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From Archimedes and the oxen problem to Pell's equation

Ilaria Veronesi [*]
Antonio Caserta [†]

Abstract

As part of the laboratory activities designed for the students of the second year of the Mathematical High School by the research group in Mathematics Education of the Department of Mathematics of the University of Salerno, the theme "From Archimedes and the oxen problem to Pell's equation" has been designed. The course has not yet been delivered to students and in this paper the didactic path to be developed in the classroom will be described. The students initially will retrace the various solution approaches proposed over the centuries. Subsequently, with a heuristic approach, they will tackle the resolution of the proposed problem by exploiting properties related to the theory of numbers. With the help of some schedules, provided by the teachers-tutors, they will conjecture and argue by answering questions aimed at stimulating didactic reflection. This workshop path involves processes such as exploring and conjecturing through the articulation of different ways of working (individual and in groups), manual and digital.

Keywords: Mathematical High School; technology advanced learning; history of mathematics; Number Theory.[‡]

[*] Department of Mathematics, University of Salerno, Italy; iveronesi@unisa.it
[†] I.I.S. Leonardo Da Vinci, Florence (FI), Italy; antonio2584@hotmail.it
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1. Introduction

> What does mathematics really consist of?
> Axioms (such as the parallel postulate)?
> Theorems (such as the fundamental theorem of algebra)?
> Proofs (such as Godel's proof of undecidability)?
> Concepts (such as sets and classes)?
> Definitions (such as the Menger definition of dimension)?
> Theories (such as category theory)?
> Formulas (such as Cauchy's integral formula)?
> Methods (such as the method of successive approximations)?
> Mathematics could surely not exist without these ingredients; they are all essential. It is nevertheless a tenable point of view that none of them is at the heart of the subject, that the mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics really consists of is problems and solutions.
>
> Paul Halmos

In this article, a proposal for a didactic path designed for the solution of a problem that spans two millennia of history through various solving processes of famous mathematicians will be illustrated and developed. It is the problem of Archimedes' oxen. Eratosthenes' epigram describes in common language a seemingly simple mathematical problem: it deals with "small" numbers and relates concrete objects. Its solution leads to algebraic challenges that are within the reach of students in the two years of upper secondary school, but with numbers not usually used in school problems. This activity makes the students reflect on the use of language, mathematical formalization, the problem-posing process and problem-solving processes on a problem that possesses an intrinsic complexity.

In fact, it requires finding integer solutions to a system of Diophantine equations, which are polynomial equations with integer coefficients and required solutions that are also integers. These types of equations are known for their complexity, making the problem a considerable challenge even for mathematicians of various eras.

In 1880, the German mathematician Hermann H. Amthor was the first to make significant progress. He did not solve the problem in the strict sense, but he showed that his solution required the solution of a gigantic quadratic Diophantine equation. Amthor published a detailed analysis that clarified the intractable nature of the problem, estimating that the solution would require very huge numbers.

In 1965, mathematician Hermann H. Hitz and his team at the *University of Illinois* finally solved the problem with the use of an IBM 7090 computer. This computer, one of the most powerful at the time, made it possible to perform the complex calculations necessary to find the solution.

Hitz used advanced algorithms to calculate the solution, which required integers of an order of magnitude that defied the known one. The result was that the solution involved numbers with hundreds of digits, impossible to manage without the help of machinery.

Dealing with a classic problem in the classroom that has an important mathematical history behind it, allows you to develop a didactic discourse starting from the point of view of those who over time have built solution processes to then face an activity of exploration, problem-posing and problem-solving.

Referring to Polya, problem-solving means finding a way out of a difficulty, to reach a goal that is not immediately attainable, solving problems is a specific undertaking of intelligence, and intelligence is the specific gift of humankind.

In fact, solving problems is a highly formative activity, whose value goes beyond Mathematics itself (hard skill), involves personal skills or transversal skills (soft skills) and, as Vygotsky states, thanks to problem-solving activities, the student can significantly exceed the level of actual development to enter the zone of proximal development. Duly consolidated, this process leads to raising the level of effective development.

Schoenfeld deals extensively with thinking, teaching, and learning. His book, Mathematical Problem-solving, describes the importance of research in solving mathematical problems by specifying what it means to think mathematically.

For Pellerey, problem-solving and metacognition are closely interconnected: the acquisition of new knowledge passes through the awareness achieved by the learner (and who reflects on his or her cultural background). All metacognitive activity therefore belongs to the zone of proximal development.

