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Forms of Crossed and Simple Polygons

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Abstract

In this paper the author presents a new form of hexagon and the solution of the open problem of classifying plane hexagons. In particular are illustrated the forms of crossed and simple n-gons for n = 3, 4, 5, 6 and also the forms of simple ones for n = 7, 8, 9. A graphic way to construct new forms of polygons is illustrated.

Keywords: polygon; simple polygon; crossed polygon; form of a polygon; hexagon.[2]

1 Introduction

Polygons represent a well-known topic in Mathematics. In Euclid's Elements (Book 1, definition 19) we read: "Rectilinear figures are those contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four" (Fitzpatrick, 2008). But many problems about polygons are still open. The issue of classifying polygons in one of them.

2 Definition of a polygon

Let n be an integer number, with $n > 2$. On the plane, we consider n different points: $P_1, P_2, \dots , P_n$, called *vertices*. The set of the segments: $[P_1, P_2]$, $[P_2, P_3]$, …, $[P_{n-1}, P_n]$, $[P_n, P_1]$ is called *n-gon* or *polygon* $P = P_1P_2 \dots P_n$. The above mentioned segments are called *sides*. So a polygon is a cyclically ordered sequence of vertices and sides (Grünbaum, 2012). This definition is

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different from Euclid's one: according to this author, a polygon is a limited part of the surface of the plane, while our definition concerns only a set of segments.

An *ordinary* polygon is one in which no point of the plane belongs to three or more sides of the polygon. A polygon is *proper* if two sides in sequence aren't aligned. A polygon is *simple* if two sides haven't internal points in common; otherwise, it is called *self-intersecting* or *crossed*. In this paper, I only consider ordinary and proper polygons (Grünbaum, 1975).

3 Classifying polygons

Two n-gons $P = P_1P_2 \dots P_n$ and $Q = Q_1Q_2 \dots Q_n$ have the same *form* if there exist a set of ordinary and proper n-gons $R(t) = R_1(t)R_2(t)\dots R_n(t)$, with $0 \le t \le 1$, so that $R(0) = P$, $R(1) = Q$ or $R(1) = Q^*$, where Q^* is a mirror image of Q and $R_1(t)$, $R_2(t)$, …, $R_n(t)$ are continuous functions for every $i = 1, 2, \dots, n$.

So the n-gons P and Q have the same form when we can obtain Q from P by means of a continuous deformation through a set of ordinary and proper n-gons; a reflection may be necessary. The *form* can be considered the equivalence class containing all the n-gons having the same form of a particular n-gon.

We *classify* n-gons when we can fix the number $T(n)$ of all different forms of n-gons existing for a particular value of the integer n , $n > 2$. It has been proved that for $n = 3, 4, 5$ the different forms of n-gons are respectively: $T(n) = 1, 3, 11$ (figure 1).

