RATIO mathematica

Journal of Foundations and Applications of Mathematics / Rivista di Fondamenti e Applicazioni della Matematica

22/12

ISSN: 1592-7415

Number 22 – 2012

RATIO MATHEMATICA

Journal of Foundations and Applications of Mathematics

Via Lucania, 1 – 64026 ROSETO DEGLI ABRUZZI (Italy)

Autorizzazione n. 9/90 del 10-07-1990 del Tribunale di Pescara

ISSN: 1592 - 7415

RATIO MATHEMATICA 22 (2012) 3-12 ISSN:1592-7415

About a definition of metric over an abelian linearly ordered group

Bice Cavallo, Livia D'Apuzzo
University Federico II, Naples, Italy
bice.cavallo@unina.it, liviadap@unina.it

Abstract

A $\mathcal{G}$-metric over an abelian linearly ordered group $\mathcal{G} = (G, \odot, \leq)$ is a binary operation, $d_{\mathcal{G}}$, verifying suitable properties. We consider a particular $\mathcal{G}$ metric derived by the group operation $\odot$ and the total weak order $\leq$, and show that it provides a base for the order topology associated to $\mathcal{G}$.

Key words: $\mathcal{G}$-metric, abelian linearly ordered group, multi-criteria decision making.

2000 AMS subject classifications: 06F20, 06A05, 90B50.

1 Introduction

The object of the investigation in our previous papers have been the pairwise comparison matrices that, in a Multicriteria Decision Making context, are a helpful tool to determine a weighted ranking on a set X of alternatives or criteria [1], [2], [3]. The pairwise comparison matrices play a basic role in the Analytic Hierarchy Process (AHP), a procedure developed by T.L. Saaty [17], [18], [19]. In [14], the authors propose an application of the AHP for reaching consensus in Multiagent Decision Making problems; other consensus models are proposed in [6], [11], [15], [16].

The entry a_{ij} of a pairwise comparison matrix $A = (a_{ij})$ can assume different meanings: a_{ij} can be a preference ratio (multiplicative case) or a preference difference (additive case) or a_{ij} is a preference degree in $[0, 1]$ (fuzzy case). In order to unify the different approaches and remove some drawbacks linked to the measure scale and a lack of an algebraic structure, in [7] we consider pairwise comparison matrices over abelian linearly ordered groups (*alo-groups*). Furthermore, we introduce a more general notion of metric over an alo-group $\mathcal{G} = (G, \odot, \leq)$, that we call $\mathcal{G}$-*metric*; it is a binary operation on G

$$d : (a, b) \in G^2 \to d(a, b) \in G,$$

verifying suitable conditions, in particular: $a = b$ if and only if the value of $d(a, b)$ coincides with the identity of $\mathcal{G}$. In [7], [8], [9], [10] we consider a particular $\mathcal{G}$-metric, based upon the group operation $\odot$ and the total order $\leq$. This metric allows us to provide, for pairwise comparison matrices over a divisible alo-group, a consistency index that has a natural meaning and it is easy to compute in the additive and multiplicative cases.

In this paper, we focus on a particular $\mathcal{G}$-metric introduced in [7] looking for a topology over the alo-group in which the $\mathcal{G}$-metric is defined. By introducing the notion of $d_{\mathcal{G}}$-*neighborhood* of an element in an alo-group $\mathcal{G} = (G, \odot, \leq)$, we show that the above $\mathcal{G}$-metric generates the order topology that is naturally induced in $\mathcal{G}$ by the total weak order $\leq$.

2 Abelian linearly ordered groups

Let G be a non empty set, $\odot : G \times G \to G$ a binary operation on G, $\leq$ a total weak order on G. Then $\mathcal{G} = (G, \odot, \leq)$ is an *alo-group*, if and only if $(G, \odot)$ is an abelian group and

$$a \leq b \Rightarrow a \odot c \leq b \odot c. \tag{1}$$

As an abelian group satisfies the cancellative law, that is $a \odot c = b \odot c \Leftrightarrow a = b$, (1) is equivalent to the strict monotonicity of $\odot$ in each variable:

$$a < b \Leftrightarrow a \odot c < b \odot c. \tag{2}$$

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, we will denote with:

- e the identity of $\mathcal{G}$;
- $x^{(-1)}$ the symmetric of $x \in G$ with respect to $\odot$;
- $\div$ the inverse operation of $\odot$ defined by $a \div b = a \odot b^{(-1)}$,
- $x^{(n)}$, with $n \in \mathbb{N}_0$, the (n)-*power* of $x \in G$:

$$x^{(n)} = \begin{cases} e, & if \quad n = 0 \\ x^{(n-1)} \odot x, & if \quad n \geq 1; \end{cases}$$

- $<$ the strict simple order defined by $x < y \Leftrightarrow x \leq y$ *and* $x \neq y$;
- $\geq$ and $>$ the opposite relations of $\leq$ and $<$ respectively.

Then

$$b^{(-1)} = e \div b, \quad (a \odot b)^{(-1)} = a^{(-1)} \odot b^{(-1)}, \quad (a \div b)^{(-1)} = b \div a; \tag{3}$$

moreover, assuming that G is no trivial, that is $G \neq \{e\}$, by (2) we get

$$a < e \Leftrightarrow a^{(-1)} > e, \quad a < b \Leftrightarrow a^{(-1)} > b^{(-1)} \tag{4}$$

$$a \odot a > a \quad \forall a > e, \qquad a \odot a < a \quad \forall a < e. \tag{5}$$

By definition, an alo-group $\mathcal{G}$ is a *lattice ordered group* [4], that is there exists $a \vee b = max\{a, b\}$, for each pair $(a, b) \in G^2$. Nevertheless, by (5), we get the following proposition.

Proposition 2.1. *A no trivial alo-group $\mathcal{G} = (G, \odot, \leq)$ has neither the greatest element nor the least element.*

Order topology. If $\mathcal{G} = (G, \odot, \leq)$ is an alo-group, then G is naturally equipped with the order topology induced by $\leq$ that we will denote with $\tau_{\mathcal{G}}$. An open set in $\tau_{\mathcal{G}}$ is union of the following open intervals:

- $]a, b[= \{x \in G : a < x < b\}$;
- $] \leftarrow, a[= \{x \in G : x < a\}$;
- $]b, \rightarrow [= \{x \in G : x > b\}$;

and a neighborhood of $c \in G$ is an open set to which c belongs. Then $G \times G$ is equipped with the related product topology. We say that $\mathcal{G}$ is a *continuous* alo-group if and only if $\odot$ is continuous.

Isomorphisms between alo- groups An *isomorphism* between two alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ is a bijection $h : G \to G'$ that is both a lattice isomorphism and a group isomorphism, that is:

$$x < y \Leftrightarrow h(x) < h(y) \quad and \quad h(x \odot y) = h(x) \circ h(y). \tag{6}$$

Thus, $h(e) = e'$, where e' is the identity in $\mathcal{G}'$, and

$$h(x^{(-1)}) = (h(x))^{(-1)}. \tag{7}$$

By applying the inverse isomorphism $h^{-1} : G' \to G$, we get:

$$h^{-1}(x' \circ y') = h^{-1}(x') \odot h^{-1}(y'), \quad h^{-1}(x'^{(-1)}) = (h^{-1}(x'))^{(-1)}. \tag{8}$$

By the associativity of the operations $\odot$ and $\circ$, the equality in (6) can be extended by induction to the n-operation $\bigodot_{i=1}^{n} x_i$, so that

$$h(\bigodot_{i=1}^{n} x_i) = \bigcirc_{i=1}^{n} h(x_i), \quad h(x^{(n)}) = h(x)^{(n)}. \tag{9}$$

3 $\mathcal{G}$-metric

Following [5], we give the following definition of norm:

Definition 3.1. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the function:*

$$|| \cdot || : a \in G \to ||a|| = a \vee a^{(-1)} \in G \tag{10}$$

is a $\mathcal{G}$-norm, *or a* norm *on $\mathcal{G}$.*

The norm $||a||$ of $a \in G$ is also called *absolute value* of a in [4].

Proposition 3.1. *[7] The $\mathcal{G}$-norm satisfies the properties:*

1. $||a|| = ||a^{(-1)}||$;
2. $a \leq ||a||$;
3. $||a|| \geq e$;
4. $||a|| = e \Leftrightarrow a = e$;
5. $||a^{(n)}|| = ||a||^{(n)}$;
6. $||a \odot b|| \leq ||a|| \odot ||b||$. *(triangle inequality)*

Definition 3.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the operation*

$$d : (a, b) \in G^2 \to d(a, b) \in G$$

is a $\mathcal{G}$-metric *or* $\mathcal{G}$-distance *if and only if:*

1. $d(a, b) \geq e$;
2. $d(a, b) = e \Leftrightarrow a = b$;

3. $d(a,b) = d(b,a)$*;*

4. $d(a,b) \leq d(a,c) \odot d(b,c)$*.*

Proposition 3.2. *[7] Let* $\mathcal{G} = (G, \odot, \leq)$ *be an alo-group. Then, the operation*

$$d_{\mathcal{G}} : (a,b) \in G^2 \to d_{\mathcal{G}}(a,b) = ||a \div b|| \in G \tag{11}$$

is a $\mathcal{G}$*-distance.*

Proposition 3.3. *[7] Let* $\mathcal{G} = (G, \odot, \leq)$ *and* $\mathcal{G}' = (G', \circ, \leq)$ *be alo-groups,* $h : G \to G'$ *an isomorphism between* $\mathcal{G}$ *and* $\mathcal{G}'$*. Then, for each choice of* $a, b \in G$ *:*

$$d_{\mathcal{G}'}(h(a), h(b)) = h(d_{\mathcal{G}}(a,b)). \tag{12}$$

Corolary 3.1. *Let* $h : G \to G'$ *be an isomorphism between the alo-groups* $\mathcal{G} = (G, \odot, \leq)$ *and* $\mathcal{G}' = (G', \circ, \leq)$*. If* $a' = h(a), b' = h(b), r' = h(r) \in G'$*, then* $r > e \Leftrightarrow r' > e'$ *and*

$$d_{\mathcal{G}'}(a', b') < r' \Leftrightarrow d_{\mathcal{G}}(a,b) < r.$$

4 Examples of continuous alo-groups over a real interval

An alo-group $\mathcal{G} = (G, \odot, \leq)$ is a *real* alo-group if and only if G is a subset of the real line R and $\leq$ is the total order on G inherited from the usual order on R. If G is a proper interval of R then, by Proposition 2.1, it is an open interval.

Examples of real divisible continuous alo-groups are the following (see [8] [9]):

Additive alo-group $\mathcal{R} = (\mathbb{R}, +, \leq)$, where $+$ is the usual addition on $\mathbb{R}$. Then, $e = 0$ and for $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$a^{(-1)} = -a, \quad a \div b = a - b, \quad a^{(n)} = na.$$

The norm $||a|| = |a| = a \vee (-a)$ generates the usual distance over R:

$$d_{\mathcal{R}}(a,b) = |a - b| = (a - b) \vee (b - a).$$

Multiplicative alo-group $]\mathbf{0,+\infty}[$ $= (]0,+\infty[,\cdot,\leq)$, where $\cdot$ is the usual multiplication on $\mathbb{R}$. Then, $e = 1$ and for $a, b \in]0,+\infty[$ and $n \in \mathbb{N}$:

$$a^{(-1)} = 1/a, \quad a \div b = \frac{a}{b}, \quad a^{(n)} = a^n.$$

The norm $||a|| = |a| = a \vee a^{-1}$ generates the following $]\mathbf{0,+\infty}[$ - distance

$$d_{]\mathbf{0,+\infty}[}(a,b) = \frac{a}{b} \vee \frac{b}{a}.$$

Fuzzy alo-group $]\mathbf{0,1}[$ $= (]0,1[,\otimes,\leq)$, where $\otimes$ is the binary operation in $]0,1[$:

$$\otimes : (a,b) \in]0,1[\times]0,1[\mapsto \frac{ab}{ab+(1-a)(1-b)} \in]0,1[, \tag{13}$$

Then, 0.5 is the identity element, $1-a$ is the inverse of $a \in]0,1[$, $a \div b = \frac{a(1-b)}{a(1-b)+(1-a)b}$, $a^{(0)} = 0.5$,

$$a^{(n)} = \frac{a^n}{a^n+(1-a)^n} \quad \forall n \in \mathbb{N} \tag{14}$$

and

$$d_{]\mathbf{0,1}[}(a,b) = \frac{a(1-b)}{a(1-b)+(1-a)b} \vee \frac{b(1-a)}{b(1-a)+(1-b)a} = \frac{a(1-b) \vee b(1-a)}{a(1-b)+b(1-a)}. \tag{15}$$

Remark 4.1. *By Proposition 2.1, the closed unit interval $[0,1]$ can not be structured as an alo-group; thus, in [7], the authors propose $\otimes$ as a suitable binary operation on $]0,1[$, satisfying the following requirements: 0.5 is the identity element with respect to $\otimes$; $1-a$ is the inverse of $a \in]0,1[$ with respect to $\otimes$; $(]0,1[,\otimes,\leq)$ is an alo-group. The operation $\otimes$ is the restriction to $]0,1[\times]0,1[$ of the uninorm:*

$$U(a,b) = \begin{cases} 0, & (a,b) \in \{(0,1),(1,0)\}; \\ \frac{ab}{ab+(1-a)(1-b)}, & \text{otherwise.} \end{cases}$$

The uninorms have been introduced in [12] as a generalization of t-norm and t-conorm [13] and are commutative and associative operations on $[0,1]$, verifying the monotonicity property (1).

5 $d_{\mathcal{G}}$- neighborhoods and order topology

In this section $\mathcal{G} = (G, \odot, \leq)$ is an alo-group and $d_{\mathcal{G}}$ the $\mathcal{G}$-distance in (11).

Definition 5.1. *Let $c, r \in G$ and $r > e$; then the $d_{\mathcal{G}}$-*neighborhood *of c with radius r is the set:*

$$N_{d_{\mathcal{G}}}(c; r) = \{x \in G : d_{\mathcal{G}}(x, c) < r\}. \tag{16}$$

Of course $c \in N_{d_{\mathcal{G}}}(c; r)$ for each $r > e$. Then, $N_{d_{\mathcal{G}}}(c)$ will denote a $d_{\mathcal{G}}$-neighborhood of c and $N_{d_{\mathcal{G}}}$ the set of the all $d_{\mathcal{G}}$-neighborhoods of the elements of $\mathcal{G}$.

Proposition 5.1. *Let $c, r \in G$ and $r > e$; then:*

$$N_{d_{\mathcal{G}}}(c; r) =]c \div r, c \odot r[$$

Proof. By properties (2), (3), (4) we get $c \div r = c \odot r^{(-1)} < c < c \odot r$ and:

$$\begin{gathered}
x \in N_{d_{\mathcal{G}}}(c; r) \\
\Updownarrow \\
\left\{ \begin{array}{l} e \leq x \div c < r \\ or \\ e < c \div x < r \end{array} \right. \\
\Updownarrow \\
\left\{ \begin{array}{l} e \leq x \div c < r \\ or \\ r^{(-1)} < x \div c < e \end{array} \right. \\
\Updownarrow \\
\left\{ \begin{array}{l} c \leq x < c \odot r \\ or \\ c \div r < x < c \end{array} \right. \\
\Updownarrow \\
x \in]c \div r, c \odot r[.
\end{gathered}$$

□

Proposition 5.2. *Let $h : G \to G'$ be an isomorphism between the alo-group $\mathcal{G} = (G, \odot, \leq)$ and the alo-group $\mathcal{G}' = (G', \circ, \leq)$. Then, for each choice of $c, r \in G$ and $c', r' \in G$ such that $c' = h(c)$, $r > e$ and $r' = h(r)$, the following equality holds:*

$$N_{d_{\mathcal{G}'}}(c'; r') = h(N_{d_{\mathcal{G}}}(c; r)). \tag{17}$$

Proof. By Proposition 3.3 and Corollary 3.1. □

Example 5.1. *The neighborhoods related to the examples in Section 4 are the following:*

- *in the additive alo-group $\mathcal{R} = (\mathbb{R}, +, \leq)$, the neighborhood of c with radius r is the open interval $]c - r, c + r[$;*
- *in the multiplicative alo-group* ***]0,+∞[*** $= (]0, +\infty[, \cdot, \leq)$*, the neighborhood of c with radius r is the interval $]\frac{c}{r}, c \cdot r[$;*
- *in the fuzzy alo-group* ***]0,1[*** $= (]0, 1[, \otimes, \leq)$*, the neighborhood of c with radius r is the open interval $]\frac{c(1-r)}{c(1-r)+(1-c)r}, \frac{cr}{cr+(1-c)(1-r)}[$.*

By Proposition 5.1, $N_{d_{\mathcal{G}}}(c; r)$ is a particular neighborhood of c in the order topology $\tau_{\mathcal{G}}$. We show by means of the following results that the set $N_{d_{\mathcal{G}}}$ generates the order topology associated to $\mathcal{G}$.

Proposition 5.3. *Let A be an open set in the order topology $\tau_{\mathcal{G}}$. Then for each $c \in A$ there exists a $d_{\mathcal{G}}$-neighborhood of c included in A.*

Proof. It is enough to prove the assertion in the case that A is an open interval $]a, b[$. Let $c \in]a, b[$ and $r = d_{\mathcal{G}}(a, c) \wedge d_{\mathcal{G}}(b, c) = (c \div a) \wedge (b \div c)$. Let us consider the cases:

1. $r = c \div a \leq b \div c$;
2. $r = b \div c < c \div a$.

In the first case, $a = c \div r$, $c \odot r \leq b$ and so $]c \div r, c \odot r[\subseteq A =]a, b[$; thus, by Proposition 5.1, $N_{d_{\mathcal{G}}}(c; r) \subseteq A$. In the second case, the inclusion $N_{d_{\mathcal{G}}}(c; r) \subseteq A$ can be proved by similar arguments. □

Corolary 5.1. *The set $N_{d_{\mathcal{G}}}$ of the all $d_{\mathcal{G}}$-neighborhoods of the elements of $\mathcal{G}$ is a base for the order topology $\tau_{\mathcal{G}}$.*

References

[1] J. Barzilai, *Consistency measures for pairwise comparison matrices.* Journal of Multi-Criteria Decision Analysis 7(1998), 123–132.

[2] L. Basile, L. D'Apuzzo, *Transitive matrices, strict preference and intensity operators*. Mathematical Methods in Economics and Finance 1(2006), 21–36.

[3] L. Basile, L. D'Apuzzo, *Transitive matrices, strict preference and ordinal evaluation operators*. Soft Computing 10(2006), 933–940.

[4] G. Birkhoff, *Lattice theory*. American Mathematical Society Colloqium Publications 25(1984).

[5] N. Burbaki, *Algèbre* II. Masson Paris MR 84d:0002 (1981).

[6] C. Carlsson, D. Ehrenberg, P. Eklund, M. Fedrizzi, P. Gustafsson, P. Lindholm, G. Merkuryeva, T. Riissanen, A.G.S. Ventre, *Consensus in distributed soft environments.* European Journal of Operational Research 61(1992), 165-185.

[7] B. Cavallo, L. D'Apuzzo, *A general unified framework for pairwise comparison matrices in multicriterial methods*. International Journal of Intelligent Systems 24(2009), 377–398.

[8] B. Cavallo, L. D'Apuzzo, *Characterizations of consistent pairwise comparison matrices over abelian linearly ordered groups.* International Journal of Intelligent Systems 25(2010), 1035–1059.

[9] B. Cavallo, L. D'Apuzzo, M. Squillante, *About a consistency index for pairwise comparison matrices over a divisible alo-group.* International Journal of Intelligent Systems 27(2012), 153–175.

[10] B. Cavallo, L. D'Apuzzo, *Deriving weights from a pairwise comparison matrix over an alo-group*. Soft Computing 16(2012), 353-366.

[11] P. Eklund, A. Rusinowska, H. De Swart, *Consensus reaching in committees.* European Journal of Operational Research 178(2007), 185-193.

[12] J. Fodor, R. Yager, A. Rybalov, *Structure of uninorms.* International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 5 (1997), 411–427.

[13] E.P. Klement, R., E. P. Mesiar, *Triangular norms* (2000).

[14] A. Maturo, A.G. S. Ventre, *An Application of the Analytic Hierarchy Process to Enhancing Consensus in Multiagent Decision Making.* In: ISAHP2009, Proceeding of the International Symposium on the Analytic Hierarchy Process for Multicriteria Decision Making, July 29- August 1, 2009, University of Pittsburg, Pittsburgh, Pennsylvania, paper 48, 1-12.

[15] A. Maturo, A.G. S. Ventre, *Aggregation and consensus in multiobjective and multiperson decision making.* International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 17 (2009), 491-499.