The decision to propose in the classroom an educational workshop such as the one presented in the article is aimed at developing significant mathematical skills in students built on curricular knowledge and acquired skills (in particular in the manipulation of algebraic objects). In fact, the activity is in line with the curriculum for 12th-grade students (4th high school year in Italy). This workshop, to be held at the Mathematical High School Project (MHS), aims to delve into mathematical topics to foster technological proficiency and bolster critical thinking skills and mathematical skills. The approach taken in this laboratory resonates with the overarching methodology employed in the diverse educational programs offered by MHS, as evidenced by studies conducted (for

example: Bimonte et al., 2021, 2022, 2023, Tortoriello & Veronesi, 2021, and Aramo et al., 2023).

2. The problem of Archimedes' oxen

At MHS, the evolution of teaching methods in a laboratory setting aligns with Vygotsky's constructivist approach to education and learning. This educational model emphasizes the integration of new knowledge through collaborative interactions among students, teachers, and researchers. In this educational environment, educators play a crucial role as mediators of meaning, guiding students toward a deeper understanding of the theme. To cultivate this development, educators should embrace a research-action methodology that prioritizes hands-on learning, teamwork, critical thinking, and the introduction of engaging content to spark students' curiosity. The project described in the paper utilizes a peer-collaboration action-research technique within small group settings, where students delve into diverse discourses and identify the fundamental framework of the hypothetical deductive method.

The progression of the activity can be delineated into multiple distinct phases, each playing a crucial role in shaping its final form:

- From the text to the problem: transformation from the first part of the epigram into a system of linear equations
- The system "gets complicated": from a mathematical point of view, the approach does not change, but the numbers involved are "larger"
- Archimedes describes a quadratic Diophantine problem
- Pell's Equation as a solution tool
- Let's find a possible solution
- Let's look for help in technology.

2.1 *Part 1. Introduction to the problem*

"*Reckon, o stranger, the number of the oxen of the Sun,*
working with care, if you possess any wisdom;
Calculate how many of them once grazed on the plains
of the Sicilian Island of Trinacria, distributed in 4 groups
of various colours: one milky white,
the second shining black,
the third is golden brown, the fourth mottled.
In each flock the bulls were distributed in considerable numbers,
In the following reports: Consider whites
as equal to the middle of the third
part of all blacks and browns;
the blacks then equal to the fourth part
And to the fifth of the mottled and all swarthy,

From an algebraic point of view, the first part is a multilinear Diophantine problem with four variables and three equations. This first part of the problem describes the proportions of bulls of the four different colors. Let W be the number of white bulls, B the number of black bulls, D the number of piebald bulls, and Y the number of yellow bulls. So, algebraically, this part of the problem can be formalized like this:

$$W = \left(\frac{1}{2} + \frac{1}{3}\right) B + Y = \frac{5}{6} B + Y$$

$$B = \left(\frac{1}{4} + \frac{1}{5}\right) D + Y = \frac{9}{20} D + Y$$

$$D = \left(\frac{1}{6} + \frac{1}{7}\right) W + Y = \frac{13}{42} W + Y$$

These three linear equations are easily solved.

$$W = \frac{5}{6} B + Y = \frac{5}{6}\left(\frac{9}{20} D + Y\right) + Y = \frac{5}{6}\left(\frac{9}{20}\left(\frac{13}{42} W + Y\right) + Y\right) + Y.$$

Therefore

$$\left(1 - \frac{5}{6} \cdot \frac{9}{20} \cdot \frac{13}{42}\right) W = \left(\frac{5}{6} \cdot \frac{9}{20} + \frac{5}{6} + 1\right) Y.$$

After some calculations, we get that 4455W=11130Y.

To obtain a general solution, it is sufficient to assume

W=11130t and Y=4455t,

from which it follows that

D=7900t and B=8010t.

These 4 coefficients are all divisible by 5, so we can reparametrize the solution as follows:

W=2226t, Y=891t, D=1580t, B=1602t

where t is a positive integer.

2.2 ***Part 2 Cows and Oxen... The problem is taking shape.***

The next part of the problem introduces four more variables and four more linear equations. In principle, it shouldn't be more difficult to solve than in part 1, but the numbers are much larger, so it's much easier to make miscalculations.