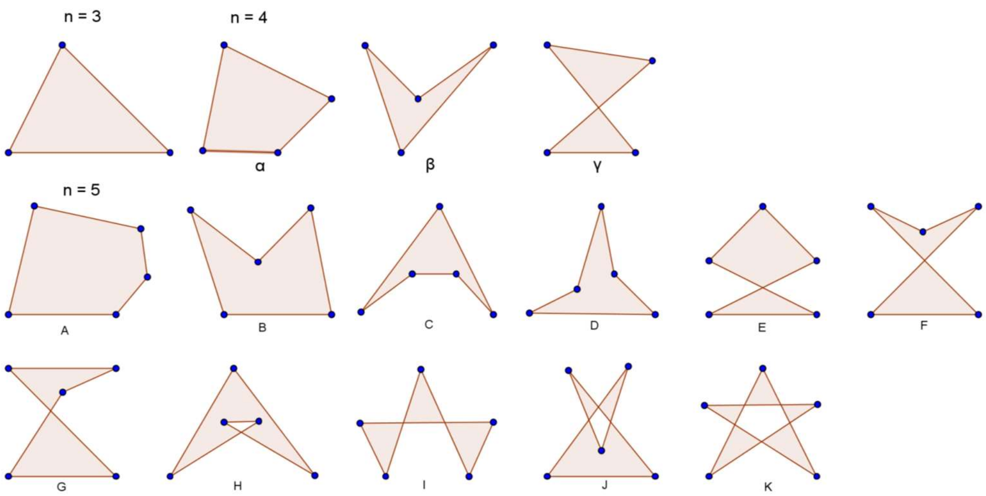
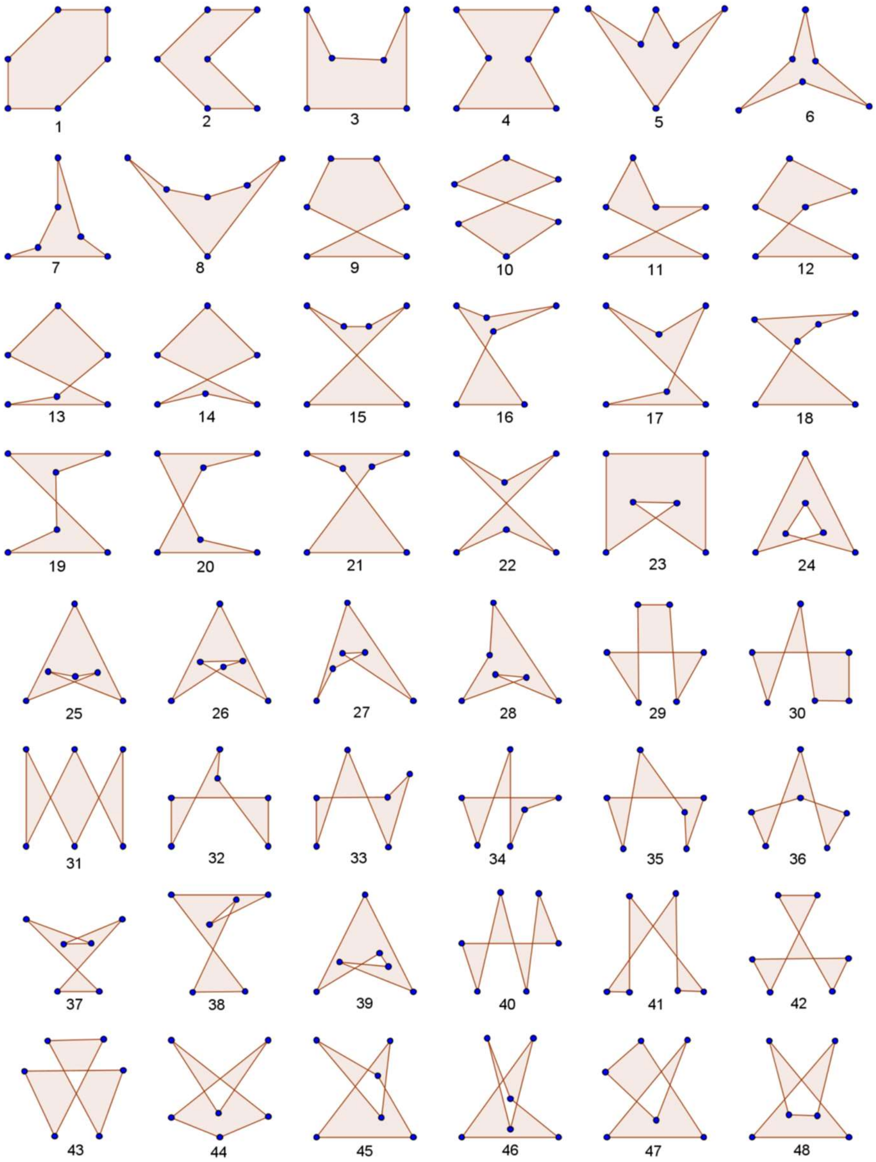


Figure 1. The forms of n-gons for $n = 3, 4, 5$.

How many different forms of hexagons are there? The answer is not simple. In 17th century Girard established that $T(6) = 69$, but three centuries later Steinitz affirmed that the number was $T(6) = 70$ (Girard, 1626; Steinitz, 1916). However, neither Girard nor Steinitz depicted the forms of hexagons in their essays. More recently Grünbaum added two new forms and conjectured that $T(6) = 72$ (Grünbaum, 1975). In 1977 I found a new form of hexagon, the 73rd one, and I demonstrated that $T(6) = 73$. In figure 2 the 73 different forms of hexagons are drawn: the hexagon number 73 represents the new and the last form of hexagon (Togliani, 1978).