[16] A. Maturo, A.G. S. Ventre. *Reaching Consensus in Multiagent Decision Making.* International Journal of Intelligent Systems 25 (2010), 266-273.

[17] T.L. Saaty, *A scaling method for priorities in hierarchical structures.* J. Math.Psychology 15 (1977), 234–281.

[18] T.L. Saaty, *The Analytic Hierarchy Process.* McGraw-Hill, New York (1980).

[19] T.L. Saaty, *Decision Making for Leaders.* University of Pittsburgh (1988).

RATIO MATHEMATICA 22 (2012) 13-35 ISSN:1592-7415

Classifications of Hyper Pseudo BCK-algebras of Order 3

R. A. Borzooei, A. Rezazadeh, R. Ameri
Department of Mathematics, Shahid Beheshti University, Tehran, Iran
Department of Mathematics, Payam Noor University, Tehran, Iran
Department of Mathematics, Tehran University, Tehran, Iran
borzooei@sbu.ac.ir, Rezazade2008@gmail.com, rez_ameri@yahoo.com

Abstract

In this paper by considering the notion of hyper pseudo BCK-algebra, we classify the set of all non-isomorphic hyper pseudo BCK-algebras of order 3. For this, we define the notion of simple and normal condition and we characterize the all of hyper pseudo BCK-algebras of order 3 that satisfies these conditions.

Key words: Hyper pseudo BCK-algebra, Simple condition, Normal condition.

2000 AMS subject classifications: 97U99.

1 Introduction

The study of BCK-algebras was initiated at 1966 by Y. Imai and K. Iséki in [5] as a generalization of the concept of set-theoretic difference and propositional calculi. In order to extend BCK-algebras in a noncommutative form, Georgescu and Iorgulescu [4] introduced the notion of pseudo BCK-algebras and studied their properties. The hyperstructure theory (called also multialgebra) was introduced in 1934 by F. Marty [10] at the 8th Congress of Scandinavian Mathematicians. Since then many researchers have worked on algebraic hyperstructures and developed it. A recent book [3] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [1, 9], R. A. Borzooei et al. applied the hyperstructures to (pseudo) BCK-algebras and introduced the notion of hyper (pseudo) BCK-algebra which is a generalization of (pseudo) BCK-algebra and investigated some related properties. In [2], R. A. Borzooei et al. classified all hyper BCK-algebras of order 3. Now, in this paper we classify the set of all non-isomorphic hyper pseudo BCK-algebras of order 3.

2 Preliminaries

Definition 2.1. *[9] By a hyper BCK-algebra we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfy the following axioms:*

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ *and* $y \ll x$ *imply* $x = y$.

for all $x, y, z \in H$, *where* $x \ll y$ *is defined by* $0 \in x \circ y$ *and for every* $A, B \subseteq H$, $A \ll B$ *is defined by* $\forall a \in A, \exists b \in B$ *such that* $a \ll b$. *In such case, we call "* $\ll$ *" the hyperorder in* H.

Definition 2.2. *[1] A hyper pseudo BCK-algebra is a structure $(H, \circ, *, 0)$ where "$*$" and "$\circ$" are hyper operations on H and "0" is a constant element, that satisfies the following: for all $x, y, z \in H$,*

(PHK1) $(x \circ z) \circ (y \circ z) \ll, x \circ y$, $(x * z) * (y * z) \ll x * y$,

(PHK2) $(x \circ y) * z = (x * z) \circ y$,

(PHK3) $x \circ H \ll \{x\}$, $x * H \ll \{x\}$,

(PHK4) $x \ll y$ *and* $y \ll x$ *imply* $x = y$.

where $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$ *and for every* $A, B \subseteq H$, $A \ll B$ *is defined by* $\forall a \in A, \exists b \in B$ *such that* $a \ll b$.

Theorem 2.1. *[2, 9] Any BCK-algebra and hyper BCK-algebra is a hyper pseudo BCK-algebra.*

Proposition 2.1. *[1] In any hyper pseudo BCK-algebra H, the following hold:*

(i) $0 \circ 0 = \{0\}$, $0 * 0 = \{0\}$, $x \circ 0 = \{x\}$, $x * 0 = \{x\}$,

(ii) $0 \ll x$, $x \ll x$, $A \ll A$,

(iii) $0 \circ x = \{0\}$, $0 * x = \{0\}$, $0 \circ A = \{0\}$, $0 * A = \{0\}$.

for all $x, y, z \in H$ and for all nonempty subsets A and B of H.

Theorem 2.2. *[2] There are 19 non-isomorphic hyper BCK-algebras of order 3.*

Note: From now on, in this paper $H = \{0, a, b\}$ is a hyper pseudo BCK-algebra of order 3, unless otherwise state.

3 Characterization of Hyper Pseudo BCK-algebras of Order 3

Definition 3.1. *[2] We say that H satisfies the normal condition if one of the conditions $a \ll b$ or $b \ll a$ holds. If no one of these conditions hold, then we say that H satisfies the simple condition.*

Definition 3.2. *Let $(H_1, \circ_1, *_1, 0_1)$ and $(H_2, \circ_2, *_2, 0_2)$ be two hyper pseudo BCK-algebras and $f : H_1 \to H_2$ be a function. Then f is said to be a homomorphism iff*

(i) $f(0_1) = 0_2$

(ii) $f(x \circ_1 y) = f(x) \circ_2 f(y), \quad \forall x, y \in H_1$

(iii) $f(x *_1 y) = f(x) *_2 f(y), \quad \forall x, y \in H_1$.

If f is one to one (onto) we say that f is a monomorphism (epimorphism) and if f is both one to one and onto, we say that f is an isomorphism.

Definition 3.3. *Let $I \subseteq H$. Then we say that I is a proper subset of H if $I \neq \{0\}$ and $I \neq H$.*

3.1 Characterization of hyper pseudo BCK-algebras of order 3 that satisfy the simple condition

Theorem 3.1. There are only 10 hyper pseudo BCK-algebras of order 3, that satisfy the simple condition.

Proof. Let H satisfy the simple condition. Now, we prove the following statements:

(i) For all $x, y \in H$ which $x \neq y$, then $x \notin y \circ x$ and $x \notin y * x$.

(ii) $a \circ b = a * b = \{a\}$ and $b \circ a = b * a = \{b\}$.

(iii) $a \circ a$ and $a * a$ are equal to $\{0\}$ or $\{0, a\}$ and $b \circ b$ and $b * b$ are equal to $\{0\}$ or $\{0, b\}$.

For the proof of (i), let $x \neq y$ and $x \in y \circ x$, by the contrary. Clearly $x \neq 0$. Because if $x = 0$, then $y \neq 0$ and $0 \in y \circ 0 = \{y\}$, which is impossible. Moreover, since $y \circ x \leq y$, then $x \leq y$ which is impossible by the simplicity of H. By the similar way, we can prove that $x \notin y * x$.

(ii) Since $a \not\leq b$, then $0 \notin a \circ b$ and $0 \notin a * b$. Hence $a \circ b$ and $a * b$ can not be equal to $\{0\}, \{0, a\}, \{0, b\}$ or $\{0, a, b\}$. Since by (i), we have $b \notin a \circ b$ and $b \notin a * b$, we conclude that $a \circ b$ and $a * b$ can not be equal to $\{b\}$ or $\{a, b\}$. Thus $a \circ b = a * b = \{a\}$. By the similar way, we can prove that $b \circ a = b * a = \{b\}$.

(iii) Since $a \ll a$, the only cases for $a \circ a$ and $a * a$ are $\{0\}, \{0, a\}, \{0, b\}$ or $\{0, a, b\}$. Also we have $a \circ a \leq a$ and $a * a \leq a$. Thus $b \notin a \circ a$ and $b \notin a * a$. Hence the only cases for $a \circ a$ and $a * a$ are $\{0\}$ or $\{0, a\}$. By the similar way, we can prove that $b \circ b$ and $b * b$ are equal to $\{0\}$ or $\{0, b\}$.

Therefore, by (i), (ii) and (iii) we conclude that there are 16 hyper pseudo BCK-algebras of order 3, which satisfy the simple condition. But some of them are isomorphic under the map $f : H \to H$ which is defined by $f(0) = 0$, $f(a) = b$ and $f(b) = a$. Hence there are 10 hyper pseudo BCK-algebras of order 3, that satisfy the simple condition. Now, we give these hyper pseudo BCK-algebras:

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$\circ_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$*_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$\circ_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$*_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$\circ_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$*_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$\circ_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$*_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0}

$\circ_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$*_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$\circ_9$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$*_9$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$\circ_{10}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$*_{10}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

3.2 Characterization of hyper pseudo BCK-algebras of order 3 that satisfy the normal condition

Note: From now on, in this section we let $H = \{0, a, b\}$ satisfies the normal condition. Since in this condition, $a \leq b$ or $b \leq a$, so without loss of generality we let $a \leq b$ and $b \not\leq a$ i.e., $0 \in a \circ b \cap a * b$, $0 \notin b \circ a$ and $0 \notin b * a$.
Lemma 3.1. Only one of the following cases hold for $b * a$ and $b \circ a$.

(NPHB1) $b \circ a = b * a = \{a\}$,

(NPHB2) $b \circ a = \{a\}, b * a = \{a, b\}$,

(NPHB3) $b \circ a = b * a = \{b\}$,

(NPHB4) $b \circ a = \{a, b\}, b * a = \{a\}$,

(NPHB5) $b \circ a = b * a = \{a, b\}$.

Proof. Since $0 \notin b \circ a$ and $0 \notin b * a$, then $b \circ a$ and $b * a$ are equal to one of the sets $\{a\}, \{b\}$ or $\{a, b\}$.

If $b \circ a = \{a\}$, then $b * a \neq \{b\}$. Since if $b * a = \{b\}$, then $(b \circ a) * a \neq (b * a) \circ a$. Hence $b * a = \{a\}$ or $\{a, b\}$.

If $b \circ a = \{b\}$, then $b * a \neq \{a, b\}$ and $\{a\}$. Since if $b * a = \{a, b\}$ or $\{a\}$, then $(b \circ a) * a \neq (b * a) \circ a$. Hence $b * a = \{b\}$.

If $b \circ a = \{a, b\}$, then $b * a \neq \{b\}$. Since if $b * a = \{b\}$, then $(b \circ a) * a \neq (b * a) \circ a$. Hence $b * a = \{a\}$ or $\{a, b\}$. Therefore, we have the above cases.
Lemma 3.2. Only one of the following cases hold for $a * b$ and $a \circ b$.

(NPHA1) $a \circ b = a * b = \{0\}$.

(NPHA2) $a \circ b = a * b = \{0, a\}$.

(NPHA3) $a \circ b = \{0\}, a * b = \{0, a\}$.

(NPHA4) $a \circ b = \{0, a\}, a * b = \{0\}$.

Proof. Since $0 \in a \circ b \cap a * b$, then $a \circ b$ and $a * b$ are equal to one of the sets $\{0\}, \{0, a\}, \{0, b\}$ or $\{0, a, b\}$. Moreover, since $a \circ b \leq a$ and $a * b \leq a$, then $b \notin a \circ b$ and $b \notin a * b$. Hence the only cases for $a \circ b$ and $a * b$ are $\{0\}$ or $\{0, a\}$. Therefore, we have the above cases.
Lemma 3.3. Only one of the following cases hold for $a * a$ and $a \circ a$.

(i) $a \circ a = a * a = \{0\}$,

(ii) $a \circ a = \{0\}, a * a = \{0, a\}$,

(iii) $a \circ a = \{0, a\}, a * a = \{0\}$,

(iv) $a \circ a = a * a = \{0, a\}$.

Proof. Since $0 \in a \circ a \cap a * a$, then $a \circ a$ and $a * a$ can be equal to the one of cases $\{0\}, \{0, a\}, \{0, b\}$ or $\{0, a, b\}$. Moreover, since $a \circ a \leq a$ and $a * a \leq a$, then $b \notin a \circ a$ and $b \notin a * a$. Hence the only cases for $a \circ a$ and $a * a$ are $\{0\}$ or $\{0, a\}$. Therefore, we have the above cases.

Theorem 3.4. There are only 5 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB1).

Proof. Since H satisfies the condition (NPHB1), then $b \circ a = b * a = \{a\}$.

Case (NPHA1): We have $a * b = a \circ b = \{0\}$. If $a \circ a = \{0, a\}$ or $a * a = \{0, a\}$, then $(a \circ a) \circ (b \circ a) \not\ll a \circ b$ and $(a * a) * (b * a) \not\ll a * b$. Therefore, $a \circ a = a * a = \{0\}$. Moreover, if $b \circ b$ and $b * b$ are equal to the one of sets $\{0, b\}$ or $\{0, a, b\}$, then $(b \circ b) \circ (a \circ b) \not\ll b \circ a$ and $(b * b) * (a * b) \not\ll b * a$. Therefore, in this case $b \circ b$ and $b * b$ are equal to the one of sets $\{0\}$ or $\{0, a\}$. Now, we consider the following cases:

(1) $b \circ b = b * b = \{0\}$. Thus in this case, we have the following tables:

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. Thus in this case, we have the following tables:

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

(3) $b \circ b = \{0, a\}$ and $b * b = \{0\}$. Thus similar to (2), we have one hyper pseudo BCK-algebra in this case.

(4) $b \circ b = b * b = \{0, a\}$. Thus in this case, we have the following tables:

$*_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

$\circ_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

Case (NPHA2): We have $a * b = a \circ b = \{0, a\}$. If $a \circ a = \{0\}$ or $a * a = \{0\}$, then $(a \circ b) \circ (a \circ b) \not\ll a \circ a$ and $(a * b) * (a * b) \not\ll a * a$. Therefore, $a \circ a = a * a = \{0, a\}$. Moreover, if $b \circ b$ and $b * b$ are equal to the one of sets $\{0, b\}$

or $\{0,a,b\}$, then $(b\circ b)\circ(a\circ b)\not\ll b\circ a$ and $(b*b)*(a*b)\not\ll b*a$ and if $b\circ b=\{0\}$ or $b*b=\{0\}$, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$ and $(b*a)*(b*a)\not\ll b*b$. Therefore, $b\circ b=b*b=\{0,a\}$. Thus we have the following case:

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a}	{0,a}

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a}	{0,a}

Case (NPHA3): We have $a*b=\{0,a\}$ and $a\circ b=\{0\}$. Since $(a*b)*(a*b)\ll a*a$, then $a*a=\{0,a\}$. Moreover, since $a\circ b=\{0\}$, then $a\circ a\neq\{0,a\}$, because $(a\circ a)\circ(b\circ a)\not\ll a\circ b$. Thus $a\circ a=\{0\}$ and in this case $(b*a)\circ a\neq(b\circ a)*a$.

Case (NPHA4): We have $a*b=\{0\}$ and $a\circ b=\{0,a\}$. In this case we can show that $(b*a)\circ a\neq(b\circ a)*a$. Therefore, we have not any hyper pseudo BCK-algebras.

We can check that all of the these 5 cases are hyper pseudo BCK-algebras and each of them are not isomorphic together.

Theorem 3.5. There are only 6 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB2).

Proof. Since H satisfies the condition (NPHB2), then $b\circ a=\{a\}$ and $b*a=\{a,b\}$.

Case (NPHA1): We have $a*b=a\circ b=\{0\}$. If one of the $a\circ a$ or $a*a$ are equal to $\{0,a\}$, then $(a\circ a)\circ(b\circ a)\not\ll a\circ b$ and $(a*a)*(b*a)\not\ll a*a$. Hence we have $a*a=a\circ a=\{0\}$. But in this case $(b*a)\circ a\neq(b\circ a)*a$. Therefore, we have not any hyper pseudo BCK-algebras in this case.

Case (NPHA2): We have $a*b=a\circ b=\{0,a\}$. If $a\circ a=\{0\}$ or $a*a=\{0\}$, then $(a\circ b)\circ(a\circ b)\not\ll a\circ a$ and $(a*b)*(a*b)\not\ll a*a$. Therefore, $a\circ a=a*a=\{0,a\}$. Moreover, if $b*b$ is equal to the one of sets $\{0\}$ or $\{0,a\}$, then $(b*a)*(b*a)\not\ll b*b$. Therefore, in this case $b*b$ is equal to the one of sets $\{0,b\}$ or $\{0,a,b\}$ and if $b\circ b$ is equal to the one of sets $\{0,b\}$ or $\{0,a,b\}$, then $(b\circ b)\circ(a\circ b)\not\ll b\circ a$. Hence in this case $b\circ b$ is equal to the one of sets $\{0\}$ or $\{0,a\}$. Moreover, since $a\circ a=\{0,a\}$, if $b\circ b=\{0\}$, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$. Thus $b\circ b=\{0,a\}$. Now, we consider the following cases:

(1) $b\circ b=\{0,a\}$ and $b*b=\{0,b\}$. Thus in this case, we have the following tables:

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a}	{0,a}

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(2) $b \circ b = \{0, a\}$ and $b * b = \{0, a, b\}$. Thus in this case, we have the following tables:

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a}	{0,a}

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

Case (NPHA3): We have $a * b = \{0, a\}$ and $a \circ b = \{0\}$. If $a * a = \{0\}$ and $a \circ a$ is equal to $\{0\}$ or $\{0, a\}$, then $(a * b) * (a * b) \not\ll a * a$. Thus $a * a = \{0, a\}$. Moreover, if $b * b$ is equal to the one of sets $\{0\}$ or $\{0, a\}$, then $(b * a) * (b * a) \not\ll b * b$. Therefore, in this case $b * b$ is equal to the one of sets $\{0, b\}$ or $\{0, a, b\}$ and if $b \circ b$ is equal to the one of sets $\{0, b\}$ or $\{0, a, b\}$, then $(b \circ b) \circ (a \circ b) \not\ll b \circ a$. Thus in this case $b \circ b$ is equal to the one of sets $\{0\}$ or $\{0, a\}$. Now, we consider the following cases:

(1) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. If $a \circ a = \{0, a\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$. Hence $a \circ a = \{0\}$. Thus in this case, we have the following tables:

$\circ_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

$*_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(2) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0, a\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$. Hence $a \circ a = \{0\}$. Thus in this case, we have the following tables:

$\circ_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

$*_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

(3) $b \circ b = \{0, a\}$ and $b * b = \{0, b\}$. If $a \circ a = \{0, a\}$, then $(a \circ a) \circ (b \circ a) \not\ll a \circ b$. Hence $a \circ a = \{0\}$. Thus in this case, we have the following tables:

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(4) $b \circ b = \{0, a\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0, a\}$, then $(a \circ a) \circ (b \circ a) \not\ll a \circ b$. Hence $a \circ a = \{0\}$. Thus in this case, we have the following tables:

$\circ_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

$*_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

Case (NPHA4): We have $a * b = \{0\}$ and $a \circ b = \{0, a\}$. If $a \circ a = \{0\}$ and $a * a$ is equal to $\{0\}$ or $\{0, a\}$, then $(a \circ b) \circ (a \circ b) \not\ll a \circ a$. Thus $a \circ a = \{0, a\}$. Moreover, if $b * b$ is equal to the one of sets $\{0\}$ or $\{0, a\}$, then $(b * a) * (b * a) \not\ll b * b$. Therefore, in this case $b * b$ is equal to the one of sets $\{0, b\}$ or $\{0, a, b\}$. Moreover, if $a * a = \{0, a\}$, then $(a * a) * (b * a) \not\ll a * b$. Hence $a * a = \{0\}$. But in this case $(b \circ a) * b \neq (b * b) \circ a$. Therefore, we have not any hyper pseudo BCK-algebras.

We can check that all of the these 6 cases are hyper pseudo BCK-algebras and each of them are not isomorphic together.

Theorem 3.6. There are only 70 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB3).

Proof. Since H satisfies the condition (NPHB3), then $b \circ a = b * a = \{b\}$.

Case (NPHA1): We have $a * b = a \circ b = \{0\}$. Now, we consider the following cases:

(1) $b \circ b = b * b = \{0\}$. In this case, we have the following tables:

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$\circ_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$\circ_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. In this case, we have the following tables:

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$\circ_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

(3) $b * b = \{0\}$ and $b \circ b = \{0, b\}$. Similar to (2), we have four hyper pseudo BCK-algebras in this case.

(4) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{13}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{13}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a}

$\circ_{14}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{14}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

(5) $b * b = \{0\}$ and $b \circ b = \{0, a\}$. If $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Thus similar to (4), we have two hyper pseudo BCK-algebras in this case.

(6) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{17}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{17}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a,b}

$\circ_{18}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{18}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

(7) $b*b=\{0\}$ and $b\circ b=\{0,a,b\}$. If $a*a=\{0\}$, then $(b*a)\circ b\neq(b\circ b)*a$. Thus similar to (6), we have two hyper pseudo BCK-algebras in this case.

