon^*(x) \boxplus \varepsilon^*(y)) = T^*(\varepsilon^*(x \oplus y)) = \varepsilon^*(T(x \oplus y)) \subseteq \varepsilon^*(T(x) \oplus T(y)) = \varepsilon^*(T(x)) \boxplus \varepsilon^*(T(y)) = T^*(\varepsilon^*(x)) \boxplus T^*(\varepsilon^*(y))$
and $T^*(a \boxdot \varepsilon^*(x)) = T^*(\varepsilon^*(a \cdot x)) = \varepsilon^*(T(a \cdot x)) = \varepsilon^*(a \cdot T(x)) = a \boxdot \varepsilon^*(T(x)) = a \boxdot T^*(\varepsilon^*(x))$
hence, T^* is a linear transformation. □

Theorem 5.5. *The map $F : \boldsymbol{HV} \longrightarrow \boldsymbol{V}$ defined by $F(V) = V/\varepsilon^*$ and $F(T) = T^*$ is a functor, where $\boldsymbol{HV}$ and $\boldsymbol{V}$ denote the category of hypervector spaces and vector spaces respectively. Moreover, F preserves the dimension.*

Proof. By Lemma 5.4 F is well-defined. Let$T : V \longrightarrow P^*(W)$ and $U : W \longrightarrow P^*(Z)$ be two linear transformations, then $F(U \circ T) = (U \circ T)^*$ such that for all $v \in V$ we have

$$\begin{aligned}(U \circ T)^*(\varepsilon^*(v)) &= \varepsilon^*((U \circ T)(v)) = \varepsilon^*(U(T(v)))\\ &= U^*\varepsilon^*(T^*(x)) = U^*T^*(\varepsilon^*(x)) = F(U)F(T)(\varepsilon^*(v))\\ &\Longrightarrow F(U \circ T) = F(U)F(T)\end{aligned}$$

Also, the identity is $F(1_V^*) : V/\varepsilon^* \longrightarrow V/\varepsilon^*$ such that $1_V^*(\varepsilon^*(v)) = \varepsilon^*(v)$. Hence, F is a functor And by Theorem 4.14 we have $dim(F(V)) = dim(V/\varepsilon^*) = dim(V)$. □

Theorem 5.6. *Let $T : V \longrightarrow P^*(W)$ be a liner transformation in $\boldsymbol{HV}$. Then the following diagram is commutative:*

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \varphi_V \downarrow & & \downarrow \varphi_W \\ V/\varepsilon^* & \xrightarrow{T^*} & W/\varepsilon^* \end{array}$$

where β_V, β_W are the canonical projections of V and W.

Proof. Let $v \in V$ then $\varphi_W(T(v)) = \varepsilon^*(T(v)) = T^*(\varepsilon^*(v)) = T^*(\varphi_V(v)) = T^*\varphi_V(v)$.
Hence, the diagram is commutative. □

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The sum of the reduced harmonic series generated by four primes determined analytically and computed by using CAS Maple

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Abstract

The paper deals with the reduced harmonic series generated by four primes. A formula for the sum of these convergent reduced harmonic series is derived. These sums (concretely 42 from all 12650 sums generated by four different primes smaller than 100) are computed by using the computer algebra system Maple 15 and its programming language, although the formula is valid not only for four arbitrary primes, but also for four integers. We can say that the reduced harmonic series generated by four primes (or by four integers) belong to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically by means of a simple formula.

Key words: reduced harmonic series, sum of convergent infinite series, computer algebra system Maple.

MSC2010: 40A05, 65B10.

1 Introduction

This paper is inspired by a small study material from the *Berkeley Math Circle* (see [5]) and it is a free continuation of the papers [2], [3], and [4].

