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On the symmetries of electrodynamic interactions

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Abstract

The development of relational electromagnetism after Gauss appears to stop around 1870. Maxwell recognised relational electromagnetism as mathematically equivalent to his own formulae and called for an explanation of why so different conceptions have such a large part in common. We reconstruct relational electromagnetism guided by the No Arbitrariness Principle. Lorenz' idea of electromagnetic waves, together with the "least action principle" proposed by Lorentz are enough to derive Maxwell's equations, the continuity equation and the Lorentz' force. We show that there must be two more symmetries in electromagnetism: a descriptive one expressing source/detector relations, and another relating perceptions of the same source by detectors moving with different (constant) relative velocities. The Poincaré group relates perceived fields by different receivers and Lorentz boosts relate source/detector perceptions. We answer Maxwell's philosophical question showing how similar theories can be abduced using different inferred entities. Each form of abduction implies an interpretation and a facilitation of the theoretical construction.

Keywords: critical epistemology; rationalism; relational electromagnetism; Lorentz transformations; Doppler effect; [1]

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1 Introduction

The notion that science, and in particular physics, does not depend on philosophical or psychological factors is usually manifested by scientists and the society at large. However, this view confuses what science should be with how science is actually practised. Following Peirce we can say that research stops when doubt is appeased and a (temporary) belief is reached. The condition for the cessation of doubt might have psychological and philosophical components. During the late XIX century and the beginning of the XX century an abrupt change in this condition can be verified (Solari and Natiello, 2022a) finally leading to new physical understanding [2] and a new epistemology (Solari and Natiello, 2022b). There is a relation of precedence: psychological needs (such as the need for analogies or to incorporate learned habits) determine, in part, physical theories which in turn determine philosophy. Denying the existence of the first link we could claim that the Truth in physics forces upon us the acceptance of some epistemologies and the rejection of others. In contrast, for a critical philosophy such as Kant's (Kant, 1798) it is philosophy the science that surveils and, if necessary, corrects all other human activities. Thus, for critical philosophy the sequence must be: philosophy controls the sciences and the contributions by psychological needs of scientists have no place and must be eliminated.

The symmetries of electromagnetic interactions played a central role in the transformation underwent by physics, and with it by science, during that period. Expectations imported from Mechanics did not fit observations of electromagnetic phenomena, in particular the propagation of electromagnetic interactions and light. Two alternatives circulated around 1850, namely local propagation through some form of physical medium in space (the ether) against delayed action at a distance. The second alternative had faded away by the turn of the century, although it was never proved wrong. The introduction (and subsequent elimination) of the ether along with a second ingredient: the expectations posed by society on science reshaped the way physicists approached Nature. The progress of the industrial revolution expected science to be the support of technological development, a goal not necessarily identical to that of exploring Nature in order to understand it. The utilitarian view of science advanced at the beginning of the second industrial revolution in the Prussian empire proclaims its success some 60 years later. With it comes an a–critical epistemology that denies philosophy the right to examine the foundations of science (Beiser, 2014) as it is actually practised: the utilitarian, capitalist, science.

Is it the same physics resulting from both forms of construction? For the case of Mechanics most results coincide (Solari and Natiello, 2018), while founda-

[2] Meaning the acceptance of a theory by a community

tional issues regarding the concept of inertial systems drastically differ (Solari and Natiello, 2021).

The ether failed to provide a sound solution to these problems and Special Relativity was advanced in 1905, being today the accepted explanatory framework. However, already in 1867 Ludwig Lorenz suggested an ether-free description of electromagnetism. While the interpretation of electrodynamics in terms of special relativity must be rejected as an acceptable theory under a rational construction (Solari and Natiello, 2022b), the success obtained by applying this theory to observable problems and the absence of an alternative (at least) equally successful, consilient and coherent (Whewell, 1840, 1858b) prevented the criticism of its foundations.

The combination of motion and coordinate description of electromagnetic phenomena has several aspects. At least three elements are usually present: Observer, Source (Emitter, Primary circuit) and Receiver (Detector, Secondary circuit). However, not all motions are equally relevant. The No Arbitrariness Principle (NAP)(Solari and Natiello, 2018) (elaborating on the idea that no knowledge about nature depends on arbitrary decisions) suggests that the only motion that actually can influence results is that between Source and Receiver. Moreover, in a relational description, there is no other motion involved and the Observer is either absent or sorted out through a group of symmetry transformations between equivalent choices.

In this work we illustrate how these setups can be fully handled. We assemble Electromagnetic theory in terms of classical epistemology; hopefully achieving a better matching with experiments than current theories and higher "consilience" (Thagard, 1978) (see also (Whewell, 1840, p. XXXIX, Aphorism XIV)). First, we derive the set of equations of electromagnetism combining Lorenz' approach with an ether-free version of Lorentz' action integral, unifying and surpassing ideas that have not been fully investigated so far. Further, we relate the electromagnetic description for the case where source and receiver are at relative rest with the corresponding description in a situation of relative motion, showing also how potentially controversial concepts such as the "velocity of light" $C = (\mu_0 \epsilon_0)^{-\frac{1}{2}}$ in different states of relative motion fit in this nineteenth century framework. From the concept of reciprocal action (which is in the philosophical basis of Newton's mechanics) we examine the arbitrariness that has to be removed in Electromagnetic theory and then, the symmetry groups that must be involved a-priori. This rational [3]theory of Electromagnetism does not require any change in space-time or epistemology.

[3]The rational epistemology was presented by William Whewell (Whewell, 1840, 1858b,a) and further developed by Charles Peirce (Peirce, 1994) and its fundaments where available by 1858 before the seminal works of Maxwell (Maxwell, 1865) and Lorenz (Lorenz, 1867).

We try to develop a method that allows all philosophers to grasp its contents, thus rescuing physics from elitism. If science is to help us to come into harmony with the universe, beginning with Planet Earth, a new perspective of exemplary science must be reached, one aiming at understanding and empathising with all living forms. Thus, the aim of this work is political, but yet it is philosophical as well as technical. If successful in our task (as we believe we are), we can claim that there is no need to abandon the goal of understanding nature and also that the utilitarian science aimed at "dominating nature" (a prediction technique whose value is given by predictive success), needs to be left behind if harmony in Planet Earth is our goal.

2 On symmetries

Physics sustains the idea that there is a world that reaches us through the senses and is independent of the observer: the sensed-real. Although every particular observation may depend on the observer, the collection of observations points towards a common idea that we call reality, or *the real*. Thus, the relation between the sensed-real and reality (the idealisation) plays a fundamental role. This starting point has been called "The fundamental antithesis of philosophy" (Whewell, 1858a, Ch. I). Going from the sensed-real to the real we must separate what belongs to reality from its circumstances that result in particularities, which quite often are the consequence of arbitrary decisions. Thus, we reserve the name of arbitrariness for the observational and descriptive decisions that we have to make when associating an ideal relation with an observable relation.

It would be desirable to present physical laws in pure abstract form, without any arbitrary element, but it would be desirable as well, for physical laws to be as accessible as possible to the mind. Since abstraction imposes difficulties in grasping the meaning of such laws, there is a trade-off that must be worked out between the two desires. This trade-off results in the introduction of some (usually small) set of arbitrary elements in the description, under the requirement that such arbitrary elements could be eventually suppressed from the presentation or, what is the same, that a change in the choice of arbitrary elements results in an equivalent presentation. These ideas lead immediately to the existence of a group of transformations relating different choices of arbitrary elements. The group structure is the result of the composition law of the transformation between presentations of the laws under different arbitrary decisions. This is the central idea under the "No arbitrariness principle" (NAP) (Solari and Natiello, 2018).

The introduction of an observer brings about the possibility of attaching to it a Cartesian space for the description of the real and at the same time it introduces the symmetries of the space (the arbitrary element).

Moving directly into electromagnetism, we observe that all its fundamental experiments reflect the influence of electromagnetic phenomena associated to a pair of bodies (one of them labelled primary circuit, source, emitter, etc., and the other secondary circuit, receiver, detector). In the same form that space is not a possible subject of experimental detection but spatial relations can be measured, electromagnetic fields can only be detected by their effects on measuring devices, i.e., detectors. If the action of a source on a receiver can be addressed with controlled degrees of influence from the rest of the universe, in the limit of no influence, the idealised law describing the universe of such relations must depend only on the relative position and motion of source and receiver. Such notions can be found all over the foundational work of Faraday (Faraday, 1839, 1844, 1855) and Maxwell (Maxwell, 1873).

Electromagnetic phenomena imply the motion of electricity (whatever electricity is, as Maxwell often said) and then, since what changes the motion of bodies has been called *forces*, we can associate forces with the action of an electromagnetic (EM) body onto another EM body. Actually, this use entails a generalisation of the concept of force, since Newtonian forces change the motional status of macroscopic bodies while microscopic (quantum) objects, such as electrons involved in conduction currents, are not what classical mechanics had in mind when Newton developed its laws. Moreover, if we envisage EM-forces as Lorentz did, by adopting Weber's view of electrical atoms (Lorentz, 1892), such forces must be identically described by observers whose motions relate by Galilean coordinate transformations, and furthermore reciprocal action must be expressed as a symmetry in some privileged systems we call "inertial frames" (Thomson, 1884). For example, the symmetry inherent to Newton's third law is expressed as the equation $F_{12} + F_{21} = 0$ being invariant in front of Galilean changes of coordinates (where F_{ij} is the force on body j originated in the interaction with body i). Yet, we know at least since Poincaré (1900) (see (Solari and Natiello, 2018) as well) that Newton's "action and reaction law" is not compatible with delayed action at distance. As far as we know, the form this symmetry takes in EM has not been shown so far. We will display its effects in the present work.