(8) $b\circ b=b*b=\{0,b\}$. Thus in this case, we have the following tables:

$\circ_{21}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$*_{21}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_{22}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$*_{22}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$\circ_{23}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$*_{23}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_{24}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$*_{24}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

(9) $b\circ b=\{0,b\}$ and $b*b=\{0,a\}$. Then $(b\circ b)*b\neq(b*b)\circ b$.

(10) $b*b=\{0,b\}$ and $b\circ b=\{0,a\}$. Then $(b\circ b)*b\neq(b*b)\circ b$.

(11) $b\circ b=\{0,b\}$ and $b*b=\{0,a,b\}$. Then $(b\circ b)*b\neq(b*b)\circ b$.

(12) $b*b=\{0,b\}$ and $b\circ b=\{0,a,b\}$. Then $(b\circ b)*b\neq(b*b)\circ b$.

(13) $b\circ b=b*b=\{0,a\}$. If $a\circ a=\{0\}$, then $(b\circ a)*b\neq(b*b)\circ a$ and if $a*a=\{0\}$, then $(b*a)\circ b\neq(b\circ b)*a$. Therefore, $a\circ a=a*a=\{0,a\}$. Thus in this case, we have the following tables:

$\circ_{25}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

$*_{25}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

(14) $b\circ b=\{0,a\}$ and $b*b=\{0,a,b\}$. Then $(b\circ b)*b\neq(b*b)\circ b$.

(15) $b*b=\{0,a\}$ and $b\circ b=\{0,a,b\}$. Then $(b\circ b)*b\neq(b*b)\circ b$.

(16) $b\circ b=b*b=\{0,a,b\}$. If $a\circ a=\{0\}$, then $(b\circ a)*b\neq(b*b)\circ a$ and if $a*a=\{0\}$, then $(b*a)\circ b\neq(b\circ b)*a$. Therefore, $a\circ a=a*a=\{0,a\}$. Thus in this case, we have the following tables:

$\circ_{26}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

$*_{26}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

Case (NPHA2): We have $a * b = a \circ b = \{0, a\}$. If one of the $a * a$ or $a \circ a$ are equal to $\{0\}$, then $(a * b) * (a * b) \not\ll a * a$ or $(a \circ b) \circ (a \circ b) \not\ll a \circ a$. Hence $a * a = a \circ a = \{0, a\}$.

(1) $b \circ b = b * b = \{0\}$. Thus in this case, we have the following tables:

$\circ_{27}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{27}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{28}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{28}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(3) $b * b = \{0\}$ and $b \circ b = \{0, b\}$. Thus similar to (2), we have one hyper pseudo BCK-algebra in this case.

(4) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(5) $b * b = \{0\}$ and $b \circ b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(6) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(7) $b * b = \{0\}$ and $b \circ b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(8) $b \circ b = b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{30}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{30}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(9) $b \circ b = \{0, b\}$ and $b * b = \{0, a\}$. Thus in this case, we have the following tables:

$\circ_{31}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{31}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

(10) Let $b*b=\{0,b\}$ and $b\circ b=\{0,a\}$. Thus similar to (9), we have one hyper pseudo BCK-algebra in this case.

(11) $b\circ b=\{0,b\}$ and $b*b=\{0,a,b\}$. Thus in this case, we have the following tables:

$\circ_{33}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{33}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

(12) $b*b=\{0,b\}$ and $b\circ b=\{0,a,b\}$. Thus similar to (11), we have one hyper pseudo BCK-algebra in this case.

(13) $b\circ b=b*b=\{0,a\}$. Thus in this case, we have the following tables:

$\circ_{35}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

$*_{35}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

(14) $b\circ b=\{0,a\}$ and $b*b=\{0,a,b\}$. Thus in this case, we have the following tables:

$\circ_{36}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

$*_{36}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

(15) $b*b=\{0,a\}$ and $b\circ b=\{0,a,b\}$. Thus similar to (14), we have one hyper pseudo BCK-algebra in this case.

(16) $b\circ b=b*b=\{0,a,b\}$. Thus in this case, we have the following tables:

$\circ_{38}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

$*_{38}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

Case (NPHA3): We have $a*b=\{0,a\}$ and $a\circ b=\{0\}$. If $a*a=\{0\}$ and $a\circ a$ is equal to $\{0\}$ or $\{0,a\}$, then $(a*b)*(a*b)\not\ll a*a$. Thus $a*a=\{0,a\}$.

(1) $b\circ b=b*b=\{0\}$. Thus in this case, we have the following tables:

$\circ_{39}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_{39}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$\circ_{40}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{40}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{41}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_{41}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$\circ_{42}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{42}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(3) $b * b = \{0\}$ and $b \circ b = \{0, b\}$. Thus similar to (2), we have two hyper pseudo BCK-algebras in this case.

(4) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. If $a \circ a = \{0\}$ then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{45}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{45}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

(5) $b * b = \{0\}$ and $b \circ b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(6) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{46}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{46}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

(7) $b * b = \{0\}$ and $b \circ b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(8) $b \circ b = b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{47}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$*_{47}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$\circ_{48}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$*_{48}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(9) $b \circ b = \{0, b\}$ and $b * b = \{0, a\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.

(10) $b * b = \{0, b\}$ and $b \circ b = \{0, a\}$. Thus in this case, we have the following tables:

$\circ_{49}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a}

$*_{49}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$\circ_{50}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

$*_{50}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(11) $b \circ b = \{0, b\}$ and $b * b = \{0, a, b\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.

(12) $b * b = \{0, b\}$ and $b \circ b = \{0, a, b\}$. Thus in this case, we have the following tables:

$\circ_{51}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a,b}

$*_{51}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$\circ_{52}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

$*_{52}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(13) $b \circ b = b * b = \{0, a\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.

(14) $b \circ b = \{0, a\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{53}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

$*_{53}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

(15) $b * b = \{0, a\}$ and $b \circ b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(16) $b \circ b = b * b = \{0, a, b\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{54}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

$*_{54}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

Case (NPHA4): We have $a \circ b = \{0, a\}$ and $a * b = \{0\}$. If $a \circ a = \{0\}$ and $a * a$ is equal to $\{0\}$ or $\{0, a\}$, then $(a \circ b) \circ (a \circ b) \not\ll a \circ a$. Thus $a \circ a = \{0, a\}$.

(1) $b \circ b = b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{55}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{55}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$\circ_{56}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{56}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{57}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{57}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_{58}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{58}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

(3) $b * b = \{0\}$ and $b \circ b = \{0, b\}$. Thus similar to (2), we have two hyper pseudo BCK-algebras in this case.

(4) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(5) $b * b = \{0\}$ and $b \circ b = \{0, a\}$. If $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Thus in this case, we have the following tables:

$\circ_{61}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

$*_{61}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

(6) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(7) $b * b = \{0\}$ and $b \circ b = \{0, a, b\}$. If $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Thus in this case, we have the following tables:

$\circ_{62}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

$*_{62}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

(8) $b \circ b = b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{63}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{63}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_{64}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{64}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

(9) $b \circ b = \{0, b\}$ and $b * b = \{0, a\}$. Thus in this case, we have the following tables:

$\circ_{65}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{65}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a}

$\circ_{66}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{66}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

(10) $b * b = \{0, b\}$ and $b \circ b = \{0, a\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.

(11) $b \circ b = \{0, b\}$ and $b * b = \{0, a, b\}$. Thus in this case, we have the following tables:

$\circ_{67}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{67}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a,b}

$\circ_{68}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{68}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

(12) $b * b = \{0, b\}$ and $b \circ b = \{0, a, b\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.
(13) $b \circ b = b * b = \{0, a\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.
(14) $b \circ b = \{0, a\}$ and $b * b = \{0, a, b\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.
(15) $b * b = \{0, a\}$ and $b \circ b = \{0, a, b\}$. If $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Thus in this case, we have the following tables:

$\circ_{69}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

$*_{69}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

(16) $b \circ b = b * b = \{0, a, b\}$. If $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Thus in this case, we have the following tables:

$\circ_{70}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

$*_{70}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

We can check that all of the these 70 cases are hyper pseudo BCK-algebras and each of them are not isomorphic together.

Theorem 3.7. There are only 6 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB4).

Proof. The proof is the similar to the proof of Theorem 3.5, by the some modification.

Theorem 3.8. There are only 9 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB5).

Proof. Since H satisfies the condition (NPHB5), then $b \circ a = b * a = \{a, b\}$.

Case (NPHA1): We have $a * b = a \circ b = \{0\}$. If one of the $b \circ b$ or $b * b$ are equal to $\{0\}$, $\{0, a\}$ or $\{0, b\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$ or $(b * a) * (b * a) \not\ll b * b$. Therefore, we have only the following case:

(1) $b \circ b = b * b = \{0, a, b\}$. If $a \circ a = \{0, a\}$, then $(a \circ a) \circ (b \circ a) \not\ll a \circ b$ and if $a * a = \{0, a\}$, then $(a * a) * (b * a) \not\ll a * b$. Therefore, $a \circ a = a * a = \{0\}$. Thus in this case, we have the following tables:

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

Case (NPHA2): We have $a * b = a \circ b = \{0, a\}$. If one of the $a * a$ or $a \circ a$ are equal to $\{0\}$, then $(a * b) * (a * b) \not\ll a * a$ or $(a \circ b) \circ (a \circ b) \not\ll a \circ a$. Hence $a * a = a \circ a = \{0, a\}$.

If $b \circ b = \{0\}$, but $b * b = \{0\}$, {0,b}, {0,a} or {0,a,b}, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$.

If $b \circ b = \{0, b\}$, but $b * b = \{0\}$ or $\{0, a\}$, then $(b * a) * (b * a) \not\ll b * b$. Hence $b * b = \{0, b\}$ or $\{0, a, b\}$.

If $b \circ b = \{0, a\}$, but $b * b = \{0\}$, $\{0, b\}$, $\{0, a\}$ or $\{0, a, b\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$.

If $b \circ b = \{0, a, b\}$, but $b * b = \{0\}$ or $\{0, a\}$, then $(b * a) * (b * a) \not\ll b * b$. Hence $b * b = \{0, b\}$ or $\{0, a, b\}$. Therefore, we have the following cases:

(1) $b \circ b = b * b = \{0, b\}$. Thus we have the following tables:

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(2) $b \circ b = \{0, b\}$ and $b * b = \{0, a, b\}$. Thus we have the following tables:

$\circ_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

$*_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

(3) $b \circ b = \{0, a, b\}$ and $b * b = \{0, b\}$. Thus similar to (2), we have one hyper pseudo BCK-algebra in this case.

(4) Let $b \circ b = b * b = \{0, a, b\}$. Thus we have the following tables:

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

Case (NPHA3): We have $a*b=\{0,a\}$ and $a\circ b=\{0\}$. If $a*a=\{0\}$ and $a\circ a$ is equal to $\{0\}$ or $\{0,a\}$, then $(a*b)*(a*b)\not\ll a*a$. Thus $a*a=\{0,a\}$.

If $b\circ b=\{0\}$, but $b*b=\{0\}$, {0,b}, {0,a} or {0,a,b}, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$.

If $b\circ b=\{0,b\}$, but $b*b=\{0\}$ or $\{0,a\}$, then $(b*a)*(b*a)\not\ll b*b$. If $b\circ b=b*b=\{0,b\}$, then $(b*a)\circ b\neq(b\circ b)*a$. If $b\circ b=\{0,b\}$ and $b*b=\{0,a,b\}$, then $(b*b)\circ b\neq(b\circ b)*b$.

If $b\circ b=\{0,a\}$, but $b*b=\{0\}$, $\{0,b\}$ or $\{0,a,b\}$, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$. If $b\circ b=b*b=\{0,a\}$, then $(b*b)\circ b\neq(b\circ b)*b$.

If $b\circ b=\{0,a,b\}$, but $b*b=\{0\}$ or $\{0,a\}$, then $(b*a)*(b*a)\not\ll b*b$. Hence $b*b=\{0,b\}$ or $\{0,a,b\}$. Therefore, we have the following cases:

(1) $b\circ b=\{0,a,b\}$ and $b*b=\{0,b\}$. If $a\circ a=\{0,a\}$, then $(a\circ a)\circ(b\circ a)\not\ll a\circ b$. Thus in this case, we have the following tables:

$\circ_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

$*_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(2) $b\circ b=b*b=\{0,a,b\}$. If $a\circ a=\{0,a\}$, then $(a\circ a)\circ(b\circ a)\not\ll a\circ b$. Thus in this case, we have the following tables:

$\circ_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

$*_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

Case (NPHA4): We have $a\circ b=\{0,a\}$ and $a*b=\{0\}$. If $a\circ a=\{0\}$ and $a*a$ is equal to $\{0\}$ or $\{0,a\}$, then $(a\circ b)\circ(a\circ b)\not\ll a\circ a$. Thus $a\circ a=\{0,a\}$.

If $b\circ b=\{0\}$, but $b*b=\{0\}$, {0,b}, {0,a} or {0,a,b}, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$.

If $b\circ b=\{0,b\}$, but $b*b=\{0\}$ or $\{0,a\}$, then $(b*a)*(b*a)\not\ll b*b$. If $b\circ b=b*b=\{0,b\}$, then $(b\circ a)*b\neq(b*b)\circ a$. Hence $b*b=\{0,a,b\}$.

If $b\circ b=\{0,a\}$, but $b*b=\{0\}$, $\{0,b\}$ or $\{0,a,b\}$, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$. If $b\circ b=b*b=\{0,a\}$, then $(b*b)\circ b\neq(b\circ b)*b$.

If $b\circ b=\{0,a,b\}$, but $b*b=\{0\}$ or $\{0,a\}$, then $(b*a)*(b*a)\not\ll b*b$. If $b\circ b=\{0,a,b\}$, but $b*b=\{0,b\}$, then $(b*b)\circ b\neq(b\circ b)*b$. Hence $b*b=\{0,a,b\}$. Therefore, we have the following cases:

(1) $b\circ b=\{0,b\}$ and $b*b=\{0,a,b\}$. If $a*a=\{0,a\}$, then $(a*a)*(b*a)\not\ll a*b$. Thus in this step, we have the following case:

$\circ_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

$*_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

(2) $b \circ b = b * b = \{0, a, b\}$. If $a * a = \{0, a\}$, then $(a * a) * (b * a) \not\ll a * b$. Thus in this step, we have the following case:

$\circ_9$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

$*_9$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

We can check that all of the these 9 cases are hyper pseudo BCK-algebras and each of them are not isomorphic together.

Theorem 3.9. There are 96 hyper pseudo BCK-algebras of order 3, that satisfies the normal condition.

Proof. By Theorems 3.4, 3.5, 3.6, 3.7 and 3.8, the proof is clear.

4 Conclusion

Theorem 4.1. *There are 106 hyper pseudo BCK-algebras of order 3 up to isomorphism.*

Proof. The proof follows by Theorems 3.1 and 3.9.

Definition 4.1. *We say that H is a proper hyper pseudo BCK-algebra if H is not a hyper BCK-algebra.*

Theorem 4.2. *There are 87 proper hyper pseudo BCK-algebras of order 3 up to isomorphism.*

Proof. The proof follows by Theorems 2.5 and 4.1.

References

[1] R. A. Borzooei, A. Rezazadeh and R. Ameri, *On Hyper Pseudo BCK-algebras*, submitted.

[2] R. A. Borzooei, M. M. Zahedi and H. Rezaei, *Classification of hyper BCK-algebras of order 3*, Ital. J. Pure Appl. Math., 12(2002), 175-184.

[3] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory*, Kluwer Academic Publications, (2003).

[4] G. Georgescu and A. Iorgulescu, *Pseudo BCK-algebras: an extension of BCK-algebras*, Proceedings of DMTCS 01: Combinatorics, Computability and Logic, Springer, London, (2001), 97-114.

[5] Y. Imai and K. Iseki, *On axiom systems of prepositional calculi*, XIV. Proc Japan Acad, 42(1996), 26-29.

[6] A. Iorgulescu, *Classes of pseudo-BCK algebras - Part I*, Journal of Multiple-Valued Logic and Soft Computing, 12(2006), 71-130.

[7] A. Iorgulescu, *Classes of pseudo-BCK algebras - Part II*, Journal of Multiple-Valued Logic and Soft Computing, 12(2006), 575-629.

[8] Y. B. Jun, M. Kondo and K. H. Kim, *Pseudo-ideals of pseudo-BCK algebras*, Scientiae Mathematicae Japonicae, 8(2003), 87-91.

[9] Y. B. Jun, M. M. Zahedi, X. L. Xin and R. A. Borzooei, *On hyper BCK-algebras*, Italian Journal of Pure and Applied Mathematics, 10(2000), 127-136.

[10] F. Marty, *Sur une generalization de la notion degroups*, 8th Congress Math. Scandinaves, Stockholm, (1934), 45-49.

RATIO MATHEMATICA 22 (2012) 37-43 ISSN:1592-7415

Operators on Weak Hypervector Spaces

Ali Taghavi and Roja Hosseinzadeh
Department of Mathematics, Faculty of Basic Sciences,
University of Mazandaran, P. O. Box 47416-1468, Babolsar, Iran.
Taghavi@umz.ac.ir, ro.hosseinzadeh@umz.ac.ir

Abstract

Let X and Y be weak hypervector spaces and $L_w(X,Y)$ be the set of all weak linear operators from X into Y. We prove some algebraic properties of $L_w(X,Y)$.
Key words: weak hypervector space, weak subhypervector space, normal weak hypervector space, weak linear operator

2000 AMS subject classifications: 46J10, 47B48.

1 Introduction

The concept of hyperstructure was first introduced by Marty [3] in 1934 and has attracted attention of many authors in last decades and has constructed some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions has been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability and etc. A wealth of applications of this concepts are given in $[1-2]$ and [12].

In 1988 the concept of hypervector space was first introduced by Scafati-Tallini. She studied more properties of this new structure in [11]. In [11], Tallini introduced the concept of norm on weak hypervector spaces. We used this definition to extend some theorems of analysis from classic vector spaces to hypervector spaces. For example see $[5-7, 9]$. Moreover, in [4] we defined the concept of dimension of weak hypervector spaces and also authors in [8] introduce the new concept hyperalgebra and quotient hyperalgebra.

Now we want to use some of our defined concepts and prove some algebraic properties of $L_w(X,Y)$, where $L_w(X,Y)$ is the set of all weak linear operators from the weak hypervector space X into the weak hypervector space Y. Note that the hypervector spaces used in this paper are the special case where there is only one hyperoperation, the external one, all the others are ordinary operations. The general hypervector spaces have all operations multivalued also in the hyperfield (see [12]).

2 Preliminaries

We need some Preliminary definitions for to state our results. In this section we state them.

Definition 2.1. [11] *A weak or weakly distributive hypervector space over a field F is a quadruple $(X,+,o,F)$ such that $(X,+)$ is an abelian group and $o : F \times X \longrightarrow P_*(X)$ is a multivalued product such that*

(i) $\forall a \in F, \forall x, y \in X,\ [ao(x+y)] \cap [aox + aoy] \neq \emptyset$,

(ii) $\forall a, b \in F, \forall x \in X,\ [(a+b)ox] \cap [aox + box] \neq \emptyset$,

(iii) $\forall a, b \in F, \forall x \in X,\ ao(box) = (ab)ox$,

(iv) $\forall a \in F, \forall x \in X,\ ao(-x) = (-a)ox = -(aox)$,

(v) $\forall x \in X,\ x \in 1ox$.

We call (i) and (ii) weak right and left distributive laws, respectively.

Note that the set $ao(box)$ in (3) is of the form $\cup_{y \in box} aoy$.

Definition 2.2. [11] *Let $(X,+,o,F)$ be a weak hypervector space over a field F, that is the field of real or complex numbers. We define a pseudonorm in X as a mapping $\|.\| : X \longrightarrow R$, of X into the reals such that:*

(i) $\|0\| = 0$,

(ii) $\forall x, y \in X,\ \|x+y\| \leq \|x\| + \|y\|$,

(iii) $\forall a \in F,\ \forall x \in X,\quad \sup \|aox\| = |a|\|x\|$.

Definition 2.3. *Let X and Y be hypervector spaces over F. A map $T : X \longrightarrow Y$ is called*

(i) linear if and only if

$$T(x+y) = T(x) + T(y),\quad T(aox) \subseteq aoT(x),\quad \forall x, y \in X, a \in F$$

(*ii*) *antilinear if and only if*

$$T(x+y) = T(x) + T(y), \quad T(aox) \supseteq aoT(x), \quad \forall x, y \in X, a \in F$$

(*iii*) *strong linear if and only if*

$$T(x+y) = T(x) + T(y), \quad T(aox) = aoT(x), \quad \forall x, y \in X, a \in F.$$

3 Main results

Before to state our results we describe some fundamental concepts and lemmas from [4]. For more details see [4]. By Lemma 3.1 in [4] we have the following definition. Throughout paper, suppose that X and Y are weak hypervector spaces over a field F.

