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Studies on the classical determinism predicted by A. Einstein, B. Podolsky and N. Rosen

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Abstract

In this paper, we continue the study initiated in preceding works of the argument by A. Einstein, B. Podolsky and N. Rosen according to which quantum mechanics could be "completed" into a broader theory recovering classical determinism. By using the previously achieved isotopic lifting of applied mathematics into isomathematics and that of of quantum mechanics into the isotopic branch of hadronic mechanics, we show that extended particles appear to progressively approach classical determinism in the interior of hadrons, nuclei and stars, and appear to recover classical determinism at the limit conditions in the interior of gravitational collapse.

Keywords: EPR argument, isomathematics, isomechanics.
2010 AMS subject classifications: 05C15, 05C60. [1]

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1. Introduction

1.1. The EPR argument

As it is well known, Albert Einstein did not consider quantum mechanical uncertainties to be final, for which reason he made his famous quote "God does not play dice with the universe."

More particularly, Einstein accepted quantum mechanics for atomic structures and other systems, but believed that quantum mechanics is an "incomplete theory," in the sense that it could be broadened into such a form to recover classical determinism at least under limit conditions.

Einstein communicated his views to B. Podolsky and N. Rosen and they jointly published in 1935 the historical paper [1] that became known as the *EPR argument.*

Soon after the appearance of paper [1], N. N. Bohr published paper [2] expressing a negative judgment on the possibility of "completing" quantum mechanics along the lines of the EPR argument.

Bohr's paper was followed by a variety of papers essentially supporting Bohr's rejection of the EPR argument, among which we recall *Bell's inequality* [3] establishing that the $SU(2)$ spin algebra does not admit limit values with an identical classical counterpart.

We should also recall *von Neumann theorem* [4] achieving a rejection of the EPR argument via the uniqueness of the eigenvalues of quantum mechanical Hermitean operators under unitary transforms.

The field became known as *local realism* and was centered on the rejection of the EPR argument via additional claims that hidden variables [5] are not admitted by quantum axioms (see the review [6]).

1.2. The 1998 apparent proof of the EPR argument

In 1998, the author published paper [7] presenting an apparent proof of the EPR argument based on the following main steps that we here outline to render this paper minimally self-sufficient:

Step 1: The proof that Bell's inequality, von Neumann's theorem and other similar objections against the EPR argument [6] are indeed correct, but under the generally tacit assumptions of point-like particles moving in vacuum under sole potential/Hamiltonian interactions (*exterior dynamical systems*) when the systems are treated via quantum mechanics and its underlying 20th century mathematics, including Lie's theory and the

Newton-Leibnitz differential calculus;

Step 2: The proof that the above treatments are not applicable for extended, therefore deformable and hyperdense particles under conditions of mutual penetration or entanglement occurring in the structure of hadrons, nuclei, stars, and gravitational collapse such as for black holes, with novel non-linear, non-local, and non-potential/non-Hamiltonian interactions (*interior dynamical systems*);

Step 3: The treatment of interior systems via the axiom-preserving lifting of 20th century applied mathematics known as *isomathematics*, whose study was initiated by the author in the late 1970's when he was at Harvard University under DOE support, Refs. [8] to [12] and then continued by various mathematicians. Isomathematics is based on:

3-A) The axiom-preserving isotopy of the conventional associative product between generic quantities a, b (numbers, functions, operators, etc.) first introduced in Eq. (5), p. 71 of Ref. [11]

$$ab \ \rightarrow a \star b = a\hat{T}b, \tag{1}$$

where $\hat{T}$ is a positive-definite quantity called the *isotopic element* providing a representation of the dimension, deformability and density of particles and physical media in which they are immersed via realizations of the type

$$\hat{T} = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2})e^{-\Gamma}, \tag{2}$$

where: n_4^2 represents the density; $n_k^2,\ k = 1, 2, 3$ represents the deformable share of particles; $n_\mu^2, \mu = 1, 2, 3, 4$, and Γ are solely restricted to be positive-definite but otherwise admit a functional dependence on any needed local variables, such as time t, coordinates r, momenta p, energy E, density d, temperature τ, pressure π, wavefunctions ψ, their derivatives $\partial\psi$, etc.

$$n_\mu = n_\mu(t, r, p, E, d, \tau, \pi, \psi, \partial\psi, ...) > 0, \ \ \mu = 1, 2, 3, 4, \tag{3}$$

$$\Gamma = \Gamma(t, r, p, E, d, \tau, \pi, \psi, \partial\psi, ...) \gg 0. \tag{4}$$

$$e^{-\ \Gamma(t,r,p,E,d,\tau,\pi,\psi,\partial\psi,....)} \ll 1. \tag{5}$$

3-B) The formulation of isoassociative algebras on an *isofield* $\hat{F}(\hat{n}, \star, \hat{I})$ first introduced in Ref. [13] (see also independent work [14]), with *isounit*

$$\hat{I} = 1\hat{T}, \tag{6}$$

and *isoreal, isocomplex and isoquaternionic isonumbers* $\hat{n} = n\hat{I}$ under isoproduct (1), with ensuing isooperations such as the isosquare

$$\hat{n}^{\hat{2}} = \hat{n} \star \hat{n}. \tag{7}$$

Isofields also imply the lifting of functions into *isofunctions* [11] [20]

$$\hat{f}(\hat{r}) = [f(r\hat{I})]\hat{I}, \tag{8}$$

among which we quote the *isoexponentiation*

$$\hat{e}^{X} = (e^{X\hat{T}})\hat{I} = \hat{I}\ (e^{\hat{T}X}), \tag{9}$$

where X is a Hermitean operator.

3-C) The ensuing axiom-preserving lifting of Lie's theory into a non-linear, non-local and non-Hamiltonian form first introduced in Ref. [11] (see also the recent paper [15] and independent work [16]), which theory is today known as the *Lie-Santilli isotheory,* with isobrackets at the foundation of Ref. [7]

$$[X\hat{,}Y] = X \star Y - Y \star X = X\hat{T}Y - Y\hat{T}X. \tag{10}$$

3-D) The isotopic lifting of the Newton-Leibnitz differential calculus, from its historical definition at isolated points, into a form defined on volumes, first introduced in Ref. [17] (see Refs. [18] for vast independent works) with *isodifferential*

$$\begin{aligned} \hat{d}\hat{r} &= \hat{T}(r,...)d\hat{r} = \\ &= \hat{T}(r,...)d[r\hat{I}(r,...)] = dr + r\hat{T}d\hat{I}(r,...), \end{aligned} \tag{11}$$

and corresponding *isoderivatives*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \tag{12}$$

Step 4: The axiom-preserving lifting of quantum mechanics into the *isotopic branch of hadronic mechanics,* or *isomechanics* for short, whose study was initiated in Refs. [8] to [12] (see the 1995 monographs [19] [20] [21] with 2008 upgrade [22] and independent studies [23][24]).

Isomechanics is formulated on a *Hilbert-Myung-Santilli (HMS) isospace* [25] $\hat{\mathcal{H}}$ over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$, and it is based on the *iso-Heisenberg isoequations* for the time evolution of a Hermitean operator $\hat{Q}$ in the infinitesimal form

$$\begin{aligned} \hat{i} \star \tfrac{\hat{d}\hat{Q}}{\hat{d}\hat{t}} = [\hat{Q}\hat{,}\hat{H}] = \hat{Q} \star \hat{H} - \hat{H} \star \hat{Q} = \\ = \hat{Q}\hat{T}\hat{H} - \hat{H}\hat{T}\hat{Q}, \end{aligned} \tag{13}$$

and the finite form

$$\begin{aligned} \hat{Q}(\hat{t}) = \hat{U}(\hat{t})^{\dagger} \star \hat{Q}(0) \star \hat{U}(\hat{t}) = \\ = \hat{e}^{\hat{H} \star \hat{t} \star \hat{i}} \star \hat{Q}(0) \star \hat{e}^{-\hat{i} \star \hat{t} \star \hat{H}} = \\ = e^{\hat{H}\hat{T}ti} Q(0) e^{-it\hat{T}\hat{H}}, \end{aligned} \tag{14}$$

with the following rules for the basic *isounitary isotransforms*

$$\hat{U}(\hat{t})^{\dagger} \star \hat{U}(\hat{t}) = \hat{U}(\hat{t}) \star \hat{U}(\hat{t})^{\dagger} = \hat{I}, \tag{15}$$

where $\hat{t} = t\hat{I}_t$ is the *isotime* which is assumed hereon to coincide with conventional time, $\hat{I}_t = 1$. Dynamical equations (13) to (15) were first presented in Eq. (4.16.49), page 752 of Ref. [9] over conventional fields and reformulated via the full use of isomathematics in Ref. [17]).

Isomechanics is also based on the *iso-Schrödinger isorepresentation* characterized by the fundamental representation of the *isomomentum* permitted by the isodifferential isocalculus, Eq. (12),

$$\begin{aligned} \hat{p}\hat{|}\psi(\hat{t},\hat{r}) >= -\hat{i} \star \hat{\partial}_{\hat{t},\hat{r}}\hat{|}\psi(\hat{t},\hat{r}) >= \\ = -i\hat{I}\partial_{\hat{r}}\hat{|}\psi(\hat{t},\hat{r}) >, \end{aligned} \tag{16}$$

from which one can derive the *iso-Schrödinger isoequation,* [12] [17] [20]

$$\begin{aligned} \hat{i} \star \hat{\partial}_{\hat{t}}|\hat{\psi}(\hat{t},\hat{r}) >= \hat{H} \star |\hat{\psi}(\hat{t},\hat{r}) >= \\ = \hat{H}(r,p)\hat{T}(t,r,p,E,d,\tau,\pi,\psi,\partial\psi,....)|\hat{\psi}(\hat{t},\hat{r} >) = \\ = \hat{E} \star |\hat{\psi}(\hat{t},\hat{r}) >= E|\hat{\psi}(\hat{t},\hat{r}) > \end{aligned} \tag{17}$$

and the *isocanonical isocommutation rules,*

$$[\hat{r}_i\hat{,}\hat{p}_j]|\hat{\psi} >= \hat{i} \star \hat{\delta}_{i.j} \star |\hat{\psi} >= i\delta_{ij}|\hat{\psi} > \tag{18}$$

$$[\hat{r}_i \hat{,} \hat{r}_j]|\hat{\psi}>=[\hat{p}_i \hat{,} \hat{p}_j]|\hat{\psi}>=0. \tag{19}$$

Note that the characterization of extended particles at mutual distances smaller than their size requires the knowledge of *two quantities,* the conventional Hamiltonian H for the representation of potential interactions, and the isotopic element $\hat{T}$ for the representation of dimension, shape, density as well as of non-linear, non-local and non-potential interactions.

Step 5: The proof in Ref. [7] that the isotopic $\hat{SU}(2)$-spin symmetry for extended particles immersed within a dense hadronic medium admits an explicit and concrete realization of *hidden variables* [5], e.g., of the type

$$\hat{T} = Diag.(\lambda, 1/\lambda), \quad Det\hat{T} = 1. \tag{20}$$

In particular, the isotopic $\hat{SU}(2)$-spin isosymmetry admits limit conditions with identical classical counterpart, Eq. (5.4) page 189 Ref. [7].

One aspect of isomathematics and isomechanics which is crucial for this paper is that *in all applications to date, the isotopic element $\hat{T}$ has values much smaller than* 1, Eqs. (4) (5), as it has been the case for: the synthesis of the neutron from the hydrogen in the core of stars; the representation of nuclear magnetic moments and spin; new clean energies; and other applications [21].

