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Derivation of Gravitational time dilation from principle of equivalence and special relativity

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Abstract

General relativity is the exact theory of gravity which has been experimentally found to be correct with extremely high accuracy. One of the most surprising predictions of the general theory is that time runs slow in a gravitational field. Its proof formally comes from Schwarzschild metric which is a solution of Einstein field equation for a spherically symmetric mass. However, as Einstein field equation is too complex, attempts have been made earlier to derive gravitational time dilation by direct use of principle of equivalence and special theory of relativity. But this objective has been accomplished partially till date as the resulting expression agrees with the exact expression only up to first order. In this paper, by using principle of equivalence and special relativity, we present a thought experiment which helps us to derive an expression that exactly matches with the expression for gravitational time delay.

Keywords: Gravitational time dilation, Principle of equivalence, Special theory of relativity.†

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1. Introduction

General theory of relativity is a geometric theory of gravity in which dynamics of the objects are governed by curvature of space-time created by distribution of energy-momentum in the universe. Surprising accuracy of the theory has been established by astronomical confirmation of its predictions like deflection of light beam by sun and precession of perihelion of mercury [1]. Another important prediction which has been experimentally confirmed is the gravitational time dilation of clocks near massive objects like planets or stars [2-3]. In simpler terms, the clocks and every natural process run slowly in a gravitational field. Time period of a clock (i.e., time taken for rotation of a pointer by 360^0) as measured by another clock placed in gravitation free region increases from T_0 to T when that clock is transferred from infinity to a location at distance R from a massive object. Mathematical expression for this change as per general theory of relativity is,

$$T = T_0 \left(1 - \frac{2GM}{c^2R}\right)^{-1/2} \qquad (1)$$

Where M=Mass of object (planet), c=Speed of light, G=Gravitational constant

The key hypothesis on which general theory of relativity stands is the principle of equivalence which states that gravitational force and inertial force are indistinguishable. A frame or observer undergoing free fall in a gravitational field acts as an inertial frame of reference. Since special relativity also involves a time dilation, few researchers [4-5] have attempted to derive the gravitational time dilation using the results of special relativity and principle of equivalence by means of thought experiments. In this regard, work of Schiff [5] is noteworthy. By simultaneously accelerating two clocks with a fixed distance between them, he derived the approximate expression for gravitational time dilation given by,

$$T = T_0 \left(1 + \frac{GM}{c^2R}\right) \qquad (2)$$

Note that if we take a power series expansion of Eq. (1), Time period of clock in gravitational field as per general relativity is given by,

$$T = T_0 \left(1 + \frac{GM}{c^2R} + terms\ of order \left(\frac{GM}{c^2R}\right)^2 and\ higher\right) \qquad (3)$$

Comparing Eq. (2) and (3), we find that derivation of gravitational time dilation by Schiff [5] is correct up-to first order term. However, in this paper, we will present a physical situation which will help us in deriving the exact form of gravitational time dilation given by Eq. (1) from the results of special theory of relativity and principle of equivalence.

2. Proof of gravitational time dilation from principle of equivalence and special relativity

Let us consider a spherical object (a planet or star) of mass *M* as shown in Figure 1. Initially let two clocks **A** and **B** are at nearly infinite distance from *M* so that gravitational force on both of them are negligible. However, clock **B** is allowed to fall freely towards *M* and clock **A** is prevented from falling by applying infinitesimal force (≈ 0 force). As initially the gravitational field near

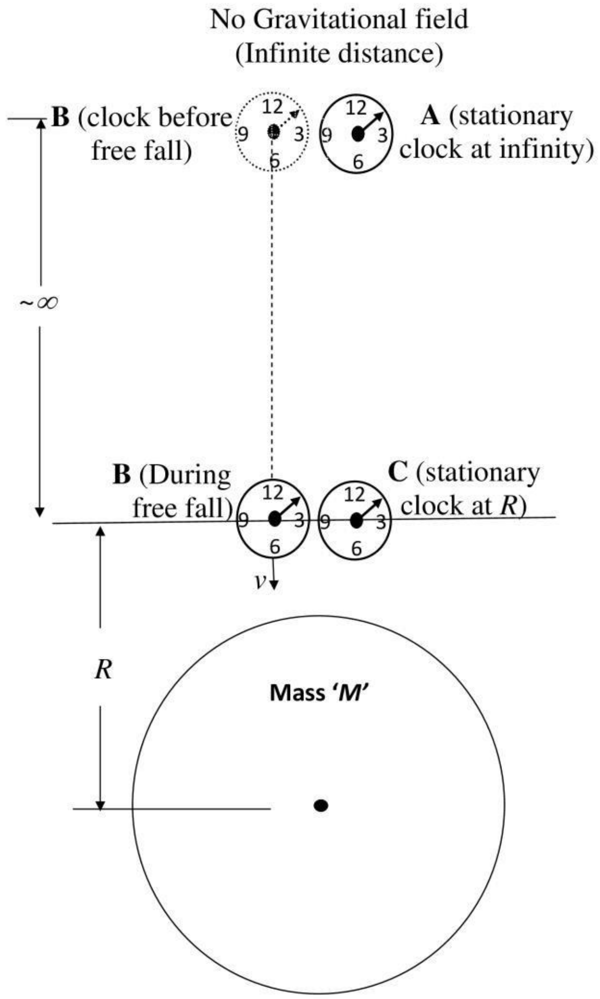

Figure 1- Pictorial representation of thought experiment for derivation of gravitational time dilation (**A** and **C** are stationary clocks, **B** is a clock freely falling from infinity)