Definition 3.1. [4] *If $a \in F$ and $x \in X$, then z_{aox} for $0 \neq a$ is that element of aox such that $x \in a^{-1}oz_{aox}$ and for $a = 0$, we define $z_{aox} = 0$.*

As the descriptions in [4], z_{aox} is not unique, necessarily. So the set of all these elements denoted by Z_{aox}. In the mentioned paper we introduced a certain category of weak hypervector spaces. These weak hypervector spaces have been called "normal". We proved that Z_{aox} is singleton in a normal weak hypervector space.

Definition 3.2. [4] *Suppose X satisfy the following conditions:*

(*i*) $(Z_{a_1ox} + Z_{a_2ox}) \cap Z_{(a_1+a_2)ox} \neq \emptyset, \ \forall x \in X, \ \forall a_1, a_2 \in F,$

(*ii*) $(Z_{aox_1} + Z_{aox_2}) \cap Z_{ao(x_1+x_2)} \neq \emptyset, \ \forall x_1, x_2 \in X, \ \forall a \in F.$

Then X is called a normal weak hypervector space.

Lemma 3.1. [4] *If $a \in F$, $0 \neq b \in F$ and $x \in X$, then the following properties hold:*

(*i*) $x \in Z_{1ox}$;

(*ii*) $aoZ_{box} = abox$;

(*iii*) $Z_{-aox} = -Z_{aox}$;

(*iv*) *If X is normal, then Z_{aox} is singleton.*

In [4], the following lemma stated a criterion for normality of a weak hypervector space.

Lemma 3.2. [4] *X is normal if and only if*

(*i*) $z_{a_1ox} + z_{a_2ox} = z_{(a_1+a_2)ox}, \ \forall x \in X, \ \forall a_1, a_2 \in F,$

(*ii*) $z_{aox_1} + z_{aox_2} = z_{ao(x_1+x_2)}, \ \forall x_1, x_2 \in X, \ \forall a \in F.$

Definition 3.3. [6] *Let $T : X \longrightarrow Y$ be an operator. T is said to be bounded if there exists a positive real number K such that we have*

$$\|Tx\| \leq K\|x\| \qquad (\forall x \in X).$$

Definition 3.4. [9] *A map $T : X \longrightarrow Y$ is called weak linear operator if T is additive and satisfies*

$$T(Z_{aox}) \subseteq aoTx, \quad (a \in F, x \in X).$$

Denote the set of all weak linear operators and the set of all bounded weak linear operators from X into Y by $L_w(X, Y)$ and $B_w(X, Y)$, respectively.

Theorem 3.1. [4] *Let X be normal. Then X with the same defined sum and the following scalar product is a classical vector space:*

$$a.x = z_{aox}, \quad \forall a \in F, x \in X.$$

Lemma 3.3. *Let Y be normal, $T \in L_w(X, Y)$ and $a \in F$. Define*

$$aT : X \to Y$$

$$x \mapsto a.Tx$$

Then aT is a weak linear operator, where the operation '.' is the defined scalar product in Theorem 3.1. Moreover, for all $a, b \in F$ and $T, S \in L_w(X, Y)$ we have

$$a(T + S) = aT + aS,$$
$$(a + b)T = aT + bT.$$

Proof. Let $a.u = z_{aou}$, where $u \in Y$. From Theorem ?, we know that Y with this scalar product is a classical vector space. Let $x, y \in X$ and $b \in F$. By Lemma 3.1 we have

$$\begin{aligned}
(aT)(z_{box}) &= a.T(z_{box}) \subseteq a.(boTx) \\
&= \{a.u : u \in boTx\} \\
&= \{z_{aou} : u \in boTx\} \\
&= z_{ao(boTx)} = z_{aboTx} \\
&= boz_{aoTx} = bo(a.Tx) = bo(aT)x
\end{aligned}$$

and also from the normality of Y, we obtain

$$\begin{aligned}(aT)(x+y)=a.T(x+y) &\subseteq a.(Tx+Ty)\\ &= z_{ao(Tx+Ty)}\\ &= z_{aoTx}+z_{aoTy}\\ &= a.Tx+a.Ty\\ &= (aT)x+(aT)y.\end{aligned}$$

Hence aT is a weak linear operator. Now let $T,S\in L_w(X,Y)$ and $x\in X$. The normality of Y yields

$$\begin{aligned}[a(T+S)]x=a.(T+S)x=z_{ao(T+S)x} &= z_{ao(Tx+Sx)}\\ &= z_{aoTx}+z_{aoSx}\\ &= a.Tx+a.Sx\\ &= (aT)x+(aS)x\\ &= (aT+aS)x\end{aligned}$$

which implies that

$$a(T+S)=aT+aS.$$

The second relation is proved in a similar way.

Theorem 3.2. *Let Y be normal. Then $L_w(X,Y)$ with the following sum and product is a weak hypervector space over F.*

$$(T+S)x=Tx+Sx \quad (T,S\in L_w(X,Y), x\in X)$$

$$aoT=\{S\in L_w(X,Y): Sx\in aoTx, \forall x\in X\} \quad (a\in F, T\in L_w(X,Y)).$$

Proof. First we show that aoT is a nonempty subset of $L_w(X,Y)$. By Lemma 3.3, $aT\in L_w(X,Y)$ and for any $x\in X$ we have

$$(aT)x=a.Tx=z_{aoTx}\in aoTx$$

which imply that $aT\in aoT$. It is easy to check that $(L_w(X,Y),+)$ is an abelian group. We show the correctness the first property of scalar product, the rest properties are obtained in a similar way. By Lemma 3.3, for all $a\in F$ and $T,S\in L_w(X,Y)$ we have

$$a(T+S)=aT+aS.$$

This together with

$$a(T+S)\in ao(T+S),\ aT\in aoT,\ aS\in aoS$$

imply that

$$[ao(T+S)] \cap [aoT + aoS] \neq \emptyset$$

and this completes the proof.

Theorem 3.3. *Let Y be normal. Then the following statements are hold.*

(i) *For all $a \in F$ and $T \in L_w(X,Y)$ we have $z_{aoT} = aT$.*

(ii) *$L_w(X,Y)$ is a normal weak hypervector space.*

Proof. (i) By Definition 3.1 we have

$$z_{aoT} \in aoT, T \in a^{-1}oz_{aoT}$$

which for all $x \in X$ implies

$$z_{aoT}x \in (aoT)x, Tx \in (a^{-1}oz_{aoT})x.$$

Since by Theorem 3.2 we have

$$(aoT)x \subseteq aoTx, (a^{-1}oz_{aoT})x \subseteq a^{-1}oz_{aoT}x,$$

we obtain

$$z_{aoT}x \in aoTx, Tx \in a^{-1}oz_{aoT}x.$$

These relations, by Definition 3.1 yield that $z_{aoT}x = z_{aoTx}$. So we obtain $z_{aoT}x = (aT)x$, for all $x \in X$ and hence $z_{aoT} = aT$.

(ii) The normality of $L_w(X,Y)$ can be concluded from Lemma 3.3 and part (i).

Theorem 3.4. *Let Y be normal. Then $B_w(X,Y)$ with the defined sum and scalar product in Theorem ? is a subhypervector space of $L_w(X,Y)$.*

Proof. It is enough to show that $T+S, aoT \in B_w(X,Y)$ for any $a \in F$ and $T, S \in B_w(X,Y)$ it is easy to check that $T+S \in B_w(X,Y)$. Let $S \in aoT$. Hence $Sx \in aoTx$ and so

$$\|Sx\| \leq |a|\|Tx\| \leq |a|\|T\|\|x\|.$$

This completes the proof.

Acknowledgements. This research is partially supported by the Research Center in Algebraic Hyperstructures and Fuzzy Mathematics, University of Mazandaran, Babolsar, Iran.

References

[1] P. Corsini, *Prolegomena of hypergroup theory*, Aviani editore, (1993).

[2] P. Corsini and V. Leoreanu, *Applications of Hyperstructure theory*, Kluwer Academic Publishers, Advances in Mathematics (Dordrecht), (2003).

[3] F. Marty, *Sur nue generalizeation de la notion de group*, 8^{th} congress of the Scandinavic Mathematics, Stockholm, (1934), 45-49.

[4] A. Taghavi and R. Hosseinzadeh, *A note on dimension of weak hypervector spaces*, Italian J. of Pure and Appl. Math, To appear.

[5] A. Taghavi and R. Hosseinzadeh, *Hahn-Banach Theorem for functionals on hypervector spaces*, The Journal of Mathematics and Computer Science, Vol .2 No.4 (2011) 682-690.

[6] A. Taghavi and R. Hosseinzadeh, *Operators on normed hypervector spaces*, Southeast Asian Bulletin of Mathematics, (2011) 35: 367-372.

[7] A. Taghavi and R. Hosseinzadeh, *Uniform Boundedness Principle for operators on hypervector spaces*, Iranian Journal of Mathematical Sciences and Informatics, Accepted.

[8] A. Taghavi and R. Parvinianzadeh, *Hyperalgebras and Quotient Hyperalgebras*, Italian J. of Pure and Appl. Math, No. 26 (2009) 17-24.

[9] A. Taghavi, T. Vougiouklis, R. Hosseinzadeh, *A note on Operators on Normed Finite Dimensional Weak Hypervector Spaces* , Scientific Bulletin, Accepted.

[10] M.Scafati-Tallini, *Characterization of remarkable Hypervector space*, Algebraic Hyperstructures and Aplications, Samotraki, Greece, (2002), Spanidis Press, Xanthi, (2003), 231-237.

[11] M.Scafati-Tallini, *Weak Hypervector space and norms in such spaces*, Algebraic Hyperstructures and Applications, Jasi, Rumania, Hadronic Press. (1994), 199–206.

[12] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, (1994).

RATIO MATHEMATICA 22 (2012) 45-59 ISSN:1592-7415

Atanassov's intuitionistic fuzzy index of hypergroupoids

Mohsen Asghari-Larimi[a], Irina Cristea[b]

[a]Department of Mathematics, Golestan University, Gorgan, Iran

[b]Centre for Systems and Information Technologies, University of Nova Gorica, Slovenia

asghari2004@yahoo.com, irinacri@yahoo.co.uk

Abstract

In this work we introduce the concept of Atanassov's intuitionistic fuzzy index of a hypergroupoid based on the notion of intuitionistic fuzzy grade of a hypergroupoid. We calculate it for some particular hypergroups, making evident some of its special properties.

Key words: (Atanassov's intuitionistic) fuzzy set, (Atanassov's intuitionistic) fuzzy grade, Hypergroup.

2000 AMS subject classifications: $20N20$; $03E72$.

1 Introduction

In 1986 Atanassov [5, 8] defined the notion of intuitionistic fuzzy set as a generalization of the elder one of fuzzy set introduced by Zadeh [30]. Since then this new tool of investigation of various uncertain problems bears his name: Atanassov's intuitionistic fuzzy set. More exactly, for any element x in a finite nonempty set X one assigns two values in the interval $[0, 1]$: the membership degree $\mu(x)$ and the non-membership degree $\lambda(x)$ such that $0 \leq \mu(x) + \lambda(x) \leq 1$. For any $x \in X$, the value $\pi(x) = 1 - \mu(x) - \lambda(x)$ is called the Atanassov's intuitionistic fuzzy index or the hesitation degree of x to X. This index is an important characteristic of the intuitionistic fuzzy sets since it provides valuable information on each element x in X (see [6, 7]). Recently, Bustince et al. [9] have given a construction method to obtain a generalized Atanassov's intuitionistic fuzzy index. For other applications of this index see [10, 11].

The concept of fuzzy grade appeared in hypergroup theory in 2003 and seven years later it was extended to the intuitionistic fuzzy case. Based on the two connections between hypergroupoids and fuzzy sets introduced by Corsini [13, 14], one may associate with any hypergroupoid H a sequence of join spaces and fuzzy sets, whose length is called the fuzzy grade of H. Corsini and Cristea determined the fuzzy grade of all i.p.s. hypergroups of order less than 8 (see [15, 16]). The same problem was treated by Cristea [21], Angheluţă and Cristea [1] for the complete hypergroups, by Corsini et al. for the hypergraphs and hypergroupoids obtained from multivalued functions [18, 19, 20].

In 2010 Cristea and Davvaz [22] introduced and studied the Atanassov's intuitionistic fuzzy grade of a hypergroupoid as the length of the sequence of join spaces and intuitionistic fuzzy sets associated with a hypergroupoid. These sequences have been determined for all i.p.s. hypergroups of order less than 8 and for the complete hypergroups of order less than 7 by Davvaz et al. [23, 24, 25].

Any hypergroupoid H may be endowed with an intuitionistic fuzzy set $\bar{A} = (\bar{\mu}, \bar{\lambda})$ in the sense of Cristea-Davvaz [22]. Based on this construction, we define here the notion of Atanassov's intuitionistic fuzzy index of a hypergroupoid H.

Throughout the paper we use, for simplicity, the term of intuitionistic fuzzy set instead of Atanassov's intuitionistic fuzzy set.

The paper is structured as follows. After some background information regarding hypergroups theory, we recall in Section 2 the construction of the sequence of join spaces and intuitionistic fuzzy sets associated with a hypergroupoid H, presenting some technical results for the membership functions $\widetilde{\mu}, \bar{\mu}, \bar{\lambda}$. In Section 3 we introduce the notion of intuitionistic fuzzy index of a hypergroupoid, giving its formula for some particular hypergroups. We conclude with final remarks and some open problems.

2 Atanassov's intuitionistic fuzzy grade of hypergroupoids

First we recall some definitions from [12, 17], needed in what follows.

Let H be a nonempty set and let $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid.*

This hyperoperation can be extended to a binary operation on $\mathcal{P}^*(H)$. If

A and B are nonempty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if, for all x, y, z in H, we have the associative law: $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if, for all $x \in H$, we have the reproducibility axiom: $x \circ H = H \circ x = H$. A hypergroup $(H, \circ)$ is called *total hypergroup* if, for any $x, y \in H$, $x \circ y = H$.

For each pair of elements $a, b \in H$, we denote: $a/b = \{x \in H \mid a \in x \circ b\}$ and $b \backslash a = \{y \in H \mid a \in b \circ y\}$.

A commutative hypergroup $(H, \circ)$ is called a *join space* if the following condition holds:

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset.$$

A commutative hypergroup $(H, \circ)$ is *canonical* if and only if it is a join space with a scalar identity.

The notion of join space has been introduced and studied for the first time by Prenowitz. Later on, together with Jantosciak, he reconstructed, from an algebraic point of view, several branches of geometry: the projective, the descriptive and the spherical geometry (see [28]).

Several connections between hypergroups and (intuitionistic) fuzzy sets have been investigated till now (see for example [2, 3, 4, 26]). Here we focus our study on that initiated by Corsini [14] in 2003, when he defined a sequence of join spaces associated with a hypergroupoid endowed with a fuzzy set. Based on this idea, Cristea and Davvaz [22] extended later on this connection to the intuitionistic fuzzy grade. We recall now briefly these constructions.

For any hypergroup $(H, \circ)$, Corsini [14] defined a fuzzy subset $\widetilde{\mu}$ of H in the following way: for any $u \in H$, one considers:

$$\widetilde{\mu}(u) = \frac{\displaystyle\sum_{(x,y) \in Q(u)} \frac{1}{|x \circ y|}}{q(u)}, \tag{1}$$

where $Q(u) = \{(a, b) \in H^2 \mid u \in a \circ b\}$, $\quad q(u) = |Q(u)|$. If $Q(u) = \emptyset$, set $\widetilde{\mu}(u) = 0$.

On the other hand, with any hypergroupoid H endowed with a fuzzy set α, we can associate a join space $(H, \circ_\alpha)$ as follows (see [13]): for any $(x, y) \in H^2$,

$$x \circ_\alpha y = \{z \in H \mid \alpha(x) \wedge \alpha(y) \leq \alpha(z) \leq \alpha(x) \vee \alpha(y)\}. \tag{2}$$

Then, from $({}^1H, \circ_1)$ we obtain, in the same way as in (1), a membership function $\widetilde{\mu}_1$ and then the join space ${}^2H$ and so on. A sequence of fuzzy sets and of join spaces $({}^rH, \widetilde{\mu}_r)$ is determined.

We denote $\tilde{\mu}_0 = \tilde{\mu}$, ${}^0H = H$. If two consecutive hypergroups of the obtained sequence are isomorphic, then the sequence stops.

The length of this sequence has been called by Corsini and Cristea [15, 16] the (strong) fuzzy grade of the hypergroupoid H.

Let now $(H, \circ)$ be a finite hypergroupoid of cardinality $n, n \in \mathbb{N}^*$. Cristea and Davvaz [22] defined on H an Atanassov's intuitionistic fuzzy set $(\overline{\mu}, \overline{\lambda})$ in the following way: for any $u \in H$, we set:

$$\begin{aligned} \overline{\mu}(u) &= \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{n^2}, \\ \overline{\lambda}(u) &= \frac{\displaystyle\sum_{(x,y)\in \overline{Q}(u)} \frac{1}{|x \circ y|}}{n^2}, \end{aligned} \tag{3}$$

where $Q(u) = \{(x, y) \mid (x, y) \in H^2, u \in x \circ y\}, \overline{Q}(u) = \{(x, y) \mid (x, y) \in H^2, u \notin x \circ y\}$. If $Q(u) = \emptyset$, then we put $\overline{\mu}(u) = 0$ and similarly, if $\overline{Q}(u) = \emptyset$, then we put $\overline{\lambda}(u) = 0$.

It follows immediately the following relation.

Corolary 2.1. *For all $u \in H$, $\tilde{\mu}(u) \geq \overline{\mu}(u)$.*

Proof. Let $|H| = n$. Then $q(u) \leq n^2$, for all $u \in H$. Thus, by definitions (1) and (3), we have $\tilde{\mu}(u) \geq \overline{\mu}(u)$. $\square$

Moreover, for all $u \in H$, we define: $A(u) = \sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}, \overline{A}(u) = \sum_{(x,y)\in \overline{Q}(u)} \frac{1}{|x \circ y|}$.

Remark 2.1. There exist hypergroups H such that there exists $u \in H$ with

$$(i) |Q(u)| = |\overline{Q}(u)| \quad \text{and} \quad \overline{\mu}(u) \neq \overline{\lambda}(u).$$

$$(ii) |Q(u)| \neq |\overline{Q}(u)| \quad \text{and} \quad \overline{\mu}(u) = \overline{\lambda}(u).$$

An example for the case (i) is the hypergroup H_1 and the case (ii) is illustrated by the hypergroup H_2 given below; both hypergroups are commutative and are represented by the following tables:

H_1	1	2	3	4
1	1,4	1,2	1,3	1,4
2		2	2,3	2,4
3			3	3,4
4				1,2,3,4

where $|Q(1)| = |\overline{Q}(1)| = 8$, $A(1) = \frac{15}{4}$, $\overline{A}(1) = \frac{20}{4}$ and by consequence $\overline{\lambda}(1) = \frac{15}{64} \neq \frac{20}{64} = \overline{\mu}(1)$.

H_2	1	2	3
1	1	1,2	1,3
2		2	2,3
3			3

where $|Q(1)| = 5 \neq 4 = |\overline{Q}(1)|$ and by consequence $\overline{\lambda}(1) = \overline{\mu}(1) = \frac{1}{3}$.

3 Atanassov's intuitionistic index of hypergroupoids

Definition 3.1. [5] Let (μ, λ) be an intuitionistic fuzzy set on the nonempty set H. We call the *hesitation degree* or the *intuitionistic index* of the element x in the set H the following expression

$$\pi(x) = 1 - \mu(x) - \lambda(x).$$

Evidently $0 \leq \pi(x) \leq 1$, for all $x \in H$.

Remark 3.1. Let $(\overline{\mu}, \overline{\lambda})$ be the intuitionistic fuzzy set associated with a hypergroupoid $(H, \circ)$ like in Cristea-Davvaz [22]. Since $\overline{\mu}(u)+\overline{\lambda}(u) = \frac{\sum_{(x,y)\in H^2} \frac{1}{|x\circ y|}}{n^2}$, for all $u \in H$, according with (ω'), it follows that $\overline{\pi}(u) = constant$, for all $u \in H$. Therefore we introduce the following definition.

Definition 3.2. Let $(H, \circ)$ be a hypergroupoid and let $\overline{\mu}$, $\overline{\lambda}$ be the membership functions defined in (w'). For all $u \in H$, we define

$$\overline{\pi}(H) = \overline{\pi}(u) = 1 - \overline{\mu}(u) - \overline{\lambda}(u),$$

and we call it the *intuitionistic index* of the hypergroupoid H.

In the following we determine the intuitionistic index for some particular hypergroupoids.

