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Fuzzy hyperalgebras and direct product

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Abstract

We introduce and study the direct product of a family of fuzzy hyperalgebras of the same type and present some properties of it.

Key words: Fuzzy hyperalgebras, Term function, Direct product.

MSC2010: 97U99.

1 Introduction

In this section we present some definitions and simple properties of hyperalgebras which will be used in the next section. In the sequel H is a fixed nonvoid set, $P^*(H)$ is the family of all nonvoid subsets of H, and for a positive integer n we denote for H^n the set of n-tuples over H (for more see [1]).

Recall that for a positive integer n a n-ary *hyperoperation* β on H is a function $\beta : H^n \quad \rightarrow \quad P^*(H)$. We say that n is the *arity* of β. A subset S of H is *closed* under the n-ary hyperoperation β if $(x_1, \ldots, x_n) \in S^n$ implies that $\beta(x_1, \ldots, x_n) \subseteq S$. A *nullary hyperoperation* on H is just an element of $P^*(H)$; i.e. a nonvoid subset of H.
A *hyperalgebra* $\mathbb{H} = \langle H, (\beta_i, \mid i \in I) \rangle$ (which is called *hyperalgebraic system* or a *multialgebra*) is the set H with together a collection $(\beta_i, \mid i \in I)$ of hyperoperations on H.

A subset S of a hyperalgebra $\mathbb{H}$=$\langle H, (\beta_i, : i \in I)\rangle$ is a *subhyperalgebra* of $\mathbb{H}$ if S is closed under each hyperoperation β_i, for all $i \in I$, that is $\beta_i(a_1, ..., a_{n_i}) \subseteq S$, whenever $(a_1, ..., a_{n_i}) \in S^{n_i}$. The *type* of $\mathbb{H}$ is the map from I into the set $\mathbb{N}^*$ of nonnegative integers assigning to each $i \in I$ the arity of β_i. Two hyperalgebras of the same type are called *similar* hyperalgrbras.
For $n > 0$ we extend an n-ary hyperoperation β on H to an n-ary operation $\overline{\beta}$ on $P^*(H)$ by setting for all $A_1, ..., A_n \in P^*(H)$

$$\overline{\beta}(A_1, ..., A_n) = \bigcup\{\beta(a_1, ..., a_n)|a_i \in A_i(i = 1, ..., n)\}$$

It is easy to see that $\langle P^*(H), (\overline{\beta}_i : i \in I)\rangle$ is an algebra of the same type of $\mathbb{H}$.

Definition 1.1 *Let* $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ *and* $\overline{\mathbb{H}}$=$\langle \overline{H}, (\overline{\beta}_i : i \in I)\rangle$ *be two similar hyperalgebras. A map* h *from* $\mathbb{H}$ *into* $\overline{\mathbb{H}}$ *is called a*
(i) A homomorphism if for every $i \in I$ *and all* $(a_1, ..., a_{n_i}) \in H^{n_i}$ *we have that*

$$h(\beta_i((a_1, ..., a_{n_i})) \subseteq \overline{\beta}_i(h(a_1), ..., h(a_{n_i}));$$

(ii) a good homomorphism if for every $i \in I$ *and all* $(a_1, ..., a_{n_i}) \in H^{n_i}$ *we have that*

$$h(\beta_i((a_1, ..., a_{n_i})) = \overline{\beta}_i(h(a_1), ..., h(a_{n_i})).$$

Definition 1.2 *Let* H *be a nonempty set. A fuzzy subset* μ *of* H *is a function*

$$\mu : H \to [0, 1].$$

Definition 1.3 *A fuzzy* n*-ary hyperoperation* f^n *on* S *is a map* $f^n : S \times \cdots \times S \longrightarrow F^*(S)$*, which associated a nonzero fuzzy subset* $f^n(a_1, \ldots, a_n)$ *with any* n*-tuple* $(a_1, \ldots, a_n)$ *of elements of* S*. The couple* (S, f^n) *is called a fuzzy* n*-ary hypergroupoid. A fuzzy nullary hyperoperation on* S *is just an element of* $F^*(S)$*; i.e. a nonzero fuzzy subset of* S*.*

Definition 1.4 *Let* H *be a nonempty set and for every* $i \in I$*,* β_i *be a fuzzy* n_i*-ary hyperoperation on* H*, Then* $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ *is called fuzzy hyperalgebra, where* $(n_i : i \in I)$ *is type of this fuzzy hyperalgebra.*

Definition 1.5 *If* $\mu_1, \ldots, \mu_{n_i}$ *be* n_i *nonzero fuzzy subsets of a fuzzy huperalgebra* $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$*, we define for all* $t \in H$

$$\beta_i(\mu_1, \ldots, \mu_{n_i})(t) = \bigvee_{(x_1,\ldots,x_{n_i})\in H^{n_i}} (\mu_1(x_1) \bigwedge \cdots \bigwedge \mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, \ldots, x_{n_i})(t))$$

Finally, for nonempty subsets $A_1, \ldots, A_{n_k}$ *of* H*, set* $A = A_1 \times \ldots \times A_{n_i}$*. Then for all* $t \in H$

$$\beta_k(A_1, \ldots, A_{n_k})(t) = \vee_{(a_1,\ldots,a_{n_k})\in A}(\beta_k(a_1, \ldots, a_{n_k})(t)).$$

For nonempty subset A of H, χ_A denote the characteristic function of A. Note that, if $f : H_1 \longrightarrow H_2$ is a map and $a \in H_1$, then $f(\chi_a) = \chi_{f(a)}$.

Definition 1.6 *Let $\mathbb{H} = \langle H, (\beta_i : i \in I)\rangle$ and $\mathbb{H}' = \langle H', (\beta'_i : i \in I)\rangle$ be two fuzzy hyperalgebras with the same type, and $f : H \longrightarrow H'$ be a map. We say that f is a homomorphism of fuzzy hyperalgebras if for every $i \in I$ and every $a_1, \dots, a_{n_i} \in H$ we have*

$$f(\beta_i(a_1, \dots, a_{n_i})) \leq \beta'_i(f(a_1), \dots, f(a_{n_i})).$$

Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra then, the set of the nonzero fuzzy subsets of H denoted by $F^*(H)$, can be organized as a universal algebra with the operations;

$$\beta_i(\mu_1, \dots, \mu_{n_i})(t) = \bigvee_{(x_1,\dots,x_{n_i})\in H^{n_i}} (\mu_1(x_1) \bigwedge \dots \bigwedge \mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, \dots, x_{n_i})(t))$$

for every $i \in I$, $\mu_1, \dots, \mu_{n_i} \in F^*(H)$ and $t \in H$. We denote this algebra by $\mathtt{F}^*(\mathbb{H})$.

In [3] Gratzer presents the algebra of the term functions of a universal algebra. If we consider an algebra $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ we call $n-$ary term functions on $\mathbb{B}$ ($n \in \mathbb{N}$) those and only those functions from B^n into B, which can be obtained by applying (i) and (ii) from bellow for finitely many times:
(i) the functions $e_i^n : B^n \to B$, $e_i^n(x_1, \dots, x_n) = x_i$, $i = 1, \dots, n$ are $n-$ary term functions on $\mathbb{B}$;
(ii) if $p_1, \dots, p_{n_i}$ are $n-$ary term functions on $\mathbb{B}$, then $\beta_i(p_1, \dots, p_{n_i}) : B^n \to B$,
$\beta_i(p_1, \dots, p_{n_i})(x_1, \dots, x_n) = \beta_i(p_1(x_1, \dots, x_n), \dots, p_{n_i}(x_1, \dots, x_n))$ is also a $n-$ary term function on $\mathbb{B}$.
We can observe that (ii) organize the set of $n-$ary term functions over $\mathbb{B}$ ($P^{(n)}(\mathbb{B})$) as a universal algebra, denoted by $\mathtt{B}^{(n)}(\mathbb{B})$.
If $\mathbb{H}$ is a fuzzy hyperalgebra then for any $n \in \mathbb{N}$, we can construct the algebra of $n-$ary term functions on $\mathtt{F}^*(\mathbb{H})$, denoted by $\mathtt{B}^{(n)}(\mathtt{F}^*(\mathbb{H})) = \langle P^{(n)}(F^*(\mathbb{H})), (\beta_i : i \in I)\rangle$.

2 On the Direct Product of Fuzzy Hyperalgebras

Proposition 2.1 *Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ are fuzzy hyperalgebras of the same type, $h : H \to B$ a fuzzy homomorphism and $p \in P^{(n)}(\mathtt{F}^*(\mathbb{H}))$. Then for all $a_1, \dots, a_n \in H$ we have $h(p(\chi_{a_1}, \dots, \chi_{a_n})) \subseteq p(h(\chi_{a_1}), \dots, h(\chi_{a_n}))$.*

Proof. The prove is by induction over the steps of construction of a term.□

Remark 2.1 *If $h : H \to B$ be fuzzy good homomorphism then*
$$h(p(\chi_{a_1}, \dots, \chi_{a_n})) = p(h(\chi_{a_1}), \dots, h(\chi_{a_n})).$$

Remark 2.2 *We can easily construct the category of the fuzzy hyperalgebras of the same type, where the morphisms are considered to be the fuzzy homomorphisms and the composition of two morphisms is the usual mapping composition and we will denote it by **FHA***

Definition 2.1 *Let $q, p \in P^{(n)}(\boldsymbol{F}^*(\mathbb{H}))$. The $n-$ary (strong) identity $p = q$ is said to be satisfied on a fuzzy hyperalgebra $\mathbb{H}$ if*
$$p(\chi_{a_1}, \dots, \chi_{a_n}) = q(\chi_{a_1}, \dots, \chi_{a_n})$$
for all $a_1, \dots, a_n \in H$. We can also consider that a weak identity $p \cap q \neq \varnothing$ is said to be satisfied on a fuzzy hyperalgebra $\mathbb{H}$ if
$$p(\chi_{a_1}, \dots, \chi_{a_n}) \wedge q(\chi_{a_1}, \dots, \chi_{a_n}) > 0$$
for all $a_1, \dots, a_n \in H$.

Definition 2.2 *Let $((H_k, (\beta_i^k : i \in I)), k \in K)$ be an indexed family of fuzzy hyperalgebras with the same type. The direct product $\prod_{k\in K} H_k$ is a fuzzy hyperalgebra with univers $\Pi_{k\in K} H_k$ and for every $i \in I$ and $(a_k^1)_{k\in K}, \dots, (a_k^{n_i})_{k\in K} \in \Pi_{k\in K} H_k$:*
$$\beta_i^{\prod}((a_k^1)_{k\in K}, \dots, (a_k^{n_i})_{k\in K})(t_k)_{k\in K} = \bigwedge_{k\in K} \beta_i^k(a_k^1, \dots, a_k^{n_i})(t_k)$$

Theorem 2.1 *The fuzzy hyperalgebra $\prod_{k\in K} H_k$ constructed this way, together with the canonical projections, is the product of the fuzzy hyperalgebras $(H_k, k \in K)$ in the category **FHA**.*

Proof. For any fuzzy hyperalgebra $(B, (\beta_i^B : i \in I))$ and for any family of fuzzy hyperalgebra homomorphisms $(\alpha_k : B \to H_k | k \in K)$ there is only one homomorphism $\alpha : B \to \Pi_{k\in K} H_k$ such that $\alpha_k = \pi_k^K \circ \alpha$ for any $k \in K$. Indeed, there exists only one mapping α such that the diagram is commutative.

$$\begin{tikzcd}
\Pi_{k\in K} H_k \arrow[r, "\pi_k^K"] & H_k \\
B \arrow[u, "\alpha"] \arrow[ur, "\alpha_k"'] &
\end{tikzcd}$$

This mapping is defined by $\alpha(b) = (\alpha_k(b))_{k\in K}$. Now we have to do is to verify that α is fuzzy hyperalgebra homomorphism. If we consider $i \in I$ and $b_1, \ldots, b_{n_i} \in B$, $(t_k)_{k\in K} \in \Pi_{k\in K} H_k$ then if $r \in \alpha^{-1}((t_k)_{k\in K})$ we have $\alpha(r) = (t_k)_{k\in K}$ and $\alpha(r) = (\alpha_k(r))_{k\in K}$, hence $\forall k \in K; t_k = \alpha_k(r)$, it means that $\forall k \in K; r \in \alpha_k^{-1}(t_k)$, therefore $\forall k \in K; \alpha^{-1}((t_k)_{k\in K}) \subseteq \alpha_k^{-1}(t_k)$. We have

$$\alpha(\beta_i^B(b_1, \ldots, b_{n_i}))(t_k)_{k\in K} = \bigvee_{r\in\alpha^{-1}((t_k)_{k\in K})} (\beta_i^B(b_1, \ldots, b_{n_i}))(r)$$
$$\leq \bigvee_{s\in\alpha_k^{-1}(t_k))} \beta_i^B(b_1, \ldots, b_{n_i}))(s) = \alpha_k(\beta_i^B(b_1, \ldots, b_{n_i}))(t_k)$$

then

$$\alpha(\beta_i^B(b_1, \ldots, b_{n_i}))(t_k)_{k\in K} \leq \bigwedge_{k\in K} \alpha_k(\beta_i^B(b_1, \ldots, b_{n_i}))(t_k)$$
$$\leq \bigwedge_{k\in K} \beta_i^k(\alpha_k(b_1), \ldots, \alpha_k(b_{n_i}))(t_k) = \beta_i^{\Pi}(\alpha(b_1), \ldots, \alpha(b_{n_i}))(t_k)_{k\in K}.$$

Which finishes the proof.□

Proposition 2.2 *For every* $n \in \mathbb{N}$, $p \in P^{(n)}(F^*(\mathbb{H}))$ *and* $(a_k^1)_{k\in K}, \ldots, (a_k^n)_{k\in K}$, *we have*

$$p(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K} = \bigwedge_{k\in K} p(\chi_{a_k^1}, \ldots, \chi_{a_k^n})(t_k)$$

Proof. We will use the steps of construction of a term.

i. If $p = e_n^j (j = 1, 2, \ldots, n)$ then

$$p(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K} = e_n^j(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K}$$
$$= \chi_{(a_k^j)_{k\in K}}(t_k)_{k\in K}$$
$$= \bigwedge_{k\in K} e_n^j(\chi_{a_k^1}, \ldots, \chi_{a_k^n})(t_k)$$
$$= \bigwedge_{k\in K} p(\chi_{a_k^1}, \ldots, \chi_{a_k^n})(t_k)$$

ii. Suppose that the statement has been proved for $p_1, \ldots, p_{n_i}$ and that $p = \beta_i(p_1, \ldots, p_{n_i})$. Then we have

$$p(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K} = \beta_i(p_1, \ldots, p_{n_i})(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K}$$
$$= \beta_i(p_1(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}}), \ldots, p_{n_i}(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}}))(t_k)_{k\in K}$$
$$= \bigvee_{(s_k^1)_{k\in K},\ldots,(s_k^{n_i})_{k\in K}} [p_1(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(s_k^1)_{k\in K} \wedge \ldots \wedge p_{n_i}(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})$$
$$(s_k^{n_i})_{k\in K} \wedge \beta_i((s_k^1)_{k\in K}, \ldots, (s_k^{n_i})_{k\in K})(t_k)_{k\in K}]$$

$$= \bigvee_{(s_k^1)_{k\in K},\ldots,(s_k^{n_i})_{k\in K}} [\bigwedge_{k\in K} p_1(\chi_{a_k^1},\ldots,\chi_{a_k^n})(s_k^1) \wedge \ldots \wedge \bigwedge_{k\in K} p_{n_i}(\chi_{a_k^1},\ldots,\chi_{a_k^n})(s_k^{n_i}) \wedge$$
$$\bigwedge_{k\in K} \beta_i(s_k^1,\ldots,s_k^{n_i})(t_k)]$$
$$= \bigwedge_{k\in K} [\bigvee_{(s_k^1)_{k\in K},\ldots,(s_k^{n_i})_{k\in K}} p_1(\chi_{a_k^1},\ldots,\chi_{a_k^n})(s_k^1) \wedge \ldots \wedge p_{n_i}(\chi_{a_k^1},\ldots,\chi_{a_k^n})(s_k^{n_i}) \wedge \beta_i(s_k^1,\ldots,s_k^{n_i})(t_k)]$$
$$= \bigwedge_{k\in K} \beta_i(p_1(\chi_{a_k^1},\ldots,\chi_{a_k^n}),\ldots,p_{n_i}(\chi_{a_k^1},\ldots,\chi_{a_k^n}))(t_k)$$
$$= \bigwedge_{k\in K} \beta_i(p_1,\ldots,p_{n_i})(\chi_{a_k^1},\ldots,\chi_{a_k^n})(t_k)$$
$$= \bigwedge_{k\in K} p(\chi_{a_k^1},\ldots,\chi_{a_k^n})(t_k).$$
which finishes the proof of the proposition.□

Theorem 2.2 *If $((H_k,(\beta_i^k : i \in I)), k \in K)$ be an indexed family of fuzzy hyperalgebras with the same type I such that $p \cap q \neq \varnothing$ is satisfied on each fuzzy hyperalgebra H_k, then is also satisfied on the fuzzy hyperalgebra $\prod_{k\in K} H_k$.*

Proof. Let $p, q \in P^{(n)}(\mathtt{F}^*(\mathbb{H}))$ and suppose that $p \cap q \neq \varnothing$ is satisfied on each fuzzy hyperalgebra H_k. This means that for all $k \in K$ and for any $a_k^1,\ldots,a_k^n \in H_k$ we have $p(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \wedge q(\chi_{a_k^1},\ldots,\chi_{a_k^n}) > 0$. By proposition 3.7 , we conclude that

$$\begin{aligned} p(\chi_{(a_k^1)_{k\in K}},\ldots,\chi_{(a_k^n)_{k\in K}}) \wedge r(\chi_{(a_k^1)_{k\in K}},\ldots,\chi_{(a_k^n)_{k\in K}}) = \\ = \bigwedge_{k\in K} p(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \wedge \bigwedge_{k\in K} q(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \\ = \bigwedge_{k\in K} (p(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \wedge q(\chi_{a_k^1},\ldots,\chi_{a_k^n})) > 0 \end{aligned}$$

and the proof is finished.□

Theorem 2.3 *If $((H_k,(\beta_i^k : i \in I)), k \in K)$ be an indexed family of fuzzy hyperalgebras with the same type I such that $p = q$ is satisfied on each fuzzy hyperalgebra H_k, then $p = q$ is also satisfied on the fuzzy hyperalgebra $\prod_{k\in K} H_k$.*

Proof. Let $p, q \in P^{(n)}(\mathtt{F}^*(\mathbb{H}))$ and suppose that $p = q$ is satisfied on each fuzzy hyperalgebra H_k. This means that for all $k \in K$ and for any $a_k^1,\ldots,a_k^n \in H_k$ we have $p(\chi_{a_k^1},\ldots,\chi_{a_k^n}) = q(\chi_{a_k^1},\ldots,\chi_{a_k^n})$. By proposition 3.7 , we conclude that

$$\begin{aligned} p(\chi_{(a_k^1)_{k\in K}},\ldots,\chi_{(a_k^n)_{k\in K}}) &= \bigwedge_{k\in K} p(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \\ &= \bigwedge_{k\in K} q(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \end{aligned}$$

$$= r(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})$$

and the proof is finished.□

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References

[1] P. Corsini, *Prolegomena of Hypergroup Theory*, Second edition, Aviani editor (1993).

[2] P. Corsini and I. Tofan, *On Fuzzy Hypergroups*, PU.M.A, **8** (1997), 29-37.

[3] G. Gratzer, *Universal algebra*, 2nd edition, Springer Verlage, 1970.

[4] J. N. Mordeson,M.S. Malik, *Fuzzy Commutative Algebra*, Word Publ. 1998.

[5] C. Pelea, *On the direct product of multialgebras*, Studia uni, Babes-bolya, Mathematica, vol. XLVIII (2003), 93-98.

[6] C. Pelea, *Multialgebras and term functions over the algebra of their non-void subsets*, Mathematica (Cluj), **43** (2001), 143-149.

[7] M.K. Sen, R.Ameri, G. Chowdhury, *Fuzzy hypersemigroups*, Soft Computing, Soft Comput, 12 (2008), 891-900.

[8] T. Vougiuklis, *Hyperstructures and their representations*, Hardonic Press (1994).

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On Fuzzy Gamma Hypermodules

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Abstract

Let R be a Γ-hyperring and M be an Γ-hypermodule over R. We introduce and study fuzzy R_Γ-hypermodules. Also, we associate a Γ-hypermodule to every fuzzy Γ-hypermodule and investigate its basic properties.

Key words: Γ-hyperring, Γ-hypermodule, fundamental relation, fuzzy Γ-hypermodule.

MSC2010: 20N20.

1 Introduction

Hyperstructure theory was born in 1934 when Marty [13] defined hypergroups, began to analysis their properties and applied them to groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. Zadeh [18] introduced the notion of a fuzzy subset of a non-empty set X, as a function from X to $[0, 1]$. Rosenfeld [15] defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra. In [16], Sen, Ameri and Chowdhury introduced the notions of fuzzy hypersemigroups and obtained a characterization of them. Then in [10], Leoreanu-Fotea and Davvaz introduced and analyzed the fuzzy hyperring notion and in [11], Leoreanu-Fotea introduced the fuzzy hypermodule notion and obtained a connection between hypermodules and fuzzy hypermodules (for more information about fuzzy hypersrtuctures see [1]-[6]). The notion of a Γ-ring was introduced by N. Nobusawa in [14]. Recently, W.E. Barnes [7], J. Luh [12], W.E. Coppage studied the structure of Γ-rings and obtained various generalization analogous of corresponding parts in ring theory. In [3] Ameri, Sadeghi introduced the notion of Γ-module over a Γ-ring.

Now in this paper we introduced and study fuzzy Γ-hypermodules as generalization of Γ-hypermodule as well as fuzzy modules. The paper has been prepared in 5 sections. In section 2, we introduce some definitions and results of Γ-hypermodules and fuzzy sets which we need to developing our paper. In section 3, we introduced and study fuzzy Γ-hypermodules and obtain its basic results. In section 4, we study fundamental relation of fuzzy Γ-hypermodules.