When EM theory is moved from its original setting as an *interaction theory* into a *field theory*, some symmetry is broken since there are no longer two EM-bodies in reciprocal action but we are thereafter concerned with only one of them, most frequently the *source*. This presentation of EM may be called the *S-field*. With equivalent arbitrariness we could shift the focus to the receiver and consider an *R-field* description. Both descriptions refer to the same EM phenomena and are therefore related.

When the S-field, the field produced by the source, is perceived by the source itself or by any extended EM-body not moving with respect to the source, we call it *S-by-S-field*. When considering the same S-field as it is perceived by the receiver, we have the *S-by-R-field* description, see Figure 1 (see Figure 2 for the corresponding *R-field* description). The operation performed on the description of the phenomenon is to identify one body or the other with an extended EM-body in the reference frame of the observer. As both approaches describe the same action, a transformation, possibly dependent on the relative velocity between the EM-bodies, must relate their expressions.

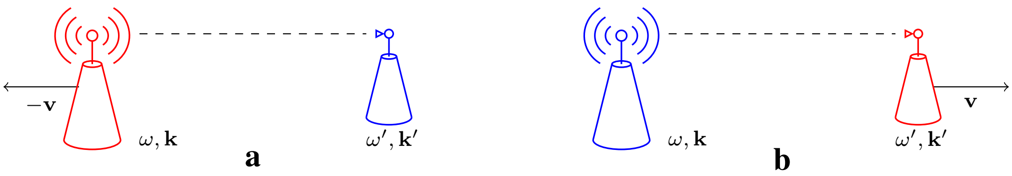

Figure 1: Field of the source (a) as seen by the receiver (S-by-R-field) and (b) as seen by the source (S-by-S-field). Source to the left of each image. In blue: the device at rest with the observer.

For the case of multiple receivers we may want to consider the relation among the different S-by-R_i-field descriptions of each receiver. To connect R_1 with R_2 corresponds to the composition of the transformations between each receiver and the source, namely $R_1 \rightarrow S$ and (the inverse of) $R_2 \rightarrow S$. The composition of transformations yields a transformation between receivers, that will depend on the relative velocities of R_1 and R_2 with respect to the source. However, receiver-receiver transformations relate objects of equivalent character, they are automorphisms and must form a group as well.

Figure 2: Field of the receiver (a) as seen by the receiver (R-by-R-field) and (b) as seen by the source (R-by-S-field). Source to the left of each image. In blue: the device at rest with the observer.

The perceived fact that electromagnetic disturbances require some time to propagate between source and receiver is acknowledged by all existing theoretical frameworks of EM. To describe this fact, the concept of *delayed action at a distance* was advanced in an organised form by the Danish scientist Ludwig Lorenz (Lorenz, 1867) after preliminary attempts (Betti, 1867; Riemann, 1867; Neumann, 1868) from the Göttingen school originated by ideas of Gauss (bd.5 p. 627-629, Gauss, 1870).

Returning to relative motion, it must be noticed that even in the case where source and receiver are in constant relative motion, the transformation between the S-by-S-field and S-by-R-field will not be an inertial transformation (i.e., a Galilean coordinate change). Galilean transformations correspond to descriptive transformations that are not concerned with the observable relative motion of the bodies. The relative motion of source and receiver is a measurable part of the physics involved and not an arbitrariness (it is there independently of the observer). Consider the following experiment: a source is producing a signal sharply peaked around a given frequency, ω_0 as perceived by a receiver not moving with respect to the source. A set of several, identically built and calibrated receivers are put in motion at various velocities, v_i, with respect to the source, see Figure3. How is the signal perceived by each receiver? Which is the perceived characteristic frequency ω_i ? Which is the relation between the signals registered by the various receivers?

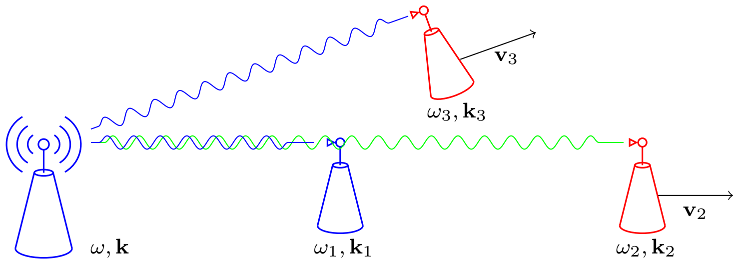

Figure 3: Sources and receivers. Blue receiver at rest relative to source, red receivers in relative motion with respect to the source.

3 Relational Electrodynamic Background

3.1 Interaction-based relational formulation.

In the presence of electromagnetic interactions, the observable effects of the interaction can be interpreted as the result of the action of the *Lorentz force* (Lorentz, 1892; Natiello and Solari, 2021) over the electrified particles that constitute matter.

The origins of the Lorentz force can be traced back to Maxwell and what he called the *Electromotive intensity* ([598], Maxwell, 1873). Similarly, Lorentz referred to Maxwell's electrokinetic and potential energies [630,631] and [634,635], Maxwell, 1873, combining them in an action integral and the principle of least action. These presentations take support in Maxwell equations,

$$\begin{aligned} B &= \nabla \times A & (1) \\ E &= -\frac{\partial A}{\partial t} - \nabla V & (2) \\ \epsilon_0 \nabla \cdot E &= \rho & (3) \\ \mu_0 j + \frac{1}{C^2}\frac{\partial E}{\partial t} &= \nabla \times B & (4) \end{aligned}$$

although their derivations some way or the other involved the ether in the argumentation: Maxwell when considering the "total current" of eq.(4) and Lorentz in the variational principle.

Ludwig Lorenz avoided to introduce the ether by acknowledging that light was a form of EM interaction and it corresponded with a transversal wave (Lorenz, 1861, 1863), later introducing retarded electromagnetic potentials (Lorenz, 1867) inspired in Franz Neumann(Neumann, 1846)[4],

$$(A, \frac{V}{C})(x,t) = \frac{\mu_0}{4\pi} \int \left(\frac{(j, \rho C)(y, t - \frac{1}{C}|x-y|)}{|x-y|} \right) d^3y, \tag{5}$$

as an expression based upon these observations, and also on Neumann's results and Kirchhoff results regarding EM waves in conductors which make ample use of the continuity equation, $\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$. The displacement equation (4) can be derived from Equation (5) and the continuity equation. It is everywhere assumed that the current-charge vanishes rapidly enough at infinity (so that the partial integrations usually present in EM theory can actually be performed).

In terms of differential equations, Eq. (1) and (2) are definitions of the magnetic and electric fields and the main constitutive equation reads

$$\Box(A, \frac{1}{C}V) = -\mu_0(j, C\rho). \tag{6}$$

where $\Box = \Delta - \frac{1}{C^2}\frac{\partial^2}{\partial t^2}$ is the D'Alembert operator. This equation is satisfied also by:

$$(\widetilde{A}, \frac{\widetilde{V}}{C})(x,t) = \frac{\mu_0}{4\pi} \int \left(\frac{(j, \rho C)(y, t + \frac{1}{C}|x-y|)}{|x-y|} \right) d^3y. \tag{7}$$

The potentials A, V describe the relation between current-density j or a charge-density ρ with their electromagnetic effect. The standard interpretation is that

[4]Maxwell's results have the same starting point in Neumann's work ([542], Maxwell, 1873)

$(j, C\rho)$ are the source (the primary circuit) of the EM action while the potentials are intermediate fields that indicate their action over the secondary circuit, corresponding to delayed action; this is, $(A, \frac{V}{C})$ are source fields, S-fields. A different association is possible for $(j, C\rho)$; they can be interpreted as those corresponding to the secondary circuit and in such case $(\widetilde{A}, \frac{\widetilde{V}}{C})$ are the R-fields that sense an EM perturbation away from the receiver and express its effect later in it, this is, they are advanced fields.

When relevant, we use the indices $1(2)$ for the source (receiver). It is possible to perform a derivation of the Lorentz force (Natiello and Solari, 2021) from the Principle of Least Action supported in Maxwell's energy considerations following Lorentz but using mathematical deduction at the few situations where Lorentz used arguments corresponding to the ether in (Lorentz, 1892). Let $\bar{x}(t)$ denote the distance between a reference point in the source and a reference point in the receiver. We will consider situations where source and receiver move as rigid bodies in relative motion (but not in relative rotation) as Lorentz did.