In two last mentioned papers the sum $S(a,b)$ of the convergent reduced harmonic series

$$\begin{aligned} G(a,b) &= \frac{1}{a} + \frac{1}{b} + \left(\frac{1}{a^2} + \frac{1}{ab} + \frac{1}{b^2}\right) + \left(\frac{1}{a^3} + \frac{1}{a^2 b} + \frac{1}{ab^2} + \frac{1}{b^3}\right) + \\ &+ \left(\frac{1}{a^4} + \frac{1}{a^3 b} + \frac{1}{a^2 b^2} + \frac{1}{ab^3} + \frac{1}{b^4}\right) + \left(\frac{1}{a^5} + \frac{1}{a^4 b} + \cdots + \frac{1}{b^5}\right) + \cdots, \end{aligned} \tag{1}$$

generated by two primes a and b, and the sum $S(a,b,c)$ of the convergent reduced harmonic series

$$\begin{aligned} G(a,b,c) &= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \\ &+ \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{a^2 b} + \frac{1}{a^2 c} + \frac{1}{b^2 a} + \frac{1}{b^2 c} + \frac{1}{c^2 a} + \frac{1}{c^2 b} + \frac{1}{abc}\right) + \\ &+ \left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{a^3 b} + \frac{1}{a^3 c} + \frac{1}{b^3 a} + \frac{1}{b^3 c} + \frac{1}{c^3 a} + \frac{1}{c^3 b} + \right. \\ &\left. + \frac{1}{a^2 b^2} + \frac{1}{a^2 c^2} + \frac{1}{b^2 c^2} + \frac{1}{a^2 bc} + \frac{1}{b^2 ac} + \frac{1}{c^2 ab}\right) + \cdots, \end{aligned} \tag{2}$$

generated by three primes a, b, and c, were derived and also computed for primes less than 100. It was shown that for arbitrary two primes (or integers) a and b it holds the formula

$$S(a,b) = \frac{a+b-1}{(a-1)(b-1)} \tag{3}$$

and for arbitrary three primes (or integers) a, b, and c it holds the formula

$$S(a,b,c) = \frac{(a+b-1)(c-1)+ab}{(a-1)(b-1)(c-1)}. \tag{4}$$

In the paper [2] the sum S of all the unit fractions that have denominators with only factors from the set $\{2,7,11,13\}$ was determined. This sum was calculated by using numeric method based on the programming language in the computer algebra system Maple 15 and also by analytical method. By these both attempts was obtained the same result: $S = 1.780\overline{5}$.

In this paper we shall deal with a certain variant of these two problems – the determination of the sum of the reduced harmonic series generated by four primes.

Let us recall the basic terms and notions. The *harmonic series* is the sum of reciprocals of all natural numbers (except zero), so this is the series

in the form $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$.
The divergence of this series can be easily proved e.g. by using the integral test or the comparison test of convergence.

The *reduced harmonic series* is defined as the subseries of the harmonic series, which arises by omitting some its terms. As an example of the reduced harmonic series we can take the series formed by reciprocals of primes and number one $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots$.
This reduced harmonic series is divergent. The first proof of its divergence was made by Leonhard Euler (15.4.1707–18.9.1783) in 1737 (see e.g. [1]).

2 Reduced harmonic series generated by four primes

Now, let us consider the reduced harmonic series $G(a,b,c,d)$ below, generated by four primes a,b,c,d:

$$\begin{aligned}
G(a,b,c,d) = {} & \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \right. \\
& \left. + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd}\right) + \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} + \frac{1}{a^2b} + \frac{1}{a^2c} + \frac{1}{a^2d} + \right. \\
& + \frac{1}{b^2a} + \frac{1}{b^2c} + \frac{1}{b^2d} + \frac{1}{c^2a} + \frac{1}{c^2b} + \frac{1}{c^2d} + \frac{1}{d^2a} + \frac{1}{d^2b} + \frac{1}{d^2c} + \\
& \left. + \frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}\right) + \left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} + \right. \\
& + \frac{1}{a^3b} + \frac{1}{a^3c} + \frac{1}{a^3d} + \frac{1}{b^3a} + \frac{1}{b^3c} + \frac{1}{b^3d} + \frac{1}{c^3a} + \frac{1}{c^3b} + \frac{1}{c^3d} + \\
& + \frac{1}{d^3a} + \frac{1}{d^3b} + \frac{1}{d^3c} + \frac{1}{a^2bc} + \frac{1}{a^2bd} + \frac{1}{a^2cd} + \frac{1}{b^2ac} + \frac{1}{b^2ad} + \\
& + \frac{1}{b^2cd} + \frac{1}{c^2ab} + \frac{1}{c^2ad} + \frac{1}{c^2bd} + \frac{1}{d^2ab} + \frac{1}{d^2ac} + \frac{1}{d^2bc} + \\
& \left. + \frac{1}{a^2b^2} + \frac{1}{a^2c^2} + \frac{1}{a^2d^2} + \frac{1}{b^2c^2} + \frac{1}{b^2d^2} + \frac{1}{c^2d^2} + \frac{1}{abcd}\right) + \cdots .
\end{aligned} \tag{5}$$

