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Mind matters in mathematics and music

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Abstract

Mathematics and music in practice and performance, and in learning and teaching, share many characteristics, such as beauty and harmony, memory and intuition (as internal senses), and mind or intellect. These raise the principles of processing information in mathematics and music and, by implication, the role of an acquaintance with the essentials of perception, abstraction, and affective connaturality in teacher education. This paper compares mathematics and music and considers the acquisition of knowledge and skills through the external and internal senses and emotions, utilizing the role of knowledge through multiple intelligences. In doing so it does not canvas the utilities of mathematics and music as fields of human endeavour so much as their role in the cultivation of serenity and knowledge in the cultured mind. This is a theoretical paper but it is based on nearly a century of teaching from the combined work of the two authors in the teaching of music and mathematics. The paper highlights the importance of inspiration in teaching, inspiration built on a thorough basis of the foundations of anthropology to include the emotions as well as the intellect. While teacher education programs rightly concern themselves with knowledge of the field of study, knowledge of pedagogy, they do not always consider the ability to inspire which is at the heart of managing and mentoring people.

Keywords: Beauty, harmony, memory, emotions, creativity, problem solving, time.[†]

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1 Introduction

"It is often said that music and mathematics are related. Essentially though, music and mathematics are poles apart. Mathematics is about the physical world. It is about the first principle of science…In contrast, music does not and cannot express the physical world…We use our ears in music and our eyes in mathematics, but both use the mind…Music is never reducible to mathematics but both disciplines are pattern rich so the temptation to draw connections can be irresistible" [14]. Here we wish to show that the mind is the key to the connections which can enrich the pedagogy of both disciplines when taught with the passion to inspire.

The purpose of this paper is simple. It aims to canvas how the mind matters in learning and teaching in mathematics and music, albeit from non-conventional considerations, free from the dictates of fashion. Thus, this paper outlines some distinct, but not separate, aspects which affect the teaching and learning of mathematics and music. While at first sight they seem very different, mathematics and music do actually possess many elements in common:

- harmony, [30]
- beauty, [22]
- notation as a tool of thought, [17]
- intuition as an affective process,
- levels of cognition,
- problem solving,
- practice, practice, practice in order to perform!
- memory as a result of thought and practice,
- appreciation of the role of intuition,
- inspiration of the teacher as motivation
- patterns and forms,
- sequences [49].

The frequently neglected connections among the experimental sciences and the fine arts are a continuing source of genuine research [33], particularly with the connections between mathematics and music and creativity [27], going back to Gottfried Wilhelm Leibnitz in the 17^{th} century: "Music is the mathematics of one who does not know that he is counting." [37]

The product of this paper is complex. It leads into questions about the nature of time [36] since both mathematics and music relate to time, albeit in different ways though both require participation [9;35] and the magnetism of beauty in their performance [15;31]. To pursue these further would require studies of the nature of time and the nature of existence [50]. We shall instead focus on the pedagogical links.

While Howard Gardner's Multiple Intelligences (or talents), MIs, has brought out the value of the various strengths, sometimes latent and frequently under-valued, in everyone, the context of these inter-relationships is too often a blank canvas [12;16]. To fill in this blank canvas, we also outline the elements of this contextual framework which can make a difference to how we teach and how we accommodate the variety of MIs in any class we teach (and at any level). For instance, ‘harmony’, ‘beauty’ and ‘problem-solving’ transcend the particular disciplines where they first appear to the novice learner in mathematics and music [26]. Beauty is a recurring theme in Hardy [15] and more recently in Gardner [13].

2 Our Approach

To do this we also need to accommodate the cognitive, affective and psychomotor taxonomies as the relate to what we teach and why we teach. In order to teach effectively we need to know what we are teaching in some depth and we should love teaching real people with all their strengths and weaknesses.