In what follows, z denotes a "local" coordinate on body 2. We consider, following Lorentz, a collection of virtual displacements parametrised by time $\delta\bar{x}(t)$ [5]. The variation of charge and current densities $\rho_2(z,t), j_2(z,t)$ on the receiver can be expressed in the coordinates of eq.(5) as:

$$\begin{aligned} \delta\rho_2(x,t) &= (-\delta\bar{x}(t)\cdot\nabla)\,\rho_2(x,t) \\ \delta j_2(x,t) &= (-\delta\bar{x}(t)\cdot\nabla)\,j_2(x,t) + \delta\dot{\bar{x}}\rho_2(x,t) \end{aligned} \tag{8}$$

The latter relates the local expression of charge and current densities in the secondary circuit and the same physical object in terms of the coordinates associated to the primary circuit.

Maxwell considers the electrokinetic and potential energies, which Lorentz further combines in the action integral

$$\mathcal{A} = \frac{1}{2}\int dt \int \left(A_1(x,t)\cdot j_2(x,t) - \rho_2(x,t)\,V_1(x,t)\right)\,d^3x \tag{9}$$

that here represents the interaction energy between a source or primary circuit labelled 1 and a receiver or secondary circuit labelled 2. The relation 6 is satisfied for fields and current-charge corresponding to the same index. The action integral in the present form corresponds to an S-by-S-field representation, namely that the

[5] As in the Lagrangian formulation, the collection of virtual displacements is differentiable, i.e., $\dot{\bar{x}}$ exists, and the variation is zero in the time extremes. Virtual displacements are not the same as time-dependent perturbations of the position, for the latter have other effects apart from the change of relative distances. Virtual displacements are closer to changes of initial conditions than to perturbations. In particular, during a virtual displacement, there is no wave progression.

fields of the source are evaluated at the position of the receiver in the coordinates x of the source and time t.

We state the result as a theorem:

Theorem 3.1. *((Natiello and Solari, 2021)) Assuming that all of $|B|^2$,$|E|^2$, A, j, V, ρ decrease faster than $\frac{1}{r^2}$ at infinity, assuming the action is given by eq. 9 and given the validity of the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$, the electromagnetic force*

$$F_{em} = \int d^3x \, [j_2(x,t) \times B_1(x,t) + \rho_2(x,t) \, E_1(x,t)]$$

on the probe can be deduced from Hamilton's principle of minimal action ($\delta_{\bar{x}(t)}\mathcal{A} = 0$) using a virtual displacement $\delta_{\bar{x}}$ of the probe (which we indicate with subindex 2), eq.(8) with respect to the primary circuit producing the fields (subindex 1).

3.2 Wave equation for the potentials

The wave equation for the potentials can be deduced from Equations (1-4).

Lemma 3.1. $A(x,t) = \frac{\mu_0}{4\pi} \int_U \left(\frac{j(y, t - \frac{1}{C}|x-y|)}{|x-y|} \right) d^3y \Rightarrow \Box A = -\mu_0 j$*, and similarly for* $\epsilon_0 \Box V = -\rho$*, where* $\Box \equiv \Delta - \frac{1}{C^2}\frac{\partial^2}{\partial t^2}$.

For a proof, see Appendix 3.1. Note that this result describes a property of eq.(5), independently of whether A, V, j, ρ are the electromagnetic vector potential and current, etc., or not. We prove now that the result holds for the electromagnetic A, V, j, ρ, via a variation of the electromagnetic action 9:

Theorem 3.2. *Let* (A, V) *be the known values of the electromagnetic potentials in a piece of matter supported on a region of space with characteristic function* χ*. Then, assuming that all of* $|B|^2, |E|^2, A, j, V, \rho$ *decrease faster than* $\frac{1}{r^2}$ *at infinity, Hamilton's principle of least action (Ch 3, 13 A p. 59, Arnold, 1989),* $\delta\mathcal{A} = 0$*, subject to the constraints given by* (A, V) *implies the relations*

$$\begin{aligned}
\frac{1}{\mu_0}\nabla \times B - \epsilon_0 \frac{\partial E}{\partial t} &= \mu_0 j \\
\epsilon_0 \nabla \cdot E &= -\frac{\rho}{\epsilon_0}. \\
\nabla \cdot j + \frac{\partial \rho}{\partial t} &= 0
\end{aligned}$$

Corollary 3.1. *In the special case when the relation* $\nabla \cdot A + \frac{1}{C^2}\frac{\partial V}{\partial t} = 0$ *(the "Lorenz gauge") is satisfied, the manifestation of the potentials outside matter obeys the wave equation, eq.(6).*

We develop the proof in Appendix 3.2.

The theorem deserves to be named Lorenz-Lorentz theorem since in Lorenz conception light was associated to the EM activity inside matter (Lorenz, 1867) and Lorentz proposed the expression for the action based on Maxwell's energy considerations.

Recasting the potentials of eq.(5) as the convolution of charge and currents with the *Lorenz kernel* hereby defined:

$$K(x-y, s-r) = \frac{1}{|y-x|}\delta(s-r-\frac{1}{C}|y-x|),$$

namely

$$(A_1, \frac{V_1}{C})(x,s) = \frac{\mu_0}{4\pi}\int \left[\int_{-\infty}^{s} K(x-y,s-r)(j_1, C\rho_1)(y,r)\right] d^3y\, dr \qquad (10)$$

a fundamental symmetry between potentials and wave operators is expressed in the following

Lemma 3.2. *The action of the kernel* $K(x-y, s-r)$ *and the differential operator* $\Box$ *are reciprocally inverse of each other.*

Proof. We discuss the proof using A to fix ideas, and write eq.(10) in shorthand as $A = \frac{\mu_0}{4\pi}K * j$ (where the star stands for convolution). Composition with $\Box$ gives:

$$\begin{aligned} \Box A &= \frac{\mu_0}{4\pi}\Box K * j = -\mu_0 j \\ K * \Box A &= -\mu_0 K * j = -4\pi A. \end{aligned}$$

Hence, in their respective domain of definition $\Box K = -4\pi Id$ (convolution identity) and $K * \Box = -4\pi Id$ (operator identity). $\Box$

3.3 Source/receiver symmetry of the action

Since the action (9) plays a fundamental role in this relational presentation we should devote some lines to consider its symmetries.

We first write the action in terms of definite integrals and the kernel $K(x-y, s-r)$

$$\mathcal{A} = \frac{1}{2}\frac{\mu_0}{4\pi}\int_{t_0}^{t} ds \int_{t_0}^{t} dr \iint K(x-y,s-r)\left(j_1 \cdot j_2 - C^2\rho_1\rho_2\right) d^3x\, d^3y \qquad (11)$$

The form of the action in eq.(11) is almost symmetric in terms of exchanging primary and secondary circuits. Interchanging primary and secondary circuit, and $(x, s) \longleftrightarrow (y, r)$ the kernel changes into

$$K(x-y, s-r) = \frac{1}{|y-x|}\delta(s-r+\frac{1}{C}|y-x|) \tag{12}$$

Thus, the action considered is always the action of the primary circuit over the secondary circuit which can be written in two forms. In one of them, the S-field (the standard form), EM changes are propagated with delay by the potentials (and their derivatives, the EM-fields) at distances away from the source. The symmetry-related form, the R-field, associates an advanced field with the receiver. In this form, the field can be seen as a sensor that will carry disturbances to the receiver that will display changes at a later time.

The symmetry of the action has the immediate consequence that all lemmas and theorems of subsections (3.1) and (3.2) have an equivalent form under this symmetry operation. In particular, there is Lorentz-force where the S-fields, R-currents and R-charges are exchanged by R-fields, S-currents and S-charges. This relation is what corresponds to the action and reaction law for actions that propagate instantaneously, since in the limit $C \to \infty$ the S-field and the R-field of a given body/device coincide.

3.4 Detection/perception in relative motion

Let us consider the potentials A, V originated in a source with current-charge $J = (j, C\rho)$ measured at (rest relative to) the source (with coordinate y). We consider further a detector extending over a variable x with reference to a distinguished point in it. In the case of source and detector at relative rest, we write

$$\begin{aligned}(A, \frac{V}{C})(x,t) &= \frac{\mu_0}{4\pi}\int d^3y \int ds \left(\frac{\delta((t-s)-\frac{1}{C}|x-y|)}{|x-y|}\right) J(y,s) \quad (13)\\ &= \frac{\mu_0}{4\pi}\int d^3z \left(\frac{J(x-z, t-\frac{1}{C}|z|)}{|z|}\right) \quad (14)\end{aligned}$$

These equations are formulated under the following premises: Coordinates y and x are described from the same spatial reference system S, whatever it is, and hence at a given time t, $x - y$ and in particular $|x - y|$ are objective invariant quantities. Moreover, since source and detector are in relative rest, these quantities are independent of t. In the present conception of electromagnetism there is another objective invariant quantity of relevance, namely the electromagnetic delay $\Delta_0 = t - s$. The index 0 highlights the situation of relative rest between source and detector. It is the state of point y on the source at the previous time s, where $C(t-s) = |x-y|$ what connects with point x of the detector at time t. Finally, the second row displays the change of variables $z = x - y$.

