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Philosophical suggestions by the single elements

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Abstract

The largest class of hyper structures is the one which satisfy the *weak properties*. These are called H_v-structures introduced in 1990 and they proved to have a lot of applications on several applied sciences. Special classes of elements appeared to have new interesting properties applicable in other sciences. We present some results on hyper structures containing '*single'* elements, and some new constructions.

Keywords: hyper structures; H_v-structures; single elements.†

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1. Some hyperstructures

Our main object is the class of hyperstructures called H_v*-structures* introduced by T. Vougiouklis in 1990 ([9], [10]), which satisfy the *weak axioms* where the non-empty intersection replaces the equality. Some basic definitions are the following:

In a set $\boldsymbol{H}$ equipped with a hyperoperation (·):

$$\cdot: \boldsymbol{H}\times\boldsymbol{H} \to P(\boldsymbol{H}) - \{\varnothing\},$$

we abbreviate by ***hope*** the *hyperoperation* (·).
In $(\boldsymbol{H},\cdot)$ the *weak associativity* is defined by

$$(xy)z\cap x(yz)\neq\varnothing,\ \forall x, y, z\in\boldsymbol{H}$$

and the *weak commutativity* by

$$xy\cap yx\neq\varnothing,\ \forall x, y\in\boldsymbol{H}.$$

The hyperstructure $(\boldsymbol{H},\cdot)$ is called an ***H_v-group*** if it is weak associative and *reproductive,* i.e.

$$x\boldsymbol{H} = \boldsymbol{H}x = \boldsymbol{H},\ \forall x\in\boldsymbol{H}.$$

Motivation. The quotient of a group by an invariant subgroup is a group. The quotient of a group by a subgroup is a hypergroup. The quotient of a group by any partition is an H_v-group.

In a similar way more complicated hyperstructures are defined:
$(\boldsymbol{R}, +,\cdot)$ is called H_v*-ring* if (+) and (·) are weak associative, the (+) is reproductive and (·) is weak *distributive* with respect to (+):

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\ \ \forall x,y,z\in\boldsymbol{R}.$$

Similarly, the H_v*-module*, the H_v*-vector space,* H_v*-algebras* and so on, are defined.

For more definitions and applications on H_v-structures one can see in books and papers as [2], [4], [5], [8], [9], [10], [13], [15].

The main tool to study hyperstructures is the *fundamental relation.* In 1970, for the hypergroups the relation β and its transitive closure β*, was defined by M. Koskas. This relation connects the hyperstructures with the corresponding classical structures and is defined in H_v-groups as well. T. Vougiouklis in 1990, introduced the γ* and ε* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations β*, γ* and ε*, *Fundamental Relations*.

Definition. The fundamental relations β^*, γ^* and ε^*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.

The basic theorem which gives a way to find the fundamental classes, as well, is the following [10]:

Theorem. Let $(\boldsymbol{H},\cdot)$ be an H_v-group and denote by $\boldsymbol{U}$ the set of all finite products of elements of $\boldsymbol{H}$. We define the relation β in $\boldsymbol{H}$ by setting $x\beta y$ iff $\{x, y\}\subset \boldsymbol{u}$ where $\boldsymbol{u}\in\boldsymbol{U}$. Then β^* is the transitive closure of β.

Analogous to the above theorem in the case of an H_v-ring is:

Theorem. Let $(\boldsymbol{R}, +,\cdot)$ be an H_v-ring, denote $\boldsymbol{U}$ the set of all finite polynomials in $\boldsymbol{R}$. We define the relation $\boldsymbol{\gamma}$ in $\boldsymbol{R}$ by:

$$x\boldsymbol{\gamma}y \text{ iff } \{x, y\} \subset u \text{ where } u\in\boldsymbol{U}.$$

Then, the relation $\boldsymbol{\gamma}^*$ is the transitive closure of the relation $\boldsymbol{\gamma}$.

An element is called ***single*** if its fundamental class is singleton. Remark that in the classical single-valued structures all elements are singles.

Fundamental relations are used for general definitions. Thus, an H_v-ring $(R, +,\cdot)$ is called *H_v-field* if R/γ^* is a field.

Let $(\boldsymbol{H},\cdot)$, $(\boldsymbol{H}, *)$ be H_v-semigroups defined on the same set $\boldsymbol{H}$. $(\cdot)$ is called ***smaller*** than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f\in \text{Aut } (\boldsymbol{H}, *) \text{ such that } xy\subset f(x*y),\ \forall x, y\in\boldsymbol{H}.$$

Then we say that $(\boldsymbol{H}, *)$ *contains* $(\boldsymbol{H},\cdot)$.

The Little Theorem. Greater hopes than the ones which are weak associative or weak commutative, are also weak associative or weak commutative, respectively.

Remark. From the Little Theorem we obtain the huge number of H_v-structures which are defined on the same set. This is the reason that the H_v-structures have a lot of applications.

An H_v-structure is called *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one.

Several large classes with one or more hopes as the *theta-hopes, P-hopes,* can be defined.