2 Preliminaries

In this section, we present some definitions which need to developing our paper. As it is well known a *hypergroupoid* is a set together with a function $\circ : H \times H \longrightarrow P^{\star}(H)$, which is called a hyperoperation, where $P^{\star}(H)$ denotes the set of all nonempty subsets of H. A hypergroupoid $(H, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$ is called a *semihypergroup*. A *hypergroup* is a semihypergroup such that for all $x \in H$ we have $x \circ H = H = H \circ x$ (called the *reproduction axiom*). We say that a hypergroup H is *canonical hypergroup* if it is *commutative*, it has a *scalar identity*, every element has a unique *inverse* and it is *reversible* (for more details of hypergroups see [9]).

Definition 2.1. *The triple $(R, +, .)$ is a hyperring (in the sense of Krasner) if the following hold: (i) $(R, +)$ is a commutative hypergroup;*
(ii) $(R, .)$ is a semihypergroup;
(iii) the hyperoperation "." is distributive over the hyperoperation "+", which means that for all r, s, t of R we have: $r.(s + t) = r.s + r.t$ and $(r + s).t = r.t + s.t$ (for more about hyperrings see [9] and [11]).

Definition 2.2. *Let $(R, \uplus, \circ)$ be a hyperring. A nonempty set M, endowed with two hyperoperations $\oplus, \odot$ is called a left hypermodule over $(R, \uplus, \circ)$ if the following conditions hold:*
(1) *$(M, \oplus)$ is a commutative hypergroup;*
(2) *$\odot : R \times M \longrightarrow P^{*}(M)$ is such that for all $a, b \in M$ and $r, s \in R$ we have*
(i) $r \odot (a \oplus b) = (r \odot a) \oplus (r \odot b)$;
(ii) $(r \uplus s) \odot a = (r \odot a) \oplus (s \odot a)$;
(iii) $(r \circ s) \odot a = r \odot (s \odot a)$.

For more details about hypermodules see [8], [9], [?] and [18]).

Definition 2.3. *([7]) Let R and Γ be additive abelian groups. We say that R is a $\Gamma-$ring if there exists a mapping*

$$\begin{aligned} \cdot : R \times \Gamma \times R &\longrightarrow R \\ (r, \gamma, r') &\longmapsto \quad r.\gamma.r' \ (= r\gamma r') \end{aligned}$$

such that for every $a, b, c \in R$ and $\alpha, \beta \in \Gamma$, the following conditions hold:
(i) $(a+b)\alpha c = a\alpha c + b\alpha c$;
$a(\alpha+\beta)c = a\alpha c + a\beta c$;
$a\alpha(b+c) = a\alpha b + a\alpha c$;
(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Definition 2.4. *Let R be a Γ-ring. A (left)gamma module over R is an additive abelian group M together with a mapping $. : R \times \Gamma \times M \longrightarrow M$ (the image of (r, γ, m) being denoted by $r\gamma m$), such that for all $m, m_1, m_2 \in M$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $r, r_1, r_2 \in R$ the following conditions are satisfied:*
(GM_1) $r.\gamma.(m_1 + m_2) = r.\gamma.m_1 + r.\gamma.m_2$;
(GM_2) $(r_1 + r_2).\gamma.m = r_1.\gamma.m + r_2.\gamma.m$;
(GM_3) $r.(\gamma_1 + \gamma_2).m = r.\gamma_1.m + r.\gamma_2.m$;
(GM_4) $r_1.\gamma_1.(r_2.\gamma_2.m) = (r_1.\gamma_1.r_2).\gamma_2.m$.
A right gamma module over R is defined in analogous manner. In this case we say that M is a left(or right) R_Γ-module (for more details about gamma modules see [2]).
Let $(H, \circ)$ be a hypergroupoid. If $\{A, B\} \subseteq P^*(H)$ and ρ is an equivalence relation on H, then we denote $A\bar{\rho}B$ if

$$\forall a \in A, \quad \exists b \in B : a\rho b, \quad and, \quad \forall b \in B, \quad \exists a \in A : a\rho b.$$

We denote $A \ \bar{\bar{\rho}} \ B$ if $\forall a \in A, \ \ \forall b \in B$ we have $a\rho b$.

An equivalence relation ρ on H is called regular (strongly regular) if for all a, a', b, b' of H. The following implication holds:

$$a\rho b, a'\rho b' \implies (a \circ a')\bar{\rho}(b \circ b')$$

$$(a\rho b, a'\rho b' \implies (a \circ a')\bar{\bar{\rho}}(b \circ b')).$$

Theorem 2.1. *([17]) Let $(M, +, .)$ be a hypermodule over a hyperring R, let δ be an equivalence relation on M and let ρ be an strongly regular relation on R. The following statements hold:*
(1) *if δ is strongly regular on M and $\forall x, y \in M$ and $\forall r \in R$ the hyperoperations:*

$$\delta(x) \oplus \delta(y) = \{\delta(z) \mid z \in x + y\} \quad and \quad \rho(r) \odot \delta(x) = \{\delta(z) \mid z \in r.x\},$$

is define a module structure on M/δ over R/ρ;
(2) *if $(M/\delta, \oplus, \odot)$ is a module over R/ρ, then δ is strongly regular on M. The relation δ^* is the smallest strongly regular relation on the hypermodule $(M, +, .)$ such that $(M/\delta, \oplus, \odot)$ the quotient structure $(M/\delta, \oplus, \odot)$ is a module over the ring R/ρ, and it is called the fundamental relation over hypermodule M.*

Hence, δ^* is the smallest equivalence relation on M, such that M/δ^* is a module over the ring R/ρ^*, where ρ^* is fundamental relation on R. If we denote by $\mathcal{U}$ the set of all expressions consisting of finite hyperoperations either on R and M or the external hyperoperation applied on finite sets of elements of R and M, then we have

$$x\delta y \Longleftrightarrow \exists u \in \mathcal{U}, \text{ such that } \{x, y\} \subset u.$$

δ^* is the transitive closure of δ. In the fundamental module $(M/\delta^*, \oplus, \odot)$ over R/ρ^*, the hyperoperations $\oplus$ and $\odot$ are defined as follows:
$\forall x, y \in M$ and $\forall z \in \delta^*(x) \oplus \delta^*(y)$, we have $\delta^*(x) \odot \delta^*(y) = \delta^*(z)$; $\forall r \in R$, $\forall x \in M$ and $\forall z \in \delta^*(r).\delta^*(x)$, we have $\rho^*(r) \odot \delta^*(x) = \delta^*(z)$, (for more details about the fundamental relation on hyperstructures see [8] and [9]).

Definition 2.5. *A multivalued system $(R, +, .)$ is a Γ-hyperring if the following hold:*
(i) $(R, +)$ and Γ are canonical hypergroups;
(ii) $(R, .)$ is semihypergroup.
(iii) $(.)$ is distributive with respect to $(+)$, i.e., for all x, y, z in R we have $x.(y + z) = (x.y) + (x.z)$ and $(x + y).z = (x.z) + (y + z)$.

Definition 2.6. *Let $(R, \uplus, \circ)$ be a Γ-hyperring and $(\Gamma, *)$ be a canonical hypergroup. We say that $(M, +, .)$ is a left $\Gamma -$ hypermodule over R, if $(M, +)$ be a canonical hypergroup and there exists a mapping*

$$\begin{aligned} \cdot : R \times \Gamma \times M &\longrightarrow P^\star(M) \\ (r, \gamma, m) &\longmapsto \quad r \cdot \gamma \cdot m \end{aligned}$$

such that for every $r, s \in R$ and $\alpha, \beta \in \Gamma$ and $a, b \in M$, the following conditions are satisfied:
(GHM_1) *(i)* $(r \uplus s).\alpha.a = r.\alpha.a + s.\alpha.a$;
(ii) $r.(\alpha * \beta).a = r.\alpha.a + r.\beta.a$;
(iii) $r.\alpha.(a + b) = r.\alpha.a + r.\alpha.b$;
(GHM_2) $(r \circ \alpha \circ s).\beta.a = r.\alpha.(s.\beta.a)$.

A right Γ-hypermodule of R is defined in a similar way. In this case we say that M is a R_Γ-hypermodule.

3 Fuzzy Gamma Subhypermodules

In the sequel R is a Γ-hyperring and all gamma hypermodules are considered over R. In [16] M.K. Sen, R. Ameri, G. Chowdhury introduced the notion of fuzzy semihypergroups, in [10] V. Leoreanu-Fotea, B. Davvaz study fuzzy hyperrings and V. Leoreanu-Fotea in [11] studied fuzzy hypermodules. Now in this section we follows these and introduce and studied fuzzy gamma hypermodules.

Let S and Γ be two nonempty sets. $F^*(S)$ denotes the set H of all nonzero fuzzy subset of S. A *Fuzzy $\Gamma - hyperoperation$* on S is a map $\circ : S \times \Gamma \times S \longrightarrow F^*(S)$, which associates a nonzero subset $a \circ \gamma \circ b$ for all $a, b \in S$ and $\gamma \in \Gamma$. $(S, \circ)$ is called a *Fuzzy $\Gamma - hypergroupoid$*.

A fuzzy Γ-*hypergroupoid* $(S, \circ)$ is called a *fuzzy* Γ-*hypersemigroup* if for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$, we have $a \circ \alpha \circ (b \circ \beta \circ c) = (a \circ \alpha \circ b) \circ \beta \circ c$, where for any $\mu \in F^*(S)$, we have $(a \circ \gamma \circ \mu)(r) = \bigvee_{t \in S}((a \circ \gamma \circ t)(r) \wedge \mu(t))$ and $(\mu \circ \gamma \circ a)(r) = \bigvee_{t \in S}(\mu(t) \wedge (t \circ \gamma \circ a)(r))$ for all $r \in S, \gamma \in \Gamma$.

If A is a nonempty subset of S and $x \in S$, then for all $r \in S, \gamma \in \Gamma$ we have:

$$(x \circ \gamma \circ A)(r) = \bigvee_{a \in A} (x \circ \gamma \circ a)(r),$$

and

$$(A \circ \gamma \circ x)(r) = \bigvee_{a \in A} (a \circ \gamma \circ x)(r).$$

A fuzzy Γ-*hypersemigroup* $(S, \circ)$ is called a *fuzzy* Γ-*hypergroup* if for all $a \in S$ and $\gamma \in \Gamma$, we have $a \circ \gamma \circ S = S \circ \gamma \circ a = \chi_S$. We say that an element e of $(S, \circ)$ is *identity (resp. scalar identity)* if for all $s, r \in S, \gamma \in \Gamma$, we have

$$(e \circ \gamma \circ r)(r) > 0, \quad \text{and} \quad (r \circ \gamma \circ e)(r) > 0,$$

$$((e \circ \gamma \circ r)(s) > 0, \ \text{and} \ \ (r \circ \gamma \circ e)(s) > 0 \ \ it follows r = s).$$

Let $(S, \circ)$ be a fuzzy hypergroup, endowed with at least an identity. An element $a' \in S$ is called an *inverse* of $a \in S$ if there is an identity $e \in S$, such that

$$(a \circ a')(e) > 0, \text{ and } (a' \circ a)(e) > 0.$$

Definition 3.1. *A fuzzy hypergroup S is regular if it has at least one identity and each element has at least one inverse.*
A regular fuzzy hypergroup $(S, \circ)$ is called reversible if for any $x, y, a \in S$, it satisfies the following conditions:
(1) if $(a \circ x)(y) > 0$, then there exists an inverse a_1 of a, such that $(a_1 \circ y)(x) > 0$;
(2) if $(x \circ a)(y) > 0$, then there exists an inverse a_2 of a, such that $(y \circ a_2)(x) > 0$.

Definition 3.2. *We say that a fuzzy hypergroup S is a fuzzy canonical if*
(1) *it is commutative;*
(2) *it has an scalar identity;*
(3) *every element has a unique inverse;*
(4) *it is reversible.*

Let μ and ν be two nonzero fuzzy subsets of a fuzzy Γ-hypergroupoid $(S, \circ)$. We define

$$(\mu \circ \gamma \circ \nu)(t) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ \gamma \circ q)(t) \wedge \nu(q), \forall t \in S, \gamma \in \Gamma.$$

In the following we introduce and study *fuzzy gamma hyperrings* .

Definition 3.3. *Let R, Γ be two nonempty sets and $\boxplus, \boxdot$ be two fuzzy hyperoperations on R and $\otimes$ be a fuzzy hyperoperation on Γ. Let $(R, \boxplus)$ and $(\Gamma, \otimes)$ be two canonical fuzzy hypergroups. R is called a fuzzy Γ-hyperring if there exists the mapping:*

$$\begin{gathered} \boxdot : R \times \Gamma \times R \longrightarrow F^*(R) \\ (r, \gamma, s) \longmapsto r \boxdot \gamma \boxdot s, \end{gathered}$$

such that for all $r, s, t \in R, \alpha, \beta \in \Gamma$, the following conditions are satisfied:
(i) $r \boxdot \alpha \boxdot (s \boxplus t) = (r \boxdot \alpha \boxdot s) \boxplus (r \boxdot \alpha \boxdot t)$;
(ii) $r \boxdot (\alpha \otimes \beta) \boxdot s = (r \boxdot \alpha \boxdot s) \boxplus (r \boxdot \beta \boxdot s)$;
(iii) $(r \boxplus s) \boxdot \alpha \boxdot t = (r \boxdot \alpha \boxdot t) \boxplus (s \boxdot \alpha \boxdot t)$;
(iv) $r \boxdot \alpha \boxdot (s \boxdot \beta \boxdot t) = (r \boxdot \alpha \boxdot s) \boxdot \beta \boxdot t$.

Definition 3.4. *Let $(\Gamma, \otimes)$ be a fuzzy canonical hypergroups. Let $(R, \boxplus, \boxdot)$ be a fuzzy Γ-hyperring. A nonempty set M, endowed with two fuzzy Γ-hyperoperation $\oplus, \odot$ is called a left fuzzy Γ-hypermodule over $(R, \boxplus . \boxdot)$ if the following conditions hold:*

(1) $(M, \oplus)$ *is a canonical fuzzy* Γ*-hypergroup;*
(2) $\odot : R \times \Gamma \times M \longrightarrow F^*(M)$ *is such that for all* $a, b \in M, r, s \in R$ *and* $\alpha, \beta \in \Gamma$ *we have*

(i) $r \odot \alpha \odot (a \oplus b) = (r \odot \alpha \odot a) \oplus (r \odot \alpha \odot b)$;
(ii) $(r \boxplus s) \odot \alpha \odot a = (r \odot \alpha \odot a) \oplus (s \odot \alpha \odot a)$;
(iii) $r \odot (\alpha \otimes \beta) \odot a = (r \odot \alpha \odot a) \oplus (r \odot \beta \odot a)$;
(iv) $r \odot \alpha \odot (s \odot \beta \odot a) = (r \cdot \alpha \cdot s) \odot \beta \odot a$.

If both $(R, \boxplus), (\Gamma, \otimes)$ *and* $(M, \oplus)$ *have scaler identities, denoted by* $0_R, 0_\Gamma$ *and* 0_M*, then the fuzzy* Γ*-hypermodule* $(M, \oplus, \odot)$ *also satisfies the condition:*

$$0_R \odot \gamma \odot a = \chi_{0_M},$$

$$r \odot 0_\Gamma \odot a = \chi_{0_\Gamma},$$

$$r \odot \gamma \odot 0_M = \chi_{0_M},$$

for all $r \in R, \gamma \in \Gamma, a \in A$*. Moreover, if* $(R, \boxdot)$ *has an identity, say* 1*, then the fuzzy* Γ*-hypermodule* $(M, \oplus, \odot)$ *is called unitary if it satisfies the condition:*
for all a *of* M*, we have* $1 \odot \gamma \odot a = \chi_a$.

Clearly, any fuzzy Γ-hyperring is a fuzzy Γ-hypermodule over itself.
Proposition 3.5. Let $(M, +, .)$ be a module over a ring $(R, \uplus, \circ)$ and $\Gamma = R$. We define the following fuzzy Γ-hyperoperations:
for a, b of M, $a \oplus b = \chi_{\{a,b\}}$,
for all a of M and $r \in R, \gamma \in \Gamma$, $r \odot \gamma \odot a = \chi_{\{r.\gamma.a\}}$,
for all r, s of R, $r \boxplus s = \chi_{\{r,s\}}$ and $r \boxdot \gamma \boxdot s = \chi_{\{r \circ \gamma \circ s\}}$.
Then $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule over the fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$.
Note that the last theorem is satisfied, when M is a Γ-module over a Γ-ring R, such that $\Gamma \neq R$.
Proposition 3.6. Let $(R, \circ)$ and $(S, \bullet)$ be two fuzzy Γ-hyperrings. Let $(M, \oplus, \odot)$ be a left fuzzy Γ-hypermodule over R and a right fuzzy Γ-hypermodule over S. Then

$A = \{\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \mid r \in R, s \in S, m \in M\}$ is a fuzzy Γ-hyperring and fuzzy Γ-hypermodule over A, under the mappings

$$\star : A \times \Gamma \times A \longrightarrow F^*(A)$$
$$(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}, \gamma, \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix}) \longmapsto$$
$$\begin{pmatrix} r \circ \gamma \circ r_1 & r \odot \gamma \odot m_1 \oplus m \odot \gamma \odot s_1 \\ 0 & s \bullet \gamma \bullet s_1 \end{pmatrix}.$$

such that

$$\begin{pmatrix} r\circ\gamma\circ r_1 & r\odot\gamma\odot m_1\oplus m\odot\gamma\odot s_1 \\ 0 & s\bullet\gamma\bullet s_1 \end{pmatrix}\begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix}=$$
$$\begin{pmatrix} (r\circ\gamma\circ r_1)(r_2) & (r\odot\gamma\odot m_1\oplus m\odot\gamma\odot s_1)(m_2) \\ 0 & (s\bullet\gamma\bullet s_1)(s_2) \end{pmatrix}=$$
$$\begin{cases} 1, & r_2,m_2,s_2\neq 0 \\ 0, & \text{otherwise.} \end{cases}$$
.

Proof. Straightforward.□

Example 3.7. Let R be a Γ-ring and $(M,+,.)$ a Γ-module. Consider the mapping $\alpha\ :\ M\longrightarrow\ R$. Then M is an fuzzy Γ-hypermodule over M, under the following operations:

$$m\oplus n=m+n.\ \text{and}\ \circ:M\times\Gamma\times M\longrightarrow F^*(M)(m,\gamma,n)\longmapsto m\circ\gamma\circ n=\chi_{\alpha(m).\gamma.n},$$

for all $m,n\in M,\gamma\in\Gamma$.

Proposition 3.8. Let $(M,+,.)$ be a Γ-module over Γ-ring R and ν be a nonzero fuzzy Γ-semigroup on M. Let μ and ρ be two nonzero fuzzy Γ-semigroups on R. For $r\in R,\ a,b\in M$ and $\gamma\in\Gamma$, define a fuzzy Γ-hyperoperation $\odot$ on M by

$$(r\odot\gamma\odot a)(t)=\begin{cases} \mu(r)\wedge\rho(\gamma)\wedge\nu(a), & if\ t=r.\gamma.a \\ 0\qquad , & otherwise. \end{cases}$$

Also, $a\oplus b=\chi_{\{a+b\}}$. It is easy to verify that $(M,\oplus,\odot)$ is a fuzzy Γ-hypermodule.

Let S,Γ be nonempty sets, and S endowed with a fuzzy Γ-hyperoperation $\circ$. For all $a,b\in S,\gamma\in\Gamma$ and $p\in[0,1]$ consider the p-cuts:

$$(a\circ\gamma\circ b)_p=\{t\in S:\ (a\circ\gamma\circ b)(t)\geq p\}$$

of $a\circ\gamma\circ b$, where $p\in[0,1]$.

For all $p\in[0,1]$, we define the following crisp Γ-hyperoperation on S:

$$a\circ_p\gamma\circ_p b=(a\circ\gamma\circ b)_p.$$

Example 3.9. Let $R=\Gamma=\mathbb{Z}$ and $M=\mathbb{Z}_n$ for $n\in\mathbb{N}$. Define following fuzzy Γ-hyperoperations for all $a,b\in M,\gamma\in\Gamma$:

$$a\oplus b=\chi_{\{a,b\}},\forall a\in M,\forall r\in R,\gamma\in\Gamma,$$

$$r \odot \gamma \odot a = \chi_{\{\overline{r\gamma a}\}}, \quad \forall r, s \in R, \forall \gamma \in \Gamma,$$

$$r.\gamma.s = \chi_{\{\overline{r\gamma s}\}} \quad \text{and} \quad r + s = \chi_{\{r,s\}}, \quad \text{for all} \quad \alpha, \beta \in \Gamma,$$

and

$$\alpha \boxplus \beta = \chi_{\{\alpha,\beta\}},$$

such that $\overline{x}$ is denote a typical element in $\mathbb{Z}_n$. Then it is easy to verify that $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule over fuzzy Γ-hyperring R and canonical fuzzy hypergroup $(\Gamma, \boxplus)$.