Analogously as in the cases of the reduced harmonic series generated by two and three primes, we assume that its sum $S(a,b,c,d)$ is finite, so the series (5) converges. Because all its terms are positive, then the series (5) converges absolutely and so we can rearrange it.

For easier determining the sum $S(a,b,c,d)$ of the series $G(a,b,c,d)$ it is necessary to rearrange it and divide it into ten subseries $G(a)$, $G(b)$, $G(c)$, $G(d)$, $G(ab)$, $G(ac)$, $G(ad)$, $G(bc)$, $G(bd)$, and $G(cd)$, where

$$G(a) = \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \cdots = \frac{1}{a}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \cdots\right), \tag{6}$$

$$G(b) = \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \frac{1}{b^4} + \cdots = \frac{1}{b}\left(1 + \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \cdots\right), \tag{7}$$

$$G(c) = \frac{1}{c} + \frac{1}{c^2} + \frac{1}{c^3} + \frac{1}{c^4} + \cdots = \frac{1}{c}\left(1 + \frac{1}{c} + \frac{1}{c^2} + \frac{1}{c^3} + \cdots\right), \tag{8}$$

$$G(d) = \frac{1}{d} + \frac{1}{d^2} + \frac{1}{d^3} + \frac{1}{d^4} + \cdots = \frac{1}{d}\left(1 + \frac{1}{d} + \frac{1}{d^2} + \frac{1}{d^3} + \cdots\right), \tag{9}$$

$$\begin{aligned} G(ab) &= \frac{1}{ab} + \frac{1}{a^2b} + \frac{1}{b^2a} + \frac{1}{abc} + \frac{1}{abd} + \frac{1}{a^3b} + \frac{1}{b^3a} + \frac{1}{a^2bc} + \frac{1}{a^2bd} + \\ &+ \frac{1}{b^2ac} + \frac{1}{b^2ad} + \frac{1}{c^2ab} + \frac{1}{d^2ab} + \frac{1}{a^2b^2} + \frac{1}{abcd} + \cdots = \\ &= \frac{1}{ab}\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \right. \\ &\left. + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd} + \cdots\right), \end{aligned} \tag{10}$$

$$\begin{aligned} G(ac) &= \frac{1}{ac} + \frac{1}{a^2c} + \frac{1}{c^2a} + \frac{1}{acd} + \frac{1}{a^3c} + \\ &+ \frac{1}{c^3a} + \frac{1}{a^2cd} + \frac{1}{c^2ad} + \frac{1}{d^2ac} + \frac{1}{a^2c^2} + \cdots = \\ &= \frac{1}{ac}\left(1 + \frac{1}{a} + \frac{1}{c} + \frac{1}{d} + \frac{1}{a^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{ac} + \frac{1}{ad} + \frac{1}{cd} + \cdots\right), \end{aligned} \tag{11}$$

$$\begin{aligned} G(ad) &= \frac{1}{ad} + \frac{1}{a^2d} + \frac{1}{d^2a} + \frac{1}{a^3d} + \frac{1}{d^3a} + \frac{1}{a^2d^2} + \cdots = \\ &= \frac{1}{ad}\left(1 + \frac{1}{a} + \frac{1}{d} + \frac{1}{a^2} + \frac{1}{d^2} + \frac{1}{ad} + \cdots\right), \end{aligned} \tag{12}$$