In order to address detection in relative motion we advance the following

Conjecture 3.1. *A detector recording solely electromagnetic information (e.g. an electromagnetic wave) cannot determine its relative velocity with respect to the source (assumed constant).*

Consequently, let us postulate that a detector in relative motion with velocity v with respect to the source perceives an EM wave which cannot be distinguished from the one originating in some current-charge *at relative rest*. We would like to show something like:

$$(A, \frac{V}{C})_v(x,t) \;=\; \frac{\mu_0}{4\pi}\int d^3y \left(\frac{1}{|x-y|}\right) J_v(y, t-\Delta) \tag{15}$$

with $\Delta = \frac{1}{C}|x-y|$.

In this new situation we still have one reference frame S to describe both source and detector. Again, $z = x - y$ is an objective quantity, only that now two differences arise: (a) $x-y$ depends on t because of the relative motion and (b) the electromagnetic delay may be modified in order to take into account the relative motion. Throughout this discussion, t is the (present) time when the electromagnetic interaction is detected, $(x-y)$ indicates the relative position of (points of) detector and source at time t, $\Delta_v = (t-s)_v$ is the electromagnetic delay and $(x-y)_v$ is the corresponding relative position at time s when the electrical disturbance in the source took place, and the index $v \in \mathbb{R}^3$ indicates a situation of relative motion between source and detector. The index v will be some function of the relative velocity u between source and detector to be determined in what follows. Moreover, $(x-y)_v$ and Δ_v are objective and invariant quantities, independent of the choice of reference frame.

We intend to find the correspondence between disturbances in the primary circuit and actions on the secondary system. We begin by considering an infinitesimal velocity δv, with $\frac{d\delta v}{dt} = 0$. In this case we have

Definition 3.1. ***(Differential delayed interaction condition)*** *In the presence of relative motion with infinitesimal velocity δv, a disturbance originated at point y and time $t - \Delta_{\delta v}$ produces an electromagnetic action at (x,t), where*

$$C\Delta_{\delta v} \;=\; |x - y - \Delta_{\delta v}\delta v|.$$

For $\delta v = 0$ the condition reduces to $C\Delta_0 = |x-y|$, corresponding to Lorenz' potentials, eqs.(5) and (10)[6]. Note that C enters in both expressions since we

[6]Letting $s = t - \Delta_{\delta V}$ we may read the definition as a consequence of: $(x-y)(s) = (x-y)(t) - (t-s)\delta v$.

postulate that the detector in relative motion registers an electromagnetic signal *as if the source were at relative rest*. This definition leads to the following

Lemma 3.3. *Let $(x-y)_v$ be the separation of source and detector at time s when the electrical disturbance at the source took place in a situation of relative motion labelled by $v \in \mathbb{R}^3$ and Δ_v the corresponding electromagnetic delay, while $(x-y)_0, \Delta_0$ are the corresponding quantities for source and detector at relative rest. Then, for each v the* ***delayed interaction condition*** *satisfies*

$$\begin{pmatrix} (x-y) \\ C\Delta \end{pmatrix}_v = \exp\left(-\begin{pmatrix} \mathbf{0} & \dfrac{v}{C} \\ \dfrac{v}{C}^T & 0 \end{pmatrix}\right)\begin{pmatrix} (x-y) \\ C\Delta \end{pmatrix}_0$$

Proof. To lowest order in δv the difference in Δ's is:

$$\begin{aligned} C\left(\Delta_{\delta v} - \Delta_0\right) &= \sqrt{|x-y|^2 - 2\,(x-y)\cdot\delta v\Delta_{\delta v} + |\delta v|^2\Delta_{\delta v}^2} - |x-y| \\ &= -\frac{(x-y)}{|x-y|}\cdot\delta v\Delta_0 + O(\delta v^2) = -\,(x-y)\cdot\frac{\delta v}{C} + O(\delta v^2) \end{aligned}$$

In this limiting case the condition reads

$$\begin{aligned} \begin{pmatrix} (x-y)_{\delta v} \\ C\Delta_{\delta v} \end{pmatrix} &= \begin{pmatrix} (x-y) - \delta v\Delta_0 \\ C\Delta_0 - (x-y)\cdot\frac{\delta v}{C} \end{pmatrix} \\ &= \left[\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} - \begin{pmatrix} 0 & \dfrac{\delta v}{C} \\ (\dfrac{\delta v}{C})^T & 0 \end{pmatrix}\right]\begin{pmatrix} (x-y) \\ C\Delta_0 \end{pmatrix}. \end{aligned} \tag{16}$$

In other words, there exists an infinitesimal transformation on $\mathbb{R}^{3+1}$ connecting the condition for $v = 0$ with that for δv. By the Trotter product formula we obtain Lie's result for finite v as a repeated composition of infinitesimal shifts,

$$\begin{aligned} TL(-v) &\equiv \exp\left(-\begin{pmatrix} \mathbf{0} & \dfrac{v}{C} \\ \dfrac{v}{C}^T & 0 \end{pmatrix}\right) \\ &= \lim_{n\to\infty}\left[\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} - \frac{1}{n}\begin{pmatrix} \mathbf{0} & \dfrac{v}{C} \\ \dfrac{v}{C}^T & 0 \end{pmatrix}\right]^n \end{aligned} \tag{17}$$

thus proving the statement. □

Remark 3.1. *Explicit formulae for the Lorentz transformations are shown in the Appendix B. The more familiar form* $L(u)$ *of the transformation is displayed in*

$$\left(\begin{array}{c} z_u \\ C\Delta_u \end{array}\right) = L(u)\left(\begin{array}{c} z \\ C\Delta_0 \end{array}\right) = \left(\begin{array}{c} z + (\gamma - 1)\hat{u}\,(\hat{u}\cdot z) + \gamma \frac{u}{C} C\Delta_0 \\ \gamma \left(C\Delta_0 + \frac{u \cdot z}{C} \right) \end{array}\right), \quad (18)$$

where $u = C\hat{v}\tanh\left|\frac{v}{C}\right|$ *and we use the shorthand* $x - y = z$*. There is a 1-to-1 correspondence in Lemma 3.3, between the two presentations of the Lorentz transformations, namely* $TL(-v) \equiv L(-u)$*. Hence, we will use only* u *in the sequel.* u *is interpreted as the relative velocity between source and detector.*
The basis for the interpretation of u *as the relative velocity is as follows. Consider the vector space* $\mathbb{R}^{3+1} \equiv \mathbb{R}^3 \times \mathbb{R}$ *associated to relative positions and relative time. A Lorentz transformation (LT), eq.*(18)*, as well as a Galilean transformation GT,*

$$\left(\begin{array}{c} Z' \\ T' \end{array}\right) = \left(\begin{array}{cc} \mathbf{1} & u \\ 0 & 1 \end{array}\right)\left(\begin{array}{c} Z \\ T \end{array}\right)$$

can be regarded as endomorphisms of $\mathbb{R}^{3+1}$ mapping a situation at relative rest onto a situation of relative motion. While the velocity u in the GT has a mechanical origin, the parameter u in LT is an abstract parameter used to classify transformations and a point of contact with the underlying physical problem is required to furnish a physical interpretation to the LT's. Considering lines on $\mathbb{R}^{3+1}$ associated to a fixed relative position, Z and different time-intervals, we obtain for the Galilean transformation the (physical) relative velocity $u = \frac{Z'(T_1)-Z'(T_0)}{T'(T_1)-T'(T_0)}$ while in the case of the Lorentz transformation we obtain

$$\frac{z'(\tau_1) - z'(\tau_0)}{\tau'(\tau_1) - \tau'(\tau_0)} = \frac{\gamma u(\tau_1 - \tau_0)}{\gamma(\tau_1 - \tau_0)} = u.$$

While the GT preserves times and as such can be viewed as a transformation in relational-space only, the LT preserves $|z|^2 - (C\tau)^2$ and, as a particular case, the condition of being in electromagnetic contact, $|z|^2 - (C\tau)^2 = 0$. We may associate the same relational velocity to both GT and LT.

Eq.(17) displays the action of a Lorentz' boost (Gilmore, 1974) in the Lie algebra (rhs) and group (lhs). The generators of the Lorentz boosts plus the generators of the rotations constitute the basis of the Lie algebra which exponentiated gives the Poincaré-Lorentz group. While the spatial rotations form a subgroup of the Poincaré-Lorentz group, the Lorentz boosts do not. Any element of the Poincaré-Lorentz group can be written as a product: $P = L(u)R(\Omega)$ as well as $P = R(\Omega)L(u')$ being Ω a 3d-rotation and $u' = R(\Omega)u$. These forms are known as left and right coset decompositions of the group (Hamermesh, 1962; Gilmore, 1974).