"These were the proportions of the cows: The white cows were exactly equal to the third part and a quarter of the whole herd of the black cows; while the blacks were once again equal to the fourth part of the piebald, and with it a fifth part, when all, including the bulls, went to graze together. Now the piebald in four parts were equal in number to one-fifth and one-sixth of the yellow flock. Finally, the yellows were equal in number to one-sixth and one-seventh of the white herd. If thou couldst tell precisely, O stranger, the number of the Sun's cattle, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be considered inexperienced or ignorant of numbers, but yet thou shalt not be numbered among the wise."

We indicate with w the number of white cows, b the number of black cows, d the number of piebald cows, and y the number of yellow cows. So, algebraically, this part can be formalized like this:

$$w = \left(\frac{1}{3} + \frac{1}{4}\right)(B + b) = \frac{7}{12}(B + b)$$

$$b = \left(\frac{1}{4} + \frac{1}{5}\right)(D + d) = \frac{9}{20}(D + d)$$

$$d = \left(\frac{1}{5} + \frac{1}{6}\right)(Y + y) = \frac{11}{30}(Y + y)$$

$$y = \left(\frac{1}{6} + \frac{1}{7}\right)(W + w) = \frac{13}{42}(W + w)$$

These four linear equations should be easily solved for the four new variables $w, b, d\ e\ y$.

So we have:

$$w = \frac{7}{12}(B + b) = \frac{7}{12}\left(B + \frac{9}{20}(D + d)\right)$$

$$= \frac{7}{12}\left(B + \frac{9}{20}\left(D + \frac{11}{30}(Y + y)\right)\right)$$

$$= \frac{7}{12}\left(B + \frac{9}{20}\left(D + \frac{11}{30}\left(Y + \frac{13}{42}(W + w)\right)\right)\right)$$

Therefore

$$\left(1 - \frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}\cdot\frac{13}{42}\right)w = \frac{7}{12}B + \frac{7}{12}\cdot\frac{9}{20}D + \frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}Y + \frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}\cdot\frac{13}{42}W$$

Simplifying the denominators, we get:

$$\begin{aligned}(12\cdot 20\cdot 30\cdot 42 - 7\cdot 9\cdot 1\cdot 13)w = {} & 7\cdot 20\cdot 30\cdot 42\, B \\ & + 7\cdot 9\cdot 30\cdot 42\, D \\ & + 7\cdot 9\cdot 11\cdot 42\, Y \\ & + 7\cdot 9\cdot 11\cdot 13\, W\end{aligned}$$

So you have,

$$293391\, w = 176400\, B + 79380\, D + 29106\, Y + 9009\, W$$

Putting them together with the previously found values we find that:

$$W = 2226t, Y = 891t, D = 1580t, e\, B = 1602t,$$

$$293391w = 454000680t.$$

Therefore

$$w = \frac{454000680}{293391}t$$

There are essentially two ways to find y, d and b. One way is to express y in terms of w, then d in terms of y and then b in terms of w. Another is to look for them directly. Using this other method, we find:

$$y = \frac{13}{42\left(W + \frac{7}{12\left(B + \frac{9}{20\left(D + \frac{11}{30(Y+y)}\right)}\right)}\right)}$$

$$d = \frac{11}{30\left(Y + \frac{13}{42\left(W + \frac{7}{12\left(B + \frac{9}{20(D+d)}\right)}\right)}\right)}$$

$$b = \frac{9}{20\left(D + \frac{11}{30\left(Y + \frac{12}{42\left(W + \frac{7}{12(B+b)}\right)}\right)}\right)}$$

$$\left(1 - \frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}\right)y = \frac{13}{42}W + \frac{13}{42}\cdot\frac{7}{12}B + \frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}D + \frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}Y$$

$$\left(1 - \frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}\right)y = \frac{11}{30}Y + \frac{11}{30}\cdot\frac{13}{42}W + \frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}B + \frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}D$$

$$\left(1-\frac{9}{20}\cdot\frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}\right)y=\frac{9}{20}\,D+\frac{9}{20}\cdot\frac{11}{30}\,Y+\frac{9}{20}\cdot\frac{11}{30}\cdot\frac{13}{42}W+\frac{9}{20}\cdot\frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}\,B$$

As before, simplifying the denominators, after a few steps you get:

$$293391y=93600\,B+54600\,D+24570\,Y+9008\,W=342670419t$$

$$293391d=110880\,Y+34320\,W20020\,B+9009\,D=221496660t$$

$$293391b=136080\,W+49896\,B+15444D+9009\,Y=308274498t$$

Ultimately, the general solution has and t.