Proposition 3.1. *Let $(H, \circ)$ be a total hypergroup of cardinality n. Then $\overline{\pi}(u) = \dfrac{n-1}{n}$, for any $u \in H$.*

Proof. Let $(H, \circ)$ be a total hypergroup and $|H| = n$. Then $x \circ y = H$, for all $x, y \in H$. Thus $A(u) = n$, $\overline{A}(u) = 0$, for all $u \in H$. Since $\overline{Q}(u) = \emptyset$, it follows that $\overline{\mu}(u) = \frac{1}{n}$, $\overline{\lambda}(u) = 0$, which imply that $\overline{\pi}(u) = \frac{n-1}{n}$. □

Proposition 3.2. *Let $H = \{x_1, x_2, x_3, ..., x_{n-1}, x_n\}$, be the hypergroupoid defined as it follows: $x_i \circ x_j = \{x_i, x_j\}$, $1 \leq i, j \leq n$. Then $\overline{\mu}(x) = \frac{1}{n}, \overline{\pi}(x) = \frac{n-1}{2n}$, for any $x \in H$.*

Proof. The table of the commutative hyperoperation $\circ$ is the following one:

H	x_1	x_2	x_3	...	x_{n-1}	x_n
x_1	x_1	x_1, x_2	x_1, x_3	...	x_1, x_{n-1}	x_1, x_n
x_2		x_2	x_2, x_3	...	x_2, x_{n-1}	x_2, x_n
x_3			x_3	...	x_3, x_{n-1}	x_3, x_n
...				...	...	...
x_{n-1}					x_{n-1}	x_{n-1}, x_n
x_n						x_n

Let $x_i \in H$. Then $|Q(x_i)| = \{(x_i, x_i), (x_i, x_s), (x_p, x_i),\ 1 \leq s, p \leq n,\ s, p \neq i\}$. It follows that $|Q(x_i)| = 2n-1$. Thus $|\overline{Q}(x_i)| = n^2-(2n-1) = (n-1)^2$. So, we have $\overline{\mu}(x_i) = \frac{1}{n^2}[1+\frac{1}{2}(2n-2)] = \frac{1}{n}$. Similarly, we obtain $\overline{\lambda}(x_i) = \overline{\pi}(x_i) = \frac{n-1}{2n}$. □

Corolary 3.1. *Let $(H, \circ)$ be the hypergroupoid defined in Proposition 3.2. Then, for any $x \in H$, we obtain $\overline{\mu}(x) = \frac{1}{n} > \overline{\lambda}(x) = \overline{\pi}(x) = \frac{n-1}{2n}$, for any $n < 3$, and $\overline{\mu}(x) = \frac{1}{n} < \overline{\lambda}(x) = \overline{\pi}(x) = \frac{n-1}{2n}$ for any $n \geq 3$.*

With any hypergroupoid H one may associate a sequence of join spaces and fuzzy sets denoted by $((^iH, \circ_i), \overline{\mu}_i(u))_{i \geq 1}$.

Then, for any $i \geq 1$, we can divide ^iH in the classes $\{^iC_j\}_{j=1}^r$, where $x, y \in {}^iC_j \Longleftrightarrow \mu_{i-1}(x) = \mu_{i-1}(y)$. Moreover, we define the following ordering relation: $j < k$ if, for elements $x \in {}^iC_j$ and $y \in {}^iC_k$, we have $\mu_{i-1}(x) < \mu_{i-1}(y)$. We need the following notations: for all j, s, set

$$k_j = |^iC_j|, \quad {}_sC = \bigcup_{1 \leq j \leq s} {}^iC_j, \quad {}^sC = \bigcup_{s \leq j \leq r} {}^iC_j, \quad {}_sk = |{}_sC|, \quad {}^sk = |{}^sC|.$$

Therefore with any ordered chain $({}^iC_1, {}^iC_2, ..., {}^iC_r)$ we may associate an ordered r-tuple $(k_1, k_2, ..., k_r)$, where $k_l = |{}^iC_l|$, for all $l,\ 1 \leq l \leq r$. Using these notations we determine the general formula for calculating the values of the membership functions $\overline{\mu}_i, \overline{\lambda}_i$.

Theorem 3.1. *[22] For any $u \in {}^iC_s,\ i \geq 1,\ s \in \{1, 2, ..., r\}$, we find that*

$$\overline{\mu}_i(z) = \frac{k_s + 2 \displaystyle\sum_{\substack{l \leq s \leq m \\ l \neq m}} \frac{k_l k_m}{\displaystyle\sum_{l \leq t \leq m} k_t}}{n^2}$$

and

$$\overline{\lambda}_i(z) = \frac{\displaystyle\sum_{l \neq s} k_l + 2 \sum_{s < l < m \leq r} \frac{k_l k_m}{\displaystyle\sum_{l \leq t \leq m} k_t} + 2 \sum_{1 \leq l < m < s} \frac{k_l k_m}{\displaystyle\sum_{l \leq t \leq m} k_t}}{n^2}.$$

We immediately obtain also the formula for calculating the intuitionistic fuzzy index.

Theorem 3.2. *For any $u \in {}^iC_s,\ i \geq 1,\ s \in \{1, 2, ..., r\}$,*

$$\overline{\pi}_i(u) = 1 - \left[\frac{\sum_{l=1}^r k_l + 2 \sum_{1 \leq l < m \leq r} \frac{k_l k_m}{\sum_{l \leq t \leq m} k_t}}{n^2} \right].$$

One of the natural problems regarding the intuitionistic fuzzy grade of a hypergroupoid is that of constructing a hypergroupoid H with $i.f.g.(H) = p$, for any arbitrary natural number $p \geq 2$. A such example is given in the next result.

Lemma 3.1. *[22] Let $H = \{x_1, x_2, x_3, ..., x_{n-1}, x_n\}$, where n is an even number, be the hypergroupoid defined as follows:*

i) $x_i \circ x_i = x_i,\ 1 \leq i \leq n,$

ii) $x_i \circ x_j = x_j \circ x_i = \{x_i, x_{i+1}, ..., x_j\},\ 1 \leq i < j \leq n.$

Then, for any $s \in \{1, 2, ..., \frac{n}{2}\}$, we have
$\overline{\mu}(x_s) = \overline{\mu}(x_{n-s+1}),\ \overline{\lambda}(x_s) = \overline{\lambda}(x_{n-s+1})$ *and* $\overline{\mu}(x_s) \leq \overline{\mu}(x_{s+1}),\ \overline{\lambda}(x_s) \geq \overline{\lambda}(x_{s+1})$.

Moreover, we obtain

$$\overline{\mu}(x_1)=\frac{1}{n^2}\left(1+2\sum_{m=2}^{n}\frac{1}{m}\right),\quad \overline{\lambda}(x_1)=\frac{1}{n^2}\left(1-n+2n\sum_{m=2}^{n}\frac{1}{m}\right)$$

$$\overline{\mu}(x_{s+1})=\overline{\mu}(x_s)+\frac{2}{n^2}\sum_{m=s+1}^{n-s}\frac{1}{m},\ \overline{\lambda}(x_s)=\overline{\lambda}(x_{s+1})+\frac{2}{n^2}\sum_{m=s+1}^{n-s}\frac{1}{m},\ \forall s, 2\le s\le\frac{n}{2}.$$

Notation 3.1. In the following we denote the hypergroup of Lemma 3.1 by $\mathcal{H}_n,\ n\in\mathbb{N}^*$. For $\mathcal{H}_n$, we denote the membership functions $\overline{\mu}$ and $\overline{\lambda}$ introduced in (ω') by $\overline{\mu}^n$ and $\overline{\lambda}^n$.

Example 3.1. Let us consider again the hypergroup $\mathcal{H}_n$ that has the following table

H	x_1	x_2	x_3	...	x_{n-1}	x_n
x_1	x_1	x_1,x_2	$x_1\to x_3$	...	$x_1\to x_{n-1}$	$x_1\to x_n$
x_2		x_2	x_2,x_3	...	$x_2\to x_{n-1}$	$x_2\to x_n$
x_3			x_3	...	$x_3\to x_{n-1}$	$x_3\to x_n$
...				...	...	...
x_{n-1}					x_{n-1}	x_{n-1},x_n
x_n						x_n

where we use the notation $x_i\to x_j=\{x_i,x_{i+1},...,x_j\}$, with $i<j$. Then we obtain

	$\overline{\mu}^n(x_1)$	$\overline{\mu}^n(x_2)$	$\overline{\mu}^n(x_3)$	$\overline{\mu}^n(x_4)$	$\overline{\lambda}^n(x_1)$	$\overline{\lambda}^n(x_2)$	$\overline{\lambda}^n(x_3)$	$\overline{\lambda}^n(x_4)$	$\overline{\pi}^n(x_k)$
$n=2$	0.500				0.250				0.250
$n=3$	0.296	0.407			0.333	0.222			0.370
$n=4$	0.197	0.302			0.354	0.250			0.447
$n=5$	0.142	0.229	0.256		0.353	0.266	0.240		0.504
$n=6$	0.108	0.179	0.212		0.342	0.273	0.240		0.547
$n=7$	0.085	0.144	0.176	0.186	0.332	0.257	0.241	0.231	0.581
$n=8$	0.075	0.125	0.145	0.159	0.226	0.176	0.156	0.141	0.698

where $k\in\{1,...,n\}$. The missing values in the previous table are equal to the other written values as in the formulas in Lemma 3.1.

Lemma 3.2. *For the hypergroup $\mathcal{H}_n$, with n an even number, for any $s\in\{1,2,...,\frac{n}{2}\}$ and $n\ge 6$, we obtain*

$$\overline{\mu}^n(x_s)<\overline{\lambda}^n(x_s).$$

Corolary 3.2. *Let us consider the hypergroup $\mathcal{H}_n$. Then, for any $s \in \{1, 2, ..., n\}$, we obtain*

$$\overline{\pi}(\mathcal{H}_n) = \overline{\pi}^n(x_s) = 1 - \frac{1}{n^2}\left[2 - n + 2(n+1)\sum_{m=2}^{n}\frac{1}{m}\right].$$

Proof. Since

$$\begin{aligned}\overline{\mu}^n(x_1) + \overline{\lambda}^n(x_1) &= \tfrac{1}{n^2}[(1 + 2\textstyle\sum_{m=2}^{n}\frac{1}{m}) + (1 - n + 2n\sum_{m=2}^{n}\frac{1}{m})] \\ &= \tfrac{1}{n^2}[2 - n + 2(n+1)\textstyle\sum_{m=2}^{n}\frac{1}{m}],\end{aligned}$$

the required result is proved. □

Theorem 3.3. *Let us consider the hypergroups $(\mathcal{H}_n, \circ)$ and $(\mathcal{H}_{n+2}, \circ)$, with n an even natural number. Then*

$$\overline{\pi}(\mathcal{H}_n) < \overline{\pi}(\mathcal{H}_{n+2}).$$

Proof. By Corollary 3.2, we will prove that

$$\frac{1}{n^2}[2 - n + 2(n+1)\sum_{m=2}^{n}\frac{1}{m}] > \frac{1}{(n+2)^2}[2 - (n+2) + 2(n+3)\sum_{m=2}^{n+2}\frac{1}{m}].$$

Denoting $\sum_{m=2}^{n}\frac{1}{m}$ by A, we will show that:

$$\frac{1}{n^2}[2 - n + 2(n+1)A] > \frac{1}{(n+2)^2}[-n + 2(n+3)(A + \frac{2n+3}{(n+1)(n+2)})],$$

that is equivalent with

$$(2-n)(n+1)(n+2)^3 + 2(n+1)^2(n+2)^3 A >$$

$$> -n^3(n+1)(n+2) + 2n^2(n+1)(n+2)(n+3)A + 2n^2(2n+3)(n+3).$$

Therefore we have

$$4(n+1)(n+2)(n^2+4n+2)A > 2(3n^4 + 10n^3 + n^2 - 16n - 8)$$

if and only if

$$2A > \frac{3n^4 + 10n^3 + n^2 - 16n - 8}{n^4 + 7n^3 + 16n^2 + 14n + 4}.$$

We prove the last relation by induction on n.
For $n=3$ the relation becomes $2(\frac{1}{2}+\frac{1}{3}) > \frac{466}{460}$, that is $\frac{5}{3} > \frac{466}{460}$, that is true.
We suppose that

$$P(n) : 2\sum_{m=2}^{n}\frac{1}{m} > \frac{3n^4+10n^3+n^2-16n-8}{n^4+7n^3+16n^2+14n+4} \text{ is true}$$

and we prove that

$$P(n) : 2\sum_{m=2}^{n+1}\frac{1}{m} > \frac{3(n+1)^4+10(n+1)^3+(n+1)^2-16(n+1)-8}{(n+1)^4+7(n+1)^3+16(n+1)^2+14(n+1)+4} \text{ is fulfilled.}$$

Since

$$2\sum_{m=2}^{n+1}\frac{1}{m} = 2\sum_{m=2}^{n}\frac{1}{m} + \frac{2}{n+1} > \frac{3n^4+10n^3+n^2-16n-8}{n^4+7n^3+16n^2+14n+4} + \frac{2}{n+1},$$

it remains to prove that

$$\frac{3n^5+15n^4+25n^3+17n^2+4n}{n^5+8n^4+23n^3+30n^2+18n+4} > \frac{3n^4+22n^3+49n^2+28n-10}{n^4+11n^3+43n^2+71n+42}.$$

After simple computations that we omit here, we prove that the last relation is true, for any natural number n.

Now the proof is complete. □

Generalizing this theorem we obtain the following result.

Corolary 3.3. *For the hypergroups $\mathcal{H}_n$ and $\mathcal{H}_{n'}$, with n, n' two even natural numbers such that $n < n'$, we have the following relation:*

$$\overline{\pi}(\mathcal{H}_n) < \overline{\pi}(\mathcal{H}_{n'}).$$

Proof. Let $n' = n+2k$, $k \in \mathbb{N}^*$. We will prove the relation by induction on k. For $k=1$, by Theorem 3.3 we have $\overline{\pi}(\mathcal{H}_n) < \overline{\pi}(\mathcal{H}_{n+2})$. Assume the corollary is true for $k=m$, i.e., $\overline{\pi}(\mathcal{H}_n) < \overline{\pi}^{(}\mathcal{H}_{n+2m})$. Again by Theorem 3.3 we have $\overline{\pi}(\mathcal{H}_{n+2m}) < \overline{\pi}(\mathcal{H}_{n+2(m+1)})$. Thus, the thesis of the corollary is true for $k=m+1$. This completes the induction and the proof. □

We conclude this section with a result regarding the membership function $\overline{\mu}^n$.

Theorem 3.4. *Let us consider the hypergroups $\mathcal{H}_n$ and $\mathcal{H}_{n+2}$, with n an even natural number. Then, for any $s \in \{1, 2, ..., n/2\}$, we have the relation*

$$\overline{\mu}^n(x_s) > \overline{\mu}^{n+2}(x_s).$$

Proof. For any $s \in \{1, 2, ..., n/2\}$, we will prove that

$$\frac{1}{n^2}[2s-1+2\sum_{m=1}^{n-2s+1}\frac{s}{s+m}+2\sum_{m=1}^{s-1}\frac{m}{n-m+1}] >$$

$$> \frac{1}{(n+2)^2}[2s-1+2\sum_{m=1}^{n-2s+1}\frac{s}{s+m}+2s(\frac{1}{n-s+2}+\frac{1}{n-s+3})+2\sum_{m=1}^{s-1}\frac{m}{n-m+3}].$$

Denote $A = 2s-1+2\sum_{m=1}^{n-2s+1}\frac{s}{s+m}$. Since

$$\frac{1}{n^2}\left(2\sum_{m=1}^{s-1}\frac{m}{n-m+1}\right) > \frac{1}{(n+2)^2}\left(2\sum_{m=1}^{s-1}\frac{m}{n-m+3}\right),$$

to prove that $\overline{\mu}^n(x_s) > \overline{\mu}^{n+2}(x_s)$ it is enough to show that

$$(n+2)^2 A > n^2\left[A + \frac{2s(2n-2s+5)}{(n-s+2)(n-s+3)}\right],$$

that is true if and only if

$$A > \frac{sn^2(2n-2s+5)}{(2n+2)(n-s+2)(n-s+3)}.$$

Since

$$\sum_{m=1}^{n-2s+1}\frac{s}{s+m} > \frac{s(n-2s+1)}{n-s+1},$$

it follows that

$$A > \frac{(4s-1)n-6s^2+5s-1}{n-s+1}.$$

It remains to prove that

$$\frac{(4s-1)n-6s^2+5s-1}{n-s+1} > \frac{sn^2(2n-2s+5)}{(2n+2)(n^2+(5-2s)n+s^2-5s+6)} =$$

$$= \frac{2sn^3+n^2(-2s^2+5s)}{2n^3+n^2(-4s+12)+n(2s^2-14s+22)+2s^2-10s+12},$$

and this is true if and only if

$$[2n^3+n^2(-4s+12)+n(2s^2-14s+22)+2s^2-10s+12][(4s-1)n-6s^2+5s-1] >$$

$$> (n-s+1)[2sn^3+n^2(-2s^2+5s)],$$

which is equivalent with

$$E(n)=n^4(6s-29+n^3(-24s^2+55s-14)+n^2(30s^3-143s^2+161s-34)+$$
$$+n(-12s^4+102s^3-246s^2+182s-34)-12s^4+70s^3-124s^2+70s-12>0,$$

whenever $2 \leq 2s \leq n$. Then we find

$$E^{(4)}(n)=144s-48>0;$$

it follows that $E'''(n)$ is a strictly increasing function, so

$$E'''(n) \geq E'''(2s)=144s^2+234s-84>0.$$

It follows that $E''(n)$ is a strictly increasing function, so

$$E''(n) \geq E''(2s)=60s^3+278s^2+154s-68>0.$$

Then $E'(n)$ is a strictly increasing function, thus

$$E'(n) \geq E'(2s)=12s^4+126s^3+230s^2+46s-34>0.$$

Thus, we obtain that $E(n)$ is a strictly increasing function, so

$$E(n) \geq E(2s)=28s^4+110s^3+104s^2+2s-12>0, \text{ for any } s \geq 1,$$

and the required result is proved. □

A generalized corollary follows.

Corolary 3.4. Let us consider the hypergroups $\mathcal{H}_n$ and $\mathcal{H}_{n'}$, with n, n' even numbers such that $n < n'$. Then

$$\overline{\mu}^n(x_s) > \overline{\mu}^{n'}(x_s), \ \forall s \in \{1,2,...,n/2\}.$$

Proof. The proof is similar to the proof of Corollary 3.3. □

4 Conclusions

Given a hypergroupoid H, one may calculate two numerical functions associated with it: the fuzzy grade and the intuitionistic fuzzy grade. In this note we define another one, called the Atanassov's intuitionistic fuzzy index of a hypergroupoid. This function depends on the values of the first membership functions in the sequence of join spaces and intuitionistic fuzzy sets associated with H as in [22]. We have determined it for some particular hypergroups. We will investigate further properties and connections with the intuitionistic fuzzy grade in a future work.

References

[1] C. Angheluţă, I. Cristea, *Atanassov's intuitionistic fuzzy grade of the complete hypergroups*, J. Mult.-Valued Logic Soft Comput., in press.

[2] M. Asghari-Larimi, *Some Properties of Intuitionistic Nil Radicals of Intuitionistic Fuzzy Ideals*, Int. Math. Forum 5 (2010), no. 29-32, 1551-1558.

[3] M. Asghari-Larimi, B. Davvaz, *Hyperstructures associated to arithmetic functions*, Ars Combin. 97 (2010), 51-63.

[4] M. Asghari-Larimi, V. Leoreanu-Fotea, *A connection between hypergroupoids and L-Fuzzy Sets of Type 2*, It. J. Pure Appl. Math. 26 (2009), 207-216.

[5] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems 20 (1986), no.1, 87-96.

[6] K. Atanassov, *More on intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 33 (1989), no.1, 37-45.

[7] K. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems 61 (1994), no.2, 137-142.

[8] K. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, Studies in Fuzziness and Soft Computing, 35, Physica-Verlag, Heidelberg, 1999.

[9] H. Bustince, E. Barrenchea, M. Pagola, J. Fernandez, C. Guerra, P. Couto, P. Melo-Pinto, *Generalized Atanassov's intuitionistic fuzzy index. Construction of Atanassov's fuzzy entropy from fuzzy implication operators*, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 19 (2011), no. 1, 5169.

[10] P. Burillo, H. Bustince, *Construction theorems for intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 84 (1996), no.3, 271-281.

[11] H. Bustince, P. Burillo, *Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets*, Fuzzy Sets and Systems 78 (1996), no. 3, 305316.

[12] P. Corsini, *Prolegomena of hypergroup theory*, Aviani Editore, 1993.

[13] P. Corsini, *Join spaces, power sets, fuzzy sets*, Algebraic hyperstructures and applications (Iasi, 1993), 4552, Hadronic Press, Palm Harbor, FL, 1994.

[14] P. Corsini, *A new connection between hypergroups and fuzzy sets*, Southeast Asian Bull. Math., 27 (2003), no.2, 221-229.