2. Single elements

Let ($\boldsymbol{H}$, $\circ$) be an H_v-group. Notice that if an element $x \in \boldsymbol{H}$ is *single* then its fundamental class is a singleton, therefore: $\beta^*(x)=\{x\}$. We denote by $\boldsymbol{S_H}$ the set of all single elements of $\boldsymbol{H}$. The following theorems are valid [5], [10]:

Theorem. Let ($\boldsymbol{H}$,$\circ$) be an H_v-group and $x \in \boldsymbol{S_H}$. Let $a \in \boldsymbol{H}$ and take any element $v \in \boldsymbol{H}$ such that $x \in a \circ v$, consequently, $x=a \circ v$. Then,

$$\beta^*(a) = \{h \in \boldsymbol{H} \mid h \circ v = x\}.$$

Theorem. Let ($\boldsymbol{H}$,$\circ$) be an H_v-group and $x \in \boldsymbol{S_H}$. Then, the core of $\boldsymbol{H}$ is

$$\omega_H = \{u \mid u \circ x = x\} = \{u \mid x \circ u = x\}.$$

Theorem. Let ($\boldsymbol{H}$,$\circ$) be an H_v-group and $x \in \boldsymbol{S_H}$. Then,

$$x \circ y = \beta^*(x \circ y) \text{ and } y \circ x = \beta^*(y \circ x), \quad \forall y \in \boldsymbol{H}.$$

This theorem proves that

'the product of a single element with any element
is a whole fundamental class'.

Therefore, if we know one single element, then we know all fundamental classes.

Two constructions, originated from the properties the single elements have, are the following:

Construction 1. *Replacement of a fundamental class by a single element*. Let ($\boldsymbol{H}$,$\circ$) be H_v-group, consider a fundamental class $\beta^*(a)$ which we want to replace by an element s, where this element is going to be single in a new H_v-group, with the same fundamental group. All hyperproducts with factors elements outside the class $\beta^*(a)$, are the same as in the H_v-group ($\boldsymbol{H}$,$\circ$). For the rest products we set

$$s \circ x = x \circ s = \beta^*(s \circ x), \ \forall x \in \boldsymbol{H}$$

and we have a new H_v-group on the set

$$(\boldsymbol{H} - \beta^*(a)) \cup \{s\}.$$

Construction 2. *Replacement of a single element by a set.* Let ($\boldsymbol{H}$,$\circ$) be an H_v-group where $\boldsymbol{S_H} \neq \varnothing$. Consider any single element $s \in \boldsymbol{S_H}$ which we want to replace by a set $\boldsymbol{S}=\{s_i : i \in \boldsymbol{I}\}$. We extend the hope ($\circ$) by setting:

$$x \circ s_i = s_i \circ x = \beta^*(s \circ x), \ \forall x \in (\boldsymbol{H} - \{s\}) \cup \boldsymbol{S}, \text{ and } i \in \boldsymbol{I}.$$

Moreover, in order to have scalar unit element e, we reduce this hope by setting

$$e \circ s_i = s_i \circ e = s_i,\ \forall i \in \boldsymbol{I}.$$

In both cases, the $((\boldsymbol{H}-\{s\})\cup\boldsymbol{S},\circ)$ is an H_v-group.

3. Suggestions for applications

Last decades hyperstructures, mainly the H_v-structures, have a variety of applications in applied sciences. These applications range from biomathematics, linguistics, physics to mention but a few [5], [11]. The applications are obtained either from the new hyperstructures or from the properties some new classes have. Mathematical modeling is the art of translating problems from an application area into tractable mathematical formulations. Mathematical models are used in empirical research because the mathematicalization of problems has several advances as: the results are clear, recognized and can be compared with other [3]. Among models there are the, so called, general models of models. The Vougiouklis & Vougiouklis in 2000, see [14], present two General Models of Models as follows:

First General Model: The stages of developing of a mathematical branch are the following five:

The choice of the basic object – the set of study.
The choice of the basic axioms – basic building rules.
Construction.
Morphisms: Maps which transfer the basic building rules.
Transformations, endomorphisms and invariant elements.

Second General Model: The ***product***, which is called Cartesian, which is based on the ordering. The ***quotient***, which is complicate and not unique procedure.

Now, we present or suggest some special applications.

An application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the V&V bar (Vougiouklis & Vougiouklis bar) [12], [14]. The suggestion by Vougiouklis & Vougiouklis, is the following:

Definition. *The V&V bar.* In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:

0 ———————————————— 1

The informants are asked to cut the bar at any point they feel expresses their answer to the specific question.

The V&V bar, has several advantages during both the filling-in and the research processing. The suggested length of the bar, according to the Golden Ratio, is 6.2cm.

The great number of H_v-structures which are defined on the same set give more models on applied sciences. In this direction we have the following application. The *isotopy* Lie-Santilli theory was born to solve Hadronic Mechanics problems (see [5] and mainly [7]). Santilli proposed a 'lifting' of the n-dimensional unit matrix of a normal theory into a new matrix. The original theory is reconstructed such as to admit the new matrix. The *isofields* needed in this theory correspond to the so *e-hyperfields*.