$$w=\frac{434000680}{293391}t,y=\frac{342670419}{293391}t,d=\frac{221496660}{293391}t\;b=\frac{308274498}{293391}$$

The numerator and denominator are divisible by 63, so we can rewrite the solution more simply:

$$w=\tfrac{7206360}{4657}t,y=\tfrac{5439213}{4657}t,d=\tfrac{3515820}{4657}t \text{ and } b=\tfrac{4893246}{4657}t.$$

4657 is a prime number so we can't reduce the fraction again. So, *t* must be divisible by 4657 by the number of cows of the 4 types of colors that are integers. We have the solution from part 2.$t=4657s$,

$$w=7206360\,s \qquad W=2226t=10366482s$$

$$y=5439213\,s \qquad Y=891t=4149387s$$

$$d=3515820\,s \qquad D=1580t=7358060s$$

$$b=4893246\,s \qquad B=1602t=7460514s$$

where s is a positive integer.

2.3 *Part 3. The last part of Archimedes' Problem.*

In the last part, Archimedes describes a quadratic Diophantine problem, which we report here:

"But come, understand also all these conditions concerning the cattle of the Sun. When the white bulls mingled with the black bulls, they stood still, equal in depth and breadth, and the plains of Trinacria, which stretched in every direction, were filled with their multitude. Moreover, when the yellow bulls and the piebald

bulls were gathered together in a herd, they stood in such a way that their number, beginning with one, slowly increased until they completed a triangular figure, there being neither bulls of other colors nor any in the middle. of them deficient. If thou art able, O stranger, to discover all these things, and to gather them together in thy mind, expounding all their relations, thou shalt depart crowned with glory, and knowing that thou hast been judged perfect in this kind of wisdom."

This part asserts that $\mathrm{W}+\mathrm{B}$ *is a square number, while* $Y+D$ *is a triangular number.*

$W+B=2226t+1602t=3828t=3828\cdot 4657\,s$, while $Y+D=891t+1580t=2471t=2471\cdot 4657\,s$. We are looking for a value for *S* such that $3280\,t=3828\cdot 4657\,s$ is a square number, that is, of the form n^2, at the same time $2471\cdot 4657\,s$ must be a triangular number, that is, of the form $\frac{k(k+1)}{2}$.

$3828=957\cdot 2^2$, so $3828\,t=3828\cdot 4657\,s$ has to be a square number. It is sufficient to show that $957\cdot 4657\,s$ is a square number. And, for our need, $s=957\cdot 4657n$ for a positive integer n.

Triangular numbers are more difficult to manage. We need $2471\cdot 4657s=2471\cdot 957\cdot 4657^2n^2=\frac{k(k+1)}{2}$ for a few whole k. The whole problem comes down to solve the quadratic Diophantine equation for k and n.

$$2471\cdot 957\cdot 4657^2n^2=\frac{k(k+1)}{2}$$

We can complete the square of the left member of the equation by writing:

$$\frac{k^2+k}{2}=\frac{\left(k+\frac{1}{2}\right)^2-\frac{1}{4}}{2}=\frac{1}{8}((2k+1)^2-1).$$

We then multiply the above Diophantine equation by 8 to bring it into the following form:

$$8\cdot 2471\cdot 957\cdot 4657^2n^2=(2k+1)^2-1,$$

By introducing a new variable $m=2k+1$, we have

$$8\cdot 2471\cdot 957\cdot 4657^2n^2=m^2-1$$

Or

$$m^2 - Cn^2 = 1$$

Where $C = 410286423278424 = 8 \cdot 2471 \cdot 957 \cdot 4657^2 = 2 \cdot 2471 \cdot 957 \cdot (2 \cdot 2471)^2$

This equation is a classic example of what goes by the name of Pell's equation.

We'll solve it in two steps. In the first, we will solve Pell's equation of the form $m^2 - C'n^2 = 1$ where $C' = 4729494$. In the second step of the solutions to this equation, we will find one in which n is a multiple of 2942.

2.4 Part 4 The Pell Equation and Continuous Fractions.

Pell's equation $m^2 - Cn^2 = 1$ has solutions that are the rational approximation of $\sqrt{C}$. Let's take two examples: C=19 and C=13. How can we find the expansion of the continuous fraction for $C = 19$ which is about 4.3588989435407? $\sqrt{19}$ is a little more than 4. If we make it so that a is reciprocal of this something then $\sqrt{19} = 4 + \frac{1}{a}$ where $a = \frac{1}{0.3588989435} = 2.7862996478$.