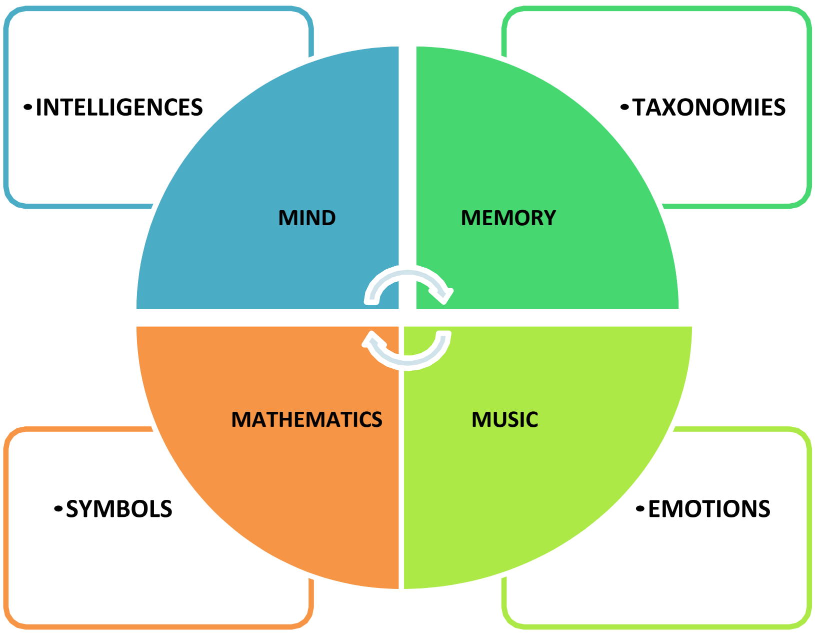

Figure 1: Structure of the paper

While the domains of Bloom and his colleagues are well-established and still fairly widely used, we shall briefly recapitulate them in the situations of mathematics and music in order to make sense of our later exposition. The word ‘taxonomy’ itself is based on the French *taxonomic* and linked to the Greek (*taxis* order *nomos* – ‘managing’). An educational taxonomy is a form of classification of the process of thinking and learning.

Historically the educational taxonomy was developed as a structure of three domains:

i) Cognitive domain [5],
ii) Affective domain [7],
iii) Psychomotor domain [8].

which have been modified slightly in the upper levels over the years. In particular, we note that (i) can be implemented as a measurement tool for the Art/Science of teaching and learning of classical piano.

COGNITIVE DOMAIN [5] **Cognitive Domain** (Knowledge) was formulated by Benjamin Bloom in 1956 as a set of six major categories, organized as a hierarchical order of cognitive progress, starting from the simplest level to the most complex.				
Level	**Category**	**Behaviour**	**Mathematics**	**Music**
1	Knowledge	Recall or recognise information	Definitions, laws, procedures; memory strategies	Playing tech niques; history and theory; memory strategies
2	Compre hension	Restate data in one's own words	Explain or interpret meaning of symbols; aware-ness of patterns	Notation; harmony structure; styles of music
3	Applicat ion	Put theory into practice	Solve a new problem, manage an activity	New techniques, finger choices
4	Analysis	Interpret internal relationships	Identify constitu ent parts and functions	Tonic and dominant keys; structure
5	Synthesis and Creativity	Develop new structures	Combine methods, develop proce-dures; choice of proofs	Compose a con-certo; thesis, an-tithesis, synthesis [29]
6	Evaluation	Assess effective-ness of whole concepts	Review strategic alternatives; crite-ria for judgements	Criteria for judgements and standards

Table 1(a): Taxonomic Domains – Cognitive Domain

AFFECTIVE DOMAIN [19] This area is concerned with feelings or emotions. Affective objectives are also divided into a hierarchy: from the simplest behaviour to the most complex.				
Level	**Category**	**Behaviour**	**Mathematics**	**Music**
1	Receive	Open to experience, willingness to hear	Take interest in learning experience	Willingness to hear and form habits

2	Respond	React and participate actively	Enthusiasm for action, interest in out- comes	Actively reacting and participating
3	Value	Attach values, express personal opinions	Decide worth and relevance of ideas	Acceptance, respect and commitment [35]
4	Organize	Reconcile internal conflicts, develop value system	Clarify, qualify and quantify personal views	Start of the student's transfor mation to be an independent learner
5	Internalize	Adopt belief system and personal philosophy	Self-reliant, behave consistently with personal vales	Values and beliefs are formed at a professional level

Table 1(b): Taxonomic Domains – Affective Domain

PSYCHOMOTOR DOMAIN [8]

This skills domain is exceptionally important in Piano Teaching! It is designed to explain the evolution of physical movement, coordination and use of the motor – skills. Dave's five major categories are listed here from the simplest behaviour to the most complex.

Level	Category	Behaviour	Mathematics	Music
1	Imitation	Copy, observe and replicate	Watch teacher and repeat action	Watch teacher and repeat (Piaget [25])
2	Manipulat ion	Reproduce activity from instruction or memory	Carry out task from written or verbal command	Gross motor control and fine motor coordination
3	Precision	Execute skill reliably, independent of help	Perform a task with quality and without assistance	Motor actions become more exact and refined
4	Articulat ion	Integrate expertise to satisfy a non-standard objective	Combine associated activities to meet novel requirements	Efficient physical mechanism; movement with reasoning
5	Naturalisati on	Automated mastery of skills at strategic level	Conjectures and strategies for use to meet needs	Musical idea and technical realization go together [47]

Table 1(c): Taxonomic Domains – Psychomotor Domain

In many senses these come together in the theory of multiple intelligences [12,16], no matter what our role in doing and enjoying mathematics or music.