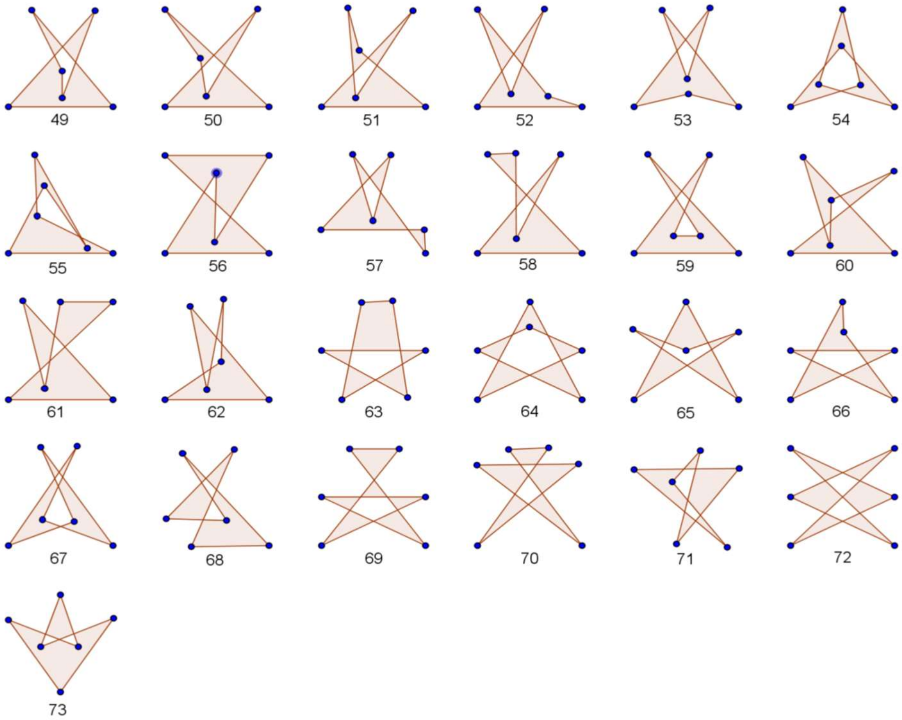


Figure 2

Before proving what has been established about hexagons, it’s necessary to introduce some new elements.

4 The kind of a polygon

The *direction* of a polygon P is established when we choose one of the two possible driving ways of P; so we a have an *oriented polygon*. In an oriented n-gon $P = P_1P_2 \dots P_n$ we introduce a *hatching* immediately on the left of the sides of P. The *internal angle* related to the vertex P_i is the one put where the hatching is. In figure 3 we see the pentagon P with its different hatchings.

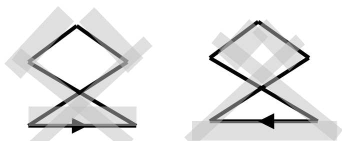


Figure 3

If J is the sum (expressed in radiant) of all internal angles and k is the number of reflex ones, we define:

$$a = \frac{n\pi - J}{2\pi} \quad , \quad a' = a + k = \frac{n\pi - J}{2\pi} + k \, ,$$

where a is the *kind of P according to Wiener* and *a'* is the *kind of P according to Hess* (Brusotti, 1936).

Obviously J may assume two different positive values for every n-gon P, depending on the chosen direction: we select the value of J corresponding to the minimum value of k. We denote with h the number of self-intersection points (*knots*) of P (Togliani, 2001). A polygon divides the plane into an unlimited part and one or more limited ones, called *cells*. Clearly a simple polygon has one cell. Now we attribute the 5-tuple (h, a, a', k, J) to every hexagon in figure 2, as shown in the following table.

hex	*h*	*a*	*a'*	*k*	*J*	hex	*h*	*a*	*a'*	*k*	*J*
1	0	1	1	0	4π	**2**	0	1	2	1	4π
3	0	1	3	2	4π	**4**	0	1	3	2	4π
5	0	1	3	2	4π	**6**	0	1	4	3	4π
7	0	1	4	3	4π	**8**	0	1	4	3	4π
9	1	0	4	4	6π	**10**	1	0	3	3	6π
11	1	0	3	3	6π	**12**	1	0	3	3	6π
13	1	0	4	4	6π	**14**	1	0	4	4	6π
15	1	0	2	2	6π	**16**	1	0	2	2	6π
17	1	0	3	3	6π	**18**	1	0	2	2	6π
19	1	0	3	3	6π	**20**	1	0	3	3	6π
21	1	0	2	2	6π	**22**	1	0	3	3	6π
23	1	2	2	0	2π	**24**	1	2	2	0	2π
25	1	2	3	1	2π	**26**	1	2	3	1	2π
27	1	2	3	1	2π	**28**	1	2	3	1	2π
29	2	1	3	2	4π	**30**	2	1	2	1	4π
31	2	1	3	2	4π	**32**	2	1	2	1	4π
33	2	1	3	2	4π	**34**	2	1	3	2	4π
35	2	1	3	2	4π	**36**	2	1	2	1	4π
37	2	1	3	2	4π	**38**	2	1	3	2	4π
39	2	1	3	2	4π	**40**	3	0	3	3	6π