Continuing, $a = 2 + \frac{1}{b}$ with $b = \frac{1}{0.7862996478} = 1.2717797887$. Successively $b = 1 + \frac{1}{c}$ with $c = \frac{1}{0.2717797887} = 3.6794494718$.

Then, $c = 3 + \frac{1}{d}$ with $d = \frac{1}{0.6794494718} = 1.4717797887$. Again, $d = 1 + \frac{1}{e}$ where $e = \frac{1}{0.4717797887} = 2.1196329812$. Finally, $e = 2 + \frac{1}{f}$ with $f = \frac{1}{0.1196329812} = 8.3588989435$. It may be noted that $f - 8 = 0.3588989435 = a - 2$, then $f = 8 + \frac{1}{b}$.

This process leads us to the continuous fraction expansion for $\sqrt{19}$ which is represented in the following way:

$$\sqrt{19} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{8 + \cdots}}}}}}$$

To save space, this expression can be written

$$\sqrt{19} = 4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{2}{1+}\frac{1}{8+\cdots}$$

Or

$$\sqrt{19} = [4,2,1,3,1,2,8,\ldots]$$

At this point, the expansion of the continuous fraction repeats, and usually the repetition is indicated by a line on the repeating part, namely:

$$\sqrt{19} = [4,2,1,3,1,2,8,2,1,3,1,2,8,\ldots] = \left[4,\overline{2,1,3,1,2,8}\right]$$

This continuous fraction expansion gives a close approximation of $\sqrt{19}$

Its first approximations are:

$$4 = 4$$

$$4 + \frac{1}{2} = \frac{9}{2}$$

$$4 + \frac{1}{2+}\frac{1}{1} = \frac{7}{2}$$

$$4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3} = \frac{11}{3}$$

$$4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1} = \frac{18}{5}$$

$$4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2} = \frac{119}{33}$$

These approximations are easy to calculate recursively, as in the table below, using the sequence $\left[4,\overline{2,1,3,1,2,8}\right]$. To create the table, you have to write the first two lines of 0 and 1 as shown, then on the left write the sequence of numbers of the continuous fraction. It calculates each new row in the table from the previous two rows so that each entry in that new row is found by adding the entry two rows above it and the factor on the left multiplied by the entry directly above it. (see Fig.1).

	m	n	m^2	$19n^2$	m^2-19n^2
	0	1			
	1	0			
4	4	1	16	19	-3
2	9	2	81	76	5
1	13	3	169	171	-2
3	48	11	2304	2299	5
1	61	14	3721	3724	-3
2	170	39	28900	28899	1
8	1421	326	2019241	2019244	-3

Fig.1 The table shows the approximations of the continuous fraction

The left-hand entry m in the row is the numerator of a rational approximation while the right-hand entry n is the denominator.

The interesting thing about these rational approximations of $\sqrt{19}$ is that the square of the numerator m is alternately less than and greater than 19 times the square of the denominator n. The above table has columns for m, $19n^2$ and m^2-19n^2. We can observe that when $(m,n)=(170,39)$, we have $170^2-19\cdot 38^2=1$, which, solves Pell's equation $m^2-19n^2=1$.

This represents the smallest solution of Pell's equation.

In addition, all the solutions of Pell's equation can be found starting from this smallest equation $(m,n)=(170,39)$.

The equation $m^2-19n^2=1$ can be interpreted as follows:

$$(m+n\sqrt{19})(m-n\sqrt{19})$$

And from this, it follows that $(m+n\sqrt{19})^k\left(m-n\sqrt{19}\right)^k=1$ for every k. This means, for example, that

$$(170+39\sqrt{19})\left(170-39\sqrt{19}\right)=1$$

implies, with k=2, that

$$(170+39\sqrt{19})^2\left(170-39\sqrt{19}\right)^2=1$$

From

$$\left(170 + 39\sqrt{19}\right)^2 = (170^2 + 19 \cdot 39^2) + (2 \cdot 170 \cdot 39)\sqrt{19}$$
$$= 57799 + 13260\sqrt{19}$$

We have a second solution, $(m, n) = (57799,13260)$.