Remark 3.2. *By construction of the $LT's$,there is an upper limit for having electromagnetic contact amenable to be related with situations at relative rest. While there is no mechanical limit to relative velocity, the present theory describes electromagnetic interactions only for $|u| < C$.*

Remark 3.3. *Eqs.16 and 18 for the detector and source points, x, y which are in electromagnetic interaction at time t, display their relative position $(x-y)_u$ at the time $t-\Delta_u$ when the disturbance in the source took place. The ratio $\frac{|(x-y)_u|}{\Delta_u} = C$ is always satisfied by construction.*

Next, we note that the propagation kernel can be more properly written as

$$K = \begin{cases} 0, & (t-s) < 0 \\ \dfrac{\delta(t-s-\frac{1}{C}|x-y|)}{|x-y|}, & (t-s) \geq 0. \end{cases}$$

Hence, we have the following

Lemma 3.4. ***(Symmetric form of the propagation kernel)*** *Lorenz propagation kernel can be rewritten as*

$$K = \begin{cases} 0, & (t-s) < 0 \\ \dfrac{2}{C}\delta((t-s)^2 - \dfrac{1}{C^2}|x-y|^2), & (t-s) \geq 0. \end{cases} \tag{19}$$

Proof. In the distribution sense $K = \dfrac{2|x-y|}{C(t-s)+|x-y|}K$. By another distributional property, for any $g(s)$ such that $g(s_0) \neq 0$ it holds that $\dfrac{\delta(s-s_0)}{|g(s)|} = \delta(g(s)(s-s_0))$. In this case, $g(s) = t - s + \dfrac{1}{C}|x-y|$. Hence, we obtain the symmetric kernel expression of eq.(19). □

Theorem 3.3. *The Lorenz propagation kernel $K(x, t; y, s)$ has the following properties in relation to Lorentz transformations*

$$\begin{aligned} K(L_u(x,t); L_u(y,s)) &= K(x,t;y,s) \\ K(L_u(x,t); y,s) &= K(x,t;L_{-u}(y,s)) \\ \int d^3y\, ds\, [K(L_u(x,t);y,s)J(y,s)] &= \int d^3y\, ds\, [K(x,t;y,s)J(L_u(y,s))] \end{aligned}$$

The last equation reads: the transformation of the potentials are the potentials associated to the transformations of the currents. We say then that the linear operator associated with K commutes with the Lorentz transformation.

Proof. It is straightforward to verify that the argument of the δ-distribution in eq.(19) is invariant upon Lorentz transformations, namely that if $((x-y)_u, C(t-s)_u)$ satisfy eq.(18), then $(t-s)^2 - \frac{1}{C^2}|x-y|^2 = (t-s)_u^2 - \frac{1}{C^2}|(x-y)_u|^2$ and also $(t-s) \geq 0 \iff (t-s)_u \geq 0$. Thus,

$$K = \begin{cases} 0, & (t-s)_u < 0 \\ \frac{2}{C}\delta((t-s)_u^2 - \frac{1}{C^2}|\,(x-y)_u\,|^2), & (t-s)_u \geq 0. \end{cases}$$

is independent of u. Using the first property it follows that $K(L_u(x,t); y,s) = K(L_u(x,t; L_u L_{-u}(y,s)) = K(x,t; L_{-u}(y,s))$. The commutation relation is the result of integrating the kernel to produce a linear operator and changing integration variables $((y,s) \mapsto L_u(y',s')$. □

Remark 3.4. *The points that are in electromagnetic connection are characterised by* $(C(t-s))^2 - |x-y|^2 = 0$*. Calling* $\tau_u \equiv (t-s_0)_u$ *and* $\chi_u \equiv (x-y)_u$*, the interaction kernel is the convolution kernel of* $\delta(\tau_u^2 - (\chi_u/C)\hat{}2)$ *which can be split in two contributions, one for* $\tau_u \geq 0$ *and another for* $\tau_u \leq 0$*. But, if* $(0,0)$ *is influencing* (τ_0, χ_0) *for* $\tau_0 \geq 0$*, it results that* $\tau_u > 0$ *(using that* $|u \cdot x/C^2| = \frac{|u\cdot\chi|}{|\chi||u|}\frac{|u||\chi|}{C^2} < \frac{|u||\chi|}{C^2}$*) hence the splitting is really in terms of* influencing*,* $\tau_u \geq 0$*, vs.* being influenced, $\tau_u \leq 0$*. This separates the sets in a form invariant with respect to* u*.*

3.4.1 Perceived fields and inferred currents-charges

Examining eq.(15), we note that it represents a convolution product with convolution kernel $\kappa(z,r)$, with $K(x,t;y,s) = \kappa(x-y, t-s)$ and that

$$(A, \frac{V}{C})_u = \frac{\mu_0}{4\pi}\kappa * J_u = \frac{\mu_0}{4\pi}J_u * \kappa$$

where the convolution is in time and space.

According to eq.(16), the arguments in the current are $(x-y, t-s)$, for $u = 0$. For $u \neq 0$ the points that are in electromagnetic relation according to Lemma 3.3 are $((x-y)_u, (t-s)_u)$, thus in $J_u * \kappa$, we propose

Conjecture 3.2. *The arguments of the effective current are* $((x-y)_u, (t-s)_u)$*, i.e.,* $J_u = L(-u)J(L(u)(x-y, t-s))$*, where* J *is the current-charge measured by the source.*

At this point we must notice that there are three forms in which current-charge can be transformed to produce a new pair satisfying the continuity equation. Two of them are Galilean:

$$(j, C\rho)(x,t) = (j - v\rho, C\rho)(x+vt,t) \tag{20}$$

$$(j, C\rho)(x,t) = (j, C\rho - \frac{v}{C}\cdot j)(x, t+\frac{v\cdot x}{C^2}) \tag{21}$$

$$(j, C\rho)_u(x,t) = L(-u)(j, C\rho)(L(u)(x,t)) \tag{22}$$

In the third form, the leftmost L acts on the charge-current $4D$-vector while the rightmost acts on the space-time coordinates.

If the form (20) is adopted, a theorem due to Maxwell ([602] Maxwell, 1873) shows that from the point of view of the receiver the transformation (21) must be applied to preserve the mechanical force but in such case the perceived potentials/fields are not waves. The empirical evidence has judged this view as not correct.

We propose to adopt eq.(22) as a definition of the inferred current. We insist at this point that the symmetry is not an a-posteriori observation of the formulae, but rather an a-priori demand of constructive reason as explained in (Solari and Natiello, 2018). The transformation of current-charge presents itself as a demand of reason to be later confronted with empirical results. That a charge density in motion can be perceived as a current is a belief firmly adopted since Weber's electrodynamic studies (Weber, 1846) and we are habituated to accept it, while that a neutral current in motion will be perceived as charge is not rooted in our beliefs in the same way, despite the fact that Maxwell's theorem already opened for that possibility.

Remark 3.5. *The symmetric form of K is especially appealing when consider the backwards propagation kernel, as in the equation pairs (5)–(7) and (10)–(12). The backward propagation kernel is the result of inverting the time inequalities in 19.*

Remark 3.6. *Which is the meaning of a successive application of Lorentz' transformations to a current? The meaning we find apt is that if $J_u = L_{-u}J(L_u(x,t))$, then $J = L_uJ_u(L_{-u}(x,t))$ (since Lorentz transformations have as inverse the transformation based on minus the velocity) and correspondingly $J_{u'} = L_{-u'}L_uJ_u(L_{u'}L_{-u}(x,t))$. Since $L_{u'}L_{-u}$ is a general element of the Poincaré-Lorentz group, $L_{u'}L_{-u} = L_{u'\ominus u}R(u',u)$ with $u'\ominus u$ the coset addition of velocities, also known as Einstein's addition (Gilmore, 1974) and $R(u',u)$ a Wigner rotation[7] . Thus, the Poincaré-Lorentz group allows to convert between inferred*

[7] Wigner was not the first to study the group structure associated to Lorentz transformations or to mention the rotation. At least Silberstein (Silberstein, 1914, p. 167) in the published notes of his 1912-1913 course on Relativity at the University College, London, preceded Wigner, who acknowledged this precedence.

currents or fields associated to different detectors in relative motion with respect to the same source. Notice that the relative velocity between both receptors is $u' - u$ but the correspondence of electromagnetic perceptions is not $L(u' - u)$ which might even not exist.

Next, we explore the consequences of this proposal. Let us define operators acting on scalar or vector functions, J, of (x, t) as

$$\begin{aligned} \widehat{K}\left[J\right](x,t) &\equiv \iint d^3z\, d\Delta\, K(z,\Delta)J(x-z,t-\Delta) \\ \widehat{L}_u\left[J\right] &\equiv J(L(u)(x,t)) \\ (\widehat{A}\circ\widehat{B})[J] &\equiv \widehat{A}\left[\widehat{B}[J]\right] \end{aligned} \tag{23}$$

The first line defines the action of the propagating kernel as a convolution, the second the action of a Lorentz transformation on the coordinates (recall that $u = Cv \tanh\left|\frac{v}{C}\right|$) while the third relation establishes notation.