It should be also noted that thanks to the new interactions represented by $\hat{T}$, isomathematics and isomechanics have permitted the first known identification of the *attractive force between identical valence electron pairs* in molecular structures [26]. A significant confirmation of values $|\hat{T}| \ll 1$ is provided by the fact that exact representations of binding energies for the hydrogen and water molecules have been achieved with *isoseries based on isoproduct (1) that are at least one thousand times faster than conventional quantum chemical series* [27] [28].

We should finally indicate that the numerical invariance of the isotopic element $\hat{T}$ and therefore, of the isounit $\hat{I} = 1/\hat{T}$, under isounitary time evolutions (14) (15) was proved in Ref. [29]. Detailed reviews and upgrades of isomathematics, isomechanics, and their applications to interior problems which are specifically written for the EPR argument should soon be available in Refs. [30] [31].

1.3. Aim of the paper

In this work, we shall attempt to complete the proof of the EPR argument of Ref. [7] by showing that extended particles in interior dynamical conditions appear to progressively recover classical determinism in interior dynamical conditions with the increase of the density and other characteristics, as indicated at the end of Ref. [7].

It should be stressed that a technical understanding of this work requires technical knowledge of hadronic mechanics, e.g., from Refs. [19] [20] [21] or from the forthcoming reviews and upgrades [30] [31].

We should indicate that the words "completion of quantum mechanics" is used in Einstein's sense for the intent of honoring his memory. For instance, the conventional associative product ab of Eq. (1), which is at the foundation of quantum mechanics, admits a "completion" into the equally associative, yet more general isoproduct $a\hat{T}b$. Under no conditions Einstein's word "completed theory" should be confused with a 'final theory,' that is a theory admitting no additional Einstein's "completions." In fact, the time-reversal invariant, Lie-isotopic isomathematics and isomechanics studied in this work admit the "completion" into the covering, irreversible *Lie-admissible genomathematics and genomechanics* (in which $\hat{T}$ is no longer Hermitean) which, in turn, admit a covering via the most general mathematics and mechanics conceived by the human mind, the multi-valued *hypermathematics and hypermechanics* [32] [33], with additional "coverings" remaining possible in due time [19] [20] [21] .

The reader should be finally aware that the isotopic element $\hat{T}$ and isounit $\hat{I} = 1/\hat{T}$ are inverted in some of the early quoted literature not dealing with determinism without affecting their consistency. An important aim of this paper has been that of achieving the final selection of isotopic element and isounit which is compatible with studies on determinism.

2. Recovering of determinism in interior conditions?

2.1. Heisenberg uncertainty principle

Consider an electron in empty space represented with the 3-dimensional Euclidean space $E(r, \delta, I)$, where r represents coordinates, $\delta = Diag.(1, 1, 1)$ represents the Euclidean metric and $I = Dian(1, 1, 1,)$ is the space unit.

Let the operator representation of said electron be done in a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ with states $\Psi(r)$ and familiar normalization

$$< \Psi(r)| \, |\Psi(r) >= \int_{-\infty}^{+\infty} \Psi(r)^{\dagger} \Psi(r) dr = 1. \tag{21}$$

As it is well known, the primary objections against the EPR argument

[2] [3] [4] were based on the *uncertainty principle* formulated by Werner Heisenberg in 1927, according to which *the position r and the momentum p of said electron cannot both be measured exactly at the same time.*

By introducing the *standard deviations* Δr and Δp, the uncertainty principle is generally written in the form (see, e.g., [5])

$$\Delta r \Delta p \ \geq \ \frac{1}{2}\hbar, \tag{22}$$

easily derivable via the vacuum expectation value of the canonical commutation rule

$$\Delta r \Delta p \ \geq \ |\frac{1}{2i} < \Psi| \, [r,p] \, |\Psi > | = \frac{1}{2}\hbar. \tag{23}$$

The standard deviations have the known form [34] (with $\hbar = 1$)

$$\begin{aligned} \Delta r &= \sqrt{< \Psi(r)|[\, r - (< \Psi(r)| \, r \, |\Psi(r) >)]^2 |\Psi(r) >}, \\ \Delta p &= \sqrt{< \Psi(p)| \, [p - (< \Psi(p)| \, p \, |\Psi(p) >)]^2 |\Psi(p) >}, \end{aligned} \tag{24}$$

where $\Psi(r)$ and $\Psi(p)$ are the wavefunctions in coordinate and momentum spaces, respectively.

2.2. Particle in interior conditions

We consider now the electron, this time, in the core of a star classically represented with the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ [17] with basic isounit $\hat{I} = 1/\hat{T} > 0$, isocoordinates $\hat{r} = r\hat{I}$, isometric

$$\hat{\delta} = \hat{T}\delta, \tag{25}$$

and isotopic element of type (2) under conditions (3) to (5).

Besides being immersed in the core of a star, the electron has no Hamiltonian interactions. Consequently, we can represent the electron in the *HMS isospace* $\hat{\mathcal{H}}$ [25] over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ [13], and introduce the time independent *isoplanewave* [20]

$$\begin{aligned} &\hat{\Psi}(\hat{r}) = \hat{\psi}(\hat{r})\hat{I} = \\ &= \hat{N} \star (\hat{e}^{\hat{i} \star \hat{k} \star \hat{r}})\hat{I} = N(e^{ik\hat{T}\hat{r}})\hat{I}, \end{aligned} \tag{26}$$

where $\hat{N} = N\hat{I}$ is an *isonormalization isoscalar,* $\hat{k} = k\hat{I}$ is the *isowavenumber*, and the isoexponentiation is given by Eq. (9).

The corresponding representation in isomomentum isospace is given by

$$\hat{\Psi}(\hat{p}) = \hat{M} \star \hat{e}^{\hat{i} \star \hat{n} \star \hat{p}}, \tag{27}$$

where $\hat{M} = M\hat{I}$ is an isonormalization isoscalar and $\hat{n} = n\hat{I}$ is the isowavenumber in isomomentum isospace.

2.3. Isodeterministic isoprinciple

The *isopropability isofunction* is given by [20]

$$\begin{gathered} \hat{\mathcal{P}} = \hat{<} | \star | \hat{>} = < \hat{\Psi}(\hat{r}) | \, T \, | \hat{\Psi}(\hat{r}) > I = \\ = [\int_{-\infty}^{+\infty} \hat{\Psi}(\hat{r})^{\dagger} \star \hat{\Psi}(\hat{r}) \star \hat{d}\hat{r}] \hat{I} = \\ = [\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{\psi}(\hat{r}) \hat{d}\hat{r}] \hat{I}, \end{gathered} \tag{28}$$

where one should keep in mind that the isodifferential $\hat{d}\hat{r}$ is now given by Eqs. (11).

The *isoexpectation isovalues* of a Hermitean operator $\hat{Q}$ are then given by [20]

$$\begin{gathered} \hat{<} | \star \hat{Q} \star | \hat{>} = < \hat{\Psi}(\hat{r}) | \star \hat{Q} \star | \hat{\Psi}(\hat{r}) > \hat{I} = \\ = [\int_{-\infty}^{+\infty} \hat{\Psi}(\hat{r})^{\dagger} \star \hat{Q} \star \hat{\Psi}(\hat{r}) \hat{d}\hat{r}] \hat{I} = \\ = [\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{Q} \hat{\psi}(\hat{r}) \hat{d}\hat{r}] \hat{I}, \end{gathered} \tag{29}$$

with corresponding expressions for the isoexpectation isovalues in isomomentum isospace.

We now introduce, apparently for the first time in this paper, the *isotopic operator*

$$\hat{\mathcal{T}} = \hat{T}\hat{I} = I, \tag{30}$$

that, despite its seemingly irrelevant value, is indeed the correct operator formulation of the isotopic element for the transition of the isoproduct from its scalar form (1) into the isoscalar form

$$\hat{n}^{\hat{2}} = \hat{n} \star \hat{n} = \hat{n} \star \hat{\mathcal{T}} \star \hat{n} = n^2 \hat{I}. \tag{31}$$

Since the identity I can be inserted anywhere in the expectation values of quantum mechanics without altering the results, realization (33) illustrates the central feature of the isotopies, namely, the property that the abstract axioms of quantum mechanics admit a "hidden" realization broader than that of the Copenhagen School whose degrees of freedom have been used in Ref.[7] for the proof of the EPR argument [1].

We now introduce the isoexpectation isovalue of the isotopic operator

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} = < \hat{\Psi}(\hat{r})| \star \hat{\mathcal{T}} \star |\hat{\Psi}(\hat{r}) > \hat{I} = \\ = [\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{T} \hat{\psi}(\hat{r}) d\hat{r}] \hat{I}, \tag{32}$$

and assume the isonormalization

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} = \\ = \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{T} \hat{\psi}(\hat{r}) d\hat{r} = \hat{T}. \tag{33}$$

We then introduce, in this paper apparently for the first time, the *isostandard isodeviation* for isocoordinates $\Delta\hat{r} = \Delta r \hat{I}$ and isomomenta $\Delta\hat{p} = \Delta p \hat{I}$, where Δr and Δp are the standard deviations in our space.

By using isocanonical isocommutation rules (18), we obtain the expression

$$\Delta\hat{r} \star \Delta\hat{p} = \Delta r \Delta p \hat{I} \approx \frac{1}{2}| < \hat{\Psi}(\hat{r})| \star [\hat{r},\hat{p}] \star \hat{\Psi}(\hat{r}) > | = \\ = \frac{1}{2}| < \hat{\Psi}(\hat{r})|\hat{T}[\hat{r},\hat{p}]\hat{T}|\hat{\Psi}(\hat{r}) > . \tag{34}$$

By eliminating the common isounit $\hat{I}$, we then have the desired *isodeterministic isoprinciple* here proposed apparently for the first time

$$\Delta r \Delta p \approx \frac{1}{2}| < \hat{\Psi}(\hat{r})| \star [\hat{r},\hat{p}] \star |\hat{\Psi}(\hat{r}) >= \\ = \frac{1}{2}| < \hat{\Psi}(\hat{r})|\hat{T}[\hat{r},\hat{p}]\hat{T}|\hat{\Psi}(\hat{r}) >= \\ \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{T} \hat{\psi}(\hat{r}) d\hat{r} = T \ll 1 \tag{35}$$

where the property $\Delta r \Delta p \ll 1$ follows from the fact that *the isotopic element $\hat{T}$ has always a value smaller than* 1 (Section 1.2).

It is now necessary to verify isoprinciple (35) by proving that the isostandard isodeviations tend to null values when $\hat{T} \to 0$.

For this purpose, we introduce the following simple isotopy of Eqs. (24) (where we ignore the common multiplication by the isounit)

$$\Delta r = \sqrt{< \hat{\Psi}(\hat{r})|[\hat{r} - < \hat{\Psi}(\hat{r})| \star \hat{r} \star |\hat{\Psi}(\hat{r}) >]^{\hat{2}}|\hat{\Psi}(\hat{r}) >}, \\ \Delta p = \sqrt{< \hat{\Psi}(\hat{p})| [\hat{p} - < \hat{\Psi}(\hat{p})| \star \hat{p} \star |\hat{\Psi}(\hat{p}) >]^2|\hat{\Psi}(\hat{p}) >}, \tag{36}$$

where the differentiation between the isotopic elements for isocoordinates and isomomenta is ignored for simplicity.

It is then easy to see that the isosquare (7) implies the covering forms of the isostandard isodeviations

$$\Delta r = \sqrt{\hat{T} < \hat{\Psi}(\hat{r})|[\,\hat{r} - < \hat{\Psi}(\hat{r})| \star \hat{r} \star |\hat{\Psi}(\hat{r}) >]^2 |\hat{\Psi}(\hat{r}) >},$$
$$\Delta p = \sqrt{\hat{T} < \hat{\Psi}(\hat{p})|\,[\hat{p} - < \hat{\Psi}(\hat{p})| \star \hat{p} \star |\hat{\Psi}(\hat{p}) >]^2 |\hat{\Psi}(\hat{p}) >}, \tag{37}$$

that indeed approach null value under the limit conditions

$$Lim_{\hat{T}=0} \Delta r = 0,$$
$$Lim_{\hat{T}=0} \Delta p = 0, \tag{38}$$

thus confirming isodeterministic isoprinciple (35).