Proposition 3.10. Let $(M, \circ)$ be a fuzzy Γ-hyperoperation. For all $a, b, c, u \in M$ and $\alpha, \beta \in \Gamma$ and for all $p \in [0, 1]$ the following equivalence holds:

$$(a \circ \alpha \circ (b \circ \beta \circ c)) \geq p \Longleftrightarrow u \in a \circ_p \alpha \circ_p (b \circ_p \beta \circ_p c).$$
$$((a \circ \alpha \circ b) \circ \beta \circ c) \geq p \Longleftrightarrow u \in (a \circ_p \alpha \circ_p b) \circ_p \beta \circ_p c.)$$

Proof. Clearly,

$$(a \circ \alpha \circ (b \circ \beta \circ c))(u) = \bigvee_{t \in M} (a \circ \alpha \circ t)(u) \wedge (b \circ \beta \circ c)(t) \geq p,$$

if and only if there exists $t_0 \in M$, such that $(a \circ \alpha \circ t_0)(u) \geq p$ and $(b \circ \beta \circ c)(t_0) \geq p$, which means that $u \in a \circ_p \alpha \circ_p t_0, t_0 \in b \circ_p \beta \circ_p c$. Therefore, $u \in a \circ_p \alpha \circ_p (b \circ_p \beta \circ_p c)$.□

Proposition 3.11. Let $(M, \oplus, \odot)$ be a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$. Then for all $a \in M, r \in R, \gamma \in \Gamma$, conditions are equivalence:
(1) $a \oplus M = \chi_M \Longleftrightarrow \forall p \in [0, 1],\ a \oplus_P M = M$;
(2) $r \odot \gamma \odot M = \chi_M \Longleftrightarrow \forall p \in [0, 1],\ r \odot_p \gamma \odot_p M = M$.

Proof. We only proof (2). Let $r \odot \gamma \odot M = \chi_M$. Then for all $t \in M$ and $p \in [0, 1]$, we have $\bigvee_{u \in M}(r \odot \gamma \odot u)(t) = 1 \geq p$, whence there exists $m \in M$, such that $(r \odot \gamma \odot m)(t) \geq p$, which means that $t \in r \odot_p \gamma \odot_p m$. Hence, $\forall p \in [0, 1],\ r \odot_p \gamma \odot_p M = M$. Conversely, for $p = 1$ we have $r \odot_1 \gamma \odot_1 M = M$, whence for all $t \in M$, there exists $u \in M$, such that $t \in r \odot_1 \gamma \odot_1 u$, which means that $(r \odot \gamma \odot u)(t) = 1$. In other words, $r \odot \gamma \odot M = \chi_M$.□

Proposition 3.12. The structure $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ if and only if $\forall p \in [0, 1]$, $(M, \oplus_p, \odot_p)$ is a Γ-hypermodule over the hyperring $(R, \boxplus_p, \boxdot_p)$.

Proof. It is straightforward.□

Consider $(M, \oplus, \odot)$ as a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy hypergroup $(\Gamma, \otimes)$. Now we follow [8], and define a new types of Γ-hyperoperations on M, R, Γ, as follows:

$$\forall a, b \in M,\ \ a + b = \{x \in M | (a \oplus b)(x) > 0\}, \quad \forall r, s \in R,$$

$$r \uplus s = \{t \in R \mid (r \boxplus s)(t) > 0\}, forall\alpha, \beta \in \Gamma,$$

$$\alpha * \beta = \{\gamma \in \Gamma \mid (\alpha * \beta)(\gamma) > 0\}, \quad \forall a \in M, \quad \forall r \in R, \forall \gamma \in \Gamma,$$

$$r.\gamma.a = \{b \in M \mid (r \odot \gamma \odot a)(b) > 0\}, \quad \forall r, s \in R, \quad \forall \gamma \in \Gamma,$$

$$r \circ \gamma \circ s = \{t \in R \mid (r \boxdot \gamma \boxdot s)(t) > 0\}.$$

Proposition 3.13. If $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy hypergroup $(\Gamma, \otimes)$, then $(M, +, .)$ is a Γ-hypermodule over the Γ-hyperring $(R, \uplus, \circ)$ and canonical hypergroup $(\Gamma, \star)$.

Proof. By [10], it is obtained that $(R, \uplus)$, $(\Gamma, *)$ and $(M, +)$ are canonical hypergroups. It is sufficient to verify $(M, .)$ is a Γ-hypermodule. We consider the following cases:

Case: (i)

$$(r \uplus s).\gamma.a = (r.\gamma.a) + (s.\gamma.a), \quad \text{for all} \quad r, s \in R, \gamma \in \Gamma, a \in M.$$

Suppose that $x \in (r \uplus s).\gamma.a = \bigcup_{y \in r \uplus s} y \odot \gamma \odot a$. Then $(y \odot \gamma \odot a)(x) > 0$ and $(r \boxplus s)(y) > 0$, for some $y \in r \uplus s$, and hence $\vee_{p \in M}$ $((r \boxplus s)(p) \wedge (p \odot \gamma \odot a)(x) > 0$. Thus $((r \boxplus s) \odot \gamma \odot a)(x) > 0$, which implies that $((r \odot \gamma \odot a) \oplus (s \odot \gamma \odot a))(x) > 0$. Thus there exist $z, t \in M$, such that $(z \oplus t)(x) > 0$, $(r \odot \gamma \odot a)(z) > 0$ and $(s \odot \gamma \odot a)(t) > 0$ i.e., $x \in z + t, z \in r.\gamma.a$ and $t \in s.\gamma.a$ and hence $x \in (r.\gamma.a) + (s.\gamma.a)$. Therefore, $(r \uplus s).\gamma.a \subseteq (r.\gamma.a) + (s.\gamma.a)$. Similarly, we can show that $(r.\gamma.a) + (s.\gamma.a)t \subseteq (r \uplus s).\gamma.a$. Therefore, $(r \uplus s).\gamma.a = (r.\gamma.a) + (s.\gamma.a)$. The other conditions are verified similarly and omitted. □

On the other hands, if $(M,+,.)$ is a Γ-hypermodule over a Γ-hyperring $(R,\uplus,\circ)$, then we define the following fuzzy Γ-hyperoperations:

$$\begin{aligned} a\oplus b &= \chi_{\{a+b\}}, \forall a,b\in M, r\boxplus s \\ &= \chi_{\{r\uplus s\}}, \forall r,s\in R, \gamma\in\Gamma, r\odot\gamma\odot a \\ &= \chi_{\{r.\gamma.a\}}, \forall a\in M, r\in R, r\boxdot\gamma\boxdot s \\ &= \chi_{\{r\circ\gamma\circ s\}}, \forall r,s\in R, \forall\gamma\in\Gamma, \beta \\ &= \chi_{\{\alpha*\beta\}}\forall\alpha,\beta\in\Gamma, \alpha\otimes\beta. \end{aligned}$$

The next result is immediately follows from above discussion and [14].

Proposition 3.14. For every hypergroup $(M,+)$, there is an associated fuzzy hypergroup.

Proposition 3.15. Let $(M,+,.)$ be a Γ-hypermodule over a Γ-hyperring. Let $(R,\uplus,\circ)$ be a canonical hypergroup $(\Gamma,\star)$. Then $(M,\oplus,\odot)$ is a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R,\boxplus,\boxdot)$ and canonical fuzzy hypergroup $(\Gamma,\otimes)$, where the fuzzy hyperoperations $\oplus,\odot,\boxplus,\boxdot$ and $\otimes$ are defined above.

Proof. By Proposition 3.14, $(M,\oplus)$ is a commutative fuzzy Γ-hypergroup. We show that $(M,\oplus)$ is canonical. Since $(M,+)$ is canonical Γ-hypergroup, then there exists $e\in M, \forall a\in M,\ a=e+a=a+e \implies (e\oplus a)(a)=\chi_{\{e+a\}}(a)>0, (a\oplus e)(a)=\chi_{\{e+a\}}(a)>0$ and because for all $a\in M$ there exists $b\in M$, such that $e\in a+b\cap b+a$, $b)$ is the inverse of a with respect to $+)$. Then

$$(a\oplus b)(e)=\chi_{\{a+b\}}(e)=\chi_{\{b+a\}}(e)=(b\oplus a)(e)>0.$$

Let $(a\oplus x)(y)=\chi_{\{a+x\}}(y)>0 \implies y\in a+x \implies \exists\, b$ (the inverse of a such that $x\in b+y \implies (b\oplus y)(x)=\chi_{\{b+y\}}(x)>0$. The other cases is can be proved in a similar way and omitted. Then $(M,\oplus)$ is a canonical fuzzy Γ-hypergroup. Now, we show that $(M,\oplus,\odot)$ is a fuzzy Γ-hypermodule. We investigate only the condition (iv) of Definition 3.4.

First , we show that that for all $r,s\in R,\alpha,\beta\in\Gamma,a\in M$, we have

$$(r\odot\alpha\odot(s\odot\beta\odot a))=(r\boxdot\alpha\boxdot s)\odot\beta\odot a,\quad \forall t\in M.$$

Then

$$(r\odot\alpha\odot(s\odot\beta\odot a))(t)=\bigvee_{p\in M},$$

$$[(r\odot\alpha\odot p)(t)\wedge(s\odot\beta\odot a)(p)]=\bigvee_{p\in M}[\chi_{r.\alpha.p}(t)\wedge\chi_{s.\beta a}(p)]=$$

$\left\{\begin{array}{ll} 1, & t \in r.\alpha.(s.\beta.a) \\ 0, & otherwise \end{array}\right. = \left\{\begin{array}{ll} 1, & t \in (r.\alpha.s).\beta.a \\ 0, & otherwise \end{array}\right.$
$= ((r \boxdot \alpha \boxdot s) \odot \beta \odot a)(t)$, for all $t \in M$.

It is easy to verify that the other conditions of Definition 3.4 can be obtained in a similar way.□

Proposition 3.16. Let M an R_Γ-module and μ be a fuzzy Γ-module of M. Then the set M will be a fuzzy Γ-hypermodule.

Proof. Let $(\Gamma, *)$ be an abelian group and $(M, +, .)$ be a Γ-module over Γ-ring $(R, \uplus, \circ)$. We define fuzzy Γ-hyperoperations on M as follows:

$$(a \oplus b)(t) = \chi_{\{a+b\}},\ (r \odot \gamma \odot a)(t) = \mu(r.\gamma.a - t),$$

$$(\alpha \otimes \beta)(\gamma) = \chi_{\{\alpha * \beta\}}(\gamma) = \chi_{\{r \uplus s\}} r \boxplus s)(z)(r \boxdot \alpha \boxdot s)(z) = \chi_{\{r \circ \alpha \circ s\}}(z),$$

$\forall a, b, t \in M, r, s, z \in R, \alpha, \beta, \gamma \in \Gamma$.

It is easy to verify that $(M, \oplus)$ is a canonical fuzzy hypergroup. Now, we show $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule with $\mu(0) = 1$.

(i)

$$\begin{array}{rcl} ((r \boxplus s) \odot \gamma \odot a)(t) & = & \vee_{p \in R}(r \boxplus s)(p) \wedge (p \odot \gamma \odot a)(t) \\ & = & \vee_{p \in R} \chi_{r \uplus s}(p) \wedge \mu(p.\gamma.a - t) \\ & = & \mu((r \uplus s).\gamma.a - t) \quad \text{if} \quad p = r \uplus s. \end{array}$$

Also, $((r \odot \gamma \odot a) \oplus (s \odot \gamma \odot a))(t) =$

$$\begin{array}{rl} = & \vee_{p,q \in M}(r \odot \gamma \odot a)(p) \wedge (p \oplus q)(t) \wedge (s \odot \gamma \odot a)(q) \\ = & \vee_{p,q \in M} \mu(r.\gamma.a - p) \wedge \chi_{\{p+q\}}(t) \wedge \mu(s.\gamma.a - q) \\ = & \vee_{p,q \in M, t=p+q} \mu(r.\gamma.a - p) \wedge \mu(s.\gamma.a - q) \\ \leq & \mu(r.\gamma.a - p + s.\gamma.a - q) \\ = & \mu((r \uplus s).\gamma.a - (p + q)), \end{array}$$

On the other hands, if $q = s.\gamma.a,\ p = t - s.\gamma.a.$, then

$$\begin{array}{rcl} \vee_{p,q \in M, t=p+q} \mu(r.\gamma.a - p) \wedge \mu(r.\gamma.a - q) & \geq & \vee_{p \in M} \mu(r.\gamma.a - p) \\ & \geq & \mu(r.\gamma.a - t + s.\gamma.a) \\ & = & \mu((r \uplus s).\gamma.q - t). \end{array}$$

(ii)

$$\begin{array}{rcl} (r \odot (\alpha \otimes \beta) \odot a)(t) & = & \vee_{\gamma \in \Gamma}[(r \odot \gamma \odot a)(t) \wedge (\alpha \otimes \beta)(\gamma)] \\ & = & \vee \mu(r.\gamma.a - t) \wedge \chi_{\{\alpha * \beta\}}(\gamma) \\ & = & \mu(r.(\alpha * \beta).a - t). \end{array}$$

Also, $((r \odot \alpha \odot a) \oplus (r \odot \beta \odot a))(t) =$

$$\begin{aligned}
&= \vee_{p,q\in M}[(r \odot \alpha \odot a)(p) \wedge (p \oplus q)(t) \wedge (r \odot \beta \odot a)(q) \\
&= \vee_{p,q\in M}[\mu(r.\alpha.a - p) \wedge \chi_{\{p+q\}}(t) \wedge \mu(r.\beta.a - q)] \\
&= \vee_{t=p+q}\ \mu(r.\alpha.a - p) \wedge \mu(r.\beta a - q) \\
&\leq \mu(r.\alpha a - p + r.\beta a - q) \\
&= \mu(r.(\alpha * \beta).a - (p+q)).
\end{aligned}$$

On the other hands, suppose that $q = r.\beta.a$, then for $p = t - r.\beta.a$ we have

$$\begin{aligned}
\vee_{t=p+q}\mu(r.\alpha.a - p) \wedge \mu(r.\beta a - q) &= \vee_{p\in M}\mu(r.\alpha a - p) \\
&\geq \mu(r.\alpha a - (t - r\beta a)) \\
&= \mu(r.(\alpha * \beta).a - (p+q)),
\end{aligned}$$

(iii)

$$\begin{aligned}
r \odot \gamma \odot (a \oplus b) &= \vee_{p\in M}(r \odot \gamma \odot p)(t) \wedge (a \oplus b)(p) \\
&= \vee_{p\in M}\mu(r.\gamma.p - t) \wedge \chi_{\{a+b\}}(p) \\
&= \mu(r.\gamma.(a+b) - t) \quad \text{and} \quad ((r \odot \gamma \odot a) \oplus (r \odot \gamma \odot b))(t) \\
&= \vee_{p,q\in M}(r \odot \gamma \odot a)(p) \wedge (p \oplus q)(t) \wedge (r \odot \gamma \odot b)(q) \\
&= \vee_{p,q\in M}\mu(r.\gamma.a - p) \wedge \chi_{\{p+q\}}(t) \wedge \mu(r.\gamma.b - q) \\
&= \vee_{p,q\in M,t=p+q}\mu(r.\gamma.a - p) \wedge \mu(r.\gamma.b - q) \\
&\leq \mu(r.\gamma.a - p + r.\gamma.b - q) = \mu(r.\gamma.(a+b) - t).
\end{aligned}$$

On the other hands, for $q = r.\gamma.b, p = t - r.\gamma.b.$ we have

$$\begin{aligned}
&\vee_{p,q\in M,t=p+q}\mu(r.\gamma.a - p) \wedge \mu(r.\gamma.b - q) \\
&\qquad \geq \vee_{p\in M}\mu(r.\gamma.a - p) \\
&\qquad \geq \mu(r.\gamma(a+b) - t).
\end{aligned}$$

(iv)

$$\begin{aligned}
(r \odot \alpha \odot (s \odot \beta \odot a))(t) &= \vee_{p\in M}(r \odot \alpha \odot p)(t) \wedge (s \odot \beta \odot a)(p) \\
&= \vee_{p\in M}\mu((r.\alpha.p) - t) \wedge \mu((s.\beta.a) - p) \\
&= \mu(r.\alpha.(s.\beta.a) - t), \quad \text{and} \quad ((r \boxdot \alpha \boxdot s) \odot \beta \odot a)(t) \\
&= \vee_{p\in R}(r \odot \alpha \odot s)(p) \wedge (p \odot \beta \odot a)(t) \\
&= \vee_{p\in R}\chi_{\{r\circ\alpha\circ s\}}(p) \wedge \mu(p.\beta.a - t) \\
&= \mu(r \circ \alpha \circ s \cdot (\beta \cdot a) - t) \quad \text{if} \quad p = r \circ \alpha \circ s.
\end{aligned}$$

□

Remark. Let $\mathbb{H} = \langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra. Denote by $F^*(H)$ the set of the nonzero fuzzy subsets of H. Then $\mathbb{H}$ can be organized as a universal algebra under the following operations:

$$\beta_i(\mu_1, ..., \mu_{n_i})(t) = \bigvee_{(x_1,...,x_{n_i})\in H^{n_i}} [(\mu_1(x_1) \bigwedge ...\mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, ..., x_{n_i})(t))],$$

for every $i \in I, \mu_1, ..., \mu_{n_i} \in F*(H)$ and $t \in H$. We denote this algebra by $F^*(\mathbb{H})$.

Proposition 3.17. If $(M, \oplus, \diamondsuit)$ is a fuzzy Γ-hypermodule, then $(F^*(M), *, \bigcirc)$ is a Γ-module.

Proof. We define operations $*, \diamondsuit$ on $F^*(M)$ by $\mu * \nu = \mu \oplus \nu$, and $r\diamondsuit\gamma\diamondsuit\mu = r \odot \gamma \odot \mu$ for all $\mu, \nu \in F^*(M), r \in R, \gamma \in \Gamma$. It is easy to see that $(F^*(M), *)$ is a group. Clearly, $(F^*(M), \oplus)$ is a semigroup.

(i) Identity: we must prove that there exists a $\nu \in F^*(M)$ such that ,$\mu * \nu = \mu$. We have

$$\begin{array}{rcl}(\mu * \nu)(t) &=& (\mu \oplus \nu)(t)\\ &=& \vee_{p,q\in M}\mu(p) \wedge (p \oplus q)(t) \wedge \nu(q)\\ &=& \vee_{p\in M}\mu(p) \wedge (p \oplus e)(t)\\ &=& \mu(t) \oplus \quad \text{if}\end{array}$$

$q = e; \nu(q) = 1, p = t.$

Thus it is enough we choose $\nu = \chi_e$.

(ii) Inverse: it must prove that for $\mu \in F^*(M)$, there exists a $\nu \in F^*(M)$, such that $\mu * \nu = \chi_e$. It is sufficient to consider $\nu = -\mu$, then we have

$$\begin{array}{rcl}(\mu * \nu)(t) &=& (\mu \oplus \nu)(t)\\ &=& \vee_{p,q\in M}\mu(p) \wedge (p \oplus q)(t) \wedge (-\mu)(q)\\ &=& \vee_{p,q\in M}\mu(p) \wedge (p \oplus q)(t) \wedge \mu(-q)\\ &\leq& \mu(p - (-q)) \wedge (p \oplus q)(t) \leq (p \oplus q)(t)\\ &=& \chi_e(t) \ \text{ where, } \ p \ \text{ is inverse of } \ q.\end{array}$$

On the other hands, we have

$$\begin{array}{rcl}\vee_{p,q\in M}\mu(p) \wedge (p \oplus q)(t) \wedge \mu(-q) &\geq& \vee_{p\in M}\mu(p) \wedge (p \oplus -p)(t)\\ &\geq& (p \oplus -p)(t)\\ &=& \chi_e(t).\end{array}$$

Other cases are easy and omitted. □

Definition 3.18. Let $(M, \oplus, \odot)$ be a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$. A nonempty subset N of M is called a *subfuzzy* Γ-hypermodule if for all $x, y \in N, r \in R$ and $\gamma \in \Gamma$, the following conditions hold:
(1) $(x \oplus y)(t) > 0 \quad \Rightarrow \quad t \in N$;
(2) $x \oplus N = \chi_N$;
(3) $(r \odot \gamma \odot x)(t) > 0 \quad \Rightarrow \quad t \in N$.

Proposition 3.19. (i) If $(N, \oplus, \odot)$ is a subfuzzy Γ-hypermodule of $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$, then the associated Γ-hypermodule $(N, +, .)$ is a Γ-hypersubmodule of $(M, +, .)$ over $(R, \uplus, \circ)$;
(ii) $(N, +, .)$ is a Γ-hypersubmodule of $(M, +, .)$ over $(R, \uplus, \circ)$ if and only if $(N, \oplus, \odot)$ is a subfuzzy Γ-hypermodule of $(M, \oplus, \odot)$ over $(R, \boxplus, \boxdot)$.

4 Fundamental Relation of Fuzzy Γ-hypermodule

In [14], fuzzy regular relations are introduced in the context of fuzzy hypersemigroups. In this section we extend this notion to fuzzy Γ-hypermodules. Let ρ be an equivalence relation on a fuzzy Γ-hypersemigroup $(M, \circ)$ and μ, ν be two fuzzy subsets on M. We say that $\mu\rho\nu$ if the following conditions hold:
(1) if $\mu(a) > 0$, then there exists $b \in M$, such that $\nu(b) > 0$ and $a\rho b$ and;
(2) if $\nu(x) > 0$, then there exists $y \in M$, such that $\mu(y) > 0$ and $x\rho y$.
An equivalence relation ρ on a fuzzy Γ-hypersemigroup $(M, \circ)$ is called a *fuzzy regular relation* (or *fuzzy hypercongruence*) on $(M, \circ)$ if, for all $a, b, c \in M, \gamma \in \Gamma$, the following implication holds:

$$a\rho b \Longrightarrow (a \circ \gamma \circ c) \ \rho \ (b \circ \gamma \circ c) \text{ and } (c \circ \gamma \circ a) \ \rho \ (c \circ \gamma \circ b).$$

This condition is equivalent to

$$a\rho a', b\rho b' \quad \Rightarrow \quad (a \circ \gamma \circ b)\rho(a' \circ \gamma \circ b'), \forall a, b, a', b' \in M, \gamma \in \Gamma.$$

Definition 4.1. An equivalence relation ρ on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and a canonical fuzzy hypergroup $(\Gamma, \otimes)$ is called a *fuzzy regular relation* on $(M, \oplus, \odot)$ if it is a fuzzy regular relation on $(M, \oplus)$ and for all $x, y \in M, r \in R, \gamma \in \Gamma$, the following implication holds:

$$x\rho y \Longrightarrow (r \odot \gamma \odot x)\rho(r \odot \gamma \odot y).$$

Let $(M, \oplus, \odot)$ be a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and a canonical fuzzy hypergroup $(\Gamma, \otimes)$. Suppose $(M, +, .)$ is the associated Γ-hypermodule over the Γ-hyperring $(R, \uplus, \circ)$ and the canonical hypergroup $(\Gamma, *)$. Then we have the next result.