Graphically, Pell's equation can be represented as a particular elliptic curve of equation $m^2 = 19n^2 + 1$ (see Fig.2)

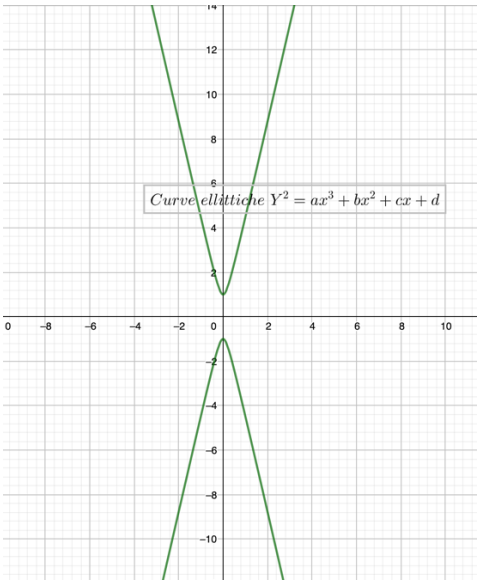

Fig. 2 Representation of the Pell equation as a particular elliptic curve

2.5 Part 5. Back to Archimedes' Problem and possible solution.

Let's go back to Archimedes' problem. We have two stages to complete. Initially, we have to solve Pell's equation of the form $m^2 - C'n^2 = 1$ with $C' = 4729494$. Second, among the solutions of this equation, we need to find one where is n a multiple of 2942 .

All we need is the continuous fraction expansion for $C' = 4729494$, which is about 2174.7399844579. At this point, we show the calculations to find it.

2.174.7399844579=2174+1/1.35137973413856

1.35137973413856=1+1/2.84592394735536

2.84592394735536=2+1/1.18213936740570

1.18213936740570=1+1/5.49030126898696

5.49030126898696=5+1/2.03956233290230

2.03956233290230=2 + 1/25.2765805450719

25.2765805450719=25+1/3.61574659004589

3.61574659004589=3 + 1/1.62404472256270

1.62404472256270=1 + 1/1.60244925378649

1.60244925378649= 1 + 1/1.65989084344423

1.65989084344423= 1 + 1/1.51540214557397

1.51540214557397= 1 + 1/1.94023251278156

1.94023251278156= 1 + 1/1.06356670972973

1.06356670972973= 1 + 1/15.7315048120601

15.7315048120601= 15 + 1/1.36704500573790

Then

$$\sqrt{4729494} = [2174, \overline{1,2,1,5,2,25,3,1,1,1,1,1,1,15,1,2,16,1,2,1,1,8,6,1,21,1,1,3,1,1,1,2,2,6,1,1,5,1,17,1,1,47,3,1,1,6,1,1,3,47,1,1,17,1,5,1,1,6,2,2,1,1,1,3,1,1,21,1,6,8,1,1,2,1,16,2,1,15,1,1,1,1,1,3,25,2,5,1,2,1,4348}].$$

We then calculate the first approximations of $\sqrt{C}$:

	m	n
	0	1
	1	0
2174	2174	1
1	2175	1
2	6524	3
1	8699	4
5	50019	23

Fig. 3 The table shows the first approximations of $\sqrt{C}$

The smallest solution of $m^2 - Cn^2 = 1$ is in the row $m = 8102213$ and $n = 4$ that gives $81022113^2 - 410286423278424 \cdot 4 = 1$

3. Use of technologies

The presented didactic path highlights a considerable intense computational activity with even high numbers that make the solution systems rather complex. This could lead one to think that the development of an activity of this type requires a lot of time and careful proximity of the teacher to the students, risking impoverishing the exploratory aspect of the research due to an excessive attention to the numerical component. To avoid this possible drift of the activity, it can be extremely effective to choose to deal with the modelling and construction of relationships between the data presented in the text, then delegate the algebraic solution to a technological tool.

In this regard, we propose approach to the solution with two technological tools currently present among teachers and their didactic choices in the classroom, namely the use of graphing calculators and a numerical approach programming in Python. The first tools, graphing calculators, has long been recognized for its utility in enhancing mathematical instruction and promoting problem-solving skills among students. With their ability to graph equations, perform complex calculations, and visualize mathematical concepts, graphing calculators have become an indispensable resource for educators seeking to facilitate deeper understanding and engagement in the learning process.

In addition to graphing calculators, the integration of numerical programming in Python presents a promising approach for enhancing the pedagogical experience in the classroom. Python, a versatile and widely-used programming language, offers educators the opportunity to introduce students to the world of computational thinking and algorithmic problem-solving in a

user-friendly manner and to apply numerical methods and algorithms to solve mathematical problems.