[15] P. Corsini, I. Cristea, *Fuzzy grade of i.p.s. hypergroups of order less or equal to* 6, Pure Math. Appl., 14 (2003), 275-288.

[16] P. Corsini, I. Cristea, *Fuzzy grade of i.p.s. hypergroups of order* 7, Iran. J. Fuzzy Syst., 1(2004), no. 2, 15-32.

[17] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory*, Advances in Mathematics (Dordrecht), 5, Kluwer Academic Publishers, Dordrecht, 2003.

[18] P. Corsini, V. Leoreanu-Fotea, *On the grade of a sequence of fuzzy sets and join spaces determined by a hypergraph*, Southeast Asian Bull. Math., 34(2010), no.2, 231-242.

[19] P. Corsini, V. Leoreanu-Fotea, A. Iranmanesh, *On the sequence of hypergroups and membership functions determined by a hypergraph*, J. Mult.-Valued Logic Soft Comput., 14(2008), no.6, 565-577.

[20] P. Corsini, R. Mahjoob, *Multivalued functions, fuzzy subsets and join spaces*, Ratio Math., 20(2010), 1-41.

[21] I. Cristea, *Complete Hypergroups,* 1*-Hypergroups and Fuzzy Sets*, An. Şt. Univ. Ovidius Constanţa, 10(2002), no.2, 25-38.

[22] I. Cristea, B. Davvaz, *Atanassov's intuitionistic fuzzy grade of hypergroups*, Inform. Sci., 180 (2010), no.8, 1506-1517.

[23] B. Davvaz, E. Hassani Sadrabadi, I. Cristea, *Atanassov's intuitionistic fuzzy grade of i.p.s. hypergroups of order less than or equal to* 6, Iran. J. Fuzzy Syst., 9(2012), vol.4, 71-97.

[24] B. Davvaz, E. Hassani Sadrabadi, I. Cristea, *Atanassov's intuitionistic fuzzy grade of i.p.s. hypergroups of order* 7, J. Mult.-Valued Logic Soft Comput., in press.

[25] B. Davvaz, E. Hassani Sadrabadi, I. Cristea, *Atanassov's intuitionistic fuzzy grade of the complete hypergroups of order less than or equal to* 6, submitted.

[26] W.A. Dudek, B. Davvaz, Y.B. Jun, *On intuitionistic fuzzy sub-quasihypergroups of quasihypergroups*, Inform. Sci., 170 (2005), no.2-4, 251-262.

[27] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandenaves, Stockholm, 1934, 45-49.

[28] W. Prenowitz, J. Jantosciak, *Geometries and Join spaces*, J. Reine und Angew Math., 257 (1972), 100-128.

[29] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, Inc, 115, Palm Harber, USA, (1994).

[30] L.A. Zadeh, *Fuzzy Sets*, Inform and Control, 8 (1965), 338-353.

RATIO MATHEMATICA 22 (2012) 61-68 ISSN:1592-7415

Codes on s-periodic errors

Pankaj Kumar Das* and Vinod Tyagi**

*Department of Mathematics, Shivaji College (University of Delhi),

Raja Garden, Delhi-110 027, India

**Department of Mathematics, Shyam Lal College (Eve.)(University of Delhi),

Shahdara, Delhi-110032, India

*pankaj4thapril@yahoo.co.in, **vinodtyagi@hotmail.com

Abstract

In this paper, we study linear codes capable of detecting and correcting s-periodic errors. Lower and upper bounds on the number of parity check digits required for codes detecting such errors are obtained. Another bound on codes correcting such errors is also obtained. An example of a code detecting such errors is provided.

Key words: parity check matrix, syndromes, standard array, periodic error.

2000 AMS subject classifications: 94B25, 94B60, 94-02.

1 Introduction

Investigations in coding theory have been made in several directions but one of the most important directions has been the detection and correction of errors. It began with Hamming codes[9] for single errors, Golay codes[10, 11] for double and triple random errors and thereafter BCH codes[12, 13, 14] were studied for multiple error correction. There is a long history towards the growth of the subject and many of the codes developed have found applications in numerous areas of practical interest. One of the areas of practical importance in which a parallel growth of the subject took place is that of burst error detecting and correcting codes. It has also been observed that in many communication channels, burst errors occur more frequently than random errors. A burst of length b may be defined as follows:

Definition 1.1. *A burst of length b is a vector whose only non-zero components are among some b consecutive components, the first and the last of which is non zero.*

Extending the work of Hamming[9], Abramson[1] developed codes which dealt with the correction of single and double adjacent errors. The work due to Fire[8] depicted a more general concept of burst errors.

Stone[19], and Bridwell and Wolf[4] considered multiple bursts. It was noted by Chien and Tang[5] that in several channels errors do occur in the form of a burst but not near the end of the vector. Channels due to Alexander, Gryb and Nast[2] fall in this category. In this light, Chien and Tang proposed a modification in the definition of a burst, now known as CT burst, according to which a CT burst of length b is defined as follows:

Definition 1.2. *A CT burst of length b is a vector whose only non zero components are confined to some b consecutive positions, the first of which is non-zero.*

Recently a new kind of error, known as repeated burst, has been observed by Berardi, Dass and Verma[3]. For further work on this type of error, one may refer to [6, 7, 18] and references therein.

It is very clear that the nature of error differ from channel to channel depending upon the behaviour of channels or the kind of errors which occur during the process of transmission. There is a need to deal with many types of error patterns and accordingly codes are to be constructed to combat such error patterns. Though the errors are generally classified mainly in two categories - random errors and burst errors, it has also been observed that the occurrence of errors may follow a pattern, different from random and burst. In certain communication channel like Astrophotography[21], small mechanical error occurs periodically in the accuracy of the tracking in a motorized mount that results small movements of the target that can spoil long-exposure images, even if the mount is perfectly polar-aligned and appears to be tracking perfectly in short tests. It repeats at a regular interval - the interval being the amount of time it takes the mount's drive gear to complete one revolution. This type of error pattern is termed as *periodic or alternate pattern*. It was in this spirit that the codes correcting s-alternate errors were developed by Tyagi and Das [20]. An s-periodic error is defined as follows:

Definition 1.3. *An s-periodic error is an n- tuple whose non zero components are located at a gap of s positions where $s = 1, 2, 3, \ldots, (n-1)$ and the number of its starting positions is among the first $s+1$ components.*

For s=1, the 1-periodic error vectors are the ones where error may occur in 1st, 3rd, 5th...positions or 2nd, 4th, 6th,... positions. For example, in a vector of length 8, 1-periodic error vectors are of the type 10101000, 00101000, 0010101, 10101010, 10001010, 01010101, 01000101, 00000101, 00000001 etc.

For s=2, the 2-periodic error vectors are those where error may occur in 1st, 4th, 7th,... positions or 2nd, 5th, 8th,...positions or 3rd, 6th, 9th,... positions. The 2-periodic error vectors may look like 10010010, 10000010, 00010010, 01000001, 01000000, 00001001, etc in a vector of length 8.

For s=3, in a code length 8, the 3-periodic errors are 10001000, 01000100, 00100010, 00010001, 10000000, 01000000 etc.

In what follows a linear code will be considered as a subspace of the space of all n-tuples over GF(q). The distance between two vectors shall be considered in the Hamming sense.

The rest of the paper is organized as follows:
In section 2, we study the linear codes that detect any s-periodic error. We obtain lower and upper bounds on the parity check digits for codes detecting such errors. It is followed by an example of such a code. In second 3, we give a bound (based on Reiger's bound[16]) on codes correcting such errors .

2 Codes detecting s-periodic errors

We consider the linear codes that are capable of detecting any s-periodic error. Clearly, the patterns to be detected should not be code words. In other words we consider codes that have no s-periodic error as a code word. Firstly, we obtain a lower bound over the number of parity-check digits required for such a code. The proof is based on the technique used in theorem 4.13, Peterson and Weldon [15].

Theorem 2.1. *Any (n, k) linear code over GF(q) that detects any s-periodic error must have at least $\left\lceil \dfrac{n}{s+1} \right\rceil$ parity-check digits.*

Proof. The result will be proved on the basis that no detectable error vector can be a code word.

Let V be an (n, k) linear code over GF(q). Consider a set X of all those vectors such that the non-zero components are located at the first position and thereafter a gap of s positions.
We claim that no two vectors of the set X can belong to the same coset of the standard array; else a code word shall be expressible as a sum or difference of two error vectors.
Assume on the contrary that there is a pair, say x_1, x_2 in X belonging to the same coset of the standard array. Their difference viz. x_1-x_2 must be a code vector. But x_1-x_2 is a vector all of whose non-zero components are located at the 1st position or after a gap of s position and so is a member of X, i.e., x_1-x_2 is an s-periodic error, which is a contradiction. Thus all the vectors in X must belong to distinct cosets of the standard array. The number of such vectors over GF(q) is clearly q^p, where $p = \left\lceil \frac{n}{s+1} \right\rceil$.
The theorem follows since there must be at least this number of cosets. □

In the following, an upper bound on the number of check digits required for the construction of a linear code discussed in theorem 2.1 is provided. This bound assures the existence of such a linear code and has been obtained by constructing a matrix under certain constraints. The proof is based on the well known technique used in Varshomov-Gilbert Sacks bound (refer Sacks[17], also theorem 4.7 Peterson and Weldon [15]).

Theorem 2.2. *There exists an (n, k) linear code over GF(q) that has no s-periodic error as a code word provided that*

$$n - k \geq \left\lceil \frac{n}{s+1} \right\rceil.$$

Proof. The existence of such a code will be shown by constructing an appropriate $(n-k) \times n$ parity-check matrix H. The requisite parity-check matrix H shall be constructed as follows.
Select any non-zero $(n-k)$-tuples as the first $j-1$ columns h_1, h_2,..., h_{j-1}; the $j^{th}(j > s+1)$ column h_j is added provided that

$$h_j \neq \textstyle\sum_{i=1}^{p} u_i h_{j-i(s+1)}$$

$$where \quad u_i \in GF(q) \quad and \quad p = \left\lceil \frac{j}{s+1} \right\rceil - 1.$$

This condition ensures that no s-periodic error will be a code word. The number of ways in which the coefficients u_i can be selected is clearly q^p.

At worst, all these linear combinations might yield a distinct sum.

Therefore a column h_j can be added to H provided that

$$q^{n-k} > q^p.$$

or,

$$n - k \geq \left\lceil \frac{j}{s+1} \right\rceil.$$

For a code of length n, replacing j by n gives the result. □

Remark: The above two theorems can be combined as follows:
For detecting s-periodic errors in a linear code of length n, $\left\lceil \frac{n}{s+1} \right\rceil$ *parity check symbols are necessary and sufficient.*

Example 2.1. Consider a (7, 4) binary code with parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This matrix has been constructed by the synthesis procedure, outlined in the proof of Theorem 2.2, by taking $s = 2$ and n =7. It can be seen from Table 1 that the syndromes of the different 2-periodic errors are nonzero, showing thereby that the code that is the null space of this matrix can detect all 2-periodic errors.

Table 1

Error patterns	Syndromes
1000000	100
0001000	010
0000001	001
1001000	110
1000001	101
0001001	011
1001001	111
0100000	110
0000100	011
0100100	101
0010000	101
0000010	111
0010010	010

3 Codes correcting s-periodic errors

The following theorem gives a bound on the number of parity-check digits for a linear code that corrects s-periodic errors. The proof is based on the technique used to establish Reiger's bound[16] (also refer Theorem 4.15, Peterson and Weldon [15]) for correction of s-periodic errors.

Theorem 3.1. *An (n, k) linear code over GF(q) that corrects all t-periodic errors, $t = 2s + 1$ must have at least $\left\lceil \dfrac{n}{s+1} \right\rceil$ parity-check digits.*

Proof. Any vector that has the form of an s-periodic error can be expressible as a sum or difference of two vectors, each of which is an t-periodic error. These component vectors must belong to different cosets of the standard array because both such errors are correctable errors. Accordingly, such a vector viz. s-periodic error can not be a code vector. In view of Theorem 2.1, such a code must have at least $\left\lceil \dfrac{n}{s+1} \right\rceil$ parity-check digits.

Acknowledgement

The authors are very much thankful to Prof. B. K. Dass, Department of Mathematics, University of Delhi for his valuable suggestions, revising the contents and bringing the paper to the current form.

References

[1] N. M. Abramson, A class of systematic codes for non-independent errors, *IRE Trans. on Information Theory*, IT-5, No. 4(1959), 150-157.

[2] A. A. Alexander, R. M. Gryb and D. M. Nast, Capabilities of the telephone network for data transmission, *Bell System Tech. J.*, Vol. 39, No. 3(1960), 431-476.

[3] L. Berardi, B. K. Dass and R. Verma, On 2-repeated burst error detecting codes, *Journal of Statistical Theory and Practice*, Vol. 3, No. 2(2009), 381-391.

[4] J. D. Bridwell and J. K. Wolf, Burst distance and multiple-burst correction, *Bell System Tech. J.,* Vol. 49(1970), 889-909.

[5] R. T. Chien and D. T. Tang, On definitions of a burst, *IBM Journal of Research and Development,* Vol. 9, No. 4(1965), 292-293.

[6] B. K. Dass and S. Madan, Repeated Burst Error Locating Linear Codes, *Discrete Mathematics, Algorithms and Applications*, Vol. 2, No. 2(2010), 181-188.

[7] B. K. Dass, and R. Verma, Repeated Burst Error Detecting Linear Codes, *Ratio Mathematica Journal of Applied Mathematics*, Vol. 19(2009), 25-30.

[8] P. Fire, A class of multiple-error-correcting binary codes for non-independent errors, *Sylvania Report RSL-E-2, Sylvania Reconnaissance Systems Laboratory,* Mountain View, Calif,(1959).

[9] R. W. Hamming, Error-detecting and error-correcting codes, *Bell System Technical Journal,* Vol. 29(1950), 147-160.

[10] M. J. E. Golay, Notes on digital coding, *Proc. IRE*, Vol. 37(1949), 657.

[11] M. J. E. Golay, Binary Coding, *IRE Trans.*, PGIT-4(1954), 23-28.

[12] R. C. Golay and D. K. Ray-Chaudhuri, On a class of Error Correcting Group Codes, *Inf. and Control,* Vol. 3(1960), 68-79.

[13] R. C. Golay and D. K. Ray-Chaudhuri, Further Results on Error Correcting Binary Group Codes, *Inf. and Control*, Vol. 3(1960), 279-290.

[14] A. Hocquenghem, Codes Corecteurs d'Erreurs, *Chiffres,* Vol. 2(1959), 147-156.

[15] W. W. Peterson and E. J. Weldon(Jr.), *Error-Correcting Codes*, 2nd edition, The MIT Press, Mass, 1972.

[16] S. H. Reiger, Codes for the correction of clustered errors, *IRE Trans. Inform. Theory,* IT-6(1960), 16-21.

[17] G. E. Sacks, Multiple error correction by means of parity-checks, *IRE Trans. Inform. Theory IT,* 4(1958), 145 - 147.

[18] B. D. Sharma, B. Rohtagi, Some Results on Weights of Vectors Having 2-Repeated Bursts, *Cybernetics and Information Technologies*, Vol. 11, No. 1(2011), 36-44.

[19] J. J. Stone, Multiple burst error correction, *Information and Control,* Vol. 4(1961), 324-331.

[20] V. Tyagi and P. K. Das, s-Alternate Error Correcting Linear Codes, *J. of Combinatorics, Information & System Sciences,* Vol. 35, No. 1-2(2010), 17-26.

[21] www.themcdonalds.net/richard/index.php?title=
Astrophotography_Mounts:_Periodic_Error_Correction.

RATIO MATHEMATICA 22 (2012) 69-84 ISSN:1592-7415

Hypergroups and Geometric Spaces

Maria Scafati Tallini
Viale Ippocrate, 97, 00161-ROMA, Italy.
tallini@mat.uniroma1.it

Abstract

We explain some links between hypergrpoups and geometric spaces. We show that for any given hypergroup it is possible to define a particular geometric space and then a canonical homomorphism between the hypergroup and a group.

Key words: hypergroup, geometric space

2000 AMS subject classifications: 22N22.

1 Hypergroups and their properties

A hypergroupoid is a pair $(G, \circ)$ where G is a non-empty set and $\circ : G \times G \to P'(G)$ is a mapping of $G \times G$ into the set of non-empty subsets of G, denoted as $P'(G)$.

A semihypergroup is a hypergroupoid satisfying the following associative property:

$$\forall x, y, z \in G,\ (x \circ y) \circ z = x \circ (y \circ z), \tag{1}$$

where the left hand side of (1) is the set $(x \circ y) \circ z = \bigcup_{u \in x \circ y} u \circ z$ and the right hand side is the set $x \circ (y \circ z) = \bigcup_{v \in y \circ z} x \circ v$. The associative property means that the two set theoretical unions coincide.

We say that $(G, \circ)$ satisfies the reproducibility property (both left and right), if

$$\forall a, b \in G, \quad \exists x \in G : \ b \in a \circ x \quad \text{and} \quad \exists y \in G : \ b \in y \circ a. \tag{2}$$

If (2) is satisfied, the family $B_2 = \{a \circ b : a, b \in G\}$ is a covering of G (the index 2 under B, means that we consider the hyper product of two elements of G).

A hypergroup is an associative hypergroupoid satisfying the reproducibility property.

We remark that a hypergroup $(G, \circ)$ having a single valued product (that is, such that $\forall x, y \in G$, $|x \circ y| = 1$) is a group. This is because the following result holds:

Theorem 1.1. *An associative groupoid* $(G, \circ)$ *is a group if, and only if,*

$$\forall a, b \in G, \quad \exists x \in G : \ ax = b \quad \textit{and} \quad \exists y \in G : \ ya = b. \tag{3}$$

((3) is also called right and left quotient axiom).

Proof. If $(G, \circ)$ is a classical group, (3) obviously holds. Assume that the associative groupoid $(G, \circ)$ satisfies (3). Let us proove that it is a group. Fix $a \in G$, let u be one of the elements $z \in G$ such that: $az = a$ (see (3)). For any $c \in G$, there is $y \in G$ such that $ya = c$. Then we have $au = a \Longrightarrow y(au) = (ya)u = ya \Longrightarrow cu = (ya)u = ya = c \Longrightarrow cu = c$. Hence,

$$\forall c \in G, \ cu = c. \tag{4}$$

Similarly, by (3), we prove that there exists $v \in G$ such that:

$$\forall c \in G, \ vc = c. \tag{5}$$

By (4), for $c = v$ and by (5), for $c = u$, we get $vu = v$, $vu = u$, that is $v = u$ and then $\forall c \in G, \ uc = cu = c$. Therefore the unity of $(G, \circ)$ exists and it is unique.

For any $a \in G$, there is at least an element $a' \in G$ and $a'' \in G$ such that: $aa' = u = a''a$, (see 3). Then $a' = ua' = (a''a)a' = a''(aa') = a''u = a''$, that is in G there is an element $a'(= a'')$, such that $a' = a'a = u$. Such an element a' is obviously unique and it is the inverse, a^{-1}, of a. Then $(G, \circ)$ is a classical group and the theorem is proved. □

A substructure of the hypergroup $(G, \circ)$ is a subset $H (\neq \emptyset)$ such that $\forall x, y \in H, \ x \circ y \in H$. The pair $(H, \circ)$ is a semihypergroup, if it satisfies (1). In particular it is a hypergroup if it satisfies also (2).