$$\begin{aligned} G(bc) &= \frac{1}{bc} + \frac{1}{b^2c} + \frac{1}{c^2b} + \frac{1}{bcd} + \frac{1}{b^3c} + \\ &+ \frac{1}{c^3b} + \frac{1}{b^2cd} + \frac{1}{c^2bd} + \frac{1}{d^2bc} + \frac{1}{b^2c^2} + \cdots = \\ &= \frac{1}{bc}\left(1 + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd} + \cdots\right), \end{aligned} \tag{13}$$

$$\begin{aligned} G(bd) &= \frac{1}{bd} + \frac{1}{b^2d} + \frac{1}{d^2b} + \frac{1}{b^3d} + \frac{1}{d^3b} + \frac{1}{b^2d^2} + \cdots = \\ &= \frac{1}{bd}\left(1 + \frac{1}{b} + \frac{1}{d} + \frac{1}{b^2} + \frac{1}{d^2} + \frac{1}{bd} + \cdots\right), \end{aligned} \tag{14}$$

$$\begin{aligned} G(cd) &= \frac{1}{cd} + \frac{1}{c^2d} + \frac{1}{d^2c} + \frac{1}{c^3d} + \frac{1}{d^3c} + \frac{1}{c^2d^2} + \cdots = \\ &= \frac{1}{cd}\left(1 + \frac{1}{c} + \frac{1}{d} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{cd} + \cdots\right). \end{aligned} \tag{15}$$

3 Analytic solution

Now, we determine by the analytic way the unknown sum $S(a,b,c,d)$ by means of the sums of the series (6) – (15). By the formula

$$s = \frac{a_1}{1-q},$$

for the sum s of the convergent infinite geometric series with the first term a_1 and with the ratio q, $|q| < 1$, we get the sums $S(a)$, $S(b)$, $S(c)$, and $S(d)$ of the series (6) – (9):

$$S(a) = \frac{1}{a} \cdot \frac{1}{1 - 1/a} = \frac{1}{a} \cdot \frac{a}{a-1} = \frac{1}{a-1} \tag{16}$$

and, analogously

$$S(b) = \frac{1}{b-1}, \qquad S(c) = \frac{1}{c-1}, \qquad S(d) = \frac{1}{d-1}. \tag{17}$$

It is clear that the sum $S(ab)$ of the series (10) we can write in the form

$$S(ab) = \frac{1}{ab}\left[1 + S(a,b,c,d)\right]. \tag{18}$$

The sum $S(ac)$ of the series (11) is the product of the fraction $1/(ac)$ and the sum of number one and the reduced harmonic series generated by three primes a, c, and d. So, by the formula (4) above, we can write

$$\begin{aligned} S(ac) &= \frac{1}{ac}\left(1+\frac{(a+c-1)(d-1)+ac}{(a-1)(c-1)(d-1)}\right) = \\ &= \frac{(d-1)\big[(a-1)(c-1)+(a+c-1)\big]+ac}{ac(a-1)(c-1)(d-1)} = \\ &= \frac{(d-1)(ac-a-c+1+a+c-1)+ac}{ac(a-1)(c-1)(d-1)} = \\ &= \frac{(d-1)ac+ac}{ac(a-1)(c-1)(d-1)} = \frac{acd}{ac(a-1)(c-1)(d-1)} = \\ &= \frac{d}{(a-1)(c-1)(d-1)}. \end{aligned} \tag{19}$$

Because the sum $S(bc)$ of the series (13) is the product of the fraction $1/(bc)$ and the sum of number one and the reduced harmonic series generated by three primes b, c, and d, we can analogously write

$$S(bc) = \frac{d}{(b-1)(c-1)(d-1)}. \tag{20}$$

Obviously, the sum $S(ad)$ of the series (12) is the product of the fraction $1/(ad)$ and the sum of number one and the reduced harmonic series generated by two primes a and d. So, by the formula (3) above, we can write