Theorem 4.2. An equivalence relation ρ is a fuzzy regular relation on $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy hypergroup $(\Gamma, \otimes)$ if and only if ρ is a regular relation on $(M, +, .)$ over the Γ-hyperring $(R, \uplus, \circ)$ and canonical hypergroup $(\Gamma, *)$.

Proof. Letting $x\rho y$ and $x'\rho y'$, where $x, x', y, y' \in M$. We have $(x \oplus x')\rho(y+y')$ if and only if the following conditions hold:

$$(x \oplus x')(u) > 0, \Rightarrow \exists v \in M : (y \oplus y')(v) > 0 \ \text{ and } \ u\rho v,$$

and

$$(y \oplus y')(t) > 0 \quad \Rightarrow \quad \exists w \in M : (x \oplus x')(w) > 0 \ \text{ and } \ at\rho w.$$

These are equivalent to:
if $u \in x + x'$, then there exists $v \in y + y'$, such that $u\rho v$;
if $t \in y + y'$, then there exists $w \in x + x'$, such that $t\rho w$;
which mean that $(x + x')\bar{\rho}(y + y')$. Hence ρ is fuzzy regular on $(M, \oplus)$ if and only if ρ is regular on $(M, +)$.
On the other hands, if $x\rho y$ and $r \in R, \gamma \in \Gamma$. We have $(r \odot \gamma \odot x)\rho(r \odot \gamma \odot y)$ if and only if the next conditions hold:
if $(r \odot \gamma \odot x)(u) > 0$, then there exists $v \in M$, such that $(r \odot \gamma \odot y)(v) > 0$ and $u\rho v$;
if $(r \odot \gamma \odot y)(t) > 0$, then there exists $w \in M$, such that $(r \odot \gamma \odot x)(w) > 0$ and $t\rho w$.
These are equivalent to:
if $u \in r.\gamma.x$, then there exists $v \in r.\gamma.y$, such that $u\rho v$;
if $t \in r.\gamma.y$, then there exists $w \in r.\gamma.x$, such that $t\rho w$;
which means that $(r.\gamma.x)\rho(r.\gamma.y)$.$\square$

Definition 4.3. An equivalence relation ρ on a fuzzy Γ-hypersemigroup $(M, \circ)$ is called a *fuzzy strongly regular relation* on $(M, \circ)$ if, for all a, a', b, b' of M and for all $\gamma \in \Gamma$, such that $a\rho b$ and $a'\rho b'$, the following condition holds:

$$(a \circ \gamma \circ c)(x) > 0, (b \circ \gamma \circ d)(y) > 0 \quad \Rightarrow \quad x\rho y,$$

for all $x, y \in M$. Note that if ρ is a fuzzy strongly relation on a fuzzy Γ-hypersemigroup $(M, \circ)$, then it is a fuzzy regular on $(M, \circ)$. An equivalence relation ρ on a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ is called a *fuzzy strongly regular* *relation* on $(R, \boxplus, \boxdot)$ if it is a fuzzy strongly regular relation both on $(R, \boxplus)$ and on $(R, \boxdot)$.

Definition 4.4. Let ρ be a fuzzy strongly regular relation on a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and θ be a fuzzy strongly regular relation on a canonical fuzzy Γ-hypergroup $(\Gamma, *)$. An equivalence relation δ on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy Γ-hypergroup $(\Gamma, \otimes)$ is called a *fuzzy strongly regular relation* on $(M, \oplus, \odot)$ if it is a fuzzy strongly regular relation on $(M, \oplus)$ and if $x\delta y$, $r\rho s$ and $\alpha\theta\beta$, then the next condition holds:
for all $u \in M$, such that $(r \odot \alpha \odot x)(u) > 0$ and for all $v \in M$, such that $(s \odot \beta \odot y)(v) > 0$, we have $u\delta v$.

Theorem 4.5. An equivalence relation δ is a fuzzy strongly regular relation on $(M, \oplus, \odot)$ if and only if δ is a strongly regular relation on $(M, +, .)$.

Proof. Set $x\delta y$ and $x'\delta y'$, where $x, x', y, y' \in M$ and set $r\rho s$, where $r, s \in R$ and $\alpha\theta\beta$, where $\alpha, \beta \in \Gamma$. The relation δ is strongly regular on $(M, \oplus, \odot)$ if and only if the following conditions are satisfied:
$\forall u \in M$, such that $(x \oplus x')(u) > 0$ and $\forall v \in M$, such that $(y \oplus y')(v) > 0$, we have $u\delta v$;
$\forall t \in M$, such that $(r \odot \alpha \odot x)(t) > 0$ and $\forall w \in M$, such that $(s \odot \beta \odot y)(w) > 0$, we have $t\delta w$.
These conditions are equivalent to the following ones:
$\forall u \in M$, such that $u \in x + x'$ and $\forall v \in M$, such that $v \in y + y'$, we have $u\delta v$;
$\forall t \in M$, such that $t \in r.\alpha.x$ and $\forall w \in M$, such that $w \in s.\beta.y$, we have $t\delta w$, which mean that $(x + x')\bar{\bar{\delta}}(y + y')$ and $(r.\alpha.x)\bar{\bar{\delta}}(s.\beta.y)$. Hence δ is strongly regular on $(M, \oplus, \odot)$ if and only if δ is strongly regular on $(M, +, .)$.
Now, Let δ be a fuzzy regular relation on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy Γ-hypergroup $(\Gamma, \otimes)$ and ρ, θ be fuzzy strongly regular relations on the Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy Γ-hypergroup. $(\Gamma, \otimes)$.
We consider the following Γ-hyperoperations on the quotient set M/δ:

$$\bar{x} \star \bar{y} = \{\bar{z} \mid z \in x + y\} = \{\bar{z} \mid (x \oplus y)(z) > 0\},$$

$$\bar{r} \circledcirc \bar{\alpha} \circledcirc \bar{x} = \{\bar{z} \mid z \in r.\alpha.x\} = \{\bar{z} \mid (r \odot \alpha \odot x)(z) > 0\}.$$

Theorem 4.6. Let $(M, \oplus, \odot)$ be a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy hypergroup $(\Gamma, *)$. Let $(M, +, .)$ be the associated Γ-hypermodule over the corresponding Γ-hypergroup $(R, \uplus, \circ)$ and canonical hypergroup $(\Gamma, *)$. Then we have:

(i) The relation δ is a fuzzy regular relation on $(M, \oplus, \odot)$ if and only if $(M/\delta, \star, \circledcirc)$ is a Γ-hypermodule over $(R, \uplus, \circ)$ and $(\Gamma, *)$.
(ii) The relation δ is a fuzzy strongly regular relation on $(M, \oplus, \odot)$ over $(R, \boxplus, \boxdot)$ and $(\Gamma, \otimes)$ if and only if $(M/\delta, \star, \circledcirc)$ is a Γ-module over R/ρ and Γ/θ.
If we denote by $\mathfrak{U}$ the set of all expressions consisting of finite fuzzy Γ-hyperoperations either on R, Γ and M or the external fuzzy Γ-hyperoperations applied on finite sets of elements of R, Γ and M, then we have
$x \epsilon y \Longleftrightarrow \exists u \in \mathfrak{U} \ : \ \{x, y\} \subset u.$
Now, we introduced *fundamental relation* on fuzzy Γ-hypemodules.
Definition 4.7. An equivalence relation ϵ^* is called *fundamental relation* on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ if ϵ^* is fundamental relation on the associated Γ-hypermodule $(M, +, .)$.
Hence, ϵ^* is fundamental relation on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ if and only if ϵ^* is the smallest fuzzy strongly equivalence relation on $(M, \oplus, \odot)$.
Denote by $\mathfrak{UF}$ the set of all expressions consisting of finite fuzzy Γ-hyperoperations either on R, Γ and M or the external fuzzy Γ-hyperoperation applied on finite sets of elements of R, Γ and M. We obtain

$$x \epsilon y \Longleftrightarrow \exists\ \mu_f \in \mathfrak{UF} : \ \{x, y\} \subseteq \mu_{f\gamma} \Longleftrightarrow \mu_{f\gamma}(x) > 0 \text{ and } \mu_{f\gamma}(y) > 0.$$

The relation ϵ^* is the transitive closure of ϵ.
Denote by $\sum^*_{\oplus}$ any finite fuzzy hypersum and by $\prod^*_{\odot}$ any finite fuzzy Γ-hyperproduct of the fuzzy Γ-hypemodule $(M, \oplus, \odot)$. As above, we obtain that

$$(\textstyle\sum^*_{i\oplus} \prod^*_{j\ \odot} a_{ji})(p) > 0 \text{ if and only if } p \in \textstyle\sum^*_{i\oplus} \prod^*_{j\odot} a_{ji}.$$

Hence, $\{x, y\} \subset \ \sum^*_{i\oplus} \prod^*_{j\ \odot} a_{ji}$ if and only if $(\sum^*_{i\oplus} \prod^*_{j\ \odot} a_{ji})(x) > 0$ and $(\sum^*_{i\oplus} \prod^*_{j\ \odot} a_{ji})(y) > 0$. Therefore, we obtain $x \epsilon y \Longleftrightarrow \exists \mu_{f\gamma} \in \mathfrak{UF}$ such that $\mu_{f\gamma}(x) > 0$ and $\mu_{f\gamma}(y) > 0$.
So, in order to obtain a fuzzy Γ-module starting from a fuzzy Γ-hypermodule, we consider first the relation ϵ, then the transitive closure ϵ^* of ϵ and finally the quotient structure $(M/\epsilon^*, \star, \circledcirc)$ of the fuzzy Γ-hypermodule $(M, \oplus, \odot)$.

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References

[1] Ameri, R., Norouzi, M., *Prime and primary hyperideals in Krasner (m,n)-ary hyperrings,* European J. Combin. 34 (2013) 379-390.

[2] R. Ameri, T. Nozari, *Complete Parts and Fundamental Relation of Fuzzy Hypersemigroups*, J. of Mult.-Valued Logic and Soft Computing, Vol. 19, (2012), 451460.

[3] R. Ameri, R. Sadeghi, *Gamma Modules*, Ratio Mathematica, No. 20,(2010), 1-14.

[4] R. Ameri, T. Nozari, *Term Functions and Fundamental Relation of Fuzzy Hyperalgebras*, Ratio Mathematica, No. 20, (2010), 1-20.

[5] R. Ameri,*On categories of hypergroups and hpermodules*, Italian Journal of Pure and Applied Mathematics, vol.6(2003),121-132.

[6] R. Ameri, M.M. Zahedi, *Hyperalgebraic system*, Italian Journal of Pure and Applied Mathematics, vol. 6(1999), 21-32.

[7] W.E. Barens, *On the Γ-ring of Nobusawa*, Pacific J. Math., 18(1966), 411-422.

[8] P. Corsini,*Prolegomena of hypergroup theory*, Aviani Editore, 1993.

[9] P. Corsini, L. Leoreanu,*Applications of hyperstructure theory*, Kluwer, Dordrecht, 2003.

[10] V. Leoreanu-Fotea, *Fuzzy Hypermodules*,Computers and Mathematics with Applications 57(2009) 466-475.

[11] V. Leoreanu-Fotea, B. Davvaz, *Fuzzy hyperrings*, Fuzzy sets and systems, vol. 160(2009) 2366-2378.

[12] J. Luh, *On the theory of simple gamma rings*, Michigan Math. J.,16 (1969), 65-75.

[13] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congres Math, Scandinaves, Stockholm,1934, pp. 45-49.

[14] N. Nobusawa, *On a generalization of the ring theory* , Osaka J.Math. 1(1964), 81-89.

[15] A. Rosenfeld, *Fuzzy group*, J. Math. Anal. Appl. 35, 512-517(1971).

[16] M.K. Sen, R. Ameri, G. Chowdhury, *Fuzzy hypersemigroup*, Soft Comput. (2007),doi:http://10.1007/s00500-007-0257-9.

[17] Th. Vogiouklis, *Fundamental relations in hyperstructure*, Bull. Greek Math. Soc. 42(1999)113-118.

[18] Zadeh. L. A. *Fuzzy sets*, Inform Control 8, 338-353(1965).

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Ordered Polygroups

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Abstract

In this paper, those polygroups which are partially ordered are introduced and some properties and related results are given.

Key words: Hypergroup, Polygroup, Ordered Polygroup.

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1 Introduction and Preliminaries

The notion of a hyperstructure and hypergroup, as a generalization of group, was introduced by F. Marty [5] in 1934 at the 8th congress of Scandinavian Mathematicians. In this definition for nonempty set H, a function $\cdot : H \times H \longrightarrow P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H, is called a *hyperoperation* on H, and the system $(H, \cdot)$ is called a *hypergroupoid*. If the hypergroupoid H satisfies $a \cdot H = H \cdot a = H$, for all $a \in H$, it is called a *hypergroup*. In a hypergroupoid H, for $A, B \subseteq H$ and $x \in H$, $A \cdot B$ and $A \cdot x$ are defined as

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b, A \cdot x = A \cdot \{x\}.$$

An element e of hypergorupoid H is called an *identity* if for all $a \in H$, $a \in a \circ e \cap e \circ a$. An element $a' \in H$ is called an inverse for $a \in H$ if there is an identity $e \in H$ such that $e \in a \circ a' \cap a' \circ a$.

By a *subhypergroupoid* of hypergroupoid H we mean a subset K of H that is closed with respect to the hyperoperation on H, and contains the unique identity of H and the inverses of its elements, provided there exist.

Hyperstructures have many applications to several sectors of both pure and applied sciences. A short review of the theory of hyperstructures appear in [2]. In [3] a wealth of applications can be found, too. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy set and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence and probabilities. Polygroups are certain subclasses of hypergroups which studied in 1981 by Ioulidis in [4] and are used to study colour algebra.

A *polygroup* is a system $< G, \cdot, ^{-1}, e >$ where $e \in G$, '$^{-1}$' is a unary operation on G and '$\cdot$' is a binary hyperoperation on H satisfying the following:

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(2) $e \cdot x = x \cdot e = \{x\}$,

(3) $x \in y \cdot z \;\Leftrightarrow\; y \in x \cdot z^{-1} \;\Leftrightarrow\; z \in y^{-1} \cdot x$.

In any polygroup the following hold:

$$e \in x \cdot x^{-1} \cap x^{-1} \cdot x,\; e^{-1} = e,\; (x^{-1})^{-1} = x,\; (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$$

where $A^{-1} = \{x^{-1} : x \in A\}$.

Some other concepts in polygroups is as follows.

A nonempty subset K of polygroup G is said to be a *subpolygroup* if and only if $e \in K$ and $< K, \cdot, ^{-1}, e >$ is itself a polygroup. Subpolygroup K of polygroup G is said to be *normal* if and only if $a^{-1}Ka \subseteq K$, for all $a \in G$.

From now on, in this paper, $G =< G, \cdot, ^{-1}, e >$ will denote a polygroup.

2 Ordered hyperstructures: Definition and properties

This section is devoted to introduce the concept of a compatible order on a polygroup. It is first introduced the concept of an ordered hypergroupoid and some basic notions. Then, the concept of ordered polygroups is introduced and some related results are given. For more details on compatible orders, specially ordered algebraic structures we refer to [1].

Definition 2.1. Let $(H, \cdot)$ be a hypergroupoid. By a *compatible order* on H we mean an order "$\leq$" with respect to which all translations $x \mapsto x \cdot y$ and $x \mapsto y \cdot x$ are isotone, that is

$$x \leq y \text{ implies } b \cdot x \cdot a \leq b \cdot y \cdot a, \text{ for all } a, b \in H \tag{2.1}$$

where for $A, B \subseteq H$, $A \leq B$ means that for all $a \in A$ there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that $a \leq b$.

Definition 2.2. By an *ordered hypergroupoid* we mean a hypergroupoid on which is defined a compatible order.

When "$\cdot$" is commutative or associative, H is said to be an *ordered commutative hypergroupoid* or an *ordered semihypergroup*, respectively.

Example 2.3. (1) Consider $\mathbb{R}_1 = [1, \infty)$, the set of all real numbers greater than 1, as a poset with the natural ordering, and define $x \cdot y$ to be the set of all upper bounds of $\{x, y\}$. Thus $(\mathbb{R}_1, \cdot, \leq)$ is an ordered commutative semihypergroup with 1 as the unique identity.

(2) Consider $\mathbb{Z}$, the additive group of all integers which is a chain with the natural ordering. For $m, n \in \mathbb{Z}$, let $m{\cdot}n$ be the subgroup of $\mathbb{Z}$ generated by $\{m, n\}$. Then $(\mathbb{Z}, \cdot, \leq)$ is an ordered commutative semihypergroup in which 0 is an identity.

(3) Let $(G, \cdot, e, \leq)$ be an ordered group, and let $x \circ y = \langle \{x, y\} \rangle$, the subgroup of G generated by $\{x, y\}$. Then, $(G, \circ, \leq)$ is an ordered commutative hypergroup with an identity e.

(4) Let $(L; \vee, \wedge, 0)$ be a lattice with the least element 0. For $a, b \in L$, let $a \circ b = F(a \wedge b)$, where $F(x)$ is the principal filter generated by $x \in L$. Then, $(L; \circ)$ is an ordered hypergroup. Also, 0 is an identity, and if $x \in L$ be such that $x \wedge y = 0$, for some $y \in L$, then y is an inverse of x.

Definition 2.4. Let H be an ordered hypergroupoid.

(1) For every $x, y \in H$ with $x \leq y$, the set $[x, y] = \{z \in H : x \leq z \leq y\}$ is said to be an *interval* in H.

(2) A subset A of H is said to be *convex* if for all $a, b \in A$, where $a \leq b$, we have $[a, b] \subseteq A$.

Definition 2.5. Let $(E; \leq)$ be an ordered set. A subset D of E is said to be a *down-set* if $y \leq x$ and $x \in D$ imply $y \in D$. Down-set D is said to be *principal* if there exists $x \in D$ such that $D = \{y \in E : y \leq x\}$ denoted by $x^{\downarrow}$.

Definition 2.6. Let $(G; \circ_G, \leq_G)$ and $(H; \circ_H, \leq_H)$ be ordered hypergroupoids and $f : G \longrightarrow H$ be an isotone map, that is $f(x) \leq_H f(y)$ whenever $x \leq_G y$. Then,

(1) f is said to be an *order homomorphism* if f is a homomorphism of hypergroupoids $(G; , \circ_G)$ and $(H; \circ_H)$,

(2) f is an *order isomorphism* if f is an isomorphism of hypergroupiods, and f^{-1} is isotone,

(3) the kernel of f is defined by $kerf = \{(x, y) \in G \times G : f(x) = f(y)\}$.

3 Ordered polygroups

In this section, we assume that $G =< G, \cdot, ^{-1}, e >$ is a polygroup unless otherwise mentioned. Hereafter, in this paper, we use xy for $x \cdot y$, and a for $\{a\}$.

Definition 3.1. By an *ordered polygroup* we mean a polygroup which is also a poset under the binary relation $\leq$ and in which (2.1) holds.

Definition 3.2. Let H be an ordered hypergroupoid with a unique identity e. An element $x \in H$ is called *positive* if $e \leq x$. The set of all positive elements of H is called the *positive cone* of H and is denoted by H^+. $x \in H$ is called *negative* if $x \leq e$. The set of all negative elements of H is called the *negative cone* of H and is denoted by H^-.

By an elementary consequence of translations we have

Proposition 3.3. *In any ordered polygroup G, for each $x, y \in G$, we have*

$$\begin{aligned} x \leq y \;\; &\Leftrightarrow \;\; x^{-1}y \cap G^+ \neq \emptyset \;\Leftrightarrow\; yx^{-1} \cap G^+ \neq \emptyset \;\Leftrightarrow\; xy^{-1} \cap G^- \neq \emptyset \\ &\Leftrightarrow \;\; y^{-1}x \cap G^- \neq \emptyset \Leftrightarrow y^- \leq x^-. \end{aligned}$$

Theorem 3.4. *A subset P of a polygroup G is the positive cone with respect to some compatible order if and only if*

(1) $P \cap P^{-1} = \{e\}$,

(2) $P^2 = P$,

(3) *for all* $x \in G$, $xPx^{-1} = P$.

Moreover, if this order is total, $P \cup P^{-1} = G$.

Proof. ($\Rightarrow$) Let $\leq$ be a compatible order on G and $P = G^+$, the associated positive cone.

(1) If $x \in P \cap P^{-1}$, on the one hand $e \leq x$, and on the other hand $x = y^{-1}$, for some $y \in P$. Since, $e \leq y$, then $x = y^{-1} \leq e$ proves that $x = e$.

(2) Since $e \in P$, $P = Pe \subseteq PP = P^2$. Now, let $x, y \in P$. Then $e \leq x$ and $e \leq y$ and so $e \leq xy$ which implies that $xy \subseteq P$. Hence, $P^2 \subseteq P$.

(3) Let $y \in P$, and $x \in G$. Then, $e \leq y$ implies that $e \in xex^{-1} \leq xyx^{-1}$ proves that $xyx^{-1} \subseteq P$. Since, this follows for all $x \in G$, replacing x by x^{-1}, we have $x^{-1}Px \subseteq P$ and so $P \subseteq xPx^{-1}$, complete the proof.