Lemma 3.5. *According to the previous discussion, the perceived potentials read*

$$(A, \frac{V}{C})_u = \widehat{K}\left[J_u\right] \tag{24}$$

In addition, we have the following identities

$$\widehat{K}\left[J_u\right] = L(-u)\widehat{L}_u\left[\widehat{K}\left[J\right]\right]$$

Proof. Note that $L(-u)$ acts on the current-charge $J = (j, C\rho)$, while $\widehat{L}_u$ acts on the spatial/temporal arguments x, Ct. Eq.(24) is just eq.(15) rewritten through eq.(23). Recalling from eq.(22) that $J_u = L(-u)\widehat{L}_u\left[J\right]$ and from 3.3 that $\widehat{L}_u\circ\widehat{K} = \widehat{K}\circ\widehat{L}_u$ and finally that the matrix $L(-u)$ commutes with the scalar operator K we obtain the result. □

3.4.2 The Doppler effect

The perception of wave frequencies in the case the waves are produced by a source in relative motion with respect to the receptor is known as *Doppler effect*. The EM Doppler effect plays a fundamental role in physics (Dingle, 1960; Mandelberg and Witten, 1962; Kaivola et al., 1985). The goal of this section is to show that the present theory provides an explanation for the experimental observations of the Doppler effect. To begin with, all Doppler experiments consist in comparing the waves perceived by a detector at rest with respect to the source against the perception of a detector moving at constant velocity (within acceptable experimental precision) relatively to the source.

In practice, the task is to obtain the Fourier transform of eq.(15). We will keep track of this process conceptually, and hence it is better to use the operator notation from Lemma 3.5. The Fourier transform of a function will be:

$$\mathcal{F}_{k,w}\left[\phi\right] = \frac{1}{(2\pi)^2}\iint d^3x\, dt\, \exp\left(-i(k\cdot x - wt)\right)\phi(x,t)$$

and is a function of (k, w), where we have made an arbitrary choice in the election of the sign preceding wt (that does not influence the conclusion). We will use the following known results:

$$\begin{aligned}
\mathcal{F}_{k,w}\left[\widehat{L}_u\phi\right] &= \mathcal{F}_{k',w'}\left[\phi\right] \text{ , with } (k', \frac{w'}{C}) = L(-u)(k, \frac{w}{C}) \\
\mathcal{F}_{k,w}\left[\widehat{K}\phi\right] &= \frac{1}{w^2 - C^2k^2}\mathcal{F}_{k,w}\left[\phi\right]
\end{aligned}$$

The first result is the immediate consequence of $L(u)$ being symmetric, while the second one can be obtained in various ways including direct integration. Applying these results to eq.((24)) we obtain

$$\begin{aligned}
\mathcal{F}_{k,w}\left[\widehat{K}[J_u]\right] &= \mathcal{F}_{k,w}\left[\widehat{K}\left[L(-u)\widehat{L}_u\left[J\right]\right]\right] \\
&= L(-u)\mathcal{F}_{k,w}\left[\widehat{L}_u\left[\widehat{K}\left[J\right]\right]\right] \\
&= L(-u)\mathcal{F}_{k',w'}\left[\widehat{K}\left[J\right]\right] \\
&= L(-u)\frac{1}{w'^2 - C^2k'^2}\mathcal{F}_{k',w'}\left[J\right] \\
&= L(-u)\frac{1}{w^2 - C^2k^2}\mathcal{F}_{k',w'}\left[J\right]
\end{aligned}$$

where $(k', \frac{w'}{C}) = L(-u)(k, \frac{w}{C})$. Thus, in terms of wave frequencies, the Fourier spectrum will have a peak at $w' = \gamma(u)(w - k\cdot u)$ associated with a source of frequency w. The primed quantities describe the characteristics of the wave as perceived by the detector while the unprimed refer to the source. When $k\cdot u = |k||u|$ the relative distance between source and detector increases, $w' < w$, and correspondingly the wavelength shifts towards higher values (red shift).

Hence, we have proved the following

Theorem 3.4. ***(Doppler effect)*** *A detector (observer) in relative motion with velocity u with respect to an electromagnetic source emitting current-charge waves* *of wavelength and frequency* (k, w) *detects electromagnetic waves of wavelength and frequency* $(k', \frac{w'}{C}) = L(-u)(k, \frac{w}{C})$.

Remark 3.7. *The symmetry* (25) *corresponds to expressing the action in terms of the inferred charge and currents by an observer. As such, it corresponds to a subjective view of EM.*

Remark 3.8. *The Galilean variation that allowed us to obtain the Lorentz force from the action, eq.*(8)*, indicates that the force experienced by the moving circuit takes the same form but the potentials to be used correspond to the perceived potentials of eq.*(15).

3.5 Mathematical presentation of the Lorentz transformation as a symmetry

Since Lorentz' transformations are well known in relation to electromagnetism, we consider their effect on the action and find their meaning in the present context.

Let $\mathcal{I}$ be the infinitesimal generator for the Lorentz transformation

$$\mathcal{I}_j = \left(Ct\frac{\partial}{\partial x_j} + \frac{x_j}{C}\frac{\partial}{\partial t} \right) \tag{25}$$

which together with the generators of the rotations

$$\mathcal{J}_i = \sum_{jk} \epsilon_{jki} \left(x_k \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_k} \right)$$

(with ϵ_{jki} Kronecker's antisymmetric tensor) complete the Lie algebra of the Poincaré-Lorentz group (Gilmore, 1974).

Theorem 3.5. *The electromagnetic action* $\mathcal{A}$ (9) *transforms into an equivalent action* $\mathcal{A}'$ *when the infinitesimal transformations*

$$\hat{\delta} = \sum_i \left(\delta\theta_i \mathcal{J}_i + \delta v_i \mathcal{I}_i \right)$$

operate on $(j, C\rho)_{1,2}$ *simultaneously and* $\frac{d\delta v_i}{dt} = 0$.

Proof. The result follows from the observation that the kernel K in (10) commutes with the six generators as a result of Theorem (3.3), and that, integrating by parts in space and time the action of $\hat{\delta}$ over $(j, C\rho)_2$ can be seen as an action over $(j, C\rho)_1$ preceded by a negative sign, and then, both actions compensate to first order. Thus, the infinitesimal action of any element of the Lie algebra acting on both subsystems (primary and secondary) corresponds to the identity. We have that

$$\mathcal{A} = \mathcal{A}' + \mathcal{F}(t)$$

with $\mathcal{F}(t)$ a functional of the potentials and currents evaluated at the time t. Since all variations are considered to be zero at the extremes of the time-interval, $\mathcal{F}(t)$ contributes to zero to the variational calculation. In terms of their variations, $\mathcal{A}$ and $\mathcal{A}'$ are equivalent. See the Appendix (A.3) for the algebraic details. □

The requirement for δv to be constant in time is familiar to any one acquainted with Lorentz' transformations. It is interesting to mention that in the present context this requirement can be lifted by defining the variation as

$$\hat{\delta}_a = \sum_i \left(\delta\theta_i \mathcal{J}_i + \delta v_i \mathcal{I}_i + \frac{1}{2C} \frac{d\delta v_i}{dt} x_i \right) \tag{26}$$

4 Discussion and conclusions

From the point of view of pragmaticist epistemology (Peirce, 1955) all currents and charges are inferred. What we know about them are their effects, hence charge and current are "that what produces this and such effects", i.e., ideas, inferred entities, not directly accessible to our senses. However, charges and currents were originally associated to forces measured by a torsion balance and deflections of needles observed in galvanometers. Such primitive methods constitute the original definition of currents and charges and are available only for an observer at rest with the measuring apparatus since they are based upon material connections of circuits. In the text, we have restricted the use of "inferred" to those measurements that are performed based on action at a distance, i.e., without a "material" connection between the circuits (in particular, when this procedure is implied by the need of measuring while the detector is moving relative to the primary circuit, if the intricacies of a circuit continuously deforming are to be avoided). Thus, the scientist can perceive (measure) currents and charges using the original defining method if at rest relative to the source and the effects (as encrypted in forces, fields and potentials) of such events if the observer is at rest relative to the receiver. Currents and charges in the source are only inferred by the observer at rest with the receiver while forces, fields and potentials are inferred by the observer at rest with respect to the source.

We have shown that Electromagnetism can be formulated in terms of fields associated to sources as well as fields associated to receivers, this symmetry is broken in a construction that focuses exclusively in S-fields rather than R-fields.

Remark 3.5 exposes the relation between S- and R- fields, while the relation between source and receiver descriptions is given for example by the pair of equations (10) and (15). The restoration of this symmetry explains how the action-reaction law of Newton's mechanics is identically broken (Subsection 3.3) in the standard construction of Electromagnetism.

Most interestingly, the present approach based on eqs. (10), (22) and (15), is consistent with normal Electrodynamics and explains two fundamental concerns of the original theory, namely that electromagnetic waves propagate with the same electromagnetic parameter C regardless of the state of relative motion between source and detector, and that the electromagnetic Doppler effect is acted by a Lorentz boost of parameter u, in agreement with the accepted description. These results are obtained within the original framework of the theory, in particular preserving the Euclidean character of the auxiliary space-time, $\mathbb{R}^{3+1} = \mathbb{R}^3 \times \mathbb{R}^1$, which fulfils the conditions imposed by spatial relations or relational space. In terms of interpretations, there is no need to regard the universal constant $C = (\mu_0\epsilon_0)^{-\frac{1}{2}}$ as a velocity, nor to have something travelling between source and detector when considering electromagnetic interactions.