2.4. Particles under pressure

To illustrate the above expressions, we consider an electron in the center of a star, thus being under extreme pressures π from the surrounding hadronic medium in all radial directions, while ignoring particle reactions in first approximation or under a sufficiently short period of time.

These conditions are here rudimentarily represented by assuming that the $\Gamma > 0$ function of the the isotopic element (2) is a constant linearly dependent on the pressure π, resulting in a realization of the isotopic element of the type

$$\hat{T} = e^{-w\pi} \ll 1, \;\; \hat{I} = e^{+w\pi} \gg 1, \tag{39}$$

where w is a positive constant.

The isodeterministic isoprinciple for the considered particle is then given by

$$\Delta r \Delta p \approx \frac{1}{2} e^{-w\pi} \;\ll\; 1, \tag{40}$$

and tends to null values for diverging pressures.

The above example illustrates the consistency of isorenormalization (33) because, a constant isotopic element implies the consistent expression

$$\hat{<} \hat{\psi}(\hat{r})|\hat{T}|\hat{\psi}(\hat{r}) > \hat{I} =$$
$$T < \hat{\psi}(\hat{r})|\,|\hat{\psi}(\hat{r}) > \hat{I} = \tag{41}$$
$$< \hat{\psi}(\hat{r})|\,|\hat{\psi}(\hat{r}) >,$$

while, by contrast, the following alternative isonormalization

$$\hat{<}\hat{\psi}(\hat{r})|\hat{T}|\hat{\psi}(\hat{r}) > \hat{I} = \hat{I}, \tag{42}$$

would imply the expression

$$< \hat{\psi}(\hat{r})||\hat{\psi}(\hat{r}) > \hat{I} = \hat{I}, \tag{43}$$

which is manifestly inconsistent since $< \hat{\psi}(\hat{r})||\hat{\psi}(\hat{r}) >$ is an ordinary number while $\hat{I}$ is a matrix with integro-differential elements.

Note that we have considered a free particle immersed in a hadronic medium, rather than a bound state of extended particles in condition of mutual penetration. Consequently, in our view, isotopic element (2) represents a *subsidiary constraint* caused by the pressure of the hadronic medium encompassing the particle considered, by therefore restricting the values of the isostandard isodeviations for isocoordinates and isomomenta.

Illustrations of the isodeterministic isoprinciple in specific structure models of hadrons and related aspects have been studied in Ref. [21] and their interpretation in terms of the isodeterministic isoprinciple will be studied in future works.

2.5. Gravitational example

To provide a gravitational illustration, recall that isotopic element (2) contains as particular cases all possible symmetric metrics in $(3+1)$-dimensions, thus including the Riemannian metric [20].

We then consider the 3-dimensional sub-case of isotopic element (2) and factorize the space component of the Schwartzchild metric $g_s(r)$ according to isotopic rule introduced in Refs. [35] [36]

$$g_s(r) = \hat{T}(r)\delta, \tag{44}$$

where δ is the Euclidean metric.

We reach in this way the following realization of the isotopic element

$$\hat{T} = \frac{1}{1 - \frac{2M}{r}} = \frac{r}{r - 2M}, \tag{45}$$

where M is the gravitational mass of the body considered, with ensuing isodeterministic isoprinciple

$$\Delta\hat{r}\Delta\hat{p} \;\approx\; \hat{T} = \frac{r}{r - 2M} \;\Rightarrow_{r\to 0}= 0, \tag{46}$$

which confirms the statement in page 190 of Ref. [7], on the possible recovering of full classical determinism in the interior of gravitational collapse (see Ref. [37], Chapter 6 in particular, for a penetrating critical analysis of black holes).

It should perhaps be indicated that Refs. [35] [36] introduced the factorization of a full Riemannian metric $g(x),\ x = (r, t)$ in $(3+1)$-dimensions

$$g(x) = \hat{T}_{gr}(x)\eta, \tag{47}$$

where $\hat{T}_{gr}$ is the *gravitational isotopic element*, and η is the Minkowski metric $\eta = Diag.(1, 1, 1, -1)$.

Refs. [35] [36] then reformulated the Riemannian geometry via the transition from a formulation over the field of real numbers $\mathcal{R}$ to that over the isofield of isoreal isonumbers $\hat{\mathcal{R}}$ where the *gravitational isounit* is evidently given by

$$\hat{I}_{gr}(x) = 1/\hat{T}_{gr}(x). \tag{48}$$

The above reformulation turns the Riemannian geometry into a new geometry called *iso-Minkowskian isogeometry,* which is locally isomorphic to the *Minkowskian* geometry, while maintaining the mathematical machinery of the Riemannian geometry (covariant derivative, connection, geodesics, etc.) us fully maintained, although reformulated in terms of the isodifferential isocalculus [38].

The apparent advantages of the *identical* iso-Minkowskian reformulation of Riemannian metrics and Einstein's field equations (see, e.g., Eqs. (2.9), page 390 of Ref. [38]) are:

1) The achievement of a consistent operator form gravity in terms of *relativistic hadronic mechanics* [39] whose axioms are those of quantum mechanics, only subjected to a broader realization;

2) The achievement of a universal *symmetry* of *all* non-singular Riemannian metrics, which symmetry is locally isomorphic to the Lorentz-Poincaré symmetry, today known as the *Lorentz-Poincaré-Santilli (LPS) isosymmetry* [40], and it is notoriously impossible on a conventional Riemannian space over the reals;

3) The achievement of clear compatibility of Einstein's field equation with 20th century sciences, such as a clear compatibility of general relativity with special relativity via the simple limit $\hat{I}_{gr} = I$ implying the transition from the universal LPS isosymmetry to the Poincaré symmetry of special relativity with ensuing recovering of conservation and other special relativity laws [41] [42]; the achievement of axiomatic compatibility of gravitation with electroweak interactions thanks to the replacement of curvature into the new notion of isoflatness with the ensuing, currently

impossible, foundations for a grand unification [43]; and other intriguing advances.

3. Concluding remarks

t In this paper, we have continued the study of the EPR argument [1] conducted in Ref. [7] and preceding works, with particular reference to the study of the uncertainties for extended particles immersed within hyperdense medias with ensuing linear and non-linear, local and non-local and Hamiltonian as well as non-Hamiltonian interactions.

This study has been conducted via the use of isomathematics and isomechanics characterized by the isotopic element $\hat{T}$ of Eq. (1) which represents the non-linear, non-local and non-Hamiltonian interactions of the particles with the medium [19] [20] [21].

The main result of this paper is that the standard deviations of coordinates and momenta for particles within hyperdense media are characterized by the isotopic element that, being always very small, $\hat{T} \ll 1$, reduces the uncertainties in a way inversely proportional to a non-linear increase of the density, pressure, temperature, and other characteristics of the medium, while admitting the value $\hat{T} = 0$ under extreme/limit conditions with ensuing recovering of full determinism as predicted by A. Einstein, B. Podolsky and N. Rosen [1].

We can, therefore, tentatively summarize the content of this paper with the following:

ISODETERMINISTIC ISOPRINCIPLE: The product of isostandard isodeviations for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$, as well as the individual isodeviations, progressively approach classical determinism for extended particles in the interior of hadrons, nuclei, and stars, and achieve classical determinism at the extreme densities in the interior of gravitational collapse.

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Some results for Volterra integro-differential equations depending on derivative in unbounded domains

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Abstract

In this paper we study the existence of continuous solutions of an integro-differential equation in unbounded interval depending on derivative. This paper extends some results obtained by the authors using the technique developed in their previous paper. This technique consists in introducing, in the given problems, a function q, belonging to a suitable space, instead of the state variable x. The fixed points of this function are the solutions of the original problem. In this investigation we use a fixed point theorem in Fréchet spaces.

Keywords: Fréchet spaces, semi-norms, acyclic sets, Ascoli-Arzelà theorem.
2010 AMS subject classification: 45G10, 47H09, 47H30*

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1 Introduction

In this paper we study, in abstract setting, the solvability of a nonlinear integro-differential equation of Volterra type with implicit derivative, defined in unbounded interval, like

$$(1) \quad x'(t) = \int_0^t k(t,s) f\big(s, x(s), x'(s)\big) ds \qquad x(0) = 0, \quad t \in J = [0, +\infty)$$

We will look for solutions of this equation in the Fréchet space of all real C^1 functions defined in the real unbounded interval $J = [0, +\infty)$.

Equation (1) is a special case of integro-differential equations. These equations have been seen as an important tool in the study of many boundary problems that we can encounter in various applications, like, for exemple, heat flow in material, kinetic theory, electrical ingeneering, vehicular traffic theory, biology, population dynamics, control theory, mechanics, mathematical economics.

The integro-differential equations have been studied in various papers with the help of several tools of functional analysis, topology and fixed point theory. For istance we can refer to [1], [2], [3], [4], [5], [11], [12] and the references therein. In [8] an Hammerstein equation, similar to (1), is consideren in the multivalued setting and bounded intervals.

Our paper extend some results obtained by the authors Anichini and Conti, using the techniques developed in previous paper (see to [1], [2], [3], [4], [5]).

The crucial key of our approach, in order to find solutions of equation (1), consists in the use of a very useful fixed point theorem for multivalued, compact, uppersemicontinuous maps with acyclic values in a Fréchet space.

2 Preliminaries and Notations

Let $C^1(J, \mathbb{R})$ be the Fréchet the of all real C^1 functions defined in the real unbounded interval $J = [0, +\infty) \subset \mathbb{R}$, equipped with the following family of semi-norms

$$\|x\|_{1,n} = \max\{\|x\|_n, \|x'\|_n\}$$

where $\|x\|_n = \sup\{|x(t)|, t \in [0, n]\}$ and $\|x'\|_n = \sup\{|x'(t)|, t \in [0, n]\}$.

We recall that the topology of $C^1(J, \mathbb{R})$ coincides with the topology of a complete metric space $\{F, d\}$ where

$$d(x, y) = \sum_{n=1}^{+\infty} \frac{2^{-n}\|x-y\|_{1,n}}{1+\|x-y\|_{1,n}}$$

A subset $A \subset C^1(J, \mathbb{R})$ is said to be bounded if, for every natural number n, there exists $M_n > 0$ such that $\|x\|_{1,n} \leq M_n \quad \forall\, x \in C^1(J, \mathbb{R})$.
A subset $A \subset C^1(J, \mathbb{R})$ is relatively compact set if and only if the functions of the set A are equicontinuous and uniformly bounded (with their derivatives) in any interval $[0, n]$.
We will denote by $C(F)$ the family of all nonempty and compact subset of a Fréchet space F.
Let M be a subset of a Fréchet space F; a multivalued map $S: M \to C(F)$ is said to be uppersemicontinuous (u.s.c.) if the graph is closed in $M \times F$, i.e. for any sequence $\{x_n\} \subset M$, $x_n \to x_0$ and $y_n \in S(x_n)$, $y_n \to y_0$, we have $y_0 \in S(x_0)$.
A multivalued map $S: M \to C(F)$ is said to be compact if it sends bounded sets into relatively compact sets. We apply the same definition for singlevalued maps.
A subset A of a metric space E is said to be an R_δ - set if A is the intersection of a countable decreasing sequence of absolute retracts contained in E (see [10]).
It is known that an R_δ - set is an acyclic set, i.e. it is acyclic with respect to any cohomology theory (see [7]).
Let M be a subset of the Fréchet space $C^1(J, \mathbb{R})$ and consider an operator $T: M \to C^1(J, \mathbb{R})$. Let $\{\epsilon_n\}$ be an infinitesimal sequence of real numbers.
A sequence $\{T_n\}$ of maps $T_n : M \to C^1(J, \mathbb{R})$ is said to be an ϵ_n-approximation of T on M if $\|T_n(x) - T(x)\|_{1,n} \leq \epsilon_n$ for every $x \in M$ and for any natural number n.
Define $U_n = \{x \in F : \|x\|_{1,n} < 1\}$.
Let T be a compact map $T: M \to C^1(J, \mathbb{R})$, where M is a closed set of the Fréchet space $C^1(J, \mathbb{R})$, and let $\{T_n\}$ be a ϵ_n-approximation of T on M, where $T_n : M \to C^1(J, \mathbb{R})$ are compact maps; then the set of fixed point of T is a compact R_δ - set if the equation $x - T_n(x) = y$ has at most a solution for every $y \in \varepsilon_n U_n$ for any natural number n (see [5]).