($\Leftarrow$) Let P be a subset of G that satisfies properties (1)-(3), and define the relation $\leq$ on G by

$$x \leq y \;\Leftrightarrow\; yx^{-1} \cap P \neq \emptyset.$$

Since, $e \in P$, by (3), $xx^{-1} = xex^{-1} \subseteq xPx^{-1} = P$ implies that $x \leq x$ and so $\leq$ is reflexive. Suppose that $x \leq y$ and $y \leq x$, for $x, y \in G$. Then $yx^{-1} \cap P \neq \emptyset$ and $xy^{-1} \cap P \neq \emptyset$ whence $xy^{-1} \cap P^{-1} \cap P \neq \emptyset$, implies that $e \in xy^{-1}$, i.e., $x = y$ proving $\leq$ is antisymmetric. Now, assume that $x \leq y$ and $y \leq z$, for $x, y, z \in G$. Then $yx^{-1} \cap P \neq \emptyset$ and $zy^{-1} \cap P \neq \emptyset$. Let $u \in yx^{-1} \cap P$ and $v \in zy^{-1} \cap P$. Then $uv \subseteq P^2 = P$. On the other hand, $\in zy^{-1}$ and $v \in yx^{-1}$ imply $y^{-1} \in z^{-1}u$ and $y \in vx$ whence $e \in y^{-1}y \subseteq z^{-1}(uv)x$. Then, there is $t \in uv$ and $s \in tx$ such that $e \in z^{-1}s$. This implies that $z = s \in tx$. Hence, $t \in zx^{-1}$, i.e., $uv \cap zx^{-1} \neq \emptyset$ whence $zx^{-1} \cap P \neq \emptyset$ proving $\leq$ is transitive. Thus, $\leq$ is an order. For compatibility, we first prove that $Px = xP$, for all $x \in G$. Let $z \in G$. Then

$$\begin{aligned} z \in Px \;&\Rightarrow\; z \in yx \text{ for some } y \in P \Rightarrow\; x^{-1}z \subseteq x^{-1}yx = x^{-1}y(x^{-1})^{-1} \subseteq P \\ &\Rightarrow\; z \in xP, \end{aligned}$$

i.e., $Px \subseteq xP$. By a similar way, we can prove that $xP \subseteq Px$. Hence, $xP = Px$, for all $x \in G$. Now, assume that $x \leq y$ and $a, b \in G$. Since, $\leq$ is reflexive, by (3)

$$\begin{aligned} ayb(axb)^{-1} &= aybb^{-1}x^{-1}a^{-1} \subseteq ayPx^{-1}a^{-1} \subseteq aPyx^{-1}a^{-1} \subseteq aP^2a^{-1} \\ &= aPa^{-1} = P \end{aligned}$$

which shows that $axb \leq ayb$. By the definition of $\leq$ we get $x \in P$ if and only if $e \leq x$ and so $P = G^+$.

If G is totally ordered, then $x \leq e$ or $e \leq x$, for all $x \in G$. So, $e \in xx^{-1} \leq ex^{-1} = x^{-1}$ and so $x \in P$ or $x \in P^{-1}$, observe that $x = (x^{-1})^{-1}$. Thus, $G = P \cup P^{-1}$. $\square$

Proposition 3.5. *If G is an ordered polygroup with $|G| > 1$, then G can not have a top element or a bottom element.*

Proof. Let $G = \{e, a\}$. If $e < a$ or $a < e$, then $a = a^{-1} < e$ or $e < a^{-1} = a$, respectively, which is a contradiction. Now, assume that

$|G| > 2$, t be the top element of G and $e \neq a \in G$. Then $a \leq t$ and so $ta \leq t$ whence $t \in te \subseteq taa^{-1} \leq ta^{-1}$. Hence, $t \in ta^{-1}$. Likewise, we conclude that $t \in a^{-1}t$. By the uniqueness, we get $a = e$ which is a contradiction.

The proof of the other case is concluded as well. □

Definition 3.6. An element x of G is said to be of *order* n, $n \in \mathbb{N}$, if $e \in x^n$ where $x^n = (\cdots((\overbrace{x \circ x) \circ x) \circ \cdots) \circ x}^{n\ times})$. If such a natural number does not exist, we say that x is of infinite order.

Theorem 3.7. *Suppose that G is an ordered polygroup in which $G^+ \neq \{e\}$. Then every element of $G^+ \setminus \{e\}$ is of infinite order.*

Proof. Suppose that $x \in G^+ \setminus \{e\}$. We first observe that if $x = x^{-1}$, x can not belong to G^+. Then, $e < x$ implies that $e < x = ex < x^2$. Moreover, this implies that $e \notin x^2$. Similarly, we conclude that $e < x^3$ and $e \notin x^3$. Continuing this process we get $e < x^n$ and $e \notin x^n$, for all $n \in \mathbb{N}$, proving x is not of finite order. □

Corollary 3.8. *Any ordered polygroup in which every nontrivial element is of finite order is an antichain.*

Proof. Let G be an ordered polygroup satisfying the hypothesis. By Theorem 3.7, we know that $G^+ = \{e\}$. Now, if $a, b \in G$ be such that $a \leq b$, then $e \in a^{-1}a \leq a^{-1}b$ and so $e \leq u$, for some $u \in a^{-1}b$. This implies that $u \in G^+$ and so $u = e$. Thus, $e \in a^{-1}b$ whence $a = b$. This means that G is an antichain. □

Corollary 3.9. *Every finite ordered polygroup is an antichain.*

Example 3.10. Let $G = \{e, a\}$. Then G is a polygroup where the hyperoperation is given by the following table:

$\circ$	e	a
e	e	a
a	a	$\{e, a\}$

in which $a^{-1} = a$ i.e., a is an idempotent. Now, if a is a positive element, so $G^+ = \{e, a\}$ and hence $(G^+)^{-1} \cap G^+ \neq \{e\}$. This contradicts Theorem 3.4. This example shows that the converse of Theorem 3.7 does not hold in general.

Definition 3.11. If G is an ordered polygroup, by a *convex subgroup* of G we shall mean a subgroup which is also a convex subset, under the order of G.

Definition 3.12. A nonempty subset H of G is said to be *S-reflexive* if $xy \cap H \neq \emptyset$ implies that $xy \subseteq H$, for all $x, y \in G$.

Theorem 3.13. *If H is a subpolygroup of an ordered polygroup G then $H^+ = H \cap G^+$. Moreover, if H^+ is S-reflexive, the following statements are equivalent:*

(1) *H is convex;*

(2) *H^+ is a down-set of G^+.*

Proof. Since, $e_H = e_G$, it is clear that $H^+ = H \cap G^+$.

(1) $\Rightarrow$ (2) Suppose that $e_H \leq y \leq x$ where $e_H, x \in H^+ \subseteq H$. Then (1) gives $y \in H \cap G^+ = H^+$ and so H^+ is a down-set of G^+.

(2) $\Rightarrow$ (1) Suppose now that $x \leq y \leq z$ where $x, z \in H$. Then $x^{-1}x \leq x^{-1}y \leq x^{-1}z$. Thus, $x^{-1}z \subseteq H^+$ and so there is $a \in H^+$ such that $a \in x^{-1}z$. Hence, there is $b \in x^{-1}y$ such that $b \leq a \in H^+$, and since H^+ is a down-set of G^+, $b \in H^+$, i.e., $x^{-1}y \cap H^+ \neq \emptyset$. Since, H^+ is S-reflexive, so $x^{-1}y \subseteq H^+ \subseteq H$ whence $y \in xH = H$, proving H is convex. $\square$

If G is an ordered polygroup and H is a normal subpolygroup of G, then a natural candidate for a positive cone of G/H is $\natural_H(G^+)$, where $\natural_H : G \longrightarrow G/H$ is the canonical projection. Precisely when this occurs is the substance of the following result.

Theorem 3.14. *Let G be an ordered polygroup and let H be a normal subpolygroup of G. Then $\natural_H(G^+) = \{pH : p \in G^+\}$ is the positive cone of a compatible order on the quotient polygroup G/H if and only if H is convex.*

Proof. Suppose that $Q = \{pH : p \in G^+\}$ is the positive cone of a compatible order on G/H. To show that H is convex, suppose that $c \leq b \leq a$ with $c, a \in H$. Then $(bH)^{-1} = (bH)^{-1} \cdot aH = b^{-1}aH$. On the other hand, $b \leq a$ implies that $b^{-1}a \cap G^+ \neq \emptyset$. Hence $(bH)^{-1} \cap Q \neq \emptyset$ and so $bH \cap Q^{-1} \neq \emptyset$. Similarly, we have $bH = bH \cdot c^{-1}H = bc^{-1}H$ and since $bc^{-1} \cap G^+ \neq \emptyset$, $bH \cap Q \neq \emptyset$. Thus, $bH \cap (Q \cap Q^{-1}) \neq \emptyset$ whence $bH = H$, i.e., $b \in H$.

Conversely, suppose that H is convex and let $Q = \{pH : p \in G^+\}$. It is clear that $Q^2 = Q$. Suppose now that $xH \in Q \cap Q^{-1}$. Then $xH = pH = q^{-1}H$ where $p, q \in G^+$. These equalities also give $pq \cap H \neq \emptyset$. Now, since $p \leq pq$, then $e_H \leq p \leq u$, where $u \in pq \cap H$ whence the convexity of H gives $p \in H$. It follows that $xH = pH = H$ and hence $Q \cap Q^{-1} = \{H\}$. Finally, since G^+ is a normal subsemihypergroup of G it is clear that $Q = \natural_H(G^+)$ is a normal subsemihypergroup of G/H. It now follows by Theorem 3.4 that Q is the positive cone of a compatible order on G/H. $\square$

If H is a convex normal subpolygroup of an ordered polygroup G then the order $\leq_H$ on G/H that corresponds to the positive cone $\{pH : p \in G^+\}$ can be described as in the proof of Theorem 3.4. We have

$$\begin{aligned} xH \leq_H yH &\Rightarrow yx^{-1}H \subseteq Q \\ &\Rightarrow (\forall a \in yx^{-1})(\exists p \in G^+) aH = pH \\ &\Rightarrow (\forall a \in yx^{-1})(\exists p \in G^+)(\exists h \in H) a \in ph \geq h \\ &\Rightarrow (\forall a \in yx^{-1})(\exists h \in H)\ a \geq h \\ &\Rightarrow yx^{-1} \geq h. \end{aligned}$$

From the last inequality and that $y \in ye \subseteq yx^{-1}x$ it follows that $y \geq u$, for some $u \in hx$. Conversely, assume that there exists $h \in H$ and $u \in hx$ such that $y \geq u$, and let $a \in yx^{-1}$. From $yx^{-1} \geq yx^{-1}$ it follows that $a \geq t$, for some $t \in ux^{-1}$ and hence $at^{-1} \geq tt^{-1}$. This implies that $v \geq e$, for some $v \in at^{-1}$ and so

$$vH \in at^{-1}H \cap Q. \tag{3.1}$$

Now, $t \in ux^{-1}$ implies that $t^{-1} \in xu^{-1} \subseteq xx^{-1}h^{-1} \subseteq xx^{-1}H$ and so $at^{-1} \subseteq axx^{-1}H = axHx^{-1} = aH$. Thus, $at^{-1}H \subseteq aH$. Combining (3.1), we get $\{aH\} \cap Q \neq \emptyset$, i.e., $aH \in Q$ and so $aH = pH$, for some $p \in G^+$. This implies $yx^{-1}H \subseteq Q$ and hence $xH \leq_H yH$, completes the proof.

Thus we see that $\leq_H$ can be described by

$$xH \leq_H yH \Leftrightarrow (\exists h \in H)(\exists u \in hx)\ y \geq u.$$

In referring to the ordered quotient polygroup G/H we shall implicitly infer that the order is $\leq_H$ as described above.

Here we give a characterization of polygroup homomorphisms that are isotone.

Theorem 3.15. *Let G and H be ordered polygroups. If $f : G \longrightarrow H$ is a polygroup homomorphism, f is isotone if and only if $f(G^+) \subseteq H^+$.*

Proof. Assume that f is isotone. If $x \in G^+$, i.e., $x \geq e$ then $f(x) \geq f(e_G) = e_H$ means that $f(x) \in H^+$.

Conversely, assume that $x \leq y$ in G. Then $yx^{-1} \subseteq G^+$ and so $f(y)f(x)^{-1} = f(yx^{-1}) \subseteq f(G^+) \subseteq H^+$. This implies that $f(y) \geq f(x)$ proving f is isotone. $\square$

Corollary 3.16. *If G is an ordered polygroup and H is a convex normal subpolygroup of G, then the natural homomorphism $\natural_H : G \longrightarrow G/H$ is isotone.*

Proof. By Theorem 3.15, it is enough to prove that $\natural(G^+) \subseteq (G/H)^+$. For this, let $yH \in \natural(G^+)$. Then $yH = gH$, for some $g \in G^+$ whence $y \in gh \geq h$ for some $h \in H$. This implies that $eH \leq_H yH$ and so $yH \in (G/H)^+$. □

Definition 3.17. Let G and H are ordered polygroups. A mapping $f : G \longrightarrow H$ is said to be *exact* if $f(G^+) = H^+$.

Definition 3.18. Two ordered polygroups G and H are said to be *isomorphic* if there is a polygroup isomorphism $f : G \longrightarrow H$ that is also an order isomorphism.

If two ordered polygroups G and H are isomorphic we write $G \doteq H$.

Theorem 3.19. *For ordered polygroups G and H, the following are equivalent:*

(1) $G \doteq H$,

(2) *there is an exact polygroup isomorphism* $f : G \longrightarrow H$.

Proof. (1) $\Rightarrow$ (2) If G and H are isomorphic, there is a polygroup isomorphism $f : G \longrightarrow H$ which is also an order isomorphism. By Theorem 3.15, $f(G^+) \subseteq H^+$. Let $g = f^{-1}$. Obviously, g satisfies the conditions of Theorem 3.15. Hence, $g(H^+) \subseteq G^+$ whence $H^+ = f(g(H^+)) \subseteq f(G^+)$. Thus $H^+ = f(G^+)$ and so (2) holds.

(2) $\Rightarrow$ (1) It is obvious. □

Theorem 3.20. *Let G and H be ordered polygroups and* $f : G \longrightarrow H$ *be an exact polygroup homomorphism. Then* $Imf \doteq G/kerf$.

Proof. We first observe that $kerf$ is a convex normal subpolygroup of G and so $G/kerf$ is an ordered polygroup. By first isomorphism theorem of polygroups there is an isomorphism $\phi : G/kerf \simeq Imf$ which $\phi(xK) = f(x)$ where $K = kerf$. It remains that we prove ϕ is exact. Let $xK \in (G/K)^+$. Then $e_G K \leq_K xK$ whence $k \leq x$, for some $k \in K$, and so $e_H = f(k) \leq f(x)$ whence $\phi(xK) = f(x) \in (Imf)^+$. Conversely, if $f(x) \in (Imf)^+ \subseteq H^+$, since f is exact, there exists $g \in G^+$ such that $f(x) = f(g)$. Consequently, $xK = gK$ and so $x \in gk \geq k$, for some $k \in K$. Thus,

$$xK \in (G/K)^+ \Leftrightarrow \phi(xK) = f(x) \in (Imf)^+$$

proving ϕ is exact. It now follows by Theorem 3.19 that $G/kerf \doteq Imf$. □

References

[1] T. S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer-Verlag, London, 2005.

[2] P. Corsini, *Prolegomena of hypergroup theory*, 2nd edition, Aviani editor, 1993.

[3] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Kluwer Academic Publishers, Dordrecht, 2003.

[4] S. Ioulidis, *Polygroups et certain de leurs properties*, Bull. Greek Math. Soc. 22 (1981), 95-104.

[5] F. Marty, *Sur une generalization de la notion de group*, 8th congress Math. Scandenaves, Stockholm, (1934), 45-49.

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Visualization of Algebraic Properties of special H_V-structures

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Abstract

The paper looks at visualization as it relates to special H_V-structures, focusing upon how it can be used to improve the perception and understanding of abstract algebraic concepts. Using position vectors into the plane IR^2, abstract algebraic properties of H_V-structures are gradually transformed into geometrical shapes.

Key words: Hyperstructures, H_V-structures, Visualization.

MSC2010: 20N20, 16Y99.

1 Introduction

In most branches of mathematical research, visualization has been an area of interest for mathematicians [1], [6], [9], specifying that visual thinking can be an alternative and powerful resource, as well as a serious tool, not only for specialists but also, for students doing mathematics. Mathematicians have always used their "mind's eye" to visualize the abstract objects and processes that arise in mathematical research. But it is only in recent years that remarkable improvements in computer technology have made it easy to externalize these vague and subjective pictures that we see in our heads, replacing them with precise and objective visualizations that can be shared with others [7]. The subject is of such recent research that searching the literature, in preparation for this paper, it was surprising to discover that no papers were specifically focused on visualization in hyperstructures.

According to [10], the term *visualization* has been used in various ways in the research literature, so it is necessary to clarify how it is used in this paper. *Thus visualization is taken to include processes of constructing and transforming both mental imagery and abstract algebraic concepts.*

This paper, looks at visualization as it relates to special H_V-structures, focusing upon how it can be used to improve the perception and understanding of abstract algebraic concepts, since, being able to "see" something in a geometrical shape, is a common metaphor for understanding it. According to Bruner [2], to understand a specific concept (algebraic), the first approach has to be intuitive. So, geometry or linear algebra into a two-dimensional real vector space, with constant references to the fundamental intuitively understood principles, are teaching and educative tools.

Using position vectors into the plane IR^2, abstract algebraic properties of H_V-structures are gradually transformed into geometrical shapes, which operate, not only as a translation of the algebraic concept but also, as a teaching process.

2 Basic definitions on hyperstructures

In 1934, F. Marty introduced the definitions of the hyperoperation and of the hypergroup as a generalization of the operation and the group respectively.

Definition 2.1 *In a set $H \neq \varnothing$, a hyperoperation is a map, such that:*

$$\circ : H \times H \to P(H) - \{\varnothing\} : (x, y) \mapsto x \circ y \subset H$$

Also, if $A, B \subset H$,then

$$A \circ B = \cup_{a \in A, b \in B}(a \circ b).$$

Properties of hyperoperations [3], [4], [12]:

i) A hyperoperation ($\circ$) in a set H is called associative, if

$$(x \circ y) \circ z = x \circ (y \circ z), \forall x, y, z \in H$$

ii) A hyperoperation ($\circ$) in a set H is called commutative, if

$$x \circ y = y \circ x, \forall x, y \in H$$

iii) A hyperoperation ($\circ$), in a set H, is having an identity or unit element if there exists $e \in H$, such that

$$x \in x \circ e \text{ and } x \in e \circ x, \forall x \in H$$

iv) A hyperoperation ($\circ$), in a set H, with a unit element e, is having an inverse element, if for every $x \in H$, there exists an element $x' \in H$, such that

$$e \in x \circ x' ande \in x' \circ x, \forall x \in H$$

v) In a set H, equipped with two hyperoperations ($\circ$) and ($*$), the ($*$) is called distributive with respect to ($\circ$), if

$$x * (y \circ z) = (x * y) \circ (x * z), \forall x, y, z \in H$$

An algebraic hyperstructure $(H, \circ)$, i.e. a set H equipped with a hyperoperation ($\circ$), is called hypergroupoid. If this hyperoperation is associative, then the hyperstructure is called semihypergroup. The semihypergroup $(H, \circ)$, is called hypergroup if it satisfies the reproduction axiom:

$$x \circ H = H \circ x, \forall x \in H.$$

One more complicated hyperstructure, is that $(H, \circ, *)$, which is called hyperring, where $(H, \circ)$ is a commutative hypergroup, the ($*$) is associative and distributive with respect to ($\circ$).

One of the topics of great interest, in the last years, is the H_v-stuctures, which was introduced by T. Vougiouklis in 1990 [11]. The class of H_v-stuctures is the largest class of algebraic hyperstructures. These structures satisfy weak axioms, where the non-empty intersection replaces the equality, as bellow [12]:
Let H be a set and $\circ : H \times H \to P(H) - \{\varnothing\}$ be a hyperoperation.

i) The ($\circ$)in H is called weak associative, we write WASS, if

$$(x \circ y) \circ z \cap x \circ (y \circ z) \neq \varnothing, \forall x, y, z \in H$$

ii) The ($\circ$) is called weak commutative, we write COW, if

$$(x \circ y) \cap (y \circ x) \neq \varnothing, \forall x, y \in H$$

iii) If H is equipped with two hyperoperations ($\circ$) and ($*$), then ($*$)is called weak distributive with respect to ($\circ$), if

$$[x * (y \circ z)] \cap [(x * y) \circ (x * z)] \neq \varnothing, \forall x, y, z \in H$$

The hyperstructure $(H, \circ)$ is called H_v-semigroup if it is WASS and it is called H_v-group if it is reproductive H_v-semigroup. It is called commutative H_v-group if $(\circ)$ is commutative and it is called H_v-commutative group if$(\circ)$ is weak commutative. The hyperstructure $(H, \circ, *)$ is called H_v-ring if both hyperstructures $(\circ)$ and $(*)$ are WASS, the reproduction axiom is valid for $(\circ)$ and $(*)$ is weak distributive with respect to $(\circ)$.
What it follows to the end of the paragraph comes from [5]:

Definition 2.2 *An H_v-ring $(R, +, \bullet)$ is called dual H_v-ring, if $(R, \bullet, +)$ is an H_v-ring, too.*

Definition 2.3 *Let V be a vector space over a field K. Then, define two hyperoperations in V as follows: For all $x, y \in V$ and $r \in K$,*

$$x \circ y = \{z/z = x + r(y - x), r \in [0, 1]\}$$

$$x \bullet y = \{z/z = x + ry, r \in [0, 1]\}$$

Remark 2.1 *Into the plane $IR^2 : x \circ y = [x, y]$, it is known as join operation [8] and $x \bullet y = [x, x+y]$. The $[\alpha, \beta]$ denotes the line segment which is bounded by the two end points α and β.*

Then, for the four hyperstructures occur, we get the following:

Proposition 2.1 *The hyperstructure $(V, *, \Box)$, where $*, \Box \in \{\circ, \bullet\}$, is a weak commutative dual H_v-ring.*

Let:

E_*	be the set of the unit elements with respect to $(*)$.
E_*^r	be the set of the right unit elements with respect to $(*)$.
E_*^l	be the set of the left unit elements with respect to $(*)$.
$I_*(x, e)$	be the set of the inverse elements of x associated with the unit e (left or right), with respect to $(*)$.
$I_*^r(x, e)$	be the set of the right inverse elements of x associated with the right unit e, with respect to $(*)$.
$I_*^l(x, e)$	be the set of the left inverse elements of x associated with the left unit e, with respect to $(*)$.