The complete set of equations of electromagnetism (Maxwell's equations, continuity equation and Lorentz' force) arise in the present form as the result of postulating Lorenz' delayed-action-at-a-distance 5 and Lorentz' action integral, to be used in the principle of least action 9. Lorenz' postulate has empirical basis while Lorentz' action is a (mathematical) organisation principle that has been considered fundamental by several authors as for example Poincaré (Poincaré, 1913). It is interesting to notice that before the irruption of the "second physicist" (Jungnickel and McCormmach, 2017), i.e., the theoretical physicist, theory in physics had a meaning close to "mathematically organised empirical observations". This is the spirit of Maxwell's work but it is as well the spirit in Newton, Ampere, Gauss and many others in the earlier times of physics. This epistemic position was heavily attacked by proponents of the ether such as Heaviside (Heaviside, 2011), Hertz (Hertz, 1893) and particularly Clausius (Clausius, 1869) who directly attacked Gauss' conception in the works by Riemann (Riemann, 1867), Betti (Betti, 1867) and Neumann (Neumann, 1868).

The present formulation addresses an issue recognised by Maxwell (1990, p. 228):

> ... According to a theory of electricity which is making great progress in Germany, two electrical particles act on one another directly at a distance, but with a force which, according to Weber, depends on their relative velocity, and according to a theory hinted at by Gauss, and developed by Riemann, Lorenz, and Neumann, acts not instantaneously, but after a time depending on the distance. The power with

> which this theory, in the hands of these eminent men, explains every kind of electrical phenomena must be studied in order to be appreciated [...]

And comparing with his preferred theory that "attributes electric action to tensions and pressures in an all-pervading medium" he writes:

> That theories apparently so fundamentally opposed should have so large a field of truth common to both is a fact the philosophical importance of which we cannot fully appreciate till we have reached a scientific altitude from which the true relation between hypotheses so different can be seen.

About one and a half century after Maxwell's conference we can discuss his philosophical inquire. Both theories are in perfect mathematical correspondence as they are with currently accepted electromagnetism but they differ in the abduction and interpretation as well as in the use of auxiliary concepts. Current electromagnetism relies heavily on the inferred idea of space-time and a mechanical analogy of the interaction. This approach is effective but leads us to embrace a new form of space-time, a necessary belief that not all of us are willing to admit. More precisely, current electromagnetism constructs first the space (relating it to Lorentz transformations) and only next spatial relations. In our view this order leads to logical inconsistencies (Solari and Natiello, 2022b) at the time of construction, despite the success achieved in terms of experimental comparisons. The present approach solves the problem by disposing of the subjective (auxiliary) space resting directly on spatio-temporal relations in such a way that rather than resting on just one transformation, a shared attribute of previous approaches, we find a harmonious coexistence of Galilean and Lorentzian transformations, a sort of reconciliatory mid-point. To achieve this views we had to accept first that space and time are not an a priori of knowledge as Kant thought (Kant, 1787) but rather a construction of the child as Piaget experimentally found (Piaget, 1999). Moreover, all these apparently conflicting approaches are needed for science to progress.

This view only uses (subjective) space and time as an auxiliary element when and if needed. The symmetry associated to the arbitrary decision of using a reference system is the one expressed by Galilean transformations (Solari and Natiello, 2018). In this context, inertial systems as auxiliary reference systems are constructed on the basis of the idea of free-bodies (Newton, 1687; Thomson, 1884; Solari and Natiello, 2021). This structure is underlying the work but not explicitly used as we have preferred to avoid reference frames.

The Lorentz transformations correspond in this construction to an endomorphism of relational space-time and are relevant only to the propagation of electromagnetic action. They are associated to the (unexpected) symmetry concerning perception and inferred charges and currents. While the full Poincaré-Lorentz group relates different but equivalent perceptions of the electromagnetic action, the particular set of Lorentz transformations relate the perception from a detector at rest with respect to the source to the perception of a detector in motion relative to the source with (invariant) velocity u. Such relation can be obtained only for $|u| < C$. We emphasise that the Poincaré-Lorentz group coexists with the Galilean symmetry of the description, although not all the equations, particularly not all differential equations, are transparent as expressions showing the symmetry. The integral presentation is in this respect more revealing.

We finally stress that symmetries as requirements of reason pre-exist physics and equations. They enter physics as a demand of reason in our quest to construct the cosmos, this is, to put in harmony our perceptions of the real-sensible.

The ether was the immediate consequence of attempts to understand electromagnetism by analogy to mechanical phenomena. Special relativity introduced an analogy of the forms, the Principle of relativity, without an understanding of the fundaments of the principle. It soon became evident that if analogies with mechanics would be preserved, the metric of space-time had to be changed. Yet, hiding the hypothesis the statement reads: electromagnetism imposes us to adopt a different metric of space-time than the Cartesian one used in its construction. Next, to accept this unmatching between the construction moment and the explanatory moment of science requires the exclusion of the first, leaving us with a science without understanding, supported only upon its predictive success, a technology of prediction, since success is the quality measure of any technology. Philosophers like Popper and Reichenbach considered their task to support the theories of scientists like Einstein. Consequently, they dropped all critical examination of matters, finally endorsing a program that was put forward by 1870, "physics must henceforth pursue the sole aim of writing down for each series of phenomena ... equations from which the course of the phenomena can be quantitatively determined; so that the sole task of physics consisted in using trial and error to find the simplest equations". Notice that even "trial and error", the method favoured by Popper, was already indicated. Such program is the instrumentalist program of science, aiming at dominating nature, a perfect mate of considering the Earth an infinite source of resources for the development of the capitalist society. We argue then that it has been forced upon us by social decisions that made nearly impossible the survival of the critical motion of reason. Conversely, by restating critical reasoning, we have been able to construct an electromagnetism that is more consilient than the received wisdom, it does not need to reform space-time and consequently makes no call for the abandonment of the construction moment. Reason can organise the chaos that reaches our senses, harmony is still an enticing possibility. If the child develops abstraction to understand the possible instead of being forced to accept the given, we need to put abstraction to work. We have been told that there is just one possible science, the given science, the science of capitalism. We have proved by presenting a counter example that the statement is wrong. Science is not only what scientists do (the given) but what humans can do as well, critical and ethical science, a science conscious of its ignorance. We close with ancient words by Chuang Tzu:

> Now you have come out beyond your banks and borders and have seen the great sea – so you realize your own pettiness. From now on it will be possible to talk to you about the Great Principle. (Chuang Tzu, 1968, Autumn Floods)

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Declaration of Interest

The authors declare that there exists no actual or potential conflict of interest including any financial, personal or other relationships with other people or organizations within three years of beginning the submitted work that could inappropriately influence, or be perceived to influence, their work.

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A Some Proofs

A.1 Proof of Lemma 3.1

Proof. We perform the calculation in detail only for A, since the other one is similar. We use the shorthand $r = |x - y|$.

$$\begin{aligned}
\nabla_x A_i &= \frac{\mu_0}{4\pi}\int d^3y\left(j_i\nabla_x\frac{1}{r} - \frac{\frac{\partial}{\partial t}j_i\nabla_x\frac{r}{C}}{r}\right)\\
\Delta A_i &= \nabla_x\cdot\nabla_x A_i\\
&= \frac{\mu_0}{4\pi}\int d^3y\left(j_i\Delta\frac{1}{r} - 2\left(\nabla_x\frac{1}{r}\right)\cdot\left(\frac{\partial}{\partial t}j_i\nabla_x\frac{r}{C}\right)\right) -\\
&\quad -\frac{\mu_0}{4\pi}\int d^3y\left(\frac{\frac{\partial}{\partial t}j_i\Delta\frac{r}{C}}{r} + \frac{\frac{\partial^2}{\partial t^2}j_i}{r}|\nabla_x\frac{r}{C}|^2\right)
\end{aligned}$$

Moreover, standard vector calculus identities give

$$\begin{aligned}
\frac{\partial}{\partial t}j_i\left(2\nabla\frac{1}{r}\cdot\nabla\frac{r}{C} + \frac{\Delta\frac{r}{C}}{r}\right) &= 0\\
|\nabla\frac{r}{C}|^2 &= \frac{1}{C^2}
\end{aligned}$$

and therefore

$$\Delta A_i(x,t) \;=\; \frac{\mu_0}{4\pi}\int d^3y\, j_i(y,t-\frac{r}{C})\Delta\left(\frac{1}{r}\right)+\left(\frac{1}{C^2}\right)\frac{\mu_0}{4\pi}\int d^3y\,\frac{\partial^2}{\partial t^2}\frac{j_i(y,t-\frac{r}{C})}{r}$$

The time derivative in the last term can be extracted outside the integral, thus yielding,

$$\begin{aligned}
\Box A_i(x,t) &= \Delta A_i(x,t)-\left(\frac{1}{C^2}\right)\frac{\mu_0}{4\pi}\int d^3y\,\frac{\partial^2}{\partial t^2}\frac{j_i(y,t-\frac{r}{C})}{r}\\
&= \Delta A_i(x,t)-\left(\frac{1}{C^2}\right)\frac{\partial^2}{\partial t^2}A_i(x,t)\\
&= \frac{\mu_0}{4\pi}\int d^3y\, j_i(y,t-\frac{|x-y|}{C})\Delta\left(\frac{1}{r}\right)\\
&= -\mu_0 j_i(x,t)
\end{aligned}$$