In the sequel we will use the following result (see [9]).

Proposition 1 (Kirszebraun's Theorem)
Let $F : M \to \mathbb{R}$ be a Lipschitz map defined on arbitrary subset M of $\mathbb{R}^n$. Then F admits a Lipschitz extension $\mathfrak{I} : \mathbb{R}^n \to \mathbb{R}$ with the same Lipschtiz constant.

The well known Gronwall's Lemma, from the standard theory of Ordinary Differential Equations, will be used.

Proposition 2 (Gronwall's Lemma)
Let g, $h : J \to J$ be continuous functions such that the following inequality:

$$g(t) \leq u(t) + \int_0^t h(s)g(s)ds \quad t \in J,$$

holds, where $u : J \to J$ is a continuous nondecreasing function.
Then we have:

$$g(t) \leq u(t)\exp\left(\int_0^t h(s)ds\right) \quad t \in J.$$

In the sequel we will use the following proposition that can be deduced from Theorem 1 of [6].

Proposition 3 (a fixed poin theorem)
Let F be a Fréchet space and $M \subset X$ be a bounded, closed and convex subset; let $S: F \to M$ be a multivalued, uppersemicontinuous map with acyclic values. If $S(F)$ is (relatively) compact, then S has a fixed point.

3 Main result

The following result holds.

Theorem
Consider integral equation (1). Assume that
i) $k : J \times J \to \mathbb{R}$ is a C^1 function; moreover we assume that there exists a continuous function $h : J \to J$ with

$$|k(t,s)| \leq h(s) \text{ and } \left|\frac{\partial k(t,s)}{\partial t}\right| \leq h(s).$$

ii) $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function; moreover we assume that there exist continuous functions a, $b : J \to J$, with $\int_0^{+\infty} a(s)ds = A < +\infty$ and $\int_0^{+\infty} b(s)ds = B < +\infty$, such that:

$$|f(s,x,y)| \leq a(s) + b(s)|y|.$$

iii) Assume that $\int_0^{+\infty} h(s)b(s)ds = \beta < 1$.

Then equation (1) has at least one solution in the space $C^1(J, \mathbb{R})$.

Proof

Let q be a function belonging to $C^1(J, \mathbb{R})$ and consider the following integral equation:

$$(2) \qquad y(t) = \int_0^t k(t,s) f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds \qquad t \in J = [0, +\infty)$$

Let $S : C^1(J, \mathbb{R}) \to C^1(J, \mathbb{R})$ be the multivalued map which associates to every $q \in C^1(J, \mathbb{R})$ the set of solutions of equation (2).

Clearly, putting $x(t) = \int_0^t y(s)ds$ (hence $x'(t) = y(t)$ and $x(0) = 0$), we have that the fixed points of the map S are the solution of equation (1).

In order to find the fixed points of multivalued map S, the following steps in the proof have to be established (Proposition 3):

a) There exists a bounded, closed and convex set $M \subset C^1(J, \mathbb{R})$ such that $S(C^1(J, \mathbb{R})) \subset M$.
b) The set $S(C^1(J, \mathbb{R}))$ is relatively compact.
c) The map S is uppersemicontinuous.
d) The set $S(q)$ is an acyclic set for every $q \in C^1(J, \mathbb{R})$.

a) Let $q \in C^1(J, \mathbb{R})$ and consider equation (2); assume that $t \in [0, n]$, from hypotheses we have, :

$$|y(t)| = \left| \int_0^t k(t,s) f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds \right| \leq$$

$$\leq \left| \int_0^t h(s)(a(s) + b(s)|y(s)|)ds \right| \leq$$

$$\leq \left| \int_0^t h(s)a(s)ds \right| + \left| \int_0^t h(s)b(s)|y(s)|ds \right|$$

$$\leq \|h\|_n A + \beta \|y\|_n.$$

So that, since $\beta < 1$, we have $\|y\|_n \leq \frac{\|h\|_n A}{1-\beta}$.

Moreover, we have for $t \in [0, n]$:

$$y'(t) = \int_0^t \frac{\partial k(t,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right)ds + k(t,t)f\left(t, \int_0^t q(s)ds\,, y(t)\right)$$

and we obtain:

$$|y'\,(t)| \leq \left|\int_0^t \frac{\partial k(t,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right)ds\right| + \left|k(t,t)f\left(t, \int_0^t q(s)ds\,, y(t)\right)\right| \leq$$

$$\leq \int_0^t h(s)a(s)ds + \int_0^t h(s)b(s)|y(s)|ds + h(t)(a(t) + b(t)|y(t)|) \leq$$

$$\leq \|h\|_n A + \beta\,\|y\|_n + \|ha\|_n + \|hb\|_n\|y\|_n \leq$$

$$\leq \|h\|_n A + \|ha\|_n + \|y\|_n(\beta + \|hb\|_n)$$

$$\leq \|h\|_n A + \|ha\|_n + \tfrac{\|h\|_n A}{1-\beta}(\beta + \|hb\|_n).$$

So that there exists $M_n > 0$ such that $\|y\|_{1,n} \leq M_n$.
Then we have $S(C^1(J, \mathbb{R})) \subset M$, where

$M = \{y \in C^1(J, \Re), \|y\|_{1,n} \leq M_n\}$.

b) Now, we want to prove that the set $S(C^1(J, \mathbb{R}))$ is relatively compact.
Let $y \in S(C^1(J, \mathbb{R}))$ and fix $\varepsilon > 0$. For any u, $w \in [0, n]$ we have:

$$y'(w) - y'(u) =$$

$$\int_0^w \frac{\partial k(w,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right)ds + k(w,w)f\left(w, \int_0^w q(s)ds\,, y(w)\right) -$$

$$\int_0^u \frac{\partial k(u,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right)ds - k(u,u)f\left(u, \int_0^u q(s)ds\,, y(u)\right)$$

$$= \int_0^u \frac{\partial k(w,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right)ds + k(w,w)f\left(w, \int_0^w q(s)ds\,, y(w)\right)$$
$$- \int_0^u \frac{\partial k(u,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right)ds$$

$$-k(u,u)f\left(u, \int_0^u q(s)ds\,, y(u)\right) + \int_u^w \frac{\partial k(w,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right)ds$$

It follows that

$$|y'(w) - y'(u)| \leq$$

$$\leq \int_0^u \left|\frac{\partial k(w,s)}{\partial t} - \frac{\partial k(u,s)}{\partial t}\right| (a(s) + b(s)|y(s)|)ds$$
$$+ \|h\|_n \left| f\left(w, \int_0^w q(\tau)d\tau, y(w)\right) - f\left(u, \int_0^u q(\tau)d\tau, y(u)\right)\right|$$
$$+ \|h\|_n \left|\int_u^w \frac{\partial k(w,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right)ds\right|$$

By continuity of the functions q, h, f and $\frac{\partial k}{\partial t}$ it follows that there exists $\delta > 0$ such that for $|w - u| < \delta$, $u, w \in [0, n]$, we have

$$|y'(w) - y'(u)| < \varepsilon$$

Since $|y(w) - y(u)| \leq M_n|w - u|$, we can conclude that the set $S(C^1(J, \mathbb{R}))$ is relatively compact.

c) Let us now show that the map S is uppersemicontinuous.
Let $\{q_m\}$ be a sequence, $q_m \in C^1(J, \mathbb{R})$, with $\|q_m - q_0\|_{1,n} \to 0$,
$y_m \in S(q_m)$, i.e.

$$y_m(t) = \int_0^t k(t,s)f\left(s, \int_0^s q_m(\tau)d\tau, y_m(s)\right)ds \quad t \in [0, n]$$

Assume that $\|y_m - y_0\|_{1,n} \to 0$. We need to show that $y_0 \in S(q_0)$.
From the Dominated Lebesgue Convergence Theorem it follows:

$$lim_{m\to+\infty}\ f\left(s,\int_0^s q_m(\tau)d\tau, y_m(s)\right) = f\left(s,\int_0^s q_0(\tau)d\tau, y_0(s)\right)$$

and

$$\begin{aligned} lim_{m\to+\infty}\ \ y_m(t) &= \\ &= lim_{m\to+\infty}\ \int_0^t k(t,s)f\left(s,\int_0^s q_m(\tau)d\tau, y_m(s)\right)ds = \\ &= \int_0^t lim_{m\to+\infty}\ k(t,s)f\left(s,\int_0^s q_m(\tau)d\tau, y_m(s)\right)ds = \\ &= \int_0^t k(t,s)f\left(s,\int_0^s q_0(\tau)d\tau, y_0(s)\right)ds\,. \end{aligned}$$

Hence, we obtain

$$y_0(t) = \int_0^t k(t,s)f\left(s,\int_0^s q_0(\tau)d\tau, y_0(s)\right)ds$$

i. e. $y_0 \in S(q_0)$

d) Now we want to show that, for every fixed $q \in C^1(J, \mathbb{R})$, the set $S(q)$ is acyclic.
Consider equation (2) (with q fixed).
Put $f\left(s,\int_0^s q(\tau)d\tau, y(s)\right) = l(s,y)$.

Then equation (2) can be written in the following way:

$$y(t) = \int_0^t k(t,s)l(s,y(s))ds \qquad t \in [0,+\infty)$$

We have:

$$|y(t)| \le \left|\int_0^t k(t,s)l(s,y(s))ds\right| \le \int_0^t h(s)a(s)ds + \int_0^t h(s)b(s)|y(s)|ds.$$

From Gronwall's Lemma it follows that:

$$|y(t)| \leq \int_0^t h(s)a(s)ds \exp(\int_0^t h(s)b(s)ds) = m(s)$$

where m is a continuous function.
Let $U: \mathbb{R} \rightarrow [0,1]$ the Uryshon (continuous) function defined by

$$U(z) = 1 \text{ if } |z| \leq 1 \text{ and } U(z) = 0 \text{ if } |z| \geq 2.$$

Now we define the function

$$g(s,y) = U\left(\frac{y}{m(s)+1}\right) l(s,y).$$

Clearly $g(s,y) = l(s,y)$ when $|y| \leq m(s)$. Hence the set of solutions of the following equation

$$y(t) = \int_0^t k(t,s)g(s,y(s))ds \quad t \in [0,+\infty)$$

coincides with the set of solutions of equation (2) with q fixed.
Consider now the integral operator $H: C^1(J,\mathbb{R}) \rightarrow C^1(J,\mathbb{R})$:

$$\big(H(y)\big)(t) = \int_0^t k(t,s)g\big(s,y(s)\big)ds \quad t \in [0,+\infty)$$

If $z = H(y)$, we have

$$z(t) = \int_0^t k(t,s)U\left(\frac{y(s)}{m(s)+1}\right) l(s,y(s))ds\,.$$

Notice that

$$U\left(\frac{y(s)}{m(s)+1}\right) l(s,y(s)) = l(s,y(s)) \text{ if } y(s) \leq m(s)+1$$

and

$$U\left(\frac{y(s)}{m(s)+1}\right) l(s, y(s)) = 0 \text{ if } y(s) \geq 2m(s) + 2.$$

So that:

$$\|z\|_n \leq \|h\|_n A + 2\beta(\|m\|_n + 1).$$

Moreover we obtain:

$$|z'(t)| \leq \int_0^t h(s) U\left(\frac{y(s)}{m(s)+1}\right) \left|l\big(s, y(s)\big)\right| ds + h(t) U\left(\frac{y(t)}{m(t)+1}\right) \left|l\big(t, y(t)\big)\right|$$

Hence

$$\|z'\|_n \leq \|h\|_n A + 2\beta(\|m\|_n + 1) + \|ha\|_n + 2\|hb\|_n(\|m\|_n + 1) = A_n$$

It follows that $\|z\|_{1,n} \leq A_n$, where $z = H(y)$.