3 Mind matters

3.1 Music and mathematics

As we can glimpse there is much in common between these two disciplines, even in performance, because neither discipline is a spectator sport, and both require active ‘timely’ memories! [35,38,44,50,51].

Schools must remember that teaching this

Beethoven 32 Variations in C Minor

www.musiciansunite.com

is just as important as teaching this

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$f(x) = a_0 + \int_{n=1}^{\infty} \left(a_n cos \frac{n\pi x}{L} + b_n sin \frac{n\pi x}{L} \right)$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Figure 2: Mathematics ‘versus’ Music

We know that when finances have to be trimmed and there is less money for instruction, music and other performing and creative arts are the first to be cut back [3]. Figure 2, partly from *Instagram*, shows this, but it also illustrates a lack of appreciation of the link between music and mathematics on the part of the educational administrators who make these decisions. The link here is beauty. The proof of the first mathematical equation in this figure is one of the most elegant and beautiful in mathematics. It also happens to be very useful in the context of the normal or Gaussian bell-shaped curve in statistics.

Moreover, Block-Schwenk in promoting the Berklee College of Music online unit "Applied Mathematics for Musicians", has this to say: "Math is a vital skill for anyone in, or aspiring to be in, the music industry. From understanding music publishing deals and royalty statements to applying music theory and music production concepts, math can help you enormously. For many of us, though, math is something that's preferably avoided or best left to someone else. *Applied Mathematics for Musicians* is designed to change that and to build your own knowledge of, and confidence in, math in practical ways that relate directly to the world of music." [4].

In the work of the Greek composer Xenakis, who applied the principles of stochastic mathematics directly into his musical composition, his claim was that this method could be used by anyone with a basic grasp of mathematical concepts, but our feeling is that he somewhat overestimated the complexities of the task. Xenakis produced some remarkable compositions, where the musical out- come is not overshadowed by the underlying processes. One of his ground-breaking works was the 1954 composition titled "Metastasises", in which Xenakis uses 12-tone methods and the Fibonacci series to explore Einstein's view of time. The music directly links to mathematics in an open and honest way that, to our knowledge, no other composer had achieved before. The results are breath-taking and uniquely beautiful [10;44].

In mathematical terms, the canon can be described as a periodic function where, for example, if f is the first voice and g is the second, then $g(t) = f(t - x)$, in which t indicates the numbers of measures and x is the interval difference between g and t. Canon 5, described as *per tonos* is $g(t) = f(t - x) + H$, where H indicates that the pitch has shifted by a perfect fifth [11].

3.2 Memory

As educational fashions rise and fall about theories such as the role of memorization in learning and applying one's learning professionally, there are certain things which one needs to know inside out in order to function well in practice.

DOMAINS	**ELEMENTS**	**BODY-MIND ISSUES**
Cognitive	Memory	Knowledge through concepts (ideas) [41]
Affective	Emotions	Affective connaturality [29]
Psychomotor	Senses	Multiple intelligences [16,46]

Table 2: Domains and sense elements

"A great deal of research on memory over the last century has been concerned with the question of *where* in the brain memories are located. It seems like a logical question, but as with many things in science, the answer is counter-intuitive: They are not stored in a particular place. Memory is a process, not a thing; it resides in spatially distributed neural circuits, not in a particular location, and those circuits are different for semantic and episodic memory, procedural and autobiographical memory" [21;48].

'Memory' is elusive, not just in terms of remembering, but in terms of classifying the ways we remember. There is also controversy about how many (and how) big are the chunks of memory we can accommodate at any one moment in order to solve problems that occur in music or mathematics [38;42.43]. See especially the work of Juan Pascual-Leone, the founder of the Neo-Piagetian approach to cognitive development [34]. Again, in practice it is a combination of intuition, emotion and memory (acquired from practice) and love of sense beauty [20]. An example of this was etched in the memory of one of us, when the great Soviet violinist, David Oistrakh, had a string break in the middle of a performance at the Sydney Town Hall in 1958, but was able to adjust immediately and complete the performance. This takes years of practice and a 'feeling' for one's performance.

3.3 Emotions

Again, the zeitgeist oscillates between an overemphasis on intelligence on the one hand or on emotion on the other hand, though it is not a question of 'either-or' but of 'both-and'! Each of us is a unity, even if we can distinguish parts and functions. Knowledge is acquired through perception and abstraction as well as through connaturality. In particular, creativity in both disciplines is often characterized by serendipity.

The teacher needs to be operating with a sensitivity to the inter-relationship of the parts and functions in order to appreciate the individual gifts of each student and the interaction of the functions of the mind and the body [44]. Thus, the organ of sight is the eye, but the organ of the intellect is not the brain: we do not think with our brain although we cannot think without our brain. We think with our mind or intellect.