Proposition 2.2 *i)* $E_\circ = V$, *ii)*$I_\circ(x, e) = \{z/z = (1 - r)x + re, r \geq 1\}$

Proposition 2.3 *i)*$E_\bullet^r = V$, *ii)*$I_\bullet^r(x, e) = \{z/z = r(e - x), r \geq 1, e \in E_\bullet^r\}$, *iii)*$E_\bullet^l = \{O\} \subset E_\bullet$, *iv)*$I_\bullet^l(x, e) = [e, e - x], e \in E_\bullet^r$.

3 Visualization in H_v-groups

Now, let us introduce a coordinate system into the IR^2. We place a given vector p so that its initial point P determines an ordered pair (a_1, a_2). Conversely, a point P with coordinates (a_1, a_2) determines the vector $p = OP$, where O the origin of the coordinate system. We shall refer to the elements x, y, z,... of the set IR^2 , as vectors whose initial point is the origin. These vectors are very well known as position vectors.

i) **The hyperoperation**: $x \bullet y = \{z/z = x + ry, r \in [0, 1]\} = [x, x + y]$

In Figure 3.1, to every point x and y of the plane, i.e. to every ordered pair (x, y) we map an infinite number of points (hyperstructure) instead of one point (operation). The infinite number of points is the line segment $[x, x+y]$ which is bounded by the two end points x and $x+y$. Graphically, having the points O, x, y, draw the parallelogram with vertices $O, x, y, x + y$. Then, the side $[x, x + y]$ is the hyperoperation $x \bullet y$.

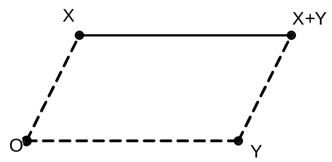


Fig.3.1

ii) **Reproduction** : $x \bullet IR^2 = \cup_{r \in IR^2}(x \bullet r) = IR^2$

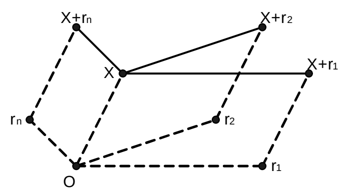


Fig.3.2

In Figure 3.2, take any point x of the plane. For any of the infinite points r_i of the plane, draw the parallelogram with vertices $O, x, r_i, x+r_i$. Unite all these infinite line segments $[x, x + r_i]$, then all these segments cover the plane.

iii) **Weak Associativity**: $x \bullet (y \bullet z) \cap (x \bullet y) \bullet z \neq \varnothing$

In Figure 3.3a, take three points x, y, z of the plane, then the side $[y, y+z]$ of the parallelogram with vertices $O, y, z, y+z$ is the hyperoperation $y \bullet z$. With the points O, x and every point r_i of the line segment $[y, y+z]$ draw, each time, the parallelogram with vertices $O, x, r_i, x+r_i$. All these infinite line segments $[x, x+r_i]$, create the triangle with vertices $x, x+y, x+y+z$. Then the area of this triangle is the first part of the above intercection, i.e. $x \bullet (y \bullet z)$. Similarly, in Figure 3.3b, the side $[x, x+y]$ of the parallelogram with vertices $O, x, y, x+y$ is the hyperoperation $x \bullet y$. With the points O, z and every point r_i of the line segment $[x, x+y]$ draw, each time, the parallelogram with vertices O, z, r_i, r_i+z. All these infinite line segments $[r_i, r_i+z]$ create the parallelogram with vertices $x, x+y, x+y+z, x+z$. Then the area of this parallelogram is the second part of the above intercection, i.e. $(x \bullet y) \bullet z$. Notice that the triangle with vertices $x, x+y, x+y+z$ is part of the parallelogram with vertices $x, x+y, x+y+z, x+z$, i.e. the intersection of these two figures is not equal to the empty set.

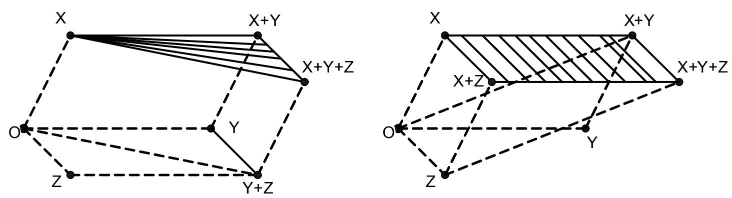


Fig.3.3a   Fig.3.3b

iv) **Weak Commutativity**: $(x \bullet y) \cap (y \bullet x) \neq \varnothing$ In Figure 3.4, take two

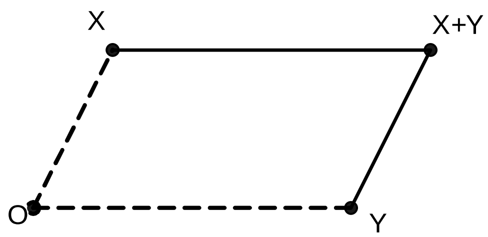


Fig.3.4

points x and y of the plane. Then draw the parallelogram with vertices

$O, x, x + y, y$. The side $[x, x + y]$ is the hyperoperation $x \bullet y$ and the side $[y, x+y]$ is the hyperoperation $y \bullet x$. Notice that the only common point of these two sides is the point $x + y$, i.e. the intersection of $x \bullet y$ and $y \bullet x$ is not equal to the empty set.

v) **The set of the right unit elements**: $(x \in x \bullet e, \forall x \in IR^2)$
$E^r_\bullet = IR^2$

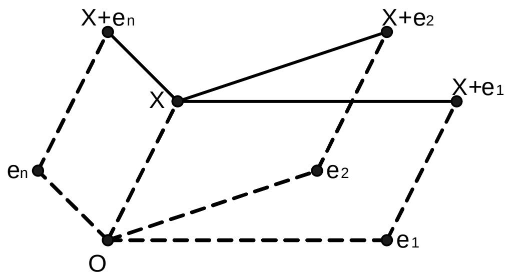


Fig.3.5

In Figure 3.5, take any point x of the plane. Then draw the parallelograms with vertices $O, x, x + e_i, e_i$, where e_i any point of the plane. The side $[x, x + e_i]$ is the hyperoperation $x \bullet e_i$. Notice that x belongs to every line segment $[x, x + e_i]$, i.e. x belongs to every $x \bullet e_i$. Since all these e_i's, having the above property, are infinite, we get that the set $E^r_\bullet$ of the right unit elements with respect to $(\bullet)$ is equal to IR^2.

vi) **The set of the right inverse elements**:

$$I^r_\bullet(x, e) = \{z/z = r(e - x), r \geq 1\}, e \in E^r_\bullet$$

Having any point x of the plane, we want to find at least one point x' of the plane, such that, for a right unit point e of the plane (i.e. any point of the plane) the following to be valid: $e \in x \bullet x'$, i.e. we want e to be point of the line segment $[x, x + x']$. In Figure 3.6a, notice that all the infinite points x' of the half-line $[e - x, +\infty)$ have the above property.

Indeed, in Figure 3.6b, for a given x and e, take any x' belonging to the half-line $[e - x, +\infty)$. Draw the parallelogram with vertices $O, x, x + x', x'$. Then e belongs to the line segment $[x, x + x']$, i.e., e belongs to the hyperoperation $x \bullet x'$.

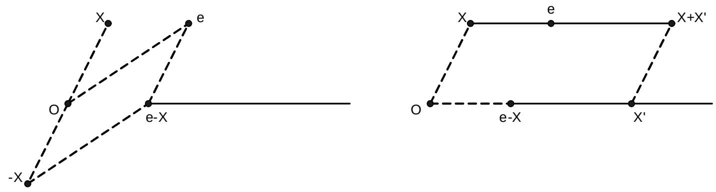
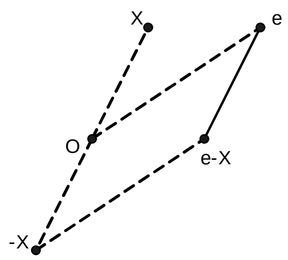


Fig.3.6



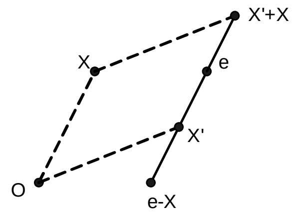


Fig.3.7a Fig.3.7b

vii) **The set of the left inverse elements**: $I^l_\bullet(x,e) = [e, e-x], e \in E^r_\bullet$

Take any point x of the plane, we want to find at least one point x' of the plane, such that, for a right unit point e of the plane (i.e. any point of the plane) the following to be valid: $e \in x' \bullet x$, i.e. we want e to be point of the line segment $[x', x'+x]$. In Figure 3.7b, notice that the points x' of the line segment $[e, e-x]$ have the above property. Indeed, in Figure 3.7b, for a given x and e, take any x' belonging to the line segment $[e, e-x]$. Draw the parallelogram with vertices $O, x, x'+x, x'$. Then e belongs to the line segment $[x', x'+x]$, i.e., e belongs to the hyperoperation $x' \bullet x$. Since, $E^l_\bullet = \{O\} \subset E_\bullet$ (that means that the origin O of the coordinate system is simultaneously left and right unit element), set $O \equiv e$, then $I^e_\bullet(x, O) = [O, O-x]$.

Remark 3.1 *Into the plane IR^2, the hyperoperation (i), together with the axioms (ii) and (iii) are giving the concept of H_v-group. Furthermore, by putting together the axiom (iv) we get the concept of H_v-commutative group.*

4 Visualization in H_v-rings

i) **Distributivity of** $(\circ)$ **with respect to** $(\circ)$:

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$$

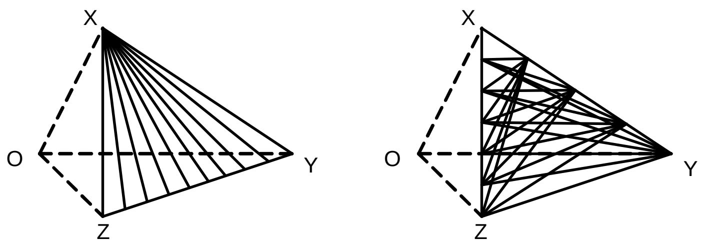


Fig.4.1a Fig.4.1b

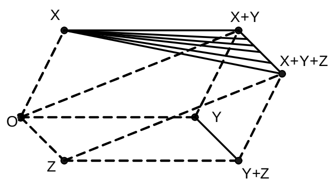
In Figure 4.1a, take three points x, y, z of the plane, then the line segment $[y, z]$ is the hyperoperation $y \circ z$. Join the point x to each point of the segment $[y, z]$. Then the area of the triangle with vertices x, y, z is the first part of the above equality, i.e. $x \circ (y \circ z)$. Similarly, in Figure 4.1b, the line segment $[x, y]$ is the hyperoperation $x \circ y$ and the line segment $[x, z]$ is the hyperoperation $x \circ z$. Join every point of the segment $[x, y]$ to every point of the segment $[x, z]$. Then the area of the triangle with vertices x, y, z is the second part of the above equality, i.e. $(x \circ y) \circ (x \circ z)$.

ii) **Weak Distributivity of** $(\bullet)$ **with respect to** $(\bullet)$:

$$x \bullet (y \bullet z) \cap (x \bullet y) \bullet (x \bullet z) \neq \varnothing$$

In Figure 4.2a, take three points x, y, z of the plane, then the side $[y, y+z]$ of the parallelogram with vertices $O, y, z, y+z$ is the hyperoperation $y \bullet z$. With the points O, x and every point r_i of the line segment $[y, y+z]$ draw, each time, the parallelogram with vertices $O, x, r_i, x+r_i$. All these infinite line segments $[x, x+r_i]$ create the triangle with vertices $x, x+y, x+y+z$. Then the area of this triangle is the first part of the above inersection, i.e. $x \bullet (y \bullet z)$.
In Figure 4.2b, the side $[x, x+y]$ of the parallelogram with vertices $O, x, y, x+y$ is the hyperoperation $x \bullet y$ and the side $[x, x+z]$ of the parallelogram with vertices $O, x, z, x+z$ is the hyperoperation $x \bullet z$.

With the points: O, every point r_i of the side $[x, x+y]$ and every point t_i of the side $[x, x+z]$ draw, each time, the parallelogram with vertices O, r_i, t_i, r_i+t_i. All these infinite line segments $[r_i, r_i+t_i]$ create the pentagon with vertices $x, 2x, 2x+y, 2x+y+z, x+y$. Then the area of this pentagon is the second part of the above intersection, i.e. $(x \bullet y) \bullet (x \bullet z)$. Notice that the line segment $[x, x+y]$ is the common part of the triangle area with vertices $x, x+y, x+y+z$ and the pentagon area with vertices $x, 2x, 2x+y, 2x+y+z, x+y$, i.e. the intersection of these two figures is not equal to the empty set.



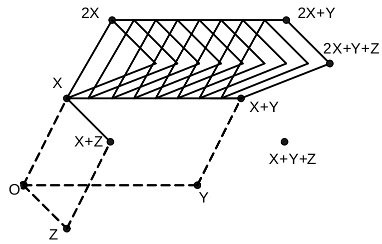
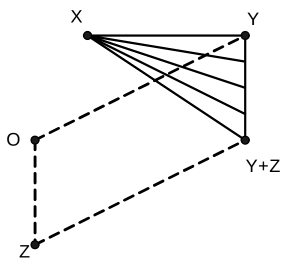


Fig.4.2a Fig.4.2b

iii) **Weak Distributivity of** $(\circ)$ **with respect to** $(\bullet)$:

$$x \circ (y \bullet z) \cap (x \circ y) \bullet (x \circ z) \neq \varnothing$$



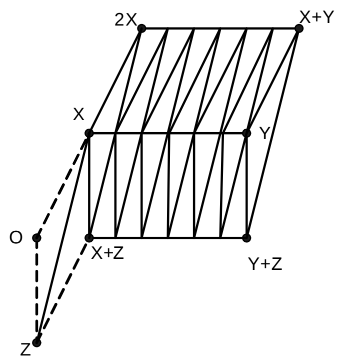


Fig.4.3a Fig.4.3b

In Figure 4.3a, take three points x, y, z of the plane, then the side $[y, y+z]$ of the parallelogram with vertices $O, y, z, y+z$ is the hyperoperation $y \bullet z$. Join the point x to each point of the segment $[y, y+z]$. Then the area of the triangle with vertices $x, y, y+z$ is the first part of the above inersection, i.e. $x \circ (y \bullet z)$. In Figure 4.3b, take three points x, y, z of the plane, then the line segment $[x, y]$ is the hyperoperation $x \circ y$ and the line segment $[x, z]$ is the hyperoperation $x \circ z$. With the points: O, every point r_i of the line segment $[x, y]$ and every point t_i of the line segment $[x, z]$ draw, each time, the parallelogram with vertices $O, r_i, t_i, r_i + t_i$. All these infinite line segments $[r_i, r_i + t_i]$ create the pentagon with vertices $x, 2x, x+y, y+z, x+z$. Then the area of this pentagon is the second part of the above intersection, i.e. $(x \circ y) \bullet (x \circ z)$. Notice that the triangle with vertices $x, y, y+z$ is part of the pentagon with vertices $x, 2x, x+y, y+z, x+z$, i.e. the intersection of these two figures is not equal to the empty set.

iv) **Distributivity of** ($\bullet$) **with respect to** ($\circ$):

$$x \bullet (y \circ z) = (x \bullet y) \circ (x \bullet z)$$

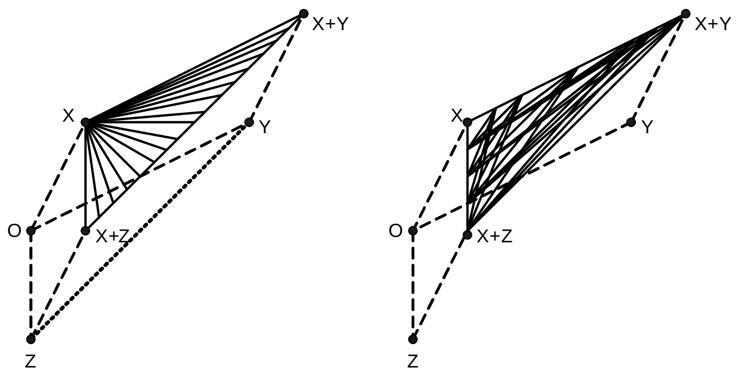


Fig.4.4a Fig.4.4b

In Figure 4.4a, take three points x, y, z of the plane, then the line segment $[y, z]$ is the hyperoperation $y \circ z$. With the points O, x and every point r_i of the line segment $[y, z]$ draw, each time, the parallelogram with vertices $O, x, r_i, x + r_i$. All these infinite line segments $[x, x + r_i]$ create the triangle with vertices $x, x+y, x+z$. Then the area of this triangle is the first part of the above equality, i.e. $x \bullet (y \circ z)$.
In Figure 4.4b, the side $[x, x+y]$ of the parallelogram with vertices $O, x, y, x+y$ is the hyperoperation $x \bullet y$ and the side $[x, x+z]$ of the parallelogram with vertices $O, x, z, x+z$ is the hyperoperation $x \bullet z$. Join every point of the side $[x, x+y]$ to every point of the side $[x, x+z]$.

Then the area of the triangle with vertices $x, x+y, x+z$ is the second part of the above equality, i.e. $(x \bullet y) \circ (x \bullet z)$.

Remark 4.1 *It is known that* $(IR^2, \circ)$ *is a commutative hypergroup. Into the plane* IR^2*, the hyperoperations* $(\circ)$ *and* $(\bullet)$ *together with the axioms 3ii), 3iii), 4i), 4ii), 4iii) and 4iv) are giving the concepts of hyperring,* H_v*-ring and dual* H_v*-ring.*

References

[1] A.J. Bishop, *A review of research on visualization in mathematics education*, In: A. Borbás (Ed.), Proceedings of the 12^{th} PME International Conference, (1988), 170-176.

[2] J.S Bruner,*The course of cognitive growth*, American psychologist, 19,(1964), 1-15.

[3] P.Corsini, V.Leoreanu, *Applications of Hypergroup Theory*,Kluwer Academic Publishers, 2003

[4] B.Davvaz, V.Leoreanu, *Hyperring Theory and Applications*,, International Academic Press, 2007

[5] A. Dramalidis, *Dual H_v-rings*, Rivista di Matematica Pura ed Applicata, 17,(1996), 55-62.

[6] A. Gutirrez, *Visualization in 3-dimensional geometry: In search of a framework*, In: Proceedings of the 20^{th} PME International Conference, L. Puig & A. Gutierrez (Eds.), 3-19,(1996), 3-19.

[7] R.Palais, *The visualization of Mathematics: Towards a Mathematical Exploratorium*, Notices of the AMS, 6,(1999), 647-658

[8] Prenowitz W., Jantosciak J., *Join Geometries. A Theory of Convex Sets and Linear Geometry*, Springer, N.Y,1979

[9] N.C Presmeg, *Visualization and mathematical giftedness*, Educational Studies in Mathematics, 17, (1986), 297-311.

[10] N.C Presmeg, *Research on visualization in learning an teaching mathematics*,In: Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future. PME 1976-2006, Ed Sense Publishers, (2006),205-235.

[11] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, In: 4^{th} AHA Congress, World Scientific,(1991), 203-211.

[12] T.Vougiouklis, *Hyperstructures and their Representations*, Monographs in Mathematics, Hadronic Press,1994

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Some algebraic properties of fuzzy S-acts

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Abstract

S-acts, a useful and important algebraic tool, have always been interest to mathematicians, specially to computer scientists. When A. Zadeh introduced the notion of the fuzzy subset in 1965, his idea opened a new direction to reserchers to provide tools in the various fields of mathematics. Here we are going to investigate some algebraic properties of fuzzy S-acts. We first make an S-act from the fuzzy subsets of an S-act A. Then we use this tool to give a characterization for fuzzy S-acts. We then introduce the notion of generated fuzzy S-act by a fuzzy subset of an S-act and give a characterization for the fuzzy actions. And then we define the notion of indecomposable fuzzy S-act and find some indecomposable fuzzy actions.

Key words: Fuzzy set, Fuzzy acts over fuzzy semigroups.

MSC2010: 08A72, 20M30.

1 Introduction and Preliminaries

No need to mention the importance of the prominent and well established Fuzzy Set Theory, introduced by Zadeh in 1965 [7], which offered tools and a new approach to model imprecision and uncertainty. Since then, very many researchers have worked on this concept and its applications to logic, set theory, algebra, analysis, topology, computer science, control engineering, information science, etc [1, 2, 3]. Actions of a semigroup (monoid or group) S on a set A have always been interest to mathematicians, specially to computer scientists and logicians. The algebraic structures so obtained are called S-sets, S-acts, and by some other terminologies [4, 6].

In [5] we have used the fuzzy concept and introduced the notion of the actions of a (fuzzy) semigroup on a fuzzy set (fuzzy S-act) and studied the relation between this structure and sheaves. Here we are going to study some of algebraic details of this structure. But first we recall that:

A set X together with a function $\mu : X \to [0,1]$ is called a *fuzzy set* (over X) and is denoted by (X,μ) or $X^{(\mu)}$. We call X the *underlying set* and μ the *membership function* of the fuzzy set $X^{(\mu)}$, and $\mu(x) \in [0,1]$ is the *grade of membership* of x in $X^{(\mu)}$.

If μ is a constant function with value $a \in [0,1]$, $X^{(\mu)}$ is denoted by $X^{(a)}$. The fuzzy set $X^{(1)}$ is called a *crisp set* and may sometimes simply be denoted by X.

For a fuzzy set $X^{(\mu)}$ and $\alpha \in [0,1]$, $X^{(\mu)}_\alpha := \{x \in X \mid \mu(x) \geq \alpha\}$ is called the α-*cut* or the α-*level set* of the fuzzy set $X^{(\mu)}$.