The proof of the main theorems for the fundamental classes has a new direction [9], [10]: The proof depends on the result! This is based in the remark that the relation β is determined from the result, that is, two elements are in relation β if they belong together in the same result. In the proof that a hyperstructure 'hide' a corresponding structure, I used an 'inverse' procedure and the length of the proof reduced drastically. With this procedure we consider that the structure we want to construct, there exists in a set of structures and then we take away all structures we do not need. Moreover, we take away all structures that they have not the desired properties. Consequently, we reverse the compose construction method, where we build with the material we have. For example, take a huge piece of marble, we carve it and reveal a statue, a structure! Therefore, the meaning of this inverse procedure is that we take off all useless material and we reveal our structure. This procedure is clear in the H_v-structures because there are too many structures, so we can take off all the useless and we obtain the desired [13].

We can present the above method in teaching a language, as follows: Suppose we want to teach a linguistic phenomenon. In order to determine this clearly, we have to take off all non-related elements. This is so, because our effort in a teaching procedure is not to give the exact definition but the fuzzy approach of it.

Finally, we present some applications derived from the properties which the single elements, have.

We remind that the basic property of the 'single' element is that it has the 'duties' of all the elements of a class. In order to see possible applications in other sciences, we must reveal what does this property means in hyperstructures. The ordinary 'operation' is a map which in every couple of elements corresponds an element, the result, of the same set. In a hope the result is a subset of the set. The definition of an operation (or a hope) and the properties of the elements, are follows. Conversely, an operation is obtained from the properties the elements have. Let us see this fact in the following: Let $(\boldsymbol{H}, \cdot)$ be a groupoid, i.e., a set equipped with an operation $(\cdot)$. We call *(left) translation* or *action*, by an element $x \in \boldsymbol{H}$, the map

$$\boldsymbol{T_x} : \boldsymbol{H} \rightarrow \boldsymbol{H} : y \rightarrow \boldsymbol{T_x}(y)=xy$$

Therefore, the translation corresponds to every y, the element xy. Analogously, in hypergroupoids the result is a subset of $\boldsymbol{H}$.

This mathematical term state that the operation can be considered as a property of the element. The elements define the operation. Thus, the character of each element depends on its behavior with respect to the others. The same is valid in Geometry where we understand the points, lines, planes, from how they behave to the rest. If we transfer tis to man, we can state the opinion that the character of each person depends on how it behaves to the others!

In the approach of the continuous, in a mathematical way, the object is to understand this using the discrete, which has absolute properties as the single element. The *continuous* appeared as obvious, as visible in the nature and our senses (Αριστοτέλους, [1]). Its understanding, its registration and its transferring can be achieved by using the *discrete*. The number, the ordering and the geometry, with the point and the segment, visualize the continuous. Therefore, in order to understand the continuous, we use the discrete in the form of an algorithm and this leads to the 'limit'.

We hear phrases as: *the last straw, the colors ofthe spectrum, the phoneme of the sound*. The mathematic expression of such phrases belongs to the fight between continuous and discrete. This fight is a game between the nature and its understanding. We know the result and we can use this fact as tool of the game. We do this when fuzzy appears on the way to the clear. It is the game of knowledge, or better, the game of 'interventions'.

The problem to start the proof procedure based on a standard, special well-known situation, is continuous in perpetuity. In early 20^{th} century, B. Russel in [6], states: "*Mathematics is a study which, when we start from its most familiar portions, may be pursued in either of two opposite directions. The more familiar direction is constructive, towards gradually increasing complexity: from integers to fractions, …, and on to higher mathematics. The other direction, which is less familiar, proceeds, by analyzing, to greater and greater abstractness and logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what ore general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. We may state the same distinction in another way. The most obvious and easy things in mathematics are not those that come logically at the beginning; they are things that, from the point of view of logical* *deduction, come somewhere in the middle. Just as the easiest bodies to see are those that are neither very near nor very far, neither very small not very great, so the easiest conceptions to grasp are those that are neither very complex nor very simple (using 'simple' in a logical sense). And as we need two sorts of instruments, the telescope and the microscope, for the enlargement of our logical powers, one to take forward to the higher mathematics, and other to take backward to the logical foundations of the things that we are inclined to make for granted in mathematics."*

There is an analogous application using a touchstone (metaphor) which has a special property. A *touchstone* is a small tablet of dark stone such as slate or lyddite, used for assaying precious metal alloys. It has a finely grained surface on which soft metals leave a visible trace.

Similarly, we know that the *diamond* has the highest hardness and thermal conductivity of any natural material, thus, its properties are utilized in major industrial applications such as cutting and polishing tools.

Therefore, in case we have elements with strong properties, i.e. they are characteristic in the operations, then we must start our study with these elements. We have to base our study on them. These elements are logical, basic, common, familiar, and convenient to start a mathematical study which leads to a total mathematical substantiation.

4. Conclusions

Basic conclusions from mathematical branches can be used in applied sciences, if one can find and focus on the substance of those conclusions. Therefore, mathematics, apart from the solutions with models it offers, might also reveal to the other sciences the exclusive methods employed, as well as its characteristic general conclusions.

We call the above suggestions 'philosophical' since it concerns the real meaning of the procedure which is not a simple, or strict, morphism in the specific science.

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