41	3	0	3	3	6π	**42**	3	2	2	0	2π
43	3	2	2	0	2π	**44**	3	2	2	0	2π
45	3	2	2	0	2π	**46**	3	2	3	1	2π
47	3	2	2	0	2π	**48**	3	2	2	0	2π
49	3	2	3	1	2π	**50**	3	2	3	1	2π
51	3	2	3	1	2π	**52**	3	2	3	1	2π
53	3	2	3	1	2π	**54**	3	2	2	0	2π
55	3	0	3	3	6π	**56**	3	0	3	3	6π
57	4	1	3	2	4π	**58**	4	1	3	2	4π
59	4	1	3	2	4π	**60**	4	1	3	2	4π
61	4	1	3	2	4π	**62**	4	1	3	2	4π
63	5	2	2	0	2π	**64**	5	2	2	0	2π
65	5	2	3	1	2π	**66**	5	2	3	1	2π
67	5	0	3	3	6π	**68**	5	0	3	3	6π
69	6	1	3	2	4π	**70**	6	1	3	2	4π
71	6	1	2	1	4π	**72**	7	0	3	3	6π
73	3	0	3	3	6π						

Table 1

5 The classification of hexagons

With reference to previous observations, it’s now possible to demonstrate the following theorems.

Theorem 1 - *The hexagon 73 in figure 2 has a form different from other ones in the same figure.*

Proof. The hexagons number 40, 41, 55, 56 are the only ones having the same 5-tuple (3, 0, 3, 3, 6π) of the hexagon 73; so 73 might have the same form of one of them. But 40 and 41 are excluded because they have only triangular cells. Moreover, for passing from hexagon 55 (or 56) to hexagon 73 using a continuous transformation, it’s necessary to pass through a form of a non-ordinary hexagon, as shown in figure 4. Furthermore, 73 has two consecutive sides without knots, contrary to what happens in 55.

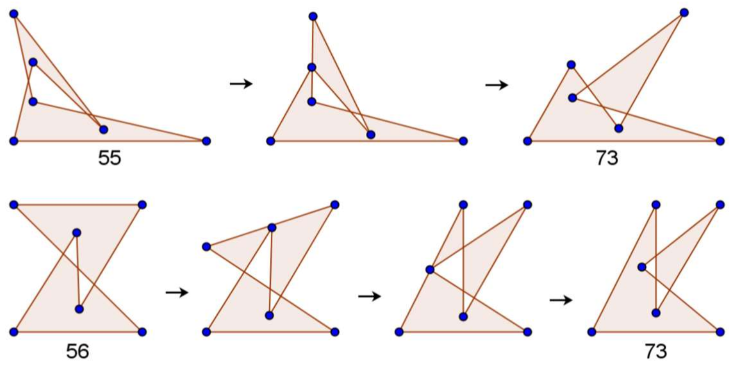


Figure 4

The open problem of classifying hexagons is closed by the following statement.

Theorem 2 – *The hexagon 73 represents the last possible form for hexagons; therefore T(6) = 73.*

To prove this theorem, it's necessary to introduce a new definition. Given the n-gon P, we call *basic graphic operations* the following ones:

a) *dulling* of a vertix P_i , using a secant straight line crossing the sides $[P_{i-1}, P_i]$ and $[P_i, P_{i+1}]$ in Z and Z' respectively;

b) *bending* of a side $[P_i, P_{i+1}]$, substituting $[P_i, P_{i+1}]$ for two new sides $[P_i, Z]$ and $[Z, P_{i+1}]$, where Z is a point of the plane not belonging to P;

c) *extension* of the sides $[P_{i-1}, P_i]$ and $[P_i, P_{i+1}]$ substituting them for $[P_i, Z]$ and $[Z', P_{i+1}]$ and introducing the new side $[Z, Z']$, where Z and Z' are points of the plane not belonging to P.