$$\begin{aligned} S(ad) &= \frac{1}{ad}\left(1+\frac{a+d-1}{(a-1)(d-1)}\right) = \frac{(a-1)(d-1)+a+d-1}{ad(a-1)(d-1)} = \\ &= \frac{ad-a-d+1+a+d-1}{ad(a-1)(d-1)} = \frac{1}{(a-1)(d-1)} \end{aligned} \tag{21}$$

and, analogously for the sums $S(bd)$ and $S(cd)$ of the series (14) and (15), we get

$$S(bd) = \frac{1}{(b-1)(d-1)}, \qquad S(cd) = \frac{1}{(c-1)(d-1)}. \tag{22}$$

By the assumption of the absolute convergence of the series (5) we can write its sum $S(a,b,c,d)$ in the form

$$S(a)+S(b)+S(c)+S(d)+S(ab)+S(ac)+S(ad)+S(bc)+S(bd)+S(cd).$$

According to (16) – (22) we get the equation

$$S(a,b,c,d) = \frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \frac{1}{d-1} + \frac{1+S(a,b,c,d)}{ab} + \\ + \frac{d}{(a-1)(c-1)(d-1)} + \frac{d}{(b-1)(c-1)(d-1)} + \\ + \frac{1}{(a-1)(d-1)} + \frac{1}{(b-1)(d-1)} + \frac{1}{(c-1)(d-1)}.$$

Multiplying both sides of this equation by $ab(a-1)(b-1)(c-1)(d-1)$, we obtain the equation

$$(ab-1)(a-1)(b-1)(c-1)(d-1)S(a,b,c,d) = ab\big[(b-1)(c-1)(d-1)+ \\ +(a-1)(c-1)(d-1)+(a-1)(b-1)(d-1)+(a-1)(b-1)(c-1)\big]+ \\ +(a-1)(b-1)(c-1)(d-1)+abd\big[(b-1)+(a-1)\big]+ \\ +ab\big[(b-1)(c-1)+(a-1)(c-1)+(a-1)(b-1)\big].$$

It is easy to derive (e.g. by means of the computer algebra system Maple 15 and its **simplify** and **factor** statements) that it holds $S(a,b,c,d) =$

$$= \frac{abc+abd+acd+bcd-ab-ac-ad-bc-bd-cd+a+b+c+d-1}{(a-1)(b-1)(c-1)(d-1)},$$

i.e.

$$S(a,b,c,d) = \frac{\big[(a+b-1)(c-1)+ab\big](d-1)+abc}{(a-1)(b-1)(c-1)(d-1)}. \tag{23}$$

This formula can be also written in another two equivalent forms:

$$S(a,b,c,d) = \frac{\big[(a+c-1)(b-1)+ac\big](d-1)+abc}{(a-1)(b-1)(c-1)(d-1)}$$

and

$$S(a,b,c,d) = \frac{\big[(b+c-1)(a-1)+bc\big](d-1)+abc}{(a-1)(b-1)(c-1)(d-1)}.$$

4 Numeric solution

For approximate calculation of the sums $S(a,b,c,d)$ for the primes $a,b,c,d < 100$, i.e. for 25 primes $a,b,c,d \in \{2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97\}$, we use the computer algebra system Maple 15. The sums $S(a,b,c,d)$ we calculate for concrete four primes by the following **for** statements and by the procedure `partabcd`:

```
partabcd:=proc(a,b,c,d)
    local s;
    s:=(((a+b-1)*(c-1)+a*b)*(d-1)+a*b*c)/((a-1)*(b-1)*(c-1)*(d-1));
print("S(a,b,c,d) for a=",a,"b=",b,"c=",c,"d=",d,"is",evalf[8](s));
end proc:

P:=[2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,
    83,89,97]:
for i in P do
    for j in P do
        for k in P do
            for l in P do
                if (i < j and j < k and k < l) then
                partabcd(i,j,k,l);
                end if;
            end do;
        end do;
    end do;
end do;
```

Fourty-two representative sums of these 12650 sums $S(a,b,c,d)$, where 12650 is the number of combinations of size 4 from a collection of size 25, i.e.