So that the set of solutions of equation

$$y(t) = \int_0^t k(t,s) g(s, y(s)) ds \quad t \in [0, +\infty)$$

coincides with the set of fixed points of operator H in the set

$$A = \left\{z \in C^1(J, \mathbb{R}), \|z\|_{1,n} \leq A_n\right\}.$$

It is easy to see (again as consequence of the Ascoli- Arzelà Theorem) that the set $H(A)$ is relatively compact set.
Moreover H is a continuous operator; to show the last assertion, let us take $y_0, y_m \in A, \|y_m - y_0\|_{1,n} \to 0, z_m \in H(y_m), \|z_m - z_0\|_{1,n} \to 0$; we are going to prove that $z_0 \in H(y_0)$.
For every $t \in [0, n]$ we have:

$$lim_{m\to+\infty}\left|\int_0^t k(t,s)g(s,y_m(s))ds-\int_0^t k(t,s)g(s,y_0(s))ds\right|\leq$$

(from the Dominated Lebesgue Convergence Theorem and the continuity of function g)

$$\leq\int_0^t lim_{m\to+\infty}\ h(s)\big|g\big(s,y_m(s)\big)-g\big(s,y_0(s)\big)\big|ds\,.$$

Hence

$$z_0(t)=\int_0^t k(t,s)g\big(s,y_0(s)\big)ds=(H(y_0))(t).$$

Fix now a natural number *n*. We know (Proposition 1) that there exists a Lipschitz function

$$g_n\colon[0,n]\times[-A_n,A_n]\to\mathbb{R}$$

such that, for every $(s,y)\in[0,n]\times[-A_n,A_n]$, we have:

$$|g_n(s,y)-g(s,y)|\leq\frac{1}{(n+1)^2\|h\|_n}$$

and

$$|g_n(s,y)-g_n(s,y_1)|\leq L_n|y-y_1|$$

for every $(s,y),(s,y_1)\in[0,n]\times[-A_n,A_n]$,

Let $G_n\colon J\times\mathbb{R}\to\mathbb{R}$ be the Lipschitz extension of the function g_n; hence

$G_n(s,y)=g_n(s,y)$ for every $(s,y)\in[0,n]\times[-A_n,A_n]$

and $|G_n(s,y)-G_n(s,y_1)|\leq L_n|y-y_1|$ for every $(s,y),(s,y_1)\in J\times\mathbb{R}$.
Let $H_n:A\to C^1(J,\mathbb{R}))$ be the operator defined as follows:

$$(H_n(y))(t)=\int_0^t k(t,s)G_n(s,y(s))ds\qquad t\in[0,+\infty)$$

Clearly this operator is compact for every natural number *n*.
Moreover, for every $t \in [0, n]$ and $y \in A$, we have:

$$\left|\left(H_n(y)\right)(t) - \left(H(y)\right)(t)\right| \leq \int_0^t k(t,s)\left|G_n\left(s, y(s)\right) - g\left(s, y(s)\right)\right| ds \leq$$

$\leq n\|h\|_n \frac{1}{(n+1)^2\|h\|_n} < \frac{1}{n}$.

So that $\|H_n(y) - H(y)\|_n < \frac{1}{n}$.
Moreover we have for every $y \in A$:

$$\begin{aligned}\left|\left(H'_n(y)\right)(t) - (H'(y))(t)\right| &\leq \\ &\leq \left|\int_0^t \frac{\partial k(t,s)}{\partial t} G_n\left(s, y(s)\right) ds - k(t,t) G_n\left(t, y(t)\right)\right. \\ &\quad \left. + \int_0^t \frac{\partial k(t,s)}{\partial t} g\left(s, y(s)\right) ds - k(t,t) g\left(t, y(t)\right)\right| \leq\end{aligned}$$

$$\int_0^t h(s)\left|G_n\left(s, y(s)\right) - g\left(s, y(s)\right)\right| ds + h(t)\left|G_n\left(t, y(t)\right) - g\left(t, y(t)\right)\right| \leq$$

$\leq n\|h\|_n \frac{1}{(n+1)^2\|h\|_n} + \|h\|_n \frac{1}{(n+1)^2\|h\|_n} = \frac{n+1}{(n+1)^2} < \frac{1}{n}$.

Hence $\|H'_n(y) - H'(y)\|_n < \frac{1}{n}$.

Let now $b \in A$. We consider the equation $y - H_n(y) = b$. We want to prove that it has at most one solution. Consider the equation $z - H_n(z) = b$; then, for every $t \in J$ and by Gronwall's Lemma we have:

$$\begin{aligned}|y(t) - z(t)| &\leq \int_0^t h(s)\left|G_n\left(s, y(s)\right) - G_n\left(s, z(s)\right)\right| ds \leq \\ &\leq \int_0^t h(s) L_n |y(s) - z(s)| ds \leq 0.\end{aligned}$$

So that we can say that $y(t) = z(t)$ for every $t \in J$.
Finally, we are able to conclude that, for every $q \in C^1(J, \mathbb{R})$, the set $S(q)$ is acyclic and the theorem is proved.

4 An example

Consider the following integro-differential equation:

$$(3) \qquad x'(t) = \int_0^t \frac{3t\, e^{-s+2}}{1+t^3}\left(\frac{3s^2\, e^{-2s}}{1+\left(\sin\left(x(s)\right)\right)^2} + s\, e^{-s-2}x'(s)\right)ds$$

$$x(0) = 0, \quad t \in J = [0, +\infty).$$

We have

$$k(t,s) = \frac{3t\, e^{-s+2}}{1+t^3},$$

$$f\left(s, x(s), x'(s)\right) = \frac{3s^2\, e^{-2s}}{1+\left(\sin\left(x(s)\right)\right)^2} + s\, e^{-s-2}x'(s)$$

$h(s) = 3\, e^{-s+2}$, $a(s) = 3s^2\, e^{-2s}$, $b(s) = s\, e^{-s-2}$.

Hence, we obtain:

$$\int_0^{+\infty} a(s)ds = \frac{3}{4}$$

$$\int_0^{+\infty} b(s)ds = e^{-2}$$

$$\int_0^{+\infty} h(s)b(s)ds = \int_0^{+\infty} 3se^{-2s}ds = \frac{3}{4} < 1$$

So that the assumptions of our theorem are satisfied and integro-differential equation (3) has solutions.

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Solving some specific tasks by Euler's and Fermat's Little theorem

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Abstract

Euler's and Fermat's Little theorems have a great use in number theory. Euler's theorem is currently widely used in computer science and cryptography, as one of the current encryption methods is an exponential cipher based on the knowledge of number theory, including the use of Euler's theorem. Therefore, knowing the theorem well and using it in specific mathematical applications is important. The aim of our paper is to show the validity of Euler's theorem by means of linear congruences and to present several specific tasks which are suitable to be solved using Euler's or Fermat's Little theorems and on which the principle of these theorems can be learned. Some tasks combine various knowledge from the field of number theory, and are specific by the fact that the inclusion of Euler's or Fermat's Little theorems to solve the task is not immediately apparent from their assignment.

Keywords: Euler's theorem, coding, Fermat's Little theorem, linearcongruences, cryptology, primality testing, Matlab

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1 Introduction

At present, mathematics provides apparatus for virtually all modern coding systems. The first coding system, with only two symbols - a dot and a comma, was Morse code which was used to send the first coding message by American inventor Samuel F. B. Morse in 1844 using an electric telegraph. Binary code encoding has become a better code for message encryption at a later time, in which each coded word consists of blocks of ones and zeroes, and this encoding is still used today [1]. Significant developments in coding occurred in the 20th century when Euler's theorem was used for coding and the coded text could be broadcasted publicly with the message kept secret. The principle of this coding is that the sender assigns a number to a coded word (e.g. 74) and encodes that word using two additional numbers (e.g. 247 and 5), which may be public in such a way that $74^5 (\mathrm{mod}\, 247) = 120$ is calculated. This will give you a message "120" that will be sent to the recipient. Since numbers 247 and 5 are public keys, anyone can encode the message "74" to "120", but only the actual recipient can decode it correctly. The essence of the key to the cipher lies in the fact that only the recipient knows that number 247 was compiled as the product of primes $p = 13$ and $q = 19$ and using Euler's theorem searches for the value x for which the congruence is $5x \equiv 1(\mathrm{mod}\, [(p-1)(q-1)])$. The recipient can easily get the result $x = 173$. Using this figure, the remainder by dividing 120^{173} by number 247 is found, thereby obtaining the original coded word 74 which can already be assigned to the message [2]. In practice, with this type of coding, the product of two very large primes is used, where the decomposition of the thus obtained number is very difficult, virtually impossible for someone who does not know the product of which two primes have been executed. Despite the fact that the principle of this coding was discovered and started to be used practically in the 20th century, it is actually derived from Euler's knowledge from the 18th century.

Most of the results in mathematics in the 18th century stemmed from efforts to solve various separate problems discovered in the 17th century. In this period, the theory of numbers remained more or less in the background, and the only mathematician who dealt with the issues of number theory after 1730 to a greater extent was Euler. In 1736, he proved Fermat's Little theorem which claims that for any natural number a and prime p, $a^{p-1} \equiv 1(\mathrm{mod}\, p)$. Later in 1760, after the introduction of Euler's totient function $\varphi(n)$ he demonstrated the validity of congruence $a^{\varphi(m)} \equiv 1(\mathrm{mod}\, m)$ which is a generalization of Fermat's Little theorem. Euler also dealt with many other Fermat's claims. He also achieved several accomplishments related to the decomposition of certain expressions with the powers of natural numbers and to perfect and friendly numbers. He was also interested in the problem of integer roots of Pell's equation, about which he published several articles, and presented his own method of solution. Euler has introduced a number of concepts into number theory, such as the quadratic residue and the quadratic nonresidue in the law of quadratic reciprocity and his work and accomplishments, despite the lack of exact evidence in several areas, were generally accepted by respected mathematicians of the 18th and 19th centuries (e.g. Gauss or Legendre) [3]. We would like to mention there's also another principle of coding using Fibonacci numbers and can be seen in [4].