3.1 Solving the problem with graphing calculators for a heuristic approach to mathematical exploration

The choice of graphing calculators is linked to the fact that in Italy since 2017 graphing calculators have been authorized for the final mathematics test of the final exam of studies and therefore many students have them and many schools develop paths and laboratory activities to help students learn their effective and conscious use. Below we see the system resolution menus with the Casio FxCG50 calculator.

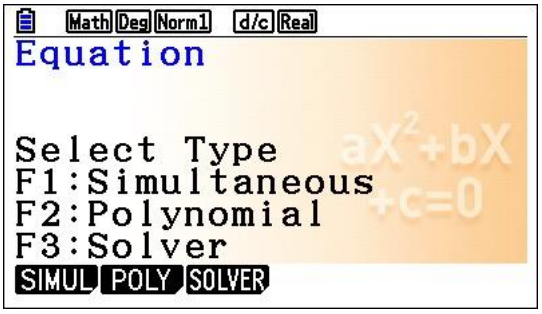

Fig. 4 Select the linear systems menu

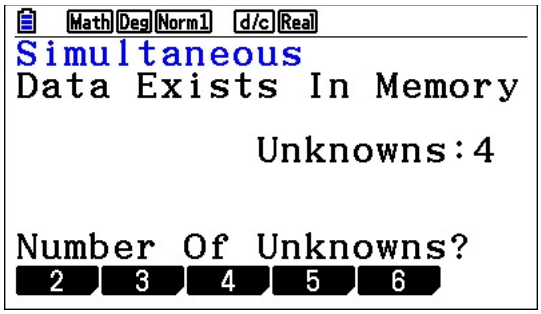

Fig. 5 Select the number of unknown

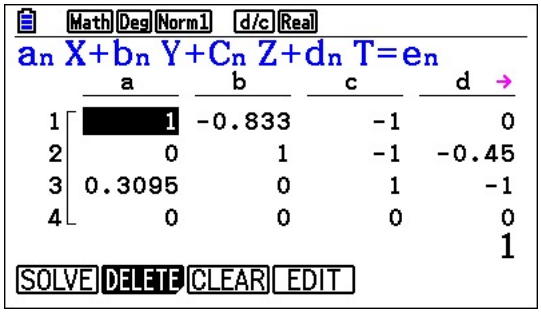

Fig. 6 Enter numerical coefficients (in fractional form)

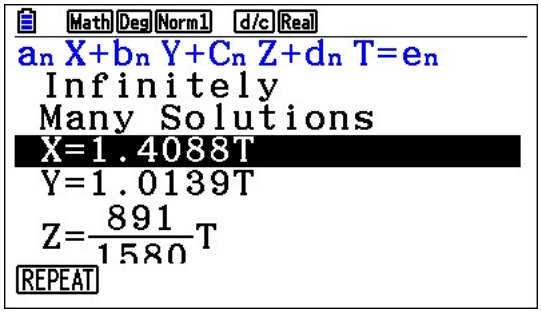

Fig. 7 The solution is expressed in parametric form

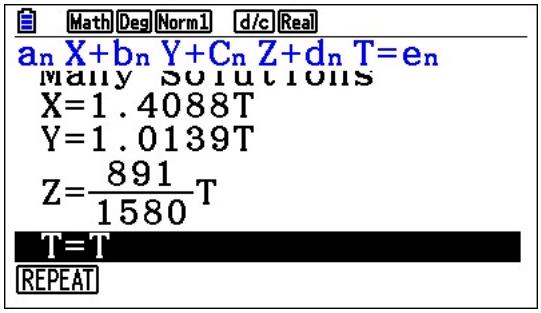

Fig. 8 The parameter is the last unknown factor you entered

It is interesting to note that the solutions are expressed in parametric form and can then be used in the numerical analysis that has been done in our article.

3.2 Solving the problem in Python with a numerical approach suggested by the very nature of the question.

The Archimedean oxen problem, as we have seen, requires solving a system of Diophantine equations involving large integers.

One version is to solve Pell's equation as follows:

$$m^2 - 410286423278424\, n^2 = 1$$

The solution of this equation involves finding a continuous fraction for the square root of the coefficient D=410286423278424 and determining the period of the continuous fraction to obtain the fundamental solutions.