both the clocks are approximately zero, both of them will tick at same speed to start with. Again, since the clock **B** falls freely through the gravitational field, net force on it (inertial plus gravitational) is zero and it constitutes a local inertial frame of reference in accordance with the principle of equivalence. As both the clocks **A** and **B** start with same time period and each of them is in (local) inertial frames of references, time periods of both of them T_A and T_B will *always* be same. In other words, due to free fall, gravitational time dilation of B by curvature of space-time is locally neutralized by its free acceleration as per principle of equivalence. Mathematically,

$$T_A = T_B = T_0 \tag{4}$$

During its free fall through the gravitational field, clock **B** will undergo non-uniform acceleration depending upon its distance from *M*. Suppose there is another stationary clock **C** at a distance of *R* from the center of planet *M*. Velocity of clock **B** when it just crosses the clock **C** can be calculated by using law of conservation of total energy i.e.

Final total energy at distance R=Initial total energy at infinity (which is 0)

$\Rightarrow$ Gravitational energy at R + Kinetic energy at R=0 (5)

Putting general relativistic expression for gravitational potential energy from [6] and expression for kinetic energy from special theory of relativity in Eq. (5), we get,

$$mc^2\left[\left(1-\frac{2GM}{c^2R}\right)^{1/2}-1\right]+(m-m_0)c^2=0$$

(where *m* is relativistic mass and m_0 is rest mass)

$$\Rightarrow \quad mc^2\left[\left(1-\frac{2GM}{c^2R}\right)^{1/2}-1\right]+\left[m-m\left(1-\frac{v^2}{c^2}\right)^{1/2}\right]c^2=0$$

$$\Rightarrow \quad v^2=\frac{2GM}{R} \tag{6}$$

It is interesting to note that Eq. (6) for velocity can also be derived by using non-relativistic form of both gravitational energy (i.e., Newton's law) and kinetic energy as shown below.

$$-G\frac{Mm_0}{R}+\frac{1}{2}m_0v^2=0$$

$$\Rightarrow \quad v^2=\frac{2GM}{R}$$

As the clock **B** is in a *local* inertial frame of reference i.e., it doesn't experience any gravitational field in its local region of space, a tiny observer standing on the clock **B** is *entitled to apply special theory of relativity to events happening nearby him*. When the clock **B** just crosses past the clock **C**, he will observe the clock **C** moving up with a velocity of *v*. So, by applying special theory of relativity, he will relate the time periods of clock **B** and **C** as measured by him by relation,

$$(T_C)_{\substack{measured \\ by\ B}} = T_B \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Using Eq. (4) and (6) in above expression, we get,

$$(T_C)_{\substack{measured \\ by\ B}} = T_0 \left(1 - \frac{2GM}{c^2 R}\right)^{-1/2} \tag{7}$$

As we have explained earlier, clocks **A** and **B** always tick at the same rate. So,

$$(T_C)_{\substack{measured \\ by\ A}} = (T_C)_{\substack{measured \\ by\ B}}$$

Putting the above in Eq. (7), we get,

$$(T_C)_{\substack{measured \\ by\ A}} = T_0 \left(1 - \frac{2GM}{c^2 R}\right)^{-1/2} \tag{8}$$

Let T=Time period of stationary clock **C** in gravitational field as measured by clock **A** (placed at infinity). Then Eq. (8) becomes,

$$T = T_0 \left(1 - \frac{2GM}{c^2 R}\right)^{-1/2} \tag{9}$$

Eq. (9) is exactly same as Eq. (1) and thus, we have proved the exact form of gravitational time dilation using the principle of equivalence and special theory of relativity.

3. Conclusion

Generally, gravitational time dilation due to a massive object is derived by using the Schwarzschild metric which is a solution of Einstein field equations. But it has been felt by many researchers that it should be possible to derive the expression for time dilation by using the principle of equivalence and special theory of relativity. However, in the past, only approximate form of exact expression could be derived [5]. In this paper, we have presented a new situation or thought experiment that proves the exact value of gravitational time dilation using principle of equivalence and special theory of relativity. Surprisingly, irrespective of the use of relativistic or non-relativistic expressions for kinetic and potential energy in our derivation, we get the exact expression for gravitational time dilation matching with Schwarzschild solution of General relativity.

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