A *fuzzy function* from $X^{(\mu)}$ to $Y^{(\eta)}$, written as $f : X^{(\mu)} \to Y^{(\eta)}$, is an ordinary function $f : X \to Y$ such that the following is a **fuzzy triangle**:

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "f"'] & {[0,1]} \\ Y \arrow[ur, "\eta"'] & \end{tikzcd}$$

meaning that $\mu \leq \eta f$ (that is, $\mu(x) \leq \eta f(x)$ for all $x \in X$). The set of all fuzzy sets with a fixed underlying set X is called the *fuzzy power* or the set of *fuzzy subsets* of X and is denoted by $\mathbf{FSub}X$. Clearly fuzzy sets together with fuzzy functions between them form a category denoted by $\mathbf{FSet}$.

To define the actions of a (fuzzy) semigroup on a fuzzy set first we note that:

Definition 1.1 *A semigroup S together with a function $\nu : S \to [0,1]$ is called a fuzzy semigroup if its multiplication is a fuzzy function: for every $s, r \in S$, $\nu(s) \wedge \nu(r) \leq \nu(sr)$; that is, the following is a fuzzy triangle:*

$$\begin{tikzcd} S \times S \arrow[r, "\nu\wedge\nu"] \arrow[d, "\lambda_S"'] & {[0,1]} \\ S \arrow[ur, "\nu"'] & \end{tikzcd}$$

If S has an identity 1, one usually add the condition $\nu(1) = 1$.

Now, recall from [5] that, for a (crisp) semigroup S, a (crisp) set A can be made into an (ordinary) S-act in the following two equivalent ways:

Universal algebraic way: The set A together with a family $(\lambda_s : A \to A)_{s\in S}$ of unary operations satisfying $(st)a = s(ta)$ (and $1a = a$, if S has an identity) where $sa = \lambda_s(a)$.

Common way: The set A together with a function $\lambda : S \times A \to A$ satisfying $(st)a = s(ta)$ (and $1a = a$, if S has an identity) where $sa = \lambda(s, a)$.

Now, having these two, so called, universal algebraic and common actions of S on A, we get the following two, not necessarily equivalent, definitions for a fuzzy act over a fuzzy monoid.

Definition 1.2 *Let $S^{(\nu)}$ be a fuzzy semigroup and $A^{(\mu)}$ be a fuzzy set such that A is an S-act, as defined above. Then, $A^{(\mu)}$ is called:*

(Universal algebraic) A fuzzy S-act (or fuzzy $S^{(1)}$-act, to emphasize fuzziness) if each λ_s is a fuzzy function; that is $\mu(a) \leq \mu(sa)$, for every $s \in S$ and $a \in A$ (with no mention of ν). That is, for every $s \in S$, the following triangle is fuzzy:

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d, "\lambda_s"'] & {[0,1]} \\ A \arrow[ur, "\mu"'] & \end{tikzcd}$$

(Common) A fuzzy $S^{(\nu)}$-act if $\lambda : S \times A \to A$ is a fuzzy function; that is, $\nu(s) \wedge \mu(a) \leq \mu(sa)$, for every $s \in S$ and $a \in A$. That is, the following triangle is fuzzy:

$$\begin{tikzcd} S \times A \arrow[r, "\nu\wedge\mu"] \arrow[d, "\lambda"'] & {[0,1]} \\ S \arrow[ur, "\mu"'] & \end{tikzcd}$$

Corolary 1.1 *(1) Note that universal algebraic definition implies common definition, and if $S^{(1)} = S$ is a (crisp) semigroup, then $\nu(s) \wedge \mu(a) = 1 \wedge \mu(a) = \mu(a)$, and so the above two definitions are equivalent.*

(2) Every fuzzy semigroup $S^{(\nu)}$ is naturally a fuzzy $S^{(\nu)}$-act and $S^{(1)} = S$ is a fuzzy $S^{(1)} = S$-act (universal algebraicly, and hence commonly). Also, if $S^{(\nu)}$ is a fuzzy left ideal, then it is a fuzzy S-act (universal algebraicly, and hence commonly).

A morphism between fuzzy $S^{(\nu)}$-acts (with both definitions), also called an $S^{(\nu)}$-map is simply an S-map as well as a fuzzy function. The set of all (fuzzy) $S^{(\nu)}$-acts with a fixed A is denoted by $S^{(\nu)}$-**FSub**A, and the category of all fuzzy $S^{(\nu)}$-acts is denoted by $S^{(\nu)}$**-FAct**.

Since an S-act A is naturally a (unary) universal algebra, the universal algebraic definition of fuzzy acts, being compatible with the definition of other fuzzy algebraic structures, may be considered to be more natural than the second one. Thus, from now on we consider the universal algebraic definition of fuzzy acts and we recall that:

Theorem 1.1 *[5] An S-act A with $\mu : A \to [0,1]$ is a fuzzy $S = S^{(1)}$-act if and only if for every $\alpha \in [0,1]$, $A_\alpha^{(\mu)}$ is an ordinary S-subact of A.*

2 Fuzzy subsets of an S-act as a fuzzy S-act

In this short section we make an S-action from the fuzzy subsets of an S-act A which is used thorough the paper and give a characterization of fuzzy S-acts defined in preliminary.

Lemma 2.1 *Let S be an commutative monoid and A be an S-act. Then fuzzy subsets of A form an S-act.*

Proof. To prove, for each fuzzy subset $A^{(\mu)}$ and each $m \in S$ we define:

$$\begin{array}{rccc} m\mu : & A & \to & [0,1] \\ & a & \rightsquigarrow & \bigvee\{\mu(x) \mid mx = a\} \end{array}$$

First we note that $m\mu$ is a fuzzy S-act, because $m\mu(na) = \bigvee\{\mu(x) \mid mx = na\}$, for every $n \in S$, and $m\mu(a) = \bigvee\{\mu(x) \mid mx = a\}$. But if $mx = a$, then $nmx = na$, and since S is commutative, $mnx = nmx = na$. Also $\mu(x) \le \mu(nx)$. So $\bigvee\{\mu(x) \mid mx = a\} \le \bigvee\{\mu(x) \mid mx = na\}$.

Now we check the S-act properties.

$$\begin{aligned} (m_1m_2)\mu(a) &= \bigvee_{x\in A}\{\mu(x) \mid (m_1m_2)x = a\} \\ &= \bigvee_{x\in A}\{\mu(x) \mid m_2x = y,\ m_1y = a\} \\ &= \bigvee_{x\in A}\{\bigvee_{y\in A}\mu(y) \mid m_2x = y,\ m_1y = a\} \\ &= \bigvee_{y\in A}\{m_2\mu(y) \mid m_1y = a\} \\ &= m_1(m_2\mu(x))(a) \end{aligned}$$

and $1_S\mu(a) = \bigvee\{\mu(x) \mid 1_Sx = a\} = \mu(a)$. Also if $\nu \le \mu$, then $(m\nu)(a) = \bigvee\{\nu(x) \mid mx = a\} \le \bigvee\{\mu(x) \mid mx = a\} = (m\mu)(a)$.$\square$

Theorem 2.1 *Let $\mu : A \to [0,1]$ be a fuzzy subset. Then $A^{(\mu)}$ is an fuzzy S-act if and only if $m\mu \le \mu$ for every $m \in S$.*

Proof. ($\Rightarrow$) Let $A^{(\mu)}$ be a fuzzy S-act and $X_a = \{x \in A \mid mx = a\}$. Then $\mu(x) \leq \mu(mx) = \mu(a)$, for every $x \in X_a$ implies that $\bigvee\{\mu(x) \mid mx = a\} \leq \mu(a)$. That is $m\mu \leq \mu$.

($\Leftarrow$) Let $A^{(\mu)}$ be a fuzzy subset. To prove we show that $\mu(a) \leq \mu(ma)$, for every $m \in S$ and $a \in A$. But we know that $m\mu \leq \mu$ and hence we have $m\mu(ma) \leq \mu(ma)$. Now since $m\mu(ma) = \bigvee_{x\in X_{ma}} \mu(x)$ and $a \in X_{na}$, we have $\mu(a) \leq m\mu(ma) \leq \mu(ma)$. □

3 Cyclic fuzzy S-acts

In this section we define a generated fuzzy S-act by a fuzzy subset of an action and then we characterize the generated fuzzy S-actions by the action introduced in Lemma 2.1. We then define the cyclic fuzzy S-acts which are a useful class of fuzzy S-acts and infact every fuzzy S-act is made of a class of cyclic ones.

Lemma 3.1 *Intersection and union of fuzzy S-acts of an S-set A is an fuzzy S-act.*

Proof. Let $\{A^{(\mu_i)}\}_{i\in I}$ be a family of fuzzy S-act. Then $(\bigcup_{i\in I}\mu_i)(ma) = \bigvee_{i\in I}\mu_i(ma) \geq \bigvee_{i\in I}\mu_i(a) = (\bigcup_{i\in I}\mu_i)(a)$ and $(\bigcap_{i\in I}\mu_i)(ma) = \bigwedge \mu_i(ma) \geq \bigwedge \mu_i(a) = (\bigcap_{i\in I}\mu_i)(a)$. □

Theorem 3.1 *Let $\mu : A \to [0,1]$ be a fuzzy S-act and $\{\mu_i\}_{i\in I\subseteq[0,1]}$ be family of i-cuts of μ. Then $\bigcup_{i\in I}\mu_i$ and $\bigcap_{i\in I}\mu_i$ are fuzzy S-acts of the form α-cut.*

Proof. By Lemma 3.1, it is enough to show that $\bigcup_{i\in I}\mu_i = \mu_{\bigvee_{i\in I} i}$ and $\bigcap_{i\in I}\mu_i = \mu_{\bigwedge_{i\in I}}$. But since $(\bigcup_{i\in I}\mu_i)(a) = \bigvee \mu_i(a) \geq i$, for every $i \in I$ and $a \in A$, hence $\bigvee \mu_i(a) \geq \bigvee i$. Also $(\bigcap_{i\in I}\mu_i)(a) = \bigwedge \mu_i(a) \geq \bigwedge i$. So $\bigcup_{i\in I}\mu_i$ and $\bigcap_{i\in I}\mu_i$ are fuzzy S-acts of the form α-cut. □

Now by the above Lemma having the following definition is natural.

Definition 3.1 *Let $\mu : A \to [0,1]$ be a fuzzy S-act. Then we take $< \mu >$ to be $\bigcap\{\nu : A \to [0,1] \mid \mu \leq \nu$ and ν is a fuzzy S-act$\}$. The fuzzy S-act $< \mu >$ is called the generated fuzzy S-act by μ.*

Theorem 3.2 *Let S be an commutative semigroup and $A^{(\mu)}$ be a fuzzy S-set of A. Then $< \mu >= \bigcup_{m\in S} m\mu$.*

Proof. First we prove that $\bigcup_{m\in S} m\mu$ is an fuzzy S-act. To do we show that $\bigcup_{m\in S} m\mu(a) \leq \bigcup_{m\in S} m\mu(na)$, for each $n \in S$. But $\bigcup_{m\in S} m\mu(a) = \bigvee_{m\in S}(\bigvee_{mx=a} \mu(x))$ and $\bigcup_{m\in S} m\mu(na) = \bigvee_{m\in S}(\bigvee_{mx=na} \mu(x))$. But if $mx = a$, then $mnx = nmx = na$. Since $\mu(x) \leq \mu(nx)$, for every $x \in A$, $\bigcup_{m\in S} m\mu(a) \leq \bigcup_{m\in S} m\mu(na)$.

Now let $A^{(\nu)}$ be a fuzzy S-act such that $\mu \leq \nu$. Then for every $m \in S$ we have $m\mu \leq \mu \leq \nu$, see Theorem 2.1 for the first inequality, and hence $\bigcup_{m\in S} m\mu(a) \leq \nu(a)$, for every $a \in A$. $\square$

In the following we have some morte propeties about generated fuzzy S acts.

Theorem 3.3 *(1)* $<< \mu >>=< \mu >$.

(2)$< \bigcup_{i\in I} \mu_i >= \bigcup_{i\in I} < \mu_i >$.

Proof. (1) It is trivial by definition of generated fuzzy S-act.

(2) By Theorem 3.2 we have

$$\begin{aligned} < \bigcup_{i\in I} \mu_i > (a) &= \bigvee\{\bigcup \mu_i(x) \mid mx = a \text{ for some } m \in M\} \\ &= \bigvee_{i\in I} \bigvee_{mx=a} \mu_i(x) \\ &= \bigcup_{i\in I} < \mu_i > (a) \end{aligned} \tag{1}$$

for every $a \in A$. $\square$

Definition 3.2 *Let A be an S-act and $\alpha \in [0,1]$ and $x \in A$. Then by cyclic fuzzy S-act $< x_\alpha >$ we mean:* $< x_\alpha > (a) = \begin{cases} \alpha & \text{if } a \in Sx \\ 0 & \text{othrewise} \end{cases}$ *for every $a \in A$.*

Corolary 3.1 *Let μ be a fuzzy S-act of an S-act A and $x \in A$. Then* $< x_{\mu(x)} > \leq \mu$.

Theorem 3.4 *Let S be a monoid and μ be a fuzzy S-act of an S-act A. Then* $\mu = \bigcup_{x\in A} < x_{\mu(x)} >$.

Proof. $\bigcup_{x\in A} < x_{\mu(x)} > (a) = \bigvee_{x\in A} < x_{\mu(x)} > (a) = \bigvee\{\mu(x) \mid a = mx \text{ for some } m \in S\}$. But since $1_S a = a$, $\mu(a) \leq \bigvee\{\mu(x) \mid a = mx \text{ for some } m \in M\}$. Also since μ is a fuzzy S-act, $\mu(x) \leq \mu(mx)$. Hence we have $\mu(a) \leq \bigvee_{mx=a} \mu(x) \leq \mu(a)$, for every $a \in A$. that is $\bigcup_{x\in A} < x_{\mu(x)} >= \mu$ $\square$

Theorem 3.5 *For every $m \in S$ and every cyclic fuzzy S-act $< x_\alpha >$ of A, $m < x_\alpha > = < x_\alpha >$.*

Proof.

$$\begin{aligned} m < x_\alpha > (a) &= \bigvee \{< x_\alpha > (y) \mid my = a\} \\ &= \begin{cases} \alpha & \text{if } a \in Sx \\ 0 & otherwise \end{cases} \\ &= < x_\alpha > (a). \end{aligned}$$ □

4 Decomposable and Indecomposable Fuzzy S-act

Here we give a definition of indecomposable Fuzzy S-act and show that the cyclic fuzzy S-acts are indecomposable. We also see some properties of indecomposable fuzzy S-acts in this section.

Definition 4.1 *An fuzzy S-act $\mu \neq 0$ of A is called decomposable whenever there exist two fuzzy S-acts $\nu, \eta \neq 0$ of A such that $\nu, \eta \leq \mu$ and $\eta \vee \nu = \mu$, and $\eta \wedge \nu = 0$. Otherwise μ is called indecomposable.*

Theorem 4.1 *Let S be a commutative monoid. Then every cyclic fuzzy S-act $< x_i >$ of A is indecomposable.*

Proof. Let $< x_i >$ be decomposable. Then there are fuzzy S-acts ν and η of A such that $\nu, \eta \leq < x_i >$ and $\eta \vee \nu = < x_i >$, and $\eta \wedge \nu = 0$. So for every $a \in A$, $\eta(a) \wedge \nu(a) = 0$ and $\eta(a) \vee \nu(a) = \begin{cases} i & \text{if} a = mx \\ 0 & \text{otherwise} \end{cases}$. Now let $\nu(m_0 x) = i$. Then we claim that for every $m \in S$, $\nu(mx) = i$ and $\eta(mx) = 0$. Because if there exists $m_1 \in S$ such that $\eta(m_1 x) = i$, then $\eta(m_1 m_0 x) = i$ and $\nu(m_1 m_0 x) = i$, so $\eta(m_1 m_0 x) \wedge \nu(m_1 m_0 x) = i \neq 0$. □

Definition 4.2 *A fuzzy S-act $A^{(\mu)}$ is called finitely generated whenever $\mu = < \bigcup_{i=1}^{n} (x_i)_{\alpha_i} >$, where $\alpha_i \in [0, 1]$.*

Theorem 4.2 *Let $f : A \to B$ be an S-act homomorphism and $A^{(\mu)}$ be an fuzzy S-act. Then $B^{f(\mu)}$ is finitely generated, if so is μ.*

Proof. To proof, we show that $f(\mu) =< \bigcup_{i=1}^{n} f((x_i)_{\alpha_i}) >$, where $\mu =< \bigcup_{i=1}^{n} (x_i)_{\alpha_i} >$. For

$$
\begin{aligned}
f(\mu)(b) &= \bigvee \{\mu(a) \mid f(a) = b\} \\
&= \begin{cases} \bigvee_{j \in J \subseteq \{1,\ldots,n\}} \alpha_j & f(x_j) = b \\ 0 & \text{otherwise} \end{cases} \\
&= \bigcup_{i \in I} f(< (x_i)_{\alpha_i} >)(b).
\end{aligned}
$$

□

Lemma 4.1 *Let G be a group. Then for every finitely generated fuzzy G-act $A^{(\mu)}$, there exists a finite subsets $\{a_1, \ldots, a_n\} \subseteq [0,1]$ and $\{x_1, \ldots, x_n\} \subseteq A$ such that $\mu(O(x_i)) = a_i$ and $O(x_i) \cap O(x_j) = \emptyset$, if $i \neq j$, where $O(x_i)$ is a notation for orbit of x_i that is the set $\{gx_i \mid g \in G$.*

Proof. Since μ is finitely generated, $\mu =< \bigcup_{i=1}^{n} (x_i)_{a_i}) >$ and since G is a group, $O(x_i) \cap O(x_j) = \emptyset$, if $i \neq j$. So

$$
\mu(x) = \begin{cases} a_i & \text{if } x \in O(x_i) \\ 0 & \text{otherwise.} \end{cases}
$$

□

Theorem 4.3 *Let $f : A \to B$ be an S-act homomorphism with commutative S, and $A^{(\mu)}$ be an fuzzy S-act. Then $f(\mu) =< f(\nu) >$, if $\mu =< \nu >$.*

Proof.

$$
\begin{aligned}
f(\mu)(b) &= \bigvee \{\mu(a) \mid f(a) = b\} \\
&= \bigvee \{(m\nu)(a) \mid m \in S,\ f(a) = b\} \quad \text{by Theorem 3.2} \\
&= \bigvee \{\nu(x) \mid m \in M, mx = a,\ f(a) = b\} \\
&= \bigvee \{f(\nu)(y) \mid m \in M,\ my = b\} \\
&= \bigvee_{m \in M} mf(\nu)(b) \\
&=< f(\nu) > (b).
\end{aligned}
$$

□

Theorem 4.4 *Let $\{A^{(\nu_i)}\}_{i \in I}$ be a family of fuzzy S-act in which there is $i_0 \in I$ such that μ_{i_0} is indecomposable. Then $\bigvee \mu_i$ is indecomposable.*

Proof. Let $\bigvee_{i \in I} \mu_i$ be decomposable. So there are $\nu_1, \nu_2 \leq \bigvee_{i \in I} \mu_i$ such that $\bigvee_{i \in I} \mu_i = \nu_1 \vee \nu_2$ and $\nu_1 \wedge \nu_2 = 0$. Then $\mu_{i_0} = \mu_{i_0} \wedge (\bigvee_{i \in I} \mu_i) = (\mu_{i_0} \wedge \nu_1) \vee (\mu_{i_0} \wedge \nu_2)$ Also $(\mu_{i_0} \wedge \nu_1) \wedge (\mu_{i_0} \wedge \nu_2) = \mu_{i_0} \wedge (\nu_1 \wedge \nu_2) = 0$. □

Corolary 4.1 *The union of indecomposable fuzzy S-acts is indecomposable.*

References

[1] Belohlavek, *Birkhoff variety theorem and fuzzy logic* Arch. Math. Logic 42 (2003), 781790.

[2] Belohlavek, R., Vychodil, V., Algebras with fuzzy equalities Fuzzy Sets and Systems 157 (2006), 161.201.

[3] Bosnjak. I, R.Madarasz, G.Vojvodic *Algebra of fuzzy sets*, Fuzzy Sets and Systems 160 (2009), 29792988.

[4] Ebrahimi, M.M., M. Mahmoudi, *The category of M-sets.* Ital. J. Pure Appl. Math. 9 (2001), 123-132.

[5] M. Haddadi *Fuzzy acts over fuzzy semigroups and* summited

[6] Kilp, M., U. Knauer, A. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York, 2000.

[7] Lotfi A. Zadeh *Fuzzy sets*. Information and Control. 8 (1965), 338353.

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On some probability concepts in fuzzy framework

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Abstract

In this paper some modalities in which the concept of probability can be fuzzified are investigated in order to obtain new tools useful in the modelization of the risks. Some papers related to this approach are [1-6]. Finally, some open problems are proposed to the reader.

Key words: probability, fuzzy set, fuzzy probability, fuzzy numbers

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1 Classical case

In the set theory the following operations are used:

- the intersection $(A \cap B)$;
- the union $(A \cup B)$;
- the complement $(\overline{A})$;
- the difference $(A \backslash B = A \cap \overline{B})$;
- the implication $(A \to B = \overline{A \backslash B} = \overline{A} \cup B)$;
- the symmetric difference $(A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \cap \overline{B}) \cup (\overline{A} \cap B))$;
- the equivalence $(A \leftrightarrow B = (A \to B) \cap (B \to A) = \overline{A \triangle B})$, where A, B are sets.

The empty set is denoted by $\emptyset$, and the set of subsets of a set S will be denoted by $\mathcal{P}(S)$.

Let $A, B, C \in \mathcal{P}(S)$.