Let $\mathcal{F}$ be the set of all the substructures of $(G, \circ)$. Two cases may occur.

$$\bigcap_{T \in \mathcal{F}} T \notin \emptyset \qquad \text{or} \qquad \bigcap_{T \in \mathcal{F}} T = \emptyset.$$

Set $\mathcal{S} = \mathcal{F}$ in the first case and $\mathcal{S} = \mathcal{F} \cup \{\emptyset\}$ in the second. In both cases it is: $G \in \mathcal{S}$ and $\forall i \in I, \ T_i \in \mathcal{S} \Longrightarrow \bigcap_{i \in I} T_i \in \mathcal{S}$, where I is a non empty set of indices. In this way, $\mathcal{S}$ is a closure system of G. Hence, for any $X \subseteq G$, the closure $\bar{X}$ of X in G is:

$$X = \bigcap_{\substack{T \in \mathcal{S}, \\ T \subseteq X}} T$$

If $\bar{X} = \emptyset$, $\bar{X}$ is the least (from the set theoretical perspective) substructure containing X. So the following closure operator is defined as follows.

$$\bar{}: X \subseteq G \longrightarrow \bar{X} \in \mathcal{S}. \tag{6}$$

Note that this closure operator satisfies the following properties:

$$X \subseteq \bar{X}; X \subseteq S,\ S \in \mathcal{S} \implies \bar{X} \subseteq S; X = \bar{X} \iff X \in \mathcal{S};$$

$$\forall\, X \subseteq G,\ \bar{X} = \bar{\bar{X}}; \quad \forall\, X, Y \subseteq G,\ X \subseteq Y \implies \bar{X} = \bar{Y};$$

$$\forall\, i \in I,\ X_i \subseteq G \implies \bigcup_{i \in I} \bar{X}_i \subseteq \overline{\bigcup_{i \in I} X_i} \quad \text{and} \quad \overline{\bigcap_{i \in I} X_i} \subseteq \bigcap_{i \in I} \bar{X}_i.$$

For all $X \subseteq G$, we define:

$$X \text{ independent } \overset{\text{def}}{\iff} \forall\, x \in X,\ x \notin \overline{X \smallsetminus \{x\}}, \tag{7}$$

$$X \text{ dependent } \overset{\text{def}}{\iff} \in x \in X,\ x \in \overline{X \smallsetminus \{x\}}, \tag{8}$$

$$X \text{ generator } \overset{\text{def}}{\iff} \bar{X} = G, \tag{9}$$

$$X \text{ base } \overset{\text{def}}{\iff} X \text{ is independent and } \bar{X} = G. \tag{10}$$

The pair $(G, \circ)$ is finitely generated if, and only if, there is a finite subset X of G such that $\bar{X} = G$. We can easily prove that

Theorem 1.2. *If $(G, \circ)$ has a finite generator X then there is a finite base contained in X.*

A hypergroup $(G, \circ)$ is called *monic* if, and only if, it does not contain any substructure different from $(G, \circ)$. If $(G, \circ)$ is a hypergroup and $n \in \mathbb{N}^+ = \mathbb{N} - \{0\}$ then we let

$$\mathcal{B}_n \overset{\text{def}}{=} \{x_1 \circ x_2 \circ \ldots \circ x_n \in G : (x_1, x_2, \ldots, x_n) \in G^n\}, \tag{11}$$

$$\mathcal{B} \overset{\text{def}}{=} \{\mathcal{B}_n : n \in \mathbb{N}^+\}. \tag{12}$$

We call complete part of the hypergoup $(G, \circ)$ a subset A such that

$$B \in \mathcal{B},\, B \cap A \neq \emptyset \implies B \subseteq A. \tag{13}$$

Obviously, $\emptyset$ and G are complete parts. Furthermore, the union and the intersection of complete parts are complete parts. Moreover the complement of a complete part is a complete part. Hence, the complete parts of $(G, \circ)$ form a topology, where every open set it is also closed. We remark that $(G, \circ)$ is a group if, and only if, every subset is a complete part (getting the discrete topology). We easily prove that

If $(G, \circ)$ is a hypergroup such that $\forall\ x, y \in G, \in B \in \mathcal{B}$ such that $x, y \in B$ then the only complete parts of $(G, \circ)$ are $\emptyset$ and G (getting the trivial topology). (14)

In Section 5, we characterize the complete parts of $(G, \circ)$ and we prove that the intersection of the subhypergroups which are also complete parts is a subhypergroup which is a complete part. We remark that the intersection of two subhypergroups may not be a subhypergroup in general. As a matter of fact, the intersection of two subhypergroups may be even the emptyset; as we will see in an example down below.

We define heart of $(G, \circ)$ and we denote it by ω, the intersection of all the subhypergroups which are also complete parts of $(G, \circ)$; that is, the least subhypergroup complete part of $(G, \circ)$. We easily prove that (see (14)):

$$\forall x, y \in G,\ \exists B \in \mathcal{B}:\ x, y \in B \implies \omega = G. \tag{15}$$

As a matter of fact, by (14), the only complete parts of $(G, \circ)$ are $\emptyset$ and G and the only subhypergroup which is a complete part is G. The following statements hold:

If $G \in \mathcal{B}$ then $\omega = G$.

If $(G, \circ)$ does not contain proper subhypergroups complete parts then $\omega = G$.

If $(G, \circ)$ is monic then $\omega = G$.

We define scalar unity of $(G, \circ)$ an element $u \in G$ such that:

$$\forall a \in G,\ a \circ u = u \circ a = \{a\}.$$

Obviously, if a scalar unity in $(G, \circ)$ exists then it is unique. Moreover, $\{u\}$ is a subhypergroup of $(G, \circ)$ and generally it is not a complete part. We get: $\{u\}$ is a complete part $\iff$ $[\forall a, b \in G : u \in a \circ b \Longrightarrow a \circ b = u]$ $\iff$ $\{u\} = \omega$. In Section 5 we prove that $u \in \omega$ in any case. Finally, we remark that the hypergroups having a scalar unity are rather difficult to discover.

2 Examples of hypergroups

Example 2.1. *Let G be any non-empty set. Define the multivalued product $\forall x, y \in G$, $x \circ y = G$. Then $(G, \circ)$ is a hypergroup which we call trivial. This hypergroup is monic, its complete parts are only $\emptyset$ and G and its heart coincides with G.*

Example 2.2. *Let G be any non-empty set. Define the product $\forall x, y \in G$, $x \circ y = \{x, y\}$. The $(G, \circ)$ is called discrete hypergroup. Every non-empty set is a subhypergroup and then there are subhypergroups whose intersection is the empty set. By (14) the only complete parts of $(G, \circ)$ are $\emptyset$ and G and then $\omega = G$.*

Example 2.3. *Let $(G, \circ)$ be a group and N a normal sugroup of $(G, \circ)$. Set $\forall\ x, y \in G$, $x \circ y = xyN$. Then $(G, \circ)$ is a hypergroup. The condition (2) is obvious (it suffices to set $x = a^{-1}b$ and $y = ba^{-1}$); as regard the associativity, we have:*

$$\forall\ a, b \in G, \quad (a \circ b) \circ c = abNc = abcN = a \circ (b \circ c).$$

Moreover, $\mathcal{B}_n$ coincides with the cosets of N in G and then $\omega = N$ and $(N, \circ)$ is the trivial hypergroup (that is $x \circ y = N$).

Example 2.4. *Let $(G_i, \circ_i)_{i \in I}$ be a family of hypergroups (in particular of groups) such that $|I| \geq 2$, $G_i \cap G_j = \emptyset$, $i \neq j$. Set $G = \bigcup_{i \in I} G_i$ and*

$$\forall x, y \in G, \quad x \circ y \overset{def}{=} \begin{cases} x \circ_i y & \text{if } x, y \in (G_i, \circ_i), \\ G & \text{if } x \in G_i \text{ and } x \in G_j. \end{cases}$$

It is easy to prove that the pair $(G, \circ)$ is a hypergroup. For any $i \in I$ we get that $(G_i, \circ_i)$ is a subhypergroup of $(G, \circ)$ and such hypergroups are two by two disjoint. The only complete parts of $(G, \circ)$ are $\emptyset$ and G and then $\omega = G$.

Example 2.5. *Let $(G, \circ)$ be any group with $|G| \geq 3$. Set:*

$$\forall x, y \in G, \quad x \circ y = G \setminus \{xy\}. \tag{16}$$

Let us prove that $(G, \circ)$ is a hypergroup. We have:

$$\forall x, y, z \in G, \quad (x \circ y) \circ z = \bigcup_{t \in G \setminus \{xy\}} G \setminus \{tz\}. \tag{17}$$

Since $|G| \geq 3$, for any $c \in G$ there is an element t' in G such that $t' \in G \setminus \{xy, cz^{-1}\}$. We have $c \in G \setminus \{t'z\}$, with $t' \in G \setminus \{xy\}$, whence, by

(17), $c \in (x \circ y) \circ z$. Therefore $(x \circ y) \circ z = G$. Similarly, we prove that $x \circ (y \circ z) = G$. So, the associative property of $(G, \circ)$ follows.

Now, let us prove the reproducibility property (2). For any $a, b \in G$ there is $x \in G \smallsetminus \{a^{-1}b\}$ and $y \in G \smallsetminus \{ba^{-1}\}$. So, $b \in G \smallsetminus \{ax\} = a \circ x$, $b \in G \smallsetminus \{ya\} = y \circ a$. This implies (2).

Let us prove that $(G, \circ)$ is monic. Let S be a subhypergroup of $(G, \circ)$ and $a \in S$. We have $a \circ a = G \smallsetminus \{a^2\} \subseteq S$, then $|S| \geq 2$ (because $|G| \geq 3$). Now, let $b \in S$, with $b \neq a$. It is $a \circ b = G \smallsetminus \{ab\} \subseteq S$, hence, if $S \neq G$ then $S = G \smallsetminus \{ab\} = G \smallsetminus \{a^2\}$ which is impossible because $a \neq b$. Therefore, $S = G$ and $(G, \circ)$ is monic.

We easily prove that the only complete parts of $(G, \circ)$ are $\emptyset$ and G, and so, $\omega = G$.

Example 2.6. *Let $G = \mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$, with $x \neq y$, set*

$$\begin{aligned} &x \circ x = \{x\}, \\ &x \circ y = \text{the closed interval whose extremal points are } x \text{ and } y. \end{aligned}$$

We prove that $(\mathbb{R}^n, \circ)$ is a hypergroup. In fact, the reproducibility property is obvious and the associativity holds because $(x \circ y) \circ z$ coincides with the triangle (eventually, degenerate), $T(x, y, z)$, with vertices x, y and z. The same happens for $x \circ (y \circ z)$. And so, $(x \circ y) \circ z = T(x, y, z) = x \circ (y \circ z)$.

In $(\mathbb{R}^n, \circ)$ every convex set is a substructure and viceversa. The open convexes are subhypergroups. Therefore, disjoint subhypergroups exist. Moreover, by (14) the only complete parts are $\emptyset$ and $G = \mathbb{R}^n$, and then $\omega = G$.

Example 2.7. *Let $G = P(d, K)$ be the d-dimensional projective space over the field K. For all $x, y \in P(d, K)$, with $x \neq y$, set*

$$\begin{aligned} &x \circ x = \{x\}, \\ &x \circ y = \text{the line through } x \text{ and } y. \end{aligned}$$

It is easy to prove that $(G, \circ)$ is a hypergroup. In fact, the reproducibility property is obvious and the associativity holds because $(x \circ y) \circ z$ coincides with the subspace spanned by x, y and z. The substructures of $(G, \circ)$ are hypergroups and coincide with the subspaces of $G = P(d, K)$. From (14), the only complete parts of $(G, \circ)$ are $\emptyset$ and $G = P(d, K)$, and then $\omega = G$.

Example 2.8. *Let $G = V_K$ be a vector space over the field K. Set*

$$\forall\ x, y \in G,\ x \circ y = \{a(x + y) :\ a \in K\}.$$

We can prove that $(G, \circ)$ is a hypergroup. The substructure of $(G, \circ)$ are the subspaces of V_K and, hence, they are hypergroups. Moreover (14) is satisfied and the only complete parts of $(G, \circ)$ are $\emptyset$ and $G = V_K$. Hence, $\omega = G$.

Example 2.9. *Let $(G_1, \circ_1)$ and $(G_2, \circ_2)$ be two hypergroups. Set $G = G_1 \times G_2$. Furthermore, if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are two elements of G then $x \circ y \stackrel{def}{=} (x_1 \circ_1 y_1, x_2 \circ_2 y_2$. It is easy to prove that such a $(G, \circ)$ is a hypergroup which we call the cartesian product of $(G_1, \circ_1)$ and $(G_2, \circ_2)$. Similarly, we define the cartesian product of a family of hypergroups $\{(G_i, \circ_i) : i \in I\}$. In particular, we set*

$G = \prod_{i \in I} G_i$,
$x \circ y = \{x_i \circ_i y_i : i \in I\}$, *where* $x = \{x_i : i \in I\}$ *and* $y = \{y_i : i \in I\}$.

We remark that if each $(G_i, \circ_i)$ has a scalar unity u_i then the cartesian product has a scalar unity $u = \{u_i : i \in I\}$.

Example 2.10. *In the examples from 2.1 to 2.8, the hypergroups do not have a scalar unity. Now we give an example of a hypergroup with a scalar unity of order 2. And this is also the simplest hypergroup which is not a group. Consider the set $G = \{u, a\}$ and set*

$$u \circ u \stackrel{def}{=} \{u\},\ u \circ a \stackrel{def}{=} a \circ u \stackrel{def}{=} \{a\},\ a \circ a \stackrel{def}{=} \{u, a\}.$$

It can be easily checked that $(G, \circ)$ is a hypergroup which as u as scalar unity. From (14), the only complete parts of $(G, \circ)$ are $\emptyset$ and G because $a \circ a = G$. Hence, $\omega = G$. The only subhypergroup of $(G, \circ)$ is $\{u\}$ which is not a complete part. The cartesian product of copies of such hypergroup is a wide class of hypergroups with scalar unity.

3 Homomorphisms of hypergroups

Let $(G, \circ)$ and $(G', \circ')$ be two hypergroupoids. We call homomorphism of $(G, \circ)$ and $(G', \circ')$ a mapping $f : G \longrightarrow G'$ such that

$$\forall x, y \in G, f(x \circ y) \subseteq f(x) \circ' f(y). \tag{18}$$

We remark that in general $f(G)$ is a substructure of $(G', \circ')$. Let us prove:

Theorem 3.1. *If $G \longrightarrow G'$ is a homomorphism of $(G, \circ)$ to $(G', \circ')$, for any substructure H' of $(G', \circ')$ such that $f^{-1}(H') \neq \emptyset$ then $H' = f^{-1}(H')$ is a substructure of $(G, \circ)$. Moreover, if $(G, \circ)$ satisfies the reproducibility property, we have:*

$$\forall a', b' \in f(G), \exists x' \in f(G) : b' \in a' \circ' x' \Longrightarrow \exists y' \in G' : b' \in y' \circ' a'. \tag{19}$$

Finally, if $(G, \circ)$ and $(G', \circ')$ are semihypergroups and A' is a complete part of $(G', \circ')$ then $A = f^{-1}(A')$ is a complete part of $(G', \circ')$ and $A = f^{-1}(A')$ is a complete part of $(G, \circ)$.

Proof. Let $x, y \in H$. It is $x' = f(x), y' = f(y) \in H'$ and then $a' \circ' y' \in H'$. By (18) we get:

$$f(x \circ y) \subseteq f(x) \circ' f(y) = x' \circ' y' \subseteq H', \quad \text{where} \quad x \circ y \subseteq f^{-1}(H') = H.$$

Therefore H is a substructure of $(G, \circ)$.

If $(G, \circ)$ satisfies the reproducibility property we get:

$$\forall a', b' \in f(G) \implies \exists a, b \in G : a = f(a), b' = f(b) \implies$$

$$\exists x \in G : b \in a \circ x, \exists y \in G : b \in y \circ a \implies$$

$$\exists x' = f(x) \in f(G) : b' \in f(a \circ x) \subseteq f(a) \circ' f(x) = a' \circ' x' \quad \text{and}$$

$$\exists y' = f(y) \in f(G) : b' = f(y \circ a) \subseteq f(y) \circ' f(a) = y' \circ' a' \implies$$

$$\exists x' \in f(G), b' \in a' \circ' x', \exists y' \in f(G) : b' \in y' \circ' a'.$$

If $(G, \circ)$ and $(G', \circ')$ are semihypergroups, that is the associativity property holds, for any complete part A' of $(G', \circ')$, setting $A = f^{-1}(A')$, we get, since $f(A) = A'$ and setting $x'_i = f(x_i)$, $x_i \in G_i$:

$$(x_1 \circ x_2 \circ \ldots \circ x_n) \cap A \neq \emptyset \implies f(x_1 \circ x_2 \circ \ldots \circ x_n) \cap f(A) \neq \emptyset \implies$$

$$\emptyset \neq f(x_1 \circ x_2 \circ \ldots \circ x_n) \cap f(A) \subseteq (x'_1 \circ' x'_2 \circ' \ldots \circ' x'_n) \subseteq A' \implies$$

$$(x_1 \circ x_2 \circ \ldots \circ x_n) \subseteq f^{-1}(x'_1 \circ' x'_2 \circ' \ldots \circ' x'_n) \subseteq A \implies (x_1 \circ x_2 \circ \ldots \circ x_n) \subseteq A,$$

therefore, A is a complete part of $(G, \circ)$ and the theorem is proved. □

A homomorphism is called strong if in (18) the equality holds. Obviously, if $(G', \circ')$ is a groupoid (that is, if the operation $\circ'$ is single valued) then every homomorphism between any two hypergroupoids $(G, \circ)$ and $(G', \circ')$ is strong. Now, let us prove

Theorem 3.2. *Let f be a strong homomorphism between the hypergroupoid $(G, \circ)$ and $(G', \circ')$. Then*

$$\mathrm{Im}(f) = f(G) \text{ is a hypergroup}; \tag{20}$$

$$\text{If } H \text{ is a substructure of } (G, \circ) \text{ then } f(H) \text{ is a substructure of } f(G); \tag{21}$$

$$\text{If } H \text{ is a subhypergroup of } (G, \circ) \text{ then } f(H) \text{ is a subhypergroup of } f(G); \tag{22}$$

$$\text{If } H' \text{ is a substructure of } f(G) \text{ then } f^{-1}(H') \text{ is a substructure of } (G, \circ); \tag{23}$$

$$\text{If } A' \text{ is a complete part of } (G', \circ') \text{ then } f^{-1}(A') \text{ is a complete part of } (G, \circ). \tag{24}$$

Proof. Since f is a strong homomorphism, if H is a substructure of $(G,\circ)$ then $\forall x',y' \in f(H) \Longrightarrow \exists x,y \in H$: $x' = f(x)$, $y' = f(y) \Longrightarrow x \circ y \subseteq H$, $x' \circ' y' = f(x) \circ' f(y) = f(x \circ y) \subseteq f(H) \Longrightarrow x' \circ' y' \subseteq f(H) \subseteq f(G)$, that is

$$\text{If } H \text{ is a substructure of } (G,\circ) \text{ then } f(H) \text{ is a substructure of } (G',\circ'). \tag{25}$$

From (25) with $H = G$, we have that $f(G)$ is a substructure of $(G',\circ')$, that is $f(G)$ is closed with respect to the product. Furthermore, from Theorem 3.1 (second part), we have that $(f(G),\circ')$ satisfies the reproducibility property.

Let us prove that $(f(G),\circ')$ is associative. In fact, $\forall x',y',z' \in f(G)$ $\exists x,y,z \in G$: $x' = f(x)$, $y' = f(y)$, $z' = f(z) \Longrightarrow (x'\circ' y')\circ' z' = f((x\circ y)\circ z) = f(x \circ (y \circ z)) = x' \circ' (y' \circ' z')$; and so, $(f(G),\circ')$ is a hypergroup. Hence, the property (20) holds. The property (21) follows from (25) and (20). The property (22) follows from (21) and Theorem 3.1 (second part). The property (23) is trivial. The property (24) follows from Theorem 3.1 (third part). Hence, the Theorem is proved. $\square$

Theorem 3.3. *Let $(G,\circ)$ be a hypergroup, ω be the heart of $(G,\circ)$, $(G',\circ)$ be a group and u be the unity of $(G',\circ)$. If $f : G \longrightarrow G'$ is a homomorphism (necessarily, strong) of $(G,\circ)$ in $(G',\circ)$ then*

1. *If H' is a subgroup of $(G',\circ)$ then $f^{-1}(H')$ is a subhypergroup of $(G,\circ)$;*
2. *$\forall A' \subseteq G'$, $f^{-1}(A')$ is a complete part of $(G,\circ)$;*
3. *$\forall x' \in f(G)$, $f^{-1}(x')$ is a complete part ($\neq \emptyset$) of $(G,\circ)$;*
4. *$\forall B \in \mathcal{B}$ (see (12), $|f(B)| = 1$;*
5. *$\omega \subseteq f^{-1}(u)$.*

Proof. If H' is a subgroup of G', from (20), we have that $H' \cap f(G)$ is a subgroup of $f(G)$ and hence, from (23), $f^{-1}(H' \cap f(G)) = f^{-1}(H')$ is a substructure of $(G,\circ)$. Let us prove that (2) holds for $f^{-1}(H')$. We have that:

$$\forall a,b \in f^{-1}(H'),\ \exists x,y \in G\colon\ b \in a \circ x,\ b \in y \circ a \ \Longrightarrow$$

$$f(a), f(b) \in H',\ f(b) = f(a) \cdot f(x) = f(y) \cdot f(a) \ \Longrightarrow$$

$$f(x) = (f(a))^{-1} \cdot f(b) \in H',\ f(y) = f(b) \cdot (f(a))^{-1} \in H' \ \Longrightarrow$$

$$\exists x,y \in f^{-1}(H')\colon\ b \in a \circ x,\ b \in y \circ a.$$

This implies that (2) holds for $f^{-1}(H')$. Hence (3.3,1) holds.

Since every subset A' of the group G' is a complete part in G', from Theorem 3.1 (third part), we have that $(3.3, 2)$ and $(3.3, 3)$ hold.