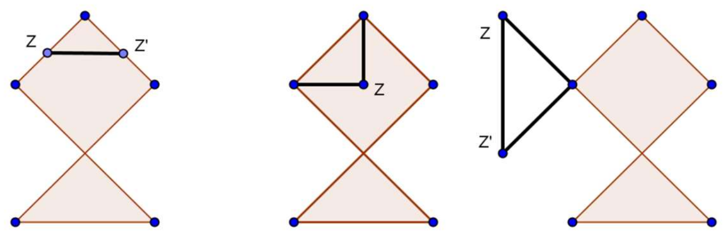


Figure 5

Proof. Each basic graphic operation allows the n-gon P to transform itself into the (n+1)-gon P'. It's not difficult to prove that using the three basic graphic operations in every possible way we can obtain all forms of hexagons starting from all forms of pentagons. Then $T(6) = 73$; in other words, starting from all the 11 forms of pentagons it's possible to obtain all the 73 forms of hexagons and no other one. q.e.d.

The introduction of the basic graphics operations is very important because it allowed me to obtain the 73^{rd} form of the hexagon in several different ways. In figure 6 some of these ways are represented, using only the operation of bending a side.

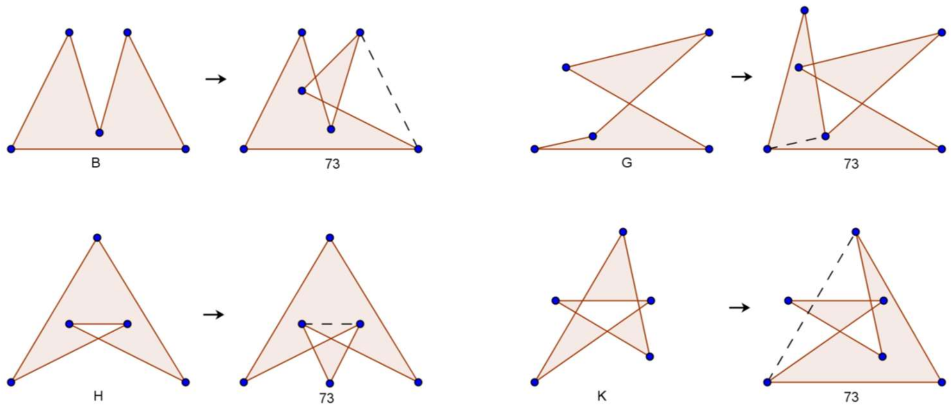


Figure 6

6 Forms of simple polygons

Let's reduce the problem of classifying polygons considering only simple ones. Now the question is: what is the number $S(n)$ of forms of simple n-gons for every $n > 2$? The problem is solved for $n = 3, 4, 5, 6$; among the well-known forms of n-gons we can only consider the simple ones and consequently we have: $S(3) = 1$, $S(4) = 2$, $S(5) = 4$, $S(6) = 8$ (figures 1 and 2).

In the research of forms of simple heptagons, octagons and nonagons I used the graphic operations of dulling and bending and also a "label" attributed to every polygon. Essentially the difference between two forms of n-gons is due to the number and to the position of their reflex angles. The *label* of a form of a simple n-gon is a sequence of n bit attributed to the interior angles of the n-gon: 0 for every reflex angle and 1 for every convex one. For example, the forms of the heptagons labelled with 0011111 and 0110111 are different: both have 2 reflex angles but these angles are differently positioned (figure 7). Obviously the label 0011111 is equivalent to other ones: 0111110, 1001111, 1111100 … ; they are all related to the same form of the heptagon.

In figures 7, 8 and 9 are respectively represented the forms of simple heptagons, octagons and nonagons found in my research. So we may conjecture that the numbers of their forms are $S(7) = 13$, $S(8) = 24$, $S(9) = 39$.

I tried to investigate simple decagons and I obtained 68 different forms. But I am not sure that this number is right. However, it's interesting to note that the found values of $S(n)$ i.e. 1, 2, 4, 8, 13, 24, 39, 68 are the same of the beginning of the sequence A096573 in Sloane's *The On-Line Encyclopedia of Integer Sequences* (https://oeis.org/A096573).

7 Conclusion and outlook

The basic graphic operations permit us to solve the problem of classifying hexagons: $T(6) = 73$. But what about the forms of heptagons? Using the same graphics operations and starting from the 73 forms of hexagons I tried to produce forms of heptagons, but the outcome was discouraging: with the only operation of dulling – used in part – the number of heptagons exceeded 200. So the basic graphic operations appear ineffective if we want to classify n-gons with $n > 6$.