$$\binom{25}{4} = \frac{25!}{(25-4)!\,4!} = \frac{25\cdot 24\cdot 23\cdot 22}{4!} = 12650\,,$$

are presented in two following tables. There are 27 sums with a finite decimal expansion (called regular numbers) with at most 4 decimals in the table 1 and another 15 sums, including the sum $S(2,7,11,13) = 1.780\overline{5}$ calculated in the paper [2] and mentioned above, rounded to 6 decimals, in the table 2:

(a,b,c,d)	$S(a,b,c,d)$	(a,b,c,d)	$S(a,b,c,d)$	(a,b,c,d)	$S(a,b,c,d)$
$(2,3,5,7)$	3.375	$(2,3,11,13)$	2.575	$(2,5,11,23)$	1.875
$(2,3,5,11)$	3.125	$(2,3,11,23)$	2.45	$(2,11,23,41)$	1.3575
$(2,3,5,13)$	3.0625	$(2,3,11,31)$	2.41	$(2,11,23,47)$	1.35
$(2,3,5,31)$	2.875	$(2,3,11,61)$	2.355	$(2,11,41,83)$	1.2825
$(2,3,5,61)$	2.8125	$(2,3,11,67)$	2.35	$(2,67,79,89)$	1.0797
$(2,3,7,11)$	2.85	$(2,3,11,89)$	2.3375	$(3,7,11,23)$	1.0125
$(2,3,7,29)$	2.625	$(2,3,13,53)$	2.3125	$(3,7,11,71)$	0.9525
$(2,3,7,41)$	2.5875	$(2,3,31,41)$	2.1775	$(3,11,23,31)$	0.7825
$(2,3,7,71)$	2.55	$(2,3,41,83)$	2.1125	$(3,11,23,43)$	0.7625

Table 1: The table with some values of the sums $S(a,b,c,d)$

(a,b,c,d)	$S(a,b,c,d)$	(a,b,c,d)	$S(a,b,c,d)$	(a,b,c,d)	$S(a,b,c,d)$
$(2,3,5,17)$	2.984375	$(2,5,7,31)$	2.013889	$(2,23,37,43)$	1.200156
$(2,3,5,73)$	2.802083	$(2,5,11,53)$	1.802885	$(3,7,13,19)$	1.001157
$(2,3,7,31)$	2.616667	$(2,7,11,13)$	1.780556	$(3,7,59,89)$	0.800402
$(2,3,11,29)$	2.417857	$(2,5,37,83)$	1.600779	$(3,31,53,79)$	0.600002
$(2,3,19,89)$	2.202652	$(2,7,59,89)$	1.400536	$(79,83,89,97)$	0.047622

Table 2: The table with another values of the sums $S(a,b,c,d)$

5 Conclusion

In this paper the sums $S(a,b,c,d)$ of the convergent reduced harmonic series $G(a,b,c,d)$ generated by four primes a, b, c and d were derived. These sums were computed for $a,b,c,d < 100$, although the formula

$$S(a,b,c,d) = \frac{\big[(a+b-1)(c-1)+ab\big](d-1)+abc}{(a-1)(b-1)(c-1)(d-1)}$$

derived above gives results for arbitrary four different primes a,b,c,d. So that, for example

$$S(101,103,107,109) = \frac{(203\cdot 106 + 101\cdot 103)\cdot 108 + 101\cdot 103\cdot 107}{100\cdot 102\cdot 106\cdot 108} \doteq 0.039056.$$

It is clear that this formula is valid not only for four primes, but also for four integers. For example the sum of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{2\cdot 4}+\frac{1}{2\cdot 6}+\frac{1}{2\cdot 8}+\frac{1}{4\cdot 6}+\frac{1}{4\cdot 8}+\frac{1}{6\cdot 8}+\frac{1}{2^3}+\frac{1}{4^3}+\cdots$$

is $S(2,4,6,8) = \dfrac{(5\cdot 5 + 2\cdot 4)\cdot 7 + 2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7} \doteq 2.657143.$

We can say that the reduced harmonic series $G(a,b,c,d)$ generated by four primes (or by four integers) belong to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically by means of a simple formula.

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