For all $B = (x_1 \circ x_2 \circ \ldots \circ x_n) \in \mathcal{B}$, let $x \in B$. If $x' \stackrel{\text{def}}{=} f(x) \in f(B)$ then $x \in f^{-1}(x')$ and hence $b \cap f^{-1}(x') \neq \emptyset$. But, $f^{-1}(x')$ is a complete part of $(G, \circ)$, hence $B \subseteq f^{-1}(x')$, and so $f(B) = \{x'\}$ and $|f(B)| = 1$. This implies $(3.3, 4)$.

The set $\{u\}$ is a subgroup of G', so from $(3.3, 1)$, $f^{-1}(u)$ is a subhypergroup of $(G, \circ)$ which is also a complete part because of $(3.3, 3)$. This implies $\omega \subseteq f^{-1}(u)$ because ω is the intersection of all the subhypergroups complete parts of $(G, \circ)$. This implies $(3.3, 5)$ and hence the theorem. □

.

The composition of two homomorphisms is a homomorphism, likewise the composition of two strong homomorphisms is a strong homomorphism. Furthermore the identity is a strong homomorphism. Hence, the hypergroups form a category with respect to the homomorphisms and a category with respect to the strong homomorphisms which is a subcategory of the first category.

4 Geometric spaces

A geometric space is a pair $(P, \mathcal{B})$, where P is a non-empty set, whose elements we call points and $\mathcal{B}$ is a family of parts of P, whose elements we call blocks. The set P is called the support of the geometric space and $\mathcal{B}$ is called the geometric structure. Let $(P, \mathcal{B})$ and $(P', \mathcal{B}')$ be two geometric spaces. We call isomorphism between them a bijection $f\colon P \to P'$, such that

$$\forall B \in \mathcal{B},\ f(B) \in \mathcal{B}', \quad \forall B' \in \mathcal{B}',\ f^{-1}(B') \in \mathcal{B}.$$

The composition of two isomorphisms is an isomorphism and the identity is an isomorphism. It follows that within the geometric spaces the relation of isomorphism is an equivalence relation. So, we study the equivalence classes of such spaces. The isomorphisms of $(P, \mathcal{B})$ onto itself are called automorphisms. The automorphisms of $(P, \mathcal{B})$ form a group under the composition which is called $Aut(P, \mathcal{B})$. This gives rise to a geometry of the geometric space $(P, \mathcal{B})$. More precisely, two subsets F and F' of $(P, \mathcal{B})$ are called "equal", if there is an automorphism of $(P, \mathcal{B})$ which changes F onto F'. Such an equality relation is an equivalence relation. The geometry of $(P, \mathcal{B})$ is the study of the properties of the subsets of $(P, \mathcal{B})$ which are invariant under the group $Aut(P, \mathcal{B})$. Two geometric spaces $(P, \mathcal{B})$ and $(P', \mathcal{B}')$ are called equivalent,

if, and only if,
$$Aut(P, \mathcal{B}) = Aut(P', \mathcal{B}'); \tag{26}$$
that is, if the geometry determined by $Aut(P, \mathcal{B})$ coincides with that arising from $Aut(P', \mathcal{B}')$.

We remark that, given a geometric space $(P, \mathcal{B})$, if $\mathcal{B}'$ consists of the complements of the elements of $\mathcal{B}$ in P, then $(P, \mathcal{B})$ and $(P, \mathcal{B}')$ are ,because of (26), two distinct geometric spaces, which have the same geometry.

Example 4.1. *Let $(S, \mathcal{A})$ be a topological space whose open sets are such that $\emptyset \in \mathcal{A}$, $S \in \mathcal{A}$, $A_{ii \in I}, A_i \in \mathcal{A} \implies \bigcup_{i \in I} A_i \in \mathcal{A}$, and $A_1, A_2 \in \mathcal{A} \implies A_1 \cap A_2 \in \mathcal{A}$.*

The complements of the open sets are the closed sets of the topology. We denote by $\mathcal{C}$ the family of the closed sets. The two structures $(P, \mathcal{A})$ and $(P, \mathcal{C})$ are two distinct geometric spaces, admitting the same geometry.

Example 4.2. *Let $\mathcal{R}$ be the family of the lines of the real plane $\mathbb{R}^2$. The pair $(\mathbb{R}^2, \mathcal{R})$ is a geometric space which is called real affine plane. The group $Aut(\mathbb{R}^2, \mathcal{R})$ is called the group of the affinities of $(\mathbb{R}^2, \mathcal{R})$ and we prove that every affinity is an invertible linear transformation; that is:*

$$x' = ax + by + c, \quad y' = a_1x + b_1y + c_1, \text{ with } ab_1 - a_1b \neq 0.$$

Example 4.3. *Let $\mathcal{C}$ be the family of the circles of $\mathbb{R}^2$. The pair $(\mathbb{R}^2, \mathcal{C})$ is a geometric space. An automorphism of such a space is a bijection which changes circles to circles and therefore changes also lines to lines (because it changes three collinear points to three collinear points, and conversely). It follows that an automorphism of $(\mathbb{R}^2, \mathcal{C})$ is an affinity which changes circles to circles and therefore it is a similitude. It follows that $Aut(\mathbb{R}^2, \mathcal{C})$ is the group of the similitudes of the real plane and the geometry of $(\mathbb{R}^2, \mathcal{C})$ is the similitude geometry.*

Example 4.4. *Let $\mathcal{C}_1$ be the family of the circles with radius 1 in the real plane $\mathbb{R}^2$. The automorphisms of the geometric space $(\mathbb{R}^2, \mathcal{C}_1)$ is the set of the bijections changing the circles of radius 1 to themselves, We can prove that such bijections change circles to circles and, then, lines to lines; hence, $Aut(\mathbb{R}^2, \mathcal{C}_1)$ is the group of the movements in the plane and the geometry of $(\mathbb{R}^2, \mathcal{C}_1)$ is the euclidean geometry.*

Example 4.5. *Let K be a field and $P(r, K)$ be the r-dimensional projective space over the field K. Its points are the $(r+1)$-tuples not all zero in K, defined up to a non-zero multiplicative factor. The lines consist of those* *points* $x = (x_0, x_1, \ldots, x_r) \in P(r, K)$ *each of which is the linear combination of two distinct fixed points* $y = (y_0, y_1, \ldots, y_r)$, $z = (z_0, z_1, \ldots, z_r) \in P(r, K)$*:*

$$x = \lambda y + \mu z \iff x_i = \lambda y_i + \mu z_i, \qquad i = 0, 1, \ldots, r$$

Let $\mathcal{L}$ *be the family of the lines of* $P(r, K)$*. The geometric space* $(P(r, K), \mathcal{L})$ *is the* r*-dimensional projective space over the field* K*. We prove that the automorphisms of such a space are of the form*

$$x'_i = \sum_{j=0}^{r} a_{ij} \vartheta x_j, \qquad i = 0, 1, \ldots, r,$$

where $\det(a_{i,j}) \neq 0$ *and* ϑ *is a collineation (that is, an automorphism* $\vartheta : K \to K$*. The geometry of* $(P(r, K), \mathcal{L})$ *is called projective geometry.*

Let $(P, \mathcal{B})$ be any geometric space. An n-tuple of blocks $(B_1, B_2, \ldots, B_n)$ is called polygonal of $(P, \mathcal{B})$ if, and only if, $B_i \cap B_{i+1} \neq \emptyset$, $i = 1, 2, \ldots, n-1$.

Assume that $\mathcal{B}$ is a covering of P. In P the following relation γ is defined (connectedness by polygonals):

$$\forall\, x, y \in P, \qquad x \,\gamma\, y \iff \begin{array}{l} \text{there is a polygonal } (B_1, B_2, \ldots, B_n) \\ \text{such that } x \in B_1 \text{ and } y \in B_n. \end{array} \tag{27}$$

The relation γ is an equivalence. In fact, γ is reflexive because $\mathcal{B}$ is a covering of P, and it is also obviously symmetric and transitive. For any $x \in P$, the equivalence class $\gamma(x)$ of x is the union of the polygonals through x and it is called connected component of x. If $\gamma(x) = P$ then the space $(P, \mathcal{B})$ is connected by polygonals. Note that, in any geometric space $(P, \mathcal{B})$, for all given $x \in P$:

$$\forall\, B \in \mathcal{B}, \qquad B \cap \gamma(x) \neq \emptyset \implies B \subseteq \gamma(x).$$

This implies that if $\mathcal{B}_{\gamma(x)}$ indicates the family of blocks contained in $\gamma(x)$ then the pair $(\gamma(x), \mathcal{B}_{\gamma(x)}$ is a connected geometric space. This space is called connected component of $(P, \mathcal{B})$. Note that $(P, \mathcal{B})$ is the disjoint union of its connected components.

Example 4.6. *Let* Ω *be a non-empty open set in* $\mathbb{R}^n$ *and let* $\mathcal{B}$ *be the family of the closed segments contained in* Ω*. Consider the geometric space* $(\Omega, \mathcal{B})$*. Actually, every classical polygonal is a polygonal according to the above definition. Conversely, every above polygonal contains a classical polygonal. Hence, the connected components of* Ω *in the classical sense coincide with the connected components of* Ω *previously defined.*

Example 4.7. *Let (V, E) be a graph. This is a geometric space $(P, \mathcal{B})$ in which $P = V$ and $\mathcal{B} = V \cup E$. Note that every block has cardinality which is less than or equal to 2. The classical notion of polygonal (or, path) of a graph coincide with the above definition of polygonal. Also, the connected components of (V, E) according to the classical definition coincide with the connected components of $(P = V, \mathcal{B} = V \cup E)$.*

Example 4.8. *A semilinear space $(P, \mathcal{L})$, where the elements of $\mathcal{L}$ are called lines, is a geometric space such that every line has at least two points and through two distinct points there is at most a line. For instance, every graph without loops is a semilinear space. Another example is given by any ruled algebraic variety of $P(r, K)$. The notion of polygonal actually coincide with the classical one (a n-tuple of lines $(l_1, l_2, \ldots, l_n)$ such that $l_i \cap l_{i+1} \neq \emptyset$) and then the notion of connected component of a semilinear space just given, coincide with the classical one.*

5 Hypergroups and geometric spaces

Let $(G, \circ)$ be a hypergroupoid and let $\mathcal{B}$ be the family of parts of G consisting of all the hyper products of more than one element in G (see (11) and (12). Then the geometric space $(P = G, \mathcal{B})$ remains defined. If in $(G, \circ)$ the reproducibility property (2) holds then $\mathcal{B}$ is a covering of G. If $x, y \in G$, x is in relation τ with y if, and only if, there is an element $B \in \mathcal{B}$ containing x and y. Equivalently, $x \,\tau\, y \overset{\text{def}}{\iff}$ there exists an hyper product containing both x and y. The relation τ is reflexive because $\mathcal{B}$ is a covering of G and obviously symmetric. However, τ may not be transitive in general.

We recall that a relation ρ defined in G can be regarded as a subset (called graph of ρ) of the cartesian product $G \times G$. This implies that it is possible to define a partial ordering relation in the set of all the relations defined in G given by the usual set-theoretical inclusion. Moreover, if $\{\rho_i : \ i \in I\}$ is a family of equivalence relations in G then $\rho \overset{\text{def}}{=} \bigcap_{i \in I} \rho_i$ is an equivalence relation defined in G. This is because if $a, b, c \in G$, then

$$a \,\rho\, b \quad \iff \quad a \,\rho_i\, b,\ \forall\ i \in I;$$

and so, ρ is reflexive, symmetric, and transitive. This implies that the equivalences in G form a closure system (also because the relation whose graph is $G \times G$ is an equivalence relation). Now, let τ^* be the intersection of all the equivalences containing τ. Note that τ^* is the smallest equivalence relation (and hence, transitive) which contains the possibly non-transitive relation τ. For this reason, τ^* is called transitive closure of τ. As a matter of fact, if G

is an hypergroup then $\tau^* = \tau$ as the following theorem states; and hence, τ is transitive.

Theorem 5.1. *Let G be a hypergroup,$N^* = N - 0$, z be an element of the heart ω of G and*

$$P(z) \stackrel{def}{=} \{A \in P'(G) : \ z \in A, \ \exists m \in N^*, \ \exists (a_1, a_2, ..., a_m) \in G^m : \ A = \prod_{j=1}^{m} a_j\}.$$

Set $M = \bigcup_{a \in P(z)} A$, then $\omega = M$, $\tau^* = \tau$.

Proof. First we prove that M is a complete part of G. Let $(z_1, z_2, ..., z_n) \in G^n$ such that $\prod_{i \in I} z_\cap M \neq \emptyset$. If $a \in \prod_{i \in I} z_i \cap M$ then a product $A \in P(z)$ such that $a \in A$ exists. From the reproducibility property of G, a pair (w, b) of elements of G exists such that $z_n \in wz$ and $z \in ab$. Hence:

$$z \in ab \subset \prod_{i=1}^{n-1} z_i b = \prod_{i=1}^{n-1} z_i z_n b \subset \prod_{i=1}^{n-1} z_i wzb \subset \prod_{i=1}^{n-1} z_i wAb;$$

and so $\prod_{i=1}^{n-1} z_i wAb \subset M$. It then follows that:

$$\prod_{i=1}^{n} z_i = \prod_{i=1}^{n-1} z_i z_n \subset \prod_{i=1}^{n-1} z_i wz \subset \prod i = 1^{n-1} z_i wab \subset \prod_{i=1}^{n-1} z_i wAb \subset M,$$

and therefore M is a complete part of G.

For any non-empty subset A of G,we denote by $C(A)$ the complete closure of A; that is, the intersection of all the complete parts of G containing A. Now, since $z \in \omega \cap M$ and M is a complete part of G, we have $\omega = C(z) \subset C(M) = M$. Moreover, for any product A of $P(z)$, it is $z \in \omega \cap A$ and, as ω is a complete part of G, then $A \subset \omega$ and consequently the inclusion $M \subset \omega$ holds; hence, $M = \omega$.

Now, let $x \, \tau^* \, y$. We have $x \in C(y) = y\omega = yM$ and so a product $A \in P(z)$ exists such that $x = yA$. By the reproducibility property of ω, there exists $b \in \omega$ such that $z \in bz$. Since $b \in \omega = M$, there exist a product $B \in P(z)$ such that $b \in B$. By the reproducibility property of G, there is $v \in G$ such that $y = vz$. Now, from $z \in A$, $\{b, z\} \subset B$, $y \in vz$, $z \in bz$ and $x \in yA$, we get:

$$x \in yA \subset vzA \subset vbzA \subset vbBA, \text{ and } y \in vz \subset vbz \subset vbbz \subset vbBA.$$

Consequently, it follows $\{x, y\} \subset vbBA$ and then $x \, \tau \, y$. So $\tau = \tau^*$. We note that a somewhat similar argument can be found in [1]. $\square$

Note that, the above theorem does not hold in the more general case of G being a semihypergroup, as the following example shows.

Example 5.1. *Let G be a set such that $|G| \geq 4$ and let a, b, c, d be four distinct elements of G. Let $a \circ a = \{b, c\}$ and, for any pair $(x, y) \in G \times G$, with $(x, y) \neq (a, a)$, let $x \circ y = \{b, d\}$. It can be easily shown that the set G equipped with the hyperproduct just defined is a semihypergroup such that, for all $n \in \mathbb{N}$, $n \geq 3$ and for all $(x_1, x_2, \ldots, x_n) \in G^n$, $\prod_{i=1}^{n} x_i = \{b, d\}$. Furthermore, $c\,\tau\,b$ because $a \circ a = \{b, c\}$, and $b\,\tau\,d$ because, say, $a \circ b = \{b, d\}$. This implies $c\,\tau^*\,b$. However, $c\,\tau\,d$ does not hold.*

If ρ is an equivalence relation in G containing τ (that is, such that $x\,\tau\,y \Longrightarrow x\,\rho\,y$) then ρ contains the connectedness by polygonals relation γ defined in (27); that is, $\rho \supseteq \tau \Longrightarrow \rho \supset \gamma$. In fact, since $\rho \supseteq \tau$ (that is, $x\,\tau\,y \Longrightarrow x\,\rho\,y$), if $x\,\gamma\,y \Longrightarrow \exists\ (B_1, B_2, \ldots, B_m)$: $B_i \in \mathcal{B}$, $B_i \cap B_{i-1} \neq \emptyset$, $x \in B_1$ and $y \in B_m \Longrightarrow x\,\tau\,x_1$, $x_1 \in B_2 \cap B_1$, $x_1\,\tau\,x_2$, $x_2 \in B_3 \cap B_2$, $\ldots$, $x_{m-1}\,\tau\,y$, $x_1 \in B_{m-1} \cap B_m \neq \emptyset \Longrightarrow x\,\rho\,x_1$, $x_1\,\rho\,x_2$, $\ldots$, $x_{m-1}\,\rho\,y \Longrightarrow x\,\rho\,y$ because ρ is an equivalence and, hence, it is transitive. It then follows that

$$\gamma = \tau^*. \tag{28}$$

Note that $\forall\ x, y \in G$, $x \circ y \in \mathcal{B}$. Hence, all the elements of $x \circ y$ belong to the same connected component of $(G, \mathcal{B})$ which we denote by $\gamma(x \circ y)$. If $x\,\tau\,x'$ and $y\,\tau\,y'$ then $x, x' \in B_1 \in \mathcal{B}$ and $y, y' \in B_2 \in \mathcal{B}$, then $x \circ y \subseteq B_1 \circ B_2$ and $x' \circ y' \subseteq B_1 \circ B_2$ and then $\gamma(x \circ y) = \gamma(x' \circ y')$. This proves,

$$x\,\tau\,x' \text{ and } y\,\tau\,y' \implies \gamma(x \circ y) = \gamma(x' \circ y'). \tag{29}$$

In the set $G/\gamma = G/\tau^*$ (see (28)) of the connected components of $(G, \mathcal{B})$ it is then possible to define the following single valued product.

$$\forall\ \gamma(x), \gamma(y) \in G/\gamma, \text{ where } x, y \in G, \qquad \gamma(x) \cdot \gamma(y) \overset{\text{def}}{=} \gamma(x \circ y). \tag{30}$$

The pair $(G/\gamma, \cdot)$ just defined is a groupoid. Furthermore, the mapping

$$\begin{array}{rcl} \varphi : (G, \circ) & \to & G/\gamma \\ x & \to & \gamma(x), \end{array} \tag{31}$$

is a surjective homomorphism because of (30). This implies that $(G/\gamma, \cdot)$ is a group, because the associativity and (2) hold in G. Let u be the unity of $(G/\gamma, \cdot)$. It can be proved that every complete part A of $(G, \circ)$ is the counterimage through φ defined in (31) of a subset of $(G/\gamma, \cdot)$, and so A is the union of connected components. Moreover, the image under φ of every subhypergroup complete part of $(G, \circ)$ is a subgroup of $(G/\gamma, \cdot)$, and viceversa.

This implies that the intersection of a set of some subhypergroups complete parts of $(G, \circ)$ is a subhypergroup complete part of $(G, \circ)$. Now, since the heart ω of $(G, \circ)$ is the intersection of all the subhypergroups complete parts of $(G, \circ)$ it follows that $\omega = \varphi^{-1}(u)$.

References

[1] Freni, D.: *Une note sur le coeur d'un hypergroupe et sur la cloture transitive β^* de β*, Riv. Mat. Pura e Applicata Univ. Udine, n. 8, (1991), 153–156.

[2] Corsini, P.: *Prolegomena of Hypergroup Theory*, Aviani Editore, Udine, 1986.

[3] Scafati Tallini, M.: *A-ipermoduli e spazi ipervettoriali*, Riv. Mat. Pura e Applicata Univ. Udine, n. 3, (1988), 39-48.

[4] Scafati Tallini, M.: *Hypervector spaces*, Proc. Fourth Int: Congress on "Algebraic Hyperstructures and Applications", Xanthi, Greece, (1990), 167-174.

[5] Scafati Tallini, M.: *Sottospazi, spazi quozienti ed omomorfismi tra spazi ipervettoriali*, Riv. Mat. Pura e Applicata Univ. Udine,n. 18, (1996), 71-84.

Contents

B. Cavallo and L. D'Apuzzo - About a definition of metric over anabelian linearly ordered group	pag.	3
R. A. Borzooei, A. Rezazadeh and R. Ameri - Classifications of Hyper Pseudo BCK-algebras of Order 3	pag.	13
A. Taghavi and R. Hosseinzadeh - Operators on Weak Hypervector Spaces	pag.	37
M. Asghari-Larimi and I.Cristea - Atanassov's intuitionistic fuzzy index of hypergroupoids	pag.	45
P. Kumar Das and V. Tyagi - Codes on *s*-periodic errors	pag.	61
M. Scafati Tallini - Hypergroups and Geometric Spaces	pag.	69