Remark 1. We have:

i) $\cap, \cup$ are commutative and associative;

ii) $A \cap A = A$, $A \cup A = A$;

iii) $A \cap (A \cup B) = A$, $A \cup (A \cap B) = A$;

iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

v) $A \cap \overline{A} = \emptyset$, $A \cup \overline{A} = S$;

vi) $\overline{A \cap B} = \overline{A} \cup \overline{B}$; $\overline{A \cup B} = \overline{A} \cap \overline{B}$;

vii) $\overline{\overline{A}} = A$.

Let $\Omega \neq \emptyset$.

Definition 2. *By field of events (in relation with the space Ω) one intend $K \subseteq \mathcal{P}(\Omega)$ such that*

i) $\Omega \in K$*;*

ii) $A, B \in K \Rightarrow A \cup B \in K$*;*

iii) $A \in K \Rightarrow \overline{A} \in K$*.*

Remark 3. Let K be a field of events. We have

i) $A, B \in K \Rightarrow A \cap B \in K$;

ii) $A, B \in K \Rightarrow A \backslash B \in K$;

iii) $A, B \in K \Rightarrow A \rightarrow B \in K$;

iv) $A, B \in K \Rightarrow A \triangle B \in K$;

v) $A, B \in K \Rightarrow A \leftrightarrow B \in K$;

vi) $\emptyset \in K$.

Let K be a field of events.

Definition 4. *By probability on K one intend*
$P : K \rightarrow [0, 1]$ *such that:*

i) $P(\Omega) = 1$;

ii) $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$.

Remark 5. We have

i) $P(\emptyset) = 0$;

ii) $P(\overline{A}) = 1 - P(A)$;

iii) $P(A \backslash B) = P(A) - P(A \cap B)$;

iv) $P(A \cap B) + P(A \cup B) = P(A) + P(B)$;

v) $P(A \to B) = 1 - P(A) + P(A \cap B)$;

v) $P(A) + P(A \to B) = P(B) + P(B \to A)$.

Remark 6. If $P : K \to [0,1]$ is an application satisfying $P(\emptyset) = 0$, $P(\Omega) = 1$, then the condition ii), from the definition and the condition vi) from the remark are equivalent.

2 Fuzzy case

For the construction which will be given in this case we need the concepts of t-norms and t-conorms.

Definition 7. *A function $t : [0,1] \times [0,1] \to [0,1]$ will be called t-norm if the following conditions are satisfied:*

i) $t(x,1) = x$, $\forall x \in [0,1]$*;*

ii) $t(x,y) = t(y,x)$*, for any* $x, y \in [0,1]$*;*

iii) $t(x, t(y,z)) = t(t(x,y), z)$*, for any* $x, y, z \in [0,1]$*;*

iv) $x \leq z \Rightarrow t(x,y) \leq t(z,y)$*,* $\forall y \in [0,1]$.

Remark 8. We have also:

v) $t(x,0) = t(1,0) = t(0,1) = 0$, $\forall x \in [0,1]$.

Example 9. i) $p : [0,1] \times [0,1] \to [0,1]$, $p(x,y) = xy$;

ii) $\min : [0,1] \times [0,1] \to [0,1]$, $\min(x,y) = \begin{cases} x, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$

ii) $t_m : [0,1] \times [0,1] \to [0,1]$, $t_m(x,y) = \max\{x + y - 1, 0\}$.

Definition 10. *A function $t^* : [0,1] \times [0,1] \to [0,1]$ will be called t-conorm if the following conditions are satisfied:*

i) $t^*(x,0) = x$, $\forall x \in [0,1]$;

ii) $t^*(x,y) = t^*(y,x)$, *for any* $x, y \in [0,1]$;

iii) $t^*(x, t^*(y,z)) = t^*(t^*(x,y), z)$, *for any* $x, y, z \in [0,1]$.

iv) $x \leq z \Rightarrow t^*(x,y) \leq t^*(z,y)$, $\forall y \in [0,1]$.

Example 11. i) $p^* : [0,1] \times [0,1] \to [0,1]$, $p^*(x,y) = x + y - xy$;

ii) $\max : [0,1] \times [0,1] \to [0,1]$, $\max(x,y) = \begin{cases} x, & \text{if } x \geq y \\ y, & \text{if } x < y \end{cases}$;

iii) $t_m^* : [0,1] \to [0,1] \to [0,1]$, $t_m^*(x,y) = \min\{x+y, 1\}$.

Definition 12. *The* t*-norm* t *and the* t*-conorm* t^* *are called dual each another if for any* $x, y \in [0,1]$

$$t(x,y) = 1 - t^*(1-x, 1-y).$$

For example, p, p^* or $\min, \max$ or t_m, t_m^* are such couples.

Definition 13. *A couple* (U, μ) *where* $U \neq \emptyset$ *and* $\mu : U \to [0,1]$ *is an application will be called fuzzy set (on the universe* U*) or fuzzy subset of* U.

The empty fuzzy set is given by $\widetilde{\phi} : U \to [0,1]$, $\widetilde{\phi}(x) = 0$, $\forall x \in U$.

We shall denote $\mu \subseteq \eta$ if $\mu(x) \leq \eta(x)$, $\forall x \in U$.

By $\widetilde{U} : U \to [0,1]$ one intend the application given by $\widetilde{U}(x) = 1$, $\forall x \in U$.

Let $\mathcal{F}(U)$ be the family of fuzzy subsets of U. The operations with fuzzy subsets can be defined in the following way:

for $\mu, \eta : \mathcal{F}(U)$, $\mu \bigcap_t \eta : U \to [0,1]$, $(\mu \bigcap_t \eta)(x) = t(\mu(x), \eta(x))$

$\mu \bigcup_t \eta \to [0,1]$, $(\mu \bigcup_t \eta)(x) = t^*(\mu(x), \eta(x))$.

The complement $\overline{\mu} : U \to [0,1]$ will be given by $\overline{\mu}(x) = 1 - \mu(x)$. In a similar way with the classical case one define $\mu \xrightarrow{t} \eta$, $\mu \xrightarrow{t} \eta$, etc.

$\mu \overset{t}{-} \eta : U \to [0,1]$, $(\mu \overset{t}{-} \eta)(x) = t(\mu(x), 1 - \eta(x))$;

and $\mu \xrightarrow{t} \eta : U \to [0,1]$, $(\mu \xrightarrow{t} \eta)(x) = t^*(1 - \mu(x), \eta(x))$.

For the couples t-norm/conorm described above we obtain: $\odot, \oplus$; $\cap, \cup$; ... More precisely for $\mu, \eta : U \to [0,1]$ we have:

A.

$\mu \odot \eta : U \to [0,1]$, $(\mu \odot \eta)(x) = \mu(x)\eta(x)$;

$\mu \oplus \eta : U \to [0,1]$, $(\mu \oplus \eta)(x) = \mu(x) + \eta(x) - \mu(x)\eta(x)$.

$\overline{\mu} : U \to [0,1]$, $\overline{\mu}(x) = 1 - \mu(x)$; and

$\mu \ominus \eta : U \to [0,1]$, $(\mu \ominus \eta)(x) = \mu(x) - \mu(x)\eta(x)$;

$\mu \bigominus \eta : U \to [0,1]$, $(\mu \bigominus \eta)(x) = 1 - \mu(x) + \mu(x)\eta(x)$;

Remark 14. We have

i) $\odot, \oplus$ are commutative and associative;

ii) $\mu \odot \mu \subseteq \mu$, $\mu \subseteq \mu \oplus \mu$;

iii) $\mu \supseteq \mu \odot (\mu \oplus \eta)$; $\mu \subseteq \mu \oplus (\mu \odot \eta)$;

iv) $\mu \oplus (\eta \odot \tau) \supseteq (\mu \oplus \eta) \odot (\mu \oplus \tau)$; $\mu \odot (\eta \oplus \tau) \subseteq (\mu \odot \eta) \oplus (\mu \odot \tau)$;

v) $(\mu \odot \overline{\mu})(x) \leq \frac{1}{4}$, $(\mu \oplus \overline{\mu})(x) \geq \frac{3}{4}$, $\forall x \in U$;

v) $\overline{\mu \oplus \eta} = \overline{\mu} \odot \overline{\eta}$; $\overline{\mu \odot \eta} = \overline{\mu} \oplus \overline{\eta}$.

B.

$\mu \cap \eta : U \to [0,1]$, $(\mu \cap \eta) = \min\{\mu(x), \eta(x)\}$;

$\mu \cup \eta : U \to [0,1]$, $(\mu \cup \eta)(x) = \max\{\mu(x), \eta(x)\}$;

$\overline{\mu} : U \to [0,1]$, $\overline{\mu}(x) = 1 - \mu(x)$;

$\mu - \eta : U \to [0,1]$, $(\mu - \eta)(x) = \min\{\mu(x), 1 - \eta(x)\}$;

$\mu \to \eta : U \to [0,1]$, $(\mu \to \eta)(x) = 1 - \min\{\mu(x), 1 - \eta(x)\}$;

Remark 15. We have

i) $\cap, \cup$ are commutative and associative;

ii) $\mu \cap \mu = \mu$, $\mu \cup \mu = \mu$

iii) $\mu \cup (\mu \cap \eta) = \mu$; $\mu \cap (\mu \cup \eta) = \mu$;

iv) $\mu \cup (\eta \cap \tau) = (\mu \cup \eta) \cap (\mu \cup \tau) = (\mu \cap \eta) \cup (\mu \cap \tau)$;

v) $(\mu \cap \overline{\mu})(x) \leq \frac{1}{2}$, $(\mu \cup \overline{\mu})(x) \geq \frac{1}{2}$, $\forall x \in U$;

vi) $\overline{\mu \cup \eta} = \overline{\mu} \cap \overline{\eta}$; $\overline{\mu \cap \eta} = \overline{\mu} \cup \overline{\eta}$.

C.

$$\mu \bigtriangledown \eta : U \to [0,1],\ (\mu \bigtriangledown \eta)(x) = \max\{\mu(x) + \eta(x) - 1, 0\};$$

$$\mu \bigtriangleup \eta : U \to [0,1],\ (\mu \bigtriangleup \eta)(x) = \min\{\mu(x) + \eta(x), 1\};$$

$$\overline{\mu} : U \to [0,1],\ \overline{\mu}(x) = 1 - \mu(x);$$

$$\mu \bullet \eta : U \to [0,1],\ (\mu \bullet \eta)(x) = \max\{\mu(x) - \eta(x), 0\};$$

$$\mu \dot{\to} \eta : U \to [0,1],\ (\mu \dot{\to} \eta)(x) = \min\{1 - \mu(x) + \eta(x), 1\}.$$

Remark 16. We have

i) $\bigtriangledown, \bigtriangleup$ are commutative and associative;

ii) $\mu \bigtriangledown \eta \subseteq \mu$, $\mu \subseteq \mu \bigtriangleup \mu$;

iii) $\mu \subseteq \mu \bigtriangleup (\mu \odot \eta)$; $\mu \supseteq \mu \bigtriangledown (\mu \bigtriangleup \eta)$;

iv) $(\mu \bigtriangledown \overline{\mu})(x) = 0$, $(\mu \bigtriangleup \overline{\mu})(x) = 1$, $\forall x \in U$;

v) $\overline{\mu \bigtriangledown \eta} = \overline{\mu \bigtriangleup \eta} = \overline{\mu} \bigtriangledown \overline{\eta}$.

Remark 17. We have $\mu \bigtriangledown \eta \subseteq \mu \odot \eta \subseteq \mu \cap \eta$; $\mu \cup \eta \subseteq \mu \oplus \eta \subseteq \mu \bigtriangleup \eta$ and $\overline{\overline{\mu}} = \mu$.

3 Fuzzy numbers

In the last section of the paper fuzzy number will be used.

Let $\mathbb{R}$ be the field of real numbers.

Definition 18. *By triangular fuzzy number one intend a triple* (a, b, c)*, where* $a, b, c \in \mathbb{R}$*,* $a \leq b \leq c$*.*

We shall denote $\mathbb{R}_t$ the set of triangular fuzzy numbers.

For $A = (a_1, b_1, c_2)$, $B = (a_2, b_2, c_2)$ from $\mathbb{R}_t$, if $c_1 \leq a_2$, or
$a_2 \leq c_1$ and $\frac{a_1+2b_1+c_1}{4} < \frac{a_2+2b_2+c_2}{4}$, or
$a_2 \leq c_1$, $\frac{a_1+2b_1+c_1}{4} = \frac{a_2+2b_2+c_2}{4}$ and $b_1 < b_2$, or
$a_2 \leq c_1$, $\frac{a_1+2b_1+c_1}{4} = \frac{a_2+2b_2+c_2}{4}$, $b_1 = b_2$ and $c_1 - a_1 < c_2 - a_2$,
we shall write $A \lesssim B$ (a special kind of "order" being obtained in this way).

Remark 19. A triangular fuzzy number $(a, b, c) \in \mathbb{R}_t$ is uniquely determined by a triple (λ, b, ρ) where $\lambda = b - a$, $\rho = c - b$ are positive reals called the left, respectively right tolerance.

We will use the notation with the central value on the first place (b, λ, ρ).

We consider the operations (these operations are introduced by the author and was presented for the first time at a conference given at the University of Chieti in 2007 and was published in [6]):

$$(a, \lambda, \rho) \boxplus (b, \lambda', \rho') = (a + b, \max\{\lambda, \lambda'\}, \max\{\rho, \rho'\})$$

$$(a, \lambda, \rho) \boxdot (b, \lambda', \rho') = (ab, \max\{\lambda, \lambda'\}, \max\{\rho, \rho'\})$$

and the relation "$\sim$" given by

$$(a, \lambda, \rho) \sim (b, \lambda', \rho') \text{if } \begin{cases} a = b \\ \lambda - \lambda' = \rho - \rho'. \end{cases}$$

One obtains:

Remark 20. We have:

i) $\boxplus, \boxdot$ are commutative and associative;

ii) $\boxdot$ is distributive with respect to $\boxplus$;

iii) $(0, 0, 0)$ is neutral element for $\boxplus$, and $(1, 0, 0)$ is neutral element for $\boxdot$;

iv) $(a, \lambda, \rho) \boxplus (-a, \rho, \lambda) \sim (0, 0, 0)$; if $a \neq 0$

$$(a, \lambda, \rho) \boxdot (\frac{1}{a}, \rho, \lambda) \sim (1, 0, 0).$$

v) "$\sim$" is an equivalence relation on $\mathcal{R}_t$.

4 Fuzzy events

Let be $\Omega \neq \emptyset$ and $\mathcal{F}(\Omega)$.

Definition 21. *By fuzzy field of events one intend $K \subseteq \mathcal{F}(\Omega)$ such that:*

i) $\widetilde{\Omega} \in K$

ii) $\mu, \eta \in K \Rightarrow \mu \bigcup_t \eta \in K$;

iii) $\mu \in K \Rightarrow \overline{\mu} \in K$.

Remark 22. We have:

i) $\widetilde{\phi} \in K$;

ii) $\mu, \eta \in K \Rightarrow \mu \bigcap_t \eta \in K$; $\mu \overset{t}{-} \eta \in K$, $\mu \overset{t}{\rightarrow} \eta \in K$;

iii) $(\mu \in K \Rightarrow \overline{\mu} \in K) \Leftrightarrow (\mu, \eta \in K \Rightarrow \mu \overset{t}{-} \eta \in K) \Leftrightarrow (\mu, \eta \in K \Rightarrow \mu \overset{t}{\rightarrow} \eta \in K)$.

Let K be a fuzzy field of events.

Definition 23. *By probability on K one intend $P : K \to [0,1]$ such that*

i) $P(\widetilde{\Omega}) = 1$

ii) $\mu \bigcap_t \eta = \phi \Rightarrow P(\mu \bigcup_t \eta) = P(\mu) + P(\eta)$.

Remark 24. Verify the following:

i) $P(\widetilde{\phi}) = 0$;

ii) $P(\overline{\mu}) = 1 - P(\mu)$;

iii) $\mu \subseteq \eta \Rightarrow P(\eta \overset{t}{-} \mu) = P(\eta)$;

iv) $P(\mu \overset{t}{-} \eta) = P(\mu) - P(\mu \bigcap_t \eta)$;

v) $P(\mu \bigcup_t \eta) + P(\mu \bigcap_t \eta) = P(\mu) + P(\eta)$;

vi) $P(\mu \overset{t}{\rightarrow} \eta) = 1 - P(\mu) + P(\mu \bigcap_t \eta)$;

vii) $P(\mu) + P(\mu \overset{t}{\rightarrow} \eta) = P(\eta) + P(\eta \overset{t}{\rightarrow} \mu)$.

In the case $t = t_m$ we suppose also that $\mu, \eta \in K \Rightarrow \mu \odot \eta \in K$. In this context we shall denote $P(\mu/\eta) = P(\mu \odot \eta)/P(\eta)(P(\eta) \neq 0)$.

Proposition 25. *In the above condition we have:*

$$P(\mu/\eta) = \frac{P(\mu)P(\eta/\mu)}{P(\mu)P(\eta/\mu) + P(\eta)P(\mu/\eta)}.$$

We have also

Proposition 26. *If $\mu_1, \ldots, \mu_n \in K$ are such that $\mu_i \bigcap_t \mu_j = \widetilde{\phi}$ for $i \neq j$, then $P(\mu \bigcup_t, \ldots, \bigcup_t \mu_n) = P(\mu_1) + \ldots + P(\mu_n)$.*

5 Fuzzy probability

The next step is to substitute $[0,1]$ in the definition of the probability (in K) with the

$$I_t = \{(a,\lambda,\rho) \in \mathbb{R}^t / \lambda \leq a, \rho \leq 1-a, a \in [0,1]\}.$$

We have two possibilities:

A. We shall use the operations and the equivalence relation given in III.

Remark 27. If (a,λ,ρ) is such that $a \in [0,1]$ then there exists $(a',\lambda',\rho') \in I_t$ such that $(a,\lambda,\rho) \sim (a',\lambda',\rho')$.

Let K be a fuzzy field of events.

Definition 28. *By fuzzy probability on K one intend an application $P : K \to I_t$ such that*

i) $P(\widetilde{\phi}) = 0$;

ii) $\mu \bigcap_t \eta = \phi \Rightarrow P(\mu \bigcup_t \eta) \sim P(\mu) \boxplus P(\eta)$;

iii) If $P(\mu) = (\alpha,\lambda,\rho)$ *then* $P(\overline{\mu}) = (1-a,\rho,\lambda)$.

Remark 29. In view to obtain more properties ii) can be replaced by ii$'$) $P(\mu) \boxplus P(\eta) \sim P(\mu \bigcap_t \eta) \boxplus P(\mu \bigcup_t \eta)$.

Problem 30. In the case i), ii$'$), iii), verify the following:

i) $P(\widetilde{\Omega}) = (1,0,0)$;

ii) $P(\mu \overset{t}{\backslash} \eta) \sim P(\mu) - P(\mu \bigcap_t \eta)$;

iii) $P(\mu \xrightarrow{t} \eta) \sim P(\overline{\mu}) + P(\mu \bigcap_t \eta)$;

iv) $P(\mu) + P(\mu \xrightarrow{t} \eta) \sim P(\eta) + P(\eta \xrightarrow{t} \mu)$.

B. In the following we propose new operations:

$(a,\lambda,\rho) \widetilde{+} (a',\lambda',\rho') = (a+a'-aa', a+a'-aa'-\max\{a+\lambda, a'+\lambda'\}, \min\{a+\rho+a'+\rho', 1\} - a - a' + aa')$

$(a,\lambda,\rho) \widetilde{\cdot} (a',\lambda',\rho) = (aa', aa' - \max\{a-\lambda+a'-\lambda'-1, 0\} \min\{a+\rho, a'+\rho'\} - aa')$

When the numbers are written in the form (a,b,c) $(a \leq b \leq c)$, the operation are defined by

$$(a,b,c) \widetilde{+} (a',b',c') = (\max\{a,a'\}, b+b'-bb', \min\{c+c',1\})$$

$$(a,b,c) \widetilde{\cdot} (a',b',c') = (\max\{a+a'-1,0\}, bb', \min\{c,c'\}).$$

Remark 31. The above operations are satisfying

$$0 \leq \max\{a, a'\} \leq b + b' - bb' \leq \min\{c + c', 1\} \leq 1$$

$$0 \leq \max\{a + a' - 1, 0\} \leq bb' \leq \min\{c, c'\} \leq 11.$$

In this frame using the form (a, b, c) we can propose the following

Definition 32. *By fuzzy probability on K one intend $P : K \to I_t$ such that*

i) $P(\widetilde{\Omega}) = (1, 1, 1)$, $P(\widetilde{\phi}) = (0, 0, 0)$;

ii) $P(\mu)\widetilde{+}P(\eta) = P(\mu \bigcap_t \eta)\widetilde{+}P(\mu \bigcup_t \eta)$;

iii) $\mu \leq \eta$, $P(\mu) \lesssim P(\eta)$.

Problem 33. Verify the following:

i) $P(\mu \overset{t}{-} \eta) = P(\mu) - P(\mu \bigcap_t \eta)$;

ii) $P(\mu \xrightarrow{t} \eta) = P(\overline{\mu}) + P(\mu \bigcap_t \eta)$;

iii) $P(\mu) + P(\mu \xrightarrow{t} \eta) = P(\eta) + P(\eta \xrightarrow{t} \mu)$.

References

[1] Bugajski, S., *Fundamentals of fuzzy probability theory*, Int. J. Theor. Phys., 35, 2229-2244, 1996.

[2] Georgescu, G., *Bosbach states on fuzzy structures*, Soft computing, 8, 217-230, 2004.

[3] Georgescu, G., *Probabilistic models for intuitionistic predicate logic,* Journal of logic and Computation Advances Access, 2010.

[4] Gudder, S., *Fuzzy probability theory*, Demonstr. Math., 31, 235-254, 1998.

[5] Gudder, S., *What is fuzzy probability theory*, Foundations of Physics, 30, 1663-1678, 2000.

[6] Tofan, I., *Some remarks about fuzzy numbers*, International Journal of Risk Theory, vol. 1, 2011, p. 87-92.

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