<table>
<tr><td colspan="2">SENSES</td><td rowspan="2">(a)
→
[18]</td><td rowspan="2">BRAIN
[23;51]</td><td rowspan="2">(b)
→
[27]</td><td rowspan="2">MIND & WILL
[6]</td><td rowspan="3">(c)
←
[28]</td></tr>
<tr><td>External
[45]</td><td>Internal
[40]</td></tr>
<tr><td colspan="2">EMOTIONS [2]</td><td colspan="3">→</td><td>↑→</td></tr>
<tr><td colspan="7">(a) Perception; (b) Abstraction; (c) non-conceptual intellectual knowledge</td></tr>
</table>

Figure 3: Elements of philosophical anthropology [24]

While there is nothing in our intellect that was not first in our senses, we can imagine things we have never seen and have a concept (idea) of things that may not exist, such as a pyramid a kilometer high made of gold. While the taxonomies, including MI, make it clear that the senses can be refined and need to be appreciated, there are aspects of perception which we really do not yet under- stand [34]. For instance, how can we know something immaterial in our mind, such as an imaginary number, even if it has no material existence, and even though its mathematical existence has an application in alternating current in physics?

<table>
<tr><td rowspan="4">Knowledge through connaturality</td><td rowspan="2">Intellective, by way of</td><td>knowing</td><td>par mode de connaissance</td></tr>
<tr><td>non-consciousness</td><td>par mode de nescience</td></tr>
<tr><td rowspan="2">Affective, by way of</td><td>practical inclination</td><td>par mode d’inclination pratique</td></tr>
<tr><td>creation</td><td>par mode de création</td></tr>
</table>

Table 3: Knowledge through affective connaturality [29]

“Mathematics, as much as music or any other art, is one of the means by which we rise to a complete self-consciousness. The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds” [47].

While we should not be prisoners of our emotions, they do help us to want to learn and to learn how to learn. There are strategies for going to emotions and going through emotions to manage unwanted emotions [40]. At the postgraduate level, emotions assist in curiosity drive research in mathematics and music by those blessed with an inspirational love for the field. Both too can engender phobias in those who cannot see their role in the development of the cultivated mind.

4 Conclusion

An example of both serendipitous outcomes and curiosity driven research is that of Roger Herz-Fischler, an eminent abstract probability theorist who was asked in 1972 to take over a course for first-year architecture students at Carleton University in Canada. He decided that the best way to engage the students was to keep himself content by talking about things that interested him. He then started to investigate in detail some of the claims of the non-mathematical manifestations of the “golden number”. Out of this arose not only texts and papers which demonstrated the scholarship of teaching, but also he engaged with the purely mathematical history in the scholarship of discovery. Neither of these might register in the current research league tables, but the reprints of his books are testimony to his erudition and the overlapping of what some people treat as separate Boyer categories. His historical research has been supported by the Social Sciences and Humanities Research Council of Canada (SSHRC) and his mathematical work by the Natural Sciences and Engineering *Research Council* of *Canada* (NSERC), a rare double!

Many of the great discoveries in science, particularly in medicine and biology, have been serendipitous by-products rather than planned assaults on a problem. The use of penicillin and lithium carbonate are two well-known examples. It is important to remind ourselves of this at a time when expository research seems to be undervalued, and curiosity driven research does not align well with the measures of research used for the university league tables.

In the (con)temporary glamour of university research 'league tables', and the concomitant obscuring of the mission of providers of higher education, serendipity cannot readily be measured as a quantitative input. Those engaged in scholarly activity in music or mathematics are well aware of these chance encounters and apparent digressions.

Similarly, the contribution to, and enrichment of, our knowledge through the emotions is an immaterial phenomenon even if our feelings 'feel' them [10]. The knowledge that twins have for each other, or a parent for a child, or long-time married couples for each other, is no less real as 'knowledge' than our knowledge of Pythagoras' Theorem. No one can deny the learning 'force' of teachers who love their field of teaching and love their students (in the sense of 'be friendly but not familiar').

Logically too we can be more convinced by a convergence of probabilities than by rigorous logic. This is an important facet of learning to learn, such as with Newman's hypothetical "illative sense" [1;32]. This is the way we are actually 'convinced' in mathematics and music, and hopefully in this paper.

Each step 'seems right', but intuition and shrewd guessing seem to be drummed out of too many children in school [40]. It takes much work to re-kindle the flame and many students who actually do quite well at examinations never really learn the warmth of love when intuition is enkindled by a teacher at ease with basic anthropology in themselves and inspiration for their students [10;23]: learning to learn how we actually learn!

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