2 Euler's theorem, Fermat's Little theorem

Let us consider two natural numbers a, m where $(a, m) = 1$. Euler's theorem [5] then states that $m|a^{\varphi(m)} - 1$, or that congruence $a^{\varphi(m)} \equiv 1(\mathrm{mod}\, m)$ applies. The symbol $\varphi(n)$ denotes the number of natural numbers smaller than n and relatively prime to n and is called *Euler's totient function* [6].
To show the validity of Euler's theorem, we will use the basic properties of congruences and residue classes. Let's write all relatively prime numbers to m less than m. These are $x_1, x_2, \cdots, x_{\varphi(m)}$. Let us further consider the sequence $ax_1, ax_2, \cdots, ax_{\varphi(m)}$ and indirectly show that all its members are relatively prime to m. If $\exists i: (ax_i, m) = d > 1$, then $d|ax_i \wedge d|m$. Then $(d, a) = 1$, because $(a, m) = 1 \wedge d|m$. In that $d|x_i$ and numbers m, x_i are commensurable which is a controversy.
Furthermore, let us indirectly show that numbers $ax_1, ax_2, \cdots, ax_{\varphi(m)}$ are non-congruent modulo m. $\exists i, j: ax_i \equiv ax_j(\mathrm{mod}\, m)$. Then $m|ax_i - ax_j = a(x_i - x_j) \wedge (a, m) = 1$, of which $m|x_i - x_j$ and then $x_i \equiv x_j(\mathrm{mod}\, m)$, which is a controversy, because x_i are differently lower from each other than m, and therefore cannot give the same remainder after division by m.
Before completing the evidence, we recall, that based on the basic properties of congruences, [7] we know that integers a and b belong to the same class R_i modulo m just when $a \equiv b\ (\mathrm{mod}\, m)$. If we first express the numbers $a, b \in R_i$ in the form $a = m \cdot q + i, b = m \cdot p + i$, then $a - b = m(q - p)$, which means $m|a - b$, and thus $a \equiv b(\mathrm{mod}\, m)$. On the other hand, let us assume that $a \equiv b\ (\mathrm{mod}\, m)$ and $a = mq + i, b = mp + j\ (0 \leq i, j < m)$. For example, it is supposed that $i > j$. Since $a \equiv b\ (\mathrm{mod}\, m)$, $m|a - b$. But then $m \mid (a - b) = [m(q - p) + (i - j)]$, of which $m \mid (i - j)$. This would be a controversy though, because $0 < i - j < m$. Similarly, a controversy arises even with the assumption $i < j$. Therefore $i = j$ must hold, hence the numbers a and b belong to the same residual class modulo m with $a \equiv b\ (\mathrm{mod}\, m)$.
As the class representative does not matter, we can write $ax_1 \cdot ax_2 \cdot \cdots \cdot ax_{\varphi(m)} \equiv x_1 \cdot x_2 \cdot \cdots \cdot x_{\varphi(m)}(\mathrm{mod}\, m)$. Then $m|ax_1 \cdot ax_2 \cdot \cdots \cdot ax_{\varphi(m)} - x_1 \cdot$

$x_2 \cdot \cdots \cdot x_{\varphi(m)} = \left(a^{\varphi(m)} - 1\right) x_1 \cdot x_2 \cdot \cdots \cdot x_{\varphi(m)}$. Since $\left(m, x_1 \cdot x_2 \cdot \cdots \cdot x_{\varphi(m)}\right) = 1$, then $m | a^{\varphi(m)} - 1$.
If m is a prime number and $p \nmid a$, then $\varphi^{(m)} = m - 1$ and we get *Fermat's Little theorem* $a^{p-1} \equiv 1 (\operatorname{mod} p)$ directly from Euler's theorem. A variation of Fermat's Little theorem can be used to test primality [8]. If there exists $a \in \{2, \cdots, n-1\}$, $n > 3$, where $a^{n-1} \not\equiv 1 (\operatorname{mod} n)$, then n is a composite number and we call it *Fermat's witness for the compositeness of number* n [9].
Fermat's primality test can be suitably algorithmically presented in a selected computational environment (e.g. Matlab). The algorithm consists of two steps:

a) we randomly select number a for which $1 < a < n$
b) it is tested whether congruence $a^{n-1} \equiv 1 (\operatorname{mod} n)$ is satisfied

If congruence $a^{n-1} \equiv 1 (\operatorname{mod} n)$ is satisfied, the number n may or may not be a prime number. If congruence is not satisfied, the number n is not a prime and number a is the Fermat's witness for the compositeness of n.
Fermat's primality test works well for numbers that are not products of prime numbers different from each other. It can be demonstrated that if we test the number n, which is not the product of different prime numbers, hence there is such a prime p where $p^2 | n$, then with a probability of at least 75% we can choose between numbers $2, \cdots, n-1$ such a number which will be the Fermat's witness for the compositeness of n [9].
First, in Matlab, we create a function that helps us test congruence $a^{n-1} \equiv 1 (\operatorname{mod} n)$ generally for two given numbers a and n. The function will calculate the value a^{n-1} mod n which we will compare with 1 within the residue classes.

```
function res = test_congruence(a, n)

expn = n - 1;
res = 1;

while expn ~= 0
  if rem(expn, 2) == 1
    res = rem(res * a, n);
  end
  expn = floor(expn / 2);
  a = rem(a^2, n);
end
```

The second function randomly generates $a \in \{2, \cdots, n-1\}$ and we look for the Fermat's witness for the compositeness of n.

```
function test_fermat(n, cnt)

fo = false;
ii = 1;
while (ii <= cnt) && (~fo)
  a = 1 + unique(ceil((n - 2) * rand(1, 1)));
  tc = test_congruence(a, n);
  if(tc ~= 1)
    fermat_witness = a;
    fo = true;
  else
    ii = ii + 1;
  end
end

if fo
  disp(['Number ' num2str(n) ' is a composite
        number.']);
  disp(['Number ' num2str(fermat_witness) ' is a
        Witness for the compositeness of '
        num2str(n) '.']);
else
  disp(['Number ' num2str(n) ' can be a prime or a
        composite number.']);
end
```

The created test function is activated through the command line for any number n.

```
>> test_fermat(223, 1)
Number 223 can be a prime or a composite number.

>> test_fermat(273, 1)
Number 273 is a composite number.
Number 220 is a Witness for the compositeness of
273.
```

3 Euler's, Fermat's Little theorem applications

In this section, we have selected and compiled a number of specific tasks [10], [11] that guide on how to solve certain types of tasks using Euler's or Fermat's

Little theorem. We remark that for a natural number n greater than 1 in canonical decomposition $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ it holds that

$$\varphi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_k}\right) \text{[6]}$$

Example 3.1. First, we demonstrate that if we divide number 17^{24} by number 39, the remainder 1 is obtained.

Solution. It is determined that $a = 17$, $m = 39$. $(39,17) = 1$ and Euler's theorem can be applied. Let us calculate $\varphi(m) = \varphi(39) = 39\left(1-\frac{1}{3}\right)\left(1-\frac{1}{13}\right) = 24$. Then according to Euler's theorem $39|17^{24}-1$, thus $\exists k \in \mathbb{Z}$: $17^{24}-1 = 39k$. Then we can write $17^{24} = 39k+1$, and 1 is obtained as a remainder.

Example 3.2. It is demonstrated that p and $8p^2+1$ are simultaneously prime just when $p = 3$.

Solution. 1. First, $p = 3$. Then $8p^2+1 = 8 \cdot 9 + 1 = 73$, which is a prime.
2. Now let p and $8p^2+1$ be prime numbers simultaneously. $8p^2+1$ is adjusted as $8p^2+1 = 8p^2-8+9 = 8(p^2-1)+9$. Let p be a prime number other than 3. Then $(p,3) = 1$ a $3|p^{\varphi(3)}-1 = p^2-1$. Since $3|p^2-1$, then $8(p^2-1) \wedge 3|9$, then $3|8(p^2-1)+9 = 8p^2+1$ and $8p^2+1$ would not be a prime number, which is a controversy, thus $p = 3$.

Example 3.3. We show if a is not divisible by 5, then only one number from a^2– 1, a^2+1 is divisible by 5.

Solution. If a is a multiple of 5, according to Euler's theorem a^4-1 is a multiple of 5. Then only one of numbers a^2-1 and a^2+1 is a multiple of 5. They both concurrently cannot be, otherwise their difference would also be divisible by number 5, which is not, since $(a^2+1)-(a^2-1) = 2$.

Example 3.4. We find all primes p for which $5^{p^2}+1 \equiv 0(mod\ p^2)$.

Solution. The prime number $p = 5$ does not satisfy the task and at the same time $(p,5) = 1$. Then according to Euler's theorem $5^{p-1} \equiv 1 \pmod{p}$. By exponentiation to $p+1$ we get $5^{p^2-1} \equiv 1 \pmod{p}$, of which $5^{p^2} \equiv 5 \pmod{p}$.
Next, the task assignment states that $5^{p^2}+1 \equiv 0 \pmod{p^2}$, that implies $5^{p^2} \equiv -1 \pmod{p^2}$ and also $5^{p^2} \equiv -1 \pmod{p}$. Then congruences $5^{p^2} \equiv$

$5 \pmod p$ and $5^{p^2} \equiv -1 \pmod p$ hold that $5 \equiv -1 \pmod p$. Then $p|6$. In that $p = 2$ or $p = 3$. For $p = 2$ it holds that $5^4 + 1 \equiv 1^4 + 1 = 2 \not\equiv 0 \pmod 4$. For $p = 3$ it holds that $5^9 + 1 = 5^6 \cdot 5^3 + 1 \equiv 5^3 + 1 = 126 \equiv 0 \pmod 9$. Then, the only prime number satisfying the task is $p = 3$.

Example 3.5. For the odd number $m > 1$ we find the remainder after division of $2^{\varphi(m)-1}$ by number m.

Solution. Euler's theorem implies that $2^{\varphi(m)} \equiv 1 \equiv 1 + m = 2 \cdot \frac{1+m}{2} = 2r(\operatorname{mod} m)$ where r is a natural number $0 \leq r < m$.
The basic properties of congruences [7] determine that if $a \equiv b(\operatorname{mod} m)$ and d is an integer with properties $d|a$, $d|b$, $(d, m) = 1$, then $\frac{a}{d} \equiv \frac{b}{d}(\operatorname{mod} m)$. Indeed $a = a_1 d$, $b = b_1 d$ and according to assumption $m|(a - b)$, it holds that $m|d(a_1 - b_1)$. Since $(d, m) = 1$, it holds that $m|(a_1 - b_1)$. Then $a_1 \equiv b_1(\operatorname{mod} m)$, thus $\frac{a}{d} \equiv \frac{b}{d}(\operatorname{mod} m)$.
Then, however, we can divide both sides of the congruence $2^{\varphi(m)} \equiv 2r(\operatorname{mod} m)$ by their common divisor, number 2, which is relatively prime to the modulo. Then $2^{\varphi(m)-1} \equiv r(\operatorname{mod} m)$, and thus the remainder sought is $r = \frac{1+m}{2}$.

Example 3.6. We find the last two digits of number 137^{42}.

Solution. The task leads to the search for the remainder when dividing number 137^{42} by number 100. Since $(137{,}100) = 1$, according to Euler's theorem it holds that $137^{\varphi(100)} - 1$ is a multiply of 100 $(100|137^{\varphi(100)} - 1)$. Next $\varphi(100) = 100\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 40$. Then $137^{40} - 1$ is a multiply of 100.
Therefore $137^{42} = 137^2 137^{40} - 137^2 + 137^2 = 137^2(137^{40} - 1) + {}$ $+137^2 = 137^2(137^{40} - 1) + (100 + 37)^2 = 100k + (100 + 37)^2$.
Next, we use the formula $(a + b)^2 = a^2 + 2ab + b^2$. Then $137^{42} = 100k + 100l + 37^2 = 100n + 1369 = 100n + 1300 + 69 = 100m + 69$. Thus, the remainder sought is 69.

Example 3.7. We find the last 2 digits of number $a = 137^{47}$.

Solution. The last 2 digits of number *a* are again obtained as the remainder after dividing the number a by 100. $(100{,}137) = 1$ and Euler's theorem can be applied. Then $100|137^{\varphi(100)} - 1$, thus $100|137^{40} - 1$. Then $137^{40} - 1$ is a multiply of 100 and $137^{40} = 100k + 1$.