Based on a conjecture taken from "*An Experimental Introduction to number theory, Benjamin Huts 2010"* which states that the fundamental solution of Pell's Equation is given by C_{K-1} for $K = r$ or $2r$ where r is the period of expansion of the continuous fraction of $\sqrt{D}$. In addition, if C_t is the fundamental solution, all solutions are given by C_{t+mr} with $m \in$ to the natural numbers if r is even and m ∈ to 2 naturals if r is odd.

Below we list the steps to Solve Archimedes' Oxen Problem with the Numerical Approach in Python.

- Calculation of the continuous fraction for $\sqrt{D}$: This involves implementing the algorithm to find the terms of the continuous fraction.

```python
import math

def continued_fraction_sqrt(D, max_terms=1000):
    """Calculates the continued fraction representation of the square root of D.

    Args:
        D (int): The integer to find the square root of.
        max_terms (int): The maximum number of terms to calculate.

    Returns:
        list: A list of the terms in the continued fraction representation of sqr
    """
    if D < 0:
        raise ValueError("D must be a non-negative integer.")

    m = 0
    d = 1
    a0 = a = int(math.sqrt(D))
```

Fig.9 Python Implementation for Continuous Fraction Computation

```
    if a * a == D:
        return [a0]

    terms = [a0]
    for _ in range(max_terms - 1):
        m = d * a - m
        d = (D - m * m) // d
        a = (a0 + m) // d
        terms.append(a)

        # Periodicity check
        if d == 1:
            break

    return terms
```

Fig.10 Python Implementation for Continuous Fraction Computation

- Calculation of convergences from the continuous fraction:

```
def continued_fraction_convergents(terms):
    """Calculates the convergents of a continued fraction given its terms.

    Args:
        terms (list): A list of terms in the continued fraction.

    Returns:
        list: A list of tuples representing the convergents (numerator, denomina
    """
    convergents = []
    n0, d0 = 1, 0
    n1, d1 = terms[0], 1

    convergents.append((n1, d1))

    for term in terms[1:]:
        n = term * n1 + n0
        d = term * d1 + d0
        convergents.append((n, d))

        n0, d0 = n1, d1
        n1, d1 = n, d

    return convergents
```

Fig.11 Python calculates the convergents of the continuous fraction.

- Finding the Period of the Continuous Fraction:

The period helps to determine the fundamental solution of Pell's equation as evidenced by the conjecture.

```python
import math

def continued_fraction_sqrt_period(D):
    """Calculates the continued fraction representation of the square root of D

    Args:
        D (int): The integer to find the square root of.

    Returns:
        list: A list of the terms in the continued fraction representation of sq
        list: The period of the continued fraction.
    """
    if D < 0:
        raise ValueError("D must be a non-negative integer.")

    m = 0
    d = 1
    a0 = a = int(math.sqrt(D))

    if a * a == D:
        return [a0], []  # No period for perfect squares

    terms = [a0]
    seen = {}

    while (m, d, a) not in seen:
        seen[(m, d, a)] = len(terms)

        m = d * a - m
        d = (D - m * m) // d
        a = (a0 + m) // d
        terms.append(a)

    period_start = seen[(m, d, a)]
    period = terms[period_start:]

    return terms, period
```

Fig.12 Program that calculates the period of the continuous fraction

Solving this problem with Python allows students to integrate mathematical knowledge with programming skills, showing how mathematics can be used to develop algorithms and solve complex problems efficiently. For example, in a secondary school, students can implement an algorithmic solution by analyzing the problem and breaking it down into simpler parts. This process develops critical thinking and problem-solving skills, which are fundamental skills in many scientific and engineering disciplines. Python also allows for more interactive and engaging teaching than traditional methods. Students can

immediately see the result of their efforts, make changes, and better understand abstract concepts through visualization and experimentation.

4. Discussion and Conclusions

In the activity presented, doing mathematics involves various levels of manipulation from the abstract to the concrete: manipulating language and transforming it into symbols, manipulating symbols, and manipulating processes. Mathematics learning therefore takes place in situations where students can develop heuristic processes and build mathematical skills through problem-posing and problem-solving.

The practical experiment offers students a unique chance to improve their understanding of how technology can be used effectively to address classical mathematical problems found in scholarly literature. Students are equipped with the essential ability to collaborate efficiently within a group, optimize the strengths of their peers, and articulate their thoughts to accomplish a shared goal. These proficiencies are fundamental not only within an academic environment but also in professional settings, where the synergy of teamwork and cooperation is paramount for achieving desired outcomes.

At the end of the classroom laboratory, a comprehensive analysis of the results will be conducted and subsequently disseminated in an upcoming publication.

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