□

A.2 Proof of Theorem 3.2

Proof. The result follows from the computation of the extremal action under the constraints

$$\begin{aligned}
(V-\mathsf{V})\chi &= 0\\
(A-\mathsf{A})\chi &= 0
\end{aligned}$$

Multiplying the constraints by the Lagrange multipliers λ and κ (the latter a vector), while we use the shorthand notations $B=\nabla\times A$ and $E=\left(-\frac{\partial A}{\partial t}-\nabla V\right)$, we need to variate the constrained electromagnetic action

$$\mathcal{A}=\frac{1}{2}\int dt\left(\int\left(\frac{1}{\mu_0}|B|^2-\epsilon_0|E|^2-\kappa\cdot(A-\mathsf{A})\chi+\lambda(V-\mathsf{V})\chi\right)d^3x\right).$$

Varying the integrand we obtain

$$\begin{aligned}
\delta\mathcal{A} \;=\; &\int dt\left(\int\left(\frac{1}{\mu_0}(\nabla\times A)\cdot(\nabla\times\delta A)-\epsilon_0\left(\nabla V\cdot\nabla\delta V+\frac{\partial A}{\partial t}\frac{\partial\delta A}{\partial t}\right)\right.\right.\\
&\left.\left.-\epsilon_0\left(\nabla V\cdot\frac{\partial\delta A}{\partial t}+\nabla\delta V\cdot\frac{\partial A}{\partial t}\right)-\chi\kappa\,\delta A+\chi\lambda\,\delta V\right)d^3x\right)
\end{aligned}$$

Partial integrations in time and standard vector calculus give the following identities:

$$\begin{aligned}
\int dt \, \frac{\partial A}{\partial t}\frac{\partial \delta A}{\partial t} &= \left[\delta A \cdot \frac{\partial A}{\partial t}\right] - \int dt \, \delta A \cdot \frac{\partial^2 A}{\partial t^2} \\
\int dt \, \nabla V \cdot \frac{\partial \delta A}{\partial t} &= [\delta A \cdot \nabla V] - \int dt \, \delta A \cdot \nabla \frac{\partial V}{\partial t} \\
(\nabla \times A) \cdot (\nabla \times \delta A) &= \nabla \times (\nabla \times A) \cdot \delta A - [\nabla \cdot ((\nabla \times A) \times \delta A)] \\
\left(\nabla V + \frac{\partial A}{\partial t}\right) \cdot \nabla \delta V &= \left[\nabla \cdot \left(\left(\nabla V + \frac{\partial A}{\partial t}\right) \delta V\right)\right] - \delta V \nabla \cdot \left(\nabla V + \frac{\partial A}{\partial t}\right)
\end{aligned}$$

The terms in square brackets vanish in the variation either for the vanishing variation at endpoints or because of Gauss theorem applied to functions decaying fast enough at infinity. Hence,

$$\begin{aligned}
\delta \mathcal{A} &= \int dt \int d^3x \left(\frac{1}{\mu_0} (\nabla \times A) \cdot (\nabla \times \delta A) - \epsilon_0 \left(-\delta V \nabla \cdot \left(\nabla V + \frac{\partial A}{\partial t} \right) \right.\right. \\
&\left.\left. -\delta A \cdot \frac{\partial^2 A}{\partial t^2} - \delta A \cdot \nabla \frac{\partial V}{\partial t} \right) - \chi \kappa \cdot \delta A + \chi \lambda \delta V \right).
\end{aligned}$$

Being δA and δV independent, we obtain

$$\begin{aligned}
\frac{1}{\mu_0} \nabla \times (\nabla \times A) + \epsilon_0 \left(\frac{\partial^2 A}{\partial t^2} + \nabla \frac{\partial V}{\partial t} \right) &= \chi \kappa \\
-\epsilon_0 \nabla \cdot \left(\nabla V + \frac{\partial A}{\partial t} \right) &= \chi \lambda
\end{aligned}$$

or equivalently

$$\begin{aligned}
\frac{1}{\mu_0} \nabla \times B - \epsilon_0 \frac{\partial E}{\partial t} &= \chi \kappa \\
\epsilon_0 \nabla \cdot E &= \chi \lambda
\end{aligned}$$

which allows us to identify $j = \chi \kappa$ (the density of current inside the material responsible for A) and $\rho = \chi \lambda$ (the density of charge responsible for V), thus proving the first result. Finally, the continuity equation follows from0

$$0 = \nabla \cdot \left(\frac{1}{\mu_0} \nabla \times B - \epsilon_0 \frac{\partial E}{\partial t} \right) + \frac{\partial}{\partial t} (\epsilon_0 \nabla \cdot E) = \nabla \cdot j + \frac{\partial \rho}{\partial t}.$$

Note also that taking curl on the first equation we verify that B satisfies a wave equation. Inserting $\nabla \times E$ in the time-derivative of the first equation and adding the gradient of the second equation, we obtain a wave equation for E.

Further, $\frac{1}{\mu_0}\nabla \cdot A + \epsilon_0 \frac{\partial V}{\partial t} = 0$ implies both that $\nabla \dfrac{\partial V}{\partial t} = -C^2 \nabla (\nabla \cdot A)$ and $\nabla \cdot \dfrac{\partial A}{\partial t} = -\dfrac{1}{C^2}\dfrac{\partial^2 V}{\partial t^2}$. Substituting each relation in the corresponding equation, we obtain eq(6), thus proving the Corollary. □

A.3 Proof of Theorem 3.5

Proof. The rotational invariance is immediate, since for any rotation matrix R,the change of coordinates $x' = Rx$ (along with the corresponding change for y), keeps the distance $|R(x-y)| = |x-y|$ invariant. Hence, for the kernel in eq.(10) and any electromagnetic kernel depending on $|x-y|$ the action integral is invariant under rotations. Let u be the velocity associated to a Lorentz transformation, which is constant by hypothesis. The proposed variation reads

$$\begin{aligned}
\hat{\delta}\mathcal{A} &= \int_{-\infty}^{t} ds \int d^3x\, \hat{\delta}\left[A^1(x,s)j_2(x,s) - V^1(x,s)\rho_2(x,s)\right] \\
&= \int_{-\infty}^{t} ds \int d^3x \left(Cs\,\delta u \cdot \nabla_x + (\frac{x}{C}\cdot \delta u)\partial_s\right)\left[A^1 j_2 - V^1\rho_2\right](x,s)
\end{aligned}$$

where $\mathcal{I}(x,s) = Cs\,\delta u \cdot \nabla_x + (x\cdot\delta u)\partial_{Cs}$ is the Lorentz generator. By Gauss Theorem the following integral vanishes for any function F inheriting the behaviour of A, j at infinity:

$$\int_K d^3x\,\delta u \cdot \nabla F(x) = \int_{\partial K} F(x)\delta u \cdot dS = 0$$

Finally,

$$\int_{-\infty}^{t} ds \frac{\partial}{\partial s}\int d^3x\,(x\cdot\delta u)G(x,s) = \mathcal{F}(t) - \mathcal{F}(t_0)$$

for some function $\mathcal{F}$ depending only of t. However by the nature of the variational process, $\mathcal{F}$ does not contribute to the variation. □

B The Lorentz transformation

The infinitesimal generator of the Lorentz transformation in eq.(16) reads

$$\mathcal{I}_j = \begin{pmatrix} \mathbf{0} & \frac{v}{C} \\ (\frac{v}{C})^T & 0 \end{pmatrix}.$$

The Lorentz transformation for finite v is obtained by exponentiation (Gilmore, 1974), yielding the 4×4 matrix expression $TL(v)$ for the transformation elements, where

$$TL(v) = \begin{pmatrix} W & X \\ X^\dagger & Y \end{pmatrix}$$

is formed by the 3-vector $X = \sinh\left(\left|\frac{v}{C}\right|\right)\dfrac{v}{|v|}$, the scalar $Y = \sqrt{1+|X|^2} = \cosh\left(\left|\dfrac{v}{C}\right|\right)$ and the 3×3 matrix $W = Id + \left(\cosh\left(\left|\frac{v}{C}\right|\right) - 1\right)\dfrac{vv^\dagger}{|v|^2}$, where $v \in \mathbb{R}^3$ is a parameter classifying the different transformations. A better known expression for the Lorentz transformation arises from the change of variables $u = C\hat{v}\tanh\left|\dfrac{v}{C}\right|$ (Gilmore, 1974). In such terms,

$$L(u) = \begin{pmatrix} Id + (\gamma - 1)\hat{u}\hat{u}^\dagger & \gamma\dfrac{u}{C} \\ \gamma\dfrac{u^\dagger}{C} & \gamma \end{pmatrix}; \quad L\left(u(v)\right) \equiv TL(v),$$

where $\gamma\left(u(v)\right) = \dfrac{1}{\sqrt{1-\left|\frac{u(v)}{C}\right|^2}} = \cosh\left(\left|\frac{v}{C}\right|\right)$. Note that for $\left|\dfrac{v}{C}\right| \ll 1$, we have that $\left|\dfrac{u}{C}\right| = \left|\dfrac{v}{C}\right| + O\left(|\dfrac{v}{C}|^3\right)$.