Let us calculate $137^{47} = 137^{40} \cdot 137^{7} = (100k + 1) \cdot 137^{7} = 100k \cdot 137^{7} + 137^{7}$ Number $100k \cdot 137^{7}$ cannot specify last 2 digits (ending with 2 zeroes), and so just number 137^{7} has the last 2 digits of the given number a. Next, the binomial theorem is applied.
$137^{7} = (130 + 7)^{7} = \binom{7}{0}130^{7} + \binom{7}{1}130^{6} \cdot 7 + \cdots + \binom{7}{6}130 \cdot 7^{6} + \binom{7}{7}7^{7}$. In this summation only the members $\binom{7}{6}130 \cdot 7^{6}$ a $\binom{7}{7}7^{7}$ decide the last two digits (other contribute zeroes in last two digits).
Their summation is calculated as $\binom{7}{6}130 \cdot 7^{6} + \binom{7}{7}7^{7} = 130 \cdot 7^{7} + 7^{7} = 131 \cdot 7^{7} = 107884133$. Overall, we get the last 2 digits of the number $a = 137^{47}$ which are 33.

Example 3.8. We find the remainder when dividing $(85^{70} + 19^{32})^{16}$ by number 21.

Solution. According to the binomial theorem $85^{70} = (84 + 1)^{70} = \binom{70}{0}84^{70} + + \binom{70}{1}84^{69} \cdot 1 + \cdots + \binom{70}{69}84 \cdot 1^{69} + \binom{70}{70}1^{70}$. We see that number 21 can be removed from every member except the last one. Then $85^{70} = (84 + 1)^{70} = 21n + 1$.
Because $\varphi(21) = 12$, $19^{12} - 1$ is a multiply of 21 (applying Euler's theorem), then $19^{32} = 19^{8}(19^{24} - 1) + 19^{8} = 21m + 19^{8}$. Therefore $(85^{70} + 19^{32})^{16} = (21n + 1 + 21m + 19^{8})^{16} = (21k + 1 + (21 - 2)^{8})^{16} = (21q + 1 + 2^{8})^{16} = (21r + 5)^{16} = 21t + 5^{16} = 21t + 5^{4}(5^{12} - 1) + 5^{4} = 21t + 21r + 625 = 21u + 16$. The remainder sought is 16.

Example 3.9. We demonstrate if $x^{p} + y^{p} = z^{p}$ where p is a prime number, then $x + y - z$ is a multiply of p.

Solution. According to Fermat's Little theorem, if p is a prime and $p \nmid x$, then $x^{p-1} \equiv 1 (\operatorname{mod} p)$, that means $p | x^{p-1} - 1$ and thus $p | x(x^{p-1} - 1) = x^{p} - x$. Similarly, $p | y^{p} - y$, $p | z^{p} - z$. Therefore we can write $x^{p} = pt_1 + x$, $y^{p} = pt_2 + y$ a $z^{p} = pt_3 + z$. If we substitute in the equation $x^{p} + y^{p} = z^{p}$, we get $p(t_3 - t_1 - t_2) = x + y - z$ after adjustment, thus $x + y - z$ is a multiply of p.

These examples are the basis for understanding the principle of working with large numbers using congruences through Euler's and Fermat's Little theorem. Congruences are a modern and irreplaceable security tool for protecting data by a public key. It is important to realize that the public key uses such large numbers for which there is no effective method of decomposing to primes even in today's modern computer age. That is why Euler's theorem plays its role in encryption even today, when encryption uses keys of up to 256 bits in length and deciphering the word while trying out all the options would probably take more years than the age of the universe is.

4 Conclusion

The paper points out some specific applications suitable for presenting and understanding the basic principle of Euler's and Fermat's Little theorems which are currently used in cryptography. Leonhard Paul Euler was such a great mathematician that many of the principles he had known almost 300 years ago were actually used by contemporary society. Euler, nicknamed as a "magician" in his time, had a great influence not only on number theory, but also on mathematical analysis or graph theory. He introduced many mathematical symbols such as the letter sigma Σ to denote the sum, or introduced numbers such as e and i, whereas e is probably the most important number of the whole mathematics [12] and occurs in various areas. When Mathematical Intelligencer in 2004 asked readers to vote for "the most beautiful theorem of mathematics", Euler's Identity $e^{i\pi} + 1 = 0$ won by a large margin [13]. It is a formula that connects the five most important symbols of mathematics. Several mathematicians have marked this equation as so mystical that it can only be reproduced and its consequences continually explored. In addition to Euler's theorem itself and its evidence by means of linear congruences, we also wanted to highlight the work and the "size" of Leonhard Euler and his key contribution to number theory.

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Mahgoub Transform on Boehmians

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Abstract

Boehmian's space is established utilizing an algebraic way that approximates identities or delta sequences and appropriate convolution. The space of distributions can be related to the proper subspace. In this paper, firstly we establish the appropriate Boehmian space, on which the Mahgoub Transformation can be described& function space K can be embedded. We add to more in this, our definitions enhance Mahgoub transform to progressively wide spaces. We additionally explain the functional axioms of Mahgoub transform on Boehmians. Lastly toward the finishing of topic, we analyze with specify axioms and properties for continuity and the enlarged Mahgoub transform, also its inverse regards to Δ-convergence and δ.

Keywords: Mahgoub Transform; The Space $\mathbb{B}(\mathfrak{X})$; The Space$\mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$; Boehmian Spaces.

2010 AMS subject classification: 44A99; 44A40;46F99; 20C20.[‡]

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1. Introduction

The Mahgoub transform [8] which is denoted by the operator $\mathfrak{M}(.)$ and Mahgoub transform of $\mathfrak{T}(t^*)$ is defined by:

$$\mathfrak{M}(\mathfrak{T}(t^*)) = \mathbb{E}(\vartheta) = \vartheta \int_0^\infty \mathfrak{T}(t^*)\, e^{-\vartheta t^*} dt^*, t^* \geq 0, \qquad (1.1)$$

and $\rho_1 \leq \vartheta \leq \rho_2$.

In a set ;

$$\mathbb{A} = \left\{ \mathfrak{T}(t^*): \exists \mathbb{M}, \rho_1, \rho_2 > 0 \,.\, |\mathfrak{T}(t^*)| < \mathbb{M} e^{\frac{|t^*|}{\rho_j}} \right\} \;, \qquad (1.2)$$

where ρ_1 and ρ_2(may be finite or infinite), the constant $\mathbb{M}$ must be finite. An existence's Mahgoub transform of $\mathfrak{T}(t^*)$ is essential for $t^* \geq 0$, a piece wise continuous and of exponential order is required, else it does not exist.

Convolution Theorem For Mahgoub Transform [9-11]:

If $\mathfrak{M}(\mathfrak{T}(t^*)) = \mathbb{E}(\vartheta)$ and $\mathfrak{M}(\mathfrak{P}(t^*)) = \mathbb{W}(\vartheta)$ then

$$\mathfrak{M}\big(\mathfrak{T}(t^*) \star \mathfrak{P}(t^*)\big) = \frac{1}{\vartheta} \mathfrak{M}\big(\mathfrak{T}(t^*)\big)\mathfrak{M}\big(\mathfrak{P}(t^*)\big) = \frac{1}{\vartheta} \mathbb{E}(\vartheta)\mathbb{W}(\vartheta) \qquad (1.3)$$

Linearity Property Of Mahgoub Transform:

If $\mathfrak{M}(\mathfrak{T}(t^*)) = \mathbb{E}(\vartheta)$, $\mathfrak{M}(\mathfrak{P}(t^*)) = \mathbb{W}(\vartheta)$ then

$$\mathfrak{M}\{\mathfrak{a}\mathfrak{T}(t^*) + \mathfrak{b}\,\mathfrak{P}(t^*)\} = \mathfrak{a}\,\mathfrak{M}\big(\mathfrak{T}(t^*)\big) + \mathfrak{b}\,\mathfrak{M}\big(\mathfrak{P}(t^*)\big) \qquad (1.4)$$

2. Boehmian Space

Boehmians was first developed as a generalization's standard mikusinski operators [2]. The formation necessary for Boehmians satisfying the following axioms.

i. a non empty set $\mathfrak{A}$;
ii. a semi group$(\varphi, \circledast)$ which is commutative;
iii. $\otimes: \mathfrak{A} \times \varphi \to \mathfrak{A}$ s.t.$\forall\, \xi \in \mathfrak{A}$ and $\eta_1, \eta_2 \in \varphi$, $\xi \otimes (\eta_1 \circledast \eta_2) = (\xi \otimes \eta_1) \otimes \eta_2$;
iv. a collection $\Delta \subset \varphi^N$such that
   a) If$\xi_1, \xi_2 \in \mathfrak{A}$, $(\eta_n) \in \Delta$, $\xi_1 \otimes \eta_n = \xi_2 \otimes \eta_n\ \forall\, n$then $\xi_1 = \xi_2$;
   b) If $(\eta_n), (\tau_n) \in \Delta$, then $(\eta_n \otimes \tau_n) \in \Delta$. where elements of Δ are known as delta sequences.

Consider

$\mathcal{H} = \{(\xi_n, \eta_n): \xi_n \in \mathfrak{A}\, , (\eta_n) \in \Delta, \xi_n \otimes \eta_m = \xi_m \otimes \eta_n \forall\, m, n \in N\}$,

Now if $(\xi_n, \eta_n), (\emptyset_n, \tau_n) \in \mathcal{H}$ then $\xi_n \otimes \tau_m = \emptyset_m \otimes \eta_n, \forall\, m, n \in N$.

We say that $(\xi_n, \eta_n) \sim (\emptyset_n, \tau_n)$. where $\sim$ is an equivalence relation in$\mathcal{H}$.The Set of equivalence classes in $\mathcal{H}$ is denoted as$\mathfrak{H}$. Elements of $\mathfrak{H}$ are said to be Boehmians.

We assume that there is a canonical embedding between $\mathfrak{H}$ and$\mathfrak{A}$, expressed as $\xi \to \frac{\xi_n \otimes \eta_n}{\eta_n}$,where$\otimes$ can also be extended in

$$\mathfrak{H} \times \mathfrak{A} \text{ by } \frac{\xi_n}{\eta_n} \otimes \tau = \frac{\xi_n \otimes \tau}{\eta_n}.$$

In $\mathfrak{H}$, there are two types of convergence is given by

i. if $\beth_n \to \beth$ as $n \to \infty$which belongs to$\mathfrak{A}$,$\mathscr{k} \in \varphi$ is any fixed element, then $\beth_n \otimes \mathscr{k} \to \beth \otimes \mathscr{k}$ as $n \to \infty$in $\mathfrak{A}$.

ii. if $\beth_n \to \beth$as $n \to \infty$ in $\mathfrak{A}$ and $\lambda_n \in \Delta$ then $\beth_n \otimes \lambda_n \to \beth$as $n \to \infty$in $\mathfrak{A}$.

An operation $\otimes$can be extended in$\mathfrak{H} \times \varphi$ as per condition:

If $\left[\frac{\beth_n}{\eta_n}\right] \in \mathfrak{H}$and $\mathscr{k} \in \varphi$ then $\left[\frac{\beth_n}{\eta_n}\right] \otimes \mathscr{k} = \left[\frac{\beth_n \otimes \mathscr{k}}{\eta_n}\right]$.

Now convergence in $\mathfrak{H}$as following:

1. A sequence (ς_n)in $\mathfrak{H}$is called$\delta-$ convergent to ς in $\mathfrak{H}$, i.e. $\varsigma_n \xrightarrow{\delta} \varsigma$ if $\exists\ (\eta_n) \in \Delta$ such that$(\varsigma_n \otimes \eta_n), (\varsigma \otimes \eta_n) \in \mathfrak{A}, \forall\, n \in N$ and $(\varsigma_n \otimes \eta_{\mathscr{k}}) \to (\varsigma \otimes \eta_{\mathscr{k}})$ as $n \to \infty$ in $\mathfrak{A}, \forall\, \mathscr{k}, n \in N$.
2. A sequence (ς_n) in $\mathfrak{H}$ is said to be Δ convergent to ς in $\mathfrak{H}$ i.e. $\varsigma_n \xrightarrow{\Delta} \varsigma$, if $\exists (\eta_n) \in \Delta$ such that$(\varsigma_n - \varsigma) \to 0$ as $n \to \infty$ which belongs to $\mathfrak{A}$.