Furthermore: how many forms of simple n-gons are there for every $n > 2$? This open problem can be graphically solved using only two basic graphic operations for small values of n; but for $n > 10$ the problem appears to be too arduous. However, the question seems to be also a combinatorial one if we associate an appropriate sequence of bit to every form of n-gon. Probably this second way to approach the problem might have interesting future developments.

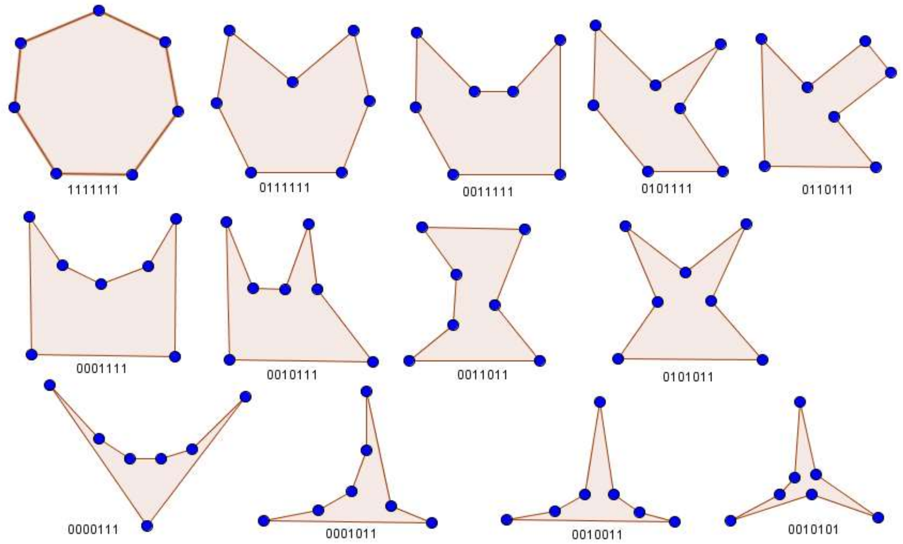


Figure 7. Forms of simple heptagons.

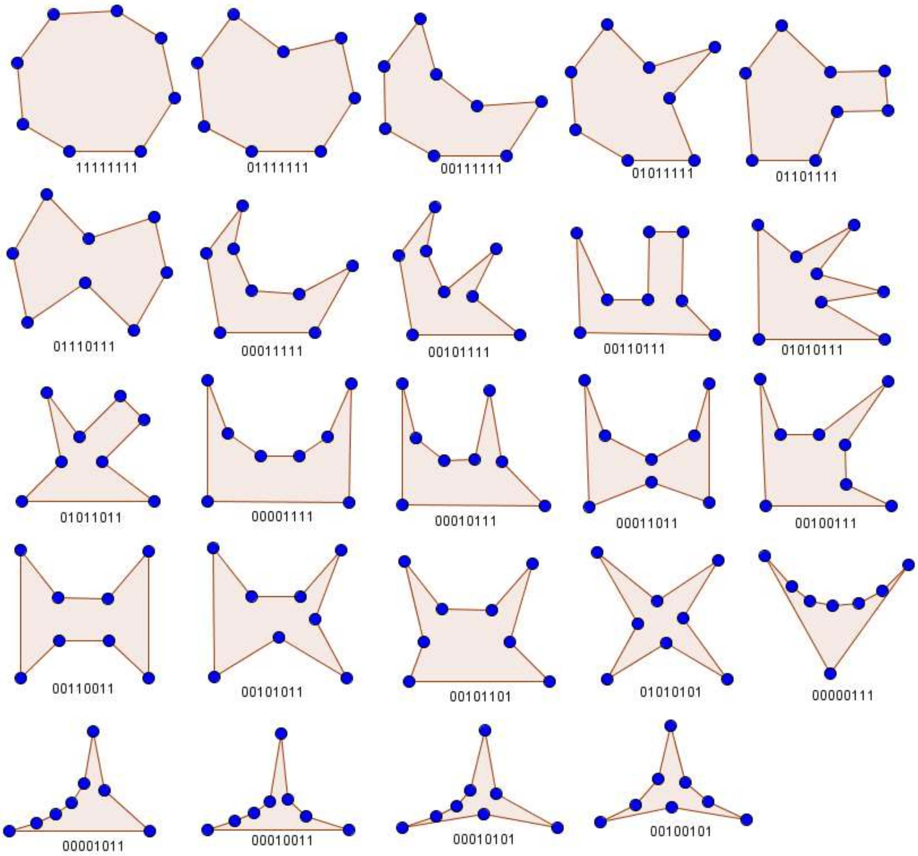


Figure 8. Forms of simple octagons.

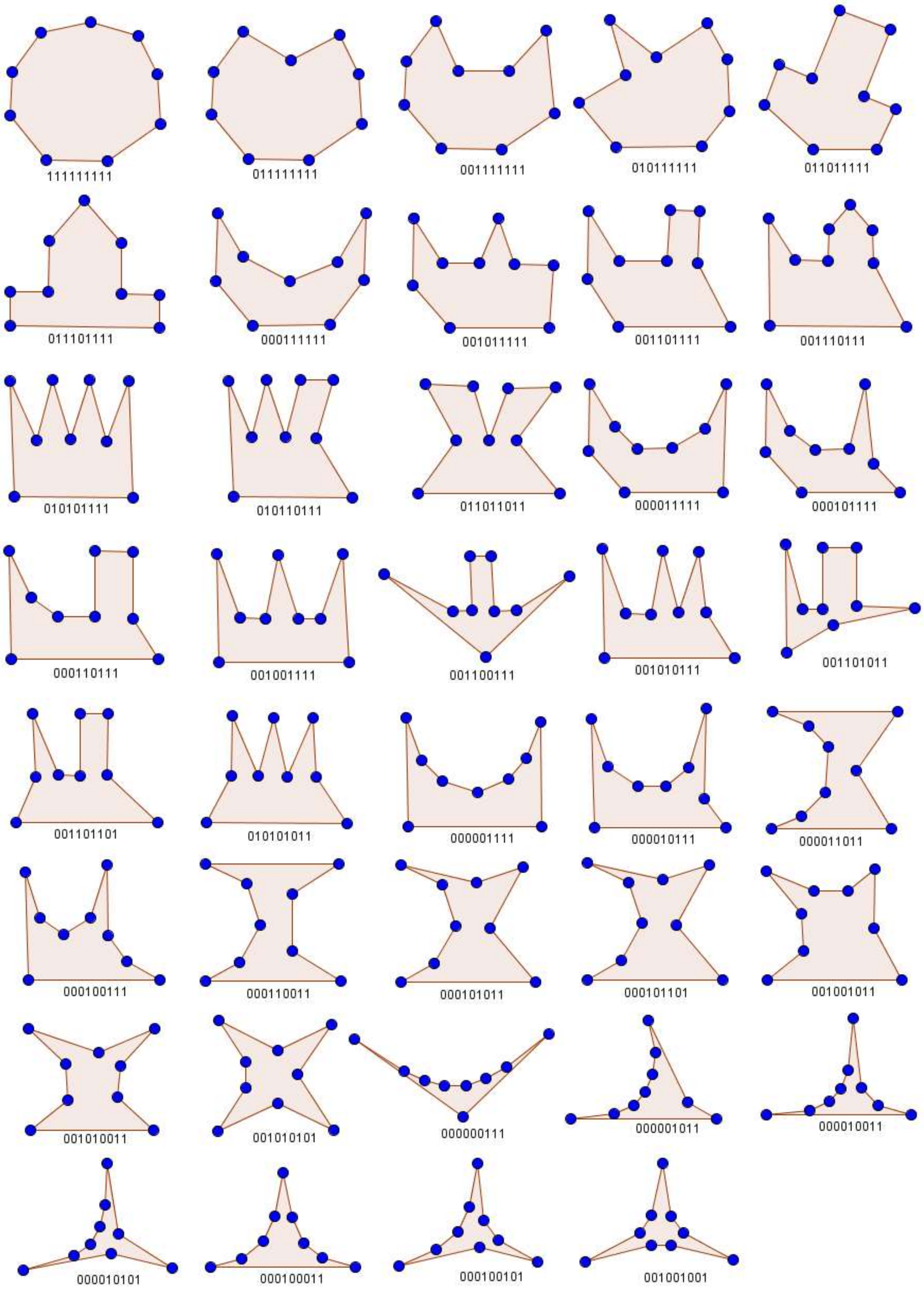


Figure 9. Forms of simple nonagons.

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