For more details, see [3-6].

3. The Boehmian Space $\mathbb{B}(\mathfrak{X})$:

Denoted by $\mathfrak{S}_+(\mathbb{R})$ and $\mathcal{C}^\infty_{0+}(\mathbb{R})$are the space's smooth function over $\mathbb{R}$ and the Schwarz space's test function's compact support over $\mathbb{R}_+$ where $\mathbb{R}_+ = (0, \infty)$ respectively. We have found vital results for the structure of Boehmian space $\mathbb{B}(\mathfrak{X})$where $\mathfrak{X} = (\mathfrak{S}_+, \mathcal{C}^\infty_{0+}, \Delta_+)$.

Lemma 3.1:

1) If $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathcal{C}^\infty_{0+}(\mathbb{R})$then $\mathfrak{d}_1 \star \mathfrak{d}_2 \in \mathcal{C}^\infty_{0+}(\mathbb{R})$(Closure).

2) If $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathfrak{S}_+(\mathbb{R})$and $\mathfrak{d}_1 \in \mathcal{C}^\infty_{0+}(\mathbb{R})$then $(\mathfrak{F}_1 + \mathfrak{F}_2) \star \mathfrak{d}_1 = \mathfrak{F}_1 \star \mathfrak{d}_1 + \mathfrak{F}_2 \star \mathfrak{d}_1$ (Distributive).
3) $\mathfrak{d}_1 \star \mathfrak{d}_2 = \mathfrak{d}_2 \star \mathfrak{d}_1 \forall \mathfrak{d}_1, \mathfrak{d}_2 \in \mathcal{C}^\infty_{0+}(\mathbb{R})$ (Commutative).

4) If $\mathfrak{F} \in \mathfrak{S}_+(\mathbb{R})$, $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathcal{C}_{0+}^{\infty}(\mathbb{R})$then $(\mathfrak{F} \star \mathfrak{d}_1) \star \mathfrak{d}_2 = \mathfrak{F} \star (\mathfrak{d}_1 \star \mathfrak{d}_2)$(Associative).

Definition3.2: A sequence (η_n) of function from $\mathcal{C}_{0+}^{\infty}(\mathbb{R})$is said to be in$\Delta_+$. If

$$\Delta_+^1 : \int_{\mathbb{R}^+} \eta_n(\xi) d\xi = 1.$$

$\Delta_+^2 : \int_{\mathbb{R}^+} |\eta_n(\xi)| d\xi \leq m$,where m is a positive integer;

$\Delta_+^3 : Supp\, \eta_n(\xi) \subset (0, \epsilon_n)$, $\quad \epsilon_n \to 0$ as $n \to \infty$.

i.e.(η_n) shrink to zero as $n \to \infty$.every member of Δ_+ is known as an approximation identity or a delta sequences. In all manners delta sequences arise in numerous parts of Mathematics, however likely the very important application are those in the presupposition's generalized functions. The fundamental application of delta sequence is the regularization's established functions and ahead we can be utilized to characterize the convolution product and its established functions.

Lemma 3.3: If(η_n),(τ_n) $\in \Delta_+$, then $supp(\eta_n \star \tau_n) \subset supp\eta_n + supp\tau_n$.

Lemma 3.4: If $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathcal{C}_{0+}^{\infty}(\mathbb{R})$ then so is $\mathfrak{d}_1 \star \mathfrak{d}_2$ and $\int_{\mathbb{R}^+} |\mathfrak{d}_1 \star \mathfrak{d}_2| \leq \int_{\mathbb{R}^+} |\mathfrak{d}_1| \int_{\mathbb{R}^+} |\mathfrak{d}_2|$.

Theorem 3.5: Let$\mathfrak{F}_1, \mathfrak{F}_2 \in \mathfrak{S}_+(\mathbb{R})$ and $(\eta_n) \in \Delta_+$ such that

$$\mathfrak{F}_1 \star \eta_n = \mathfrak{F}_2 \star \eta_n.$$

where $n = 1,2,3, \ldots$, then $\mathfrak{F}_1 = \mathfrak{F}_2$ in $\mathfrak{S}_+(\mathbb{R})$.

Proof: To prove that $\mathfrak{F}_1 \star \eta_n = \mathfrak{F}_1$.

Let K be a compact support accommodating the $supp\eta_n$ for each$n \in N$. By using Δ_+^1, we write

$$\left|\xi^k D^m(\mathfrak{F}_1 \star \eta_n - \mathfrak{F}_1)(\xi)\right| \leq \int_K |\eta_n(\tau)| \left|\xi^k D^m\big(\mathfrak{F}_1(\xi - \tau) - \mathfrak{F}_1(\xi)\big)\right| d\tau \tag{3.1}$$

The mapping $\tau \to \mathfrak{F}_1^\tau$where$\mathfrak{F}_1^\tau = \mathfrak{F}_1(\xi - \tau)$, is Uniformly continuous from $\mathbb{R}^+ \to \mathbb{R}^+$. By using axiom$\Delta_+^3$that $supp\eta_n \to 0$ as $n \to \infty$, now we choose $r > 0$;$supp\eta_n \subseteq [0, r]$ for large n and $\tau < r$, that is

$$|\mathfrak{F}_1(\xi - \tau) - \mathfrak{F}_1(\xi)| = |\mathfrak{F}_1^\tau - \mathfrak{F}_1| < \frac{\epsilon_n}{M} \tag{3.2}$$

Hence using Δ_+^2 and Eq's. (3.2), (3.1) we get

$|\xi^k D^m(\mathfrak{F}_1 \star \eta_n - \mathfrak{F}_1)(\xi)| < \epsilon_n \to 0$ as $n \to \infty$.
Thus $\mathfrak{F}_1 \star \eta_n \to \mathfrak{F}_1$ in $\mathfrak{S}_+(\mathbb{R})$. Similarly,
we prove that $\mathfrak{F}_2 \star \eta_n \to \mathfrak{F}_2$ in $\mathfrak{S}_+(\mathbb{R})$ ☖

Theorem 3.6: if $\mathfrak{F}_n \to \mathfrak{F}$ in $\mathfrak{S}_+(\mathbb{R})$ as $n \to \infty$ and $\mathfrak{d} \in \mathcal{C}_{0+}^{\infty}(\mathbb{R})$ then $\lim_{n\to\infty} \mathfrak{F}_n \star \mathfrak{d} = \mathfrak{F} \star \mathfrak{d}$.

Proof: Using Theorem we get

$$\left|\xi^k D^m\big((\mathfrak{F}_n \star \mathfrak{d}) - (\mathfrak{F} \star \mathfrak{d})\big)(\xi)\right| = \left|\xi^k (D^m(\mathfrak{F}_n - \mathfrak{F}) \star \mathfrak{d})(\xi)\right| \quad (3.3)$$

The equation follows from [3]

$$D^m\mathfrak{F} \star \mathfrak{d} = D^m\mathfrak{F} \star \mathfrak{d} = \mathfrak{F} \star D^m\mathfrak{d}$$

for all $\mathfrak{d} \in \mathcal{C}_{0+}^{\infty}(\mathbb{R})$, we have

$$\left|\xi^k D^m\big((\mathfrak{F}_n \star \mathfrak{d}) - (\mathfrak{F} \star \mathfrak{d})\big)(\xi)\right| \le \int_K \xi^k |D^m(\mathfrak{F}_n - \mathfrak{F})(\xi - \tau)||\mathfrak{d}(\tau)|d\tau$$

$\le M\gamma_k(\mathfrak{F}_n - \mathfrak{F})$ for some constant M$\to 0$ as $n \to \infty$. ☖

Theorem 3.7: In $\mathfrak{S}_+(\mathbb{R})$, Let $\lim_{n\to\infty} \mathfrak{F}_n = \mathfrak{F}$ and
$(\eta_n) \in \Delta_+ \Rightarrow \lim_{n\to\infty} \mathfrak{F}_n \star \eta_n = \mathfrak{F}$.

Proof: By the hypothesis of the Theorem 3.5, we get $\lim_{n\to\infty} \mathfrak{F}_n \star \eta_n = \mathfrak{F}_n \to \mathfrak{F}$ as $n \to \infty$.

Hence , we arrive,

$\lim_{n\to\infty} \mathfrak{F}_n \star \eta_n = \mathfrak{F}$ as $n \to \infty$. ☖

The Canonical embedding between $\mathbb{B}(\mathfrak{X})$ and $\mathfrak{S}_+(\mathbb{R})$, defined as $\xi \to \left[\frac{\xi \star \eta_n}{\eta_n}\right]$.
The extension of $\star$ to $\mathbb{B}(\mathfrak{X}) \times \mathfrak{S}_+(\mathbb{R})$ is given by $\left[\frac{\xi_n}{\eta_n}\right] \star \tau = \left[\frac{\xi_n \star \tau}{\eta_n}\right]$.
Convergence in $\mathbb{B}(\mathfrak{X})$ is followed:

***δ*– Convergence:** A sequence (ς_n) in $\mathbb{B}(\mathfrak{X})$ is called $\delta-$ convergent to ς in $\mathbb{B}(\mathfrak{X})$ denoted by $\varsigma_n \xrightarrow{\delta} \varsigma$ if $\exists\, (\eta_n) \in \Delta$ such that $(\varsigma_n \star \eta_n), (\varsigma \star \eta_n) \in \mathfrak{S}_+(\mathbb{R}), \forall\, n \in N$ and $(\varsigma_n \star \eta_{k}) \to (\varsigma \star \eta_{k})$ as $n \to \infty$ in $\mathfrak{S}_+(\mathbb{R}), \forall\, k, n \in N$.

$\Delta_+ -$ Convergence: A sequence (ς_n) in $\mathbb{B}(\mathfrak{X})$ is said to be $\Delta_+ -$ convergent to ς in $\mathbb{B}(\mathfrak{X})$ i.e. $\varsigma_n \xrightarrow{\Delta} \varsigma$, if $\exists(\eta_n) \in \Delta_+$ such that $(\varsigma_n - \varsigma) \otimes \eta_n \in \mathfrak{S}_+(\mathbb{R}) \forall\, n \in N$ and $(\varsigma_n - \varsigma) \otimes \eta_n \to 0$ as $n \to \infty$ in $\mathfrak{S}_+(\mathbb{R})$.

Theorem 3.8: Define $\mathfrak{F} \rightarrow \left[\frac{\mathfrak{F}\star\eta_n}{\eta_n}\right]$ is continuous mapping which is embedding from $\mathfrak{S}_+(\mathbb{R})$ into $\mathbb{B}(\mathfrak{X})$.

Proof: To show**:** The mapping is one - one.

We have $\left[\frac{\mathfrak{F}_1\star\eta_n}{\eta_n}\right] = \left[\frac{\mathfrak{F}_2\star\tau_n}{\tau_n}\right]$, then

$$(\mathfrak{F}_1 \star \eta_n) \star \tau_m = (\mathfrak{F}_2 \star \tau_m) \star \eta_n \,, m, n \in N.$$

$\because (\tau_n), (\eta_n) \in \Delta_+, \mathfrak{F}_1 \star (\eta_m \star \tau_n) = \mathfrak{F}_2 \star (\tau_n \star \eta_m) = \mathfrak{F}_2 \star (\eta_m \star \tau_n)$.

Using Theorem 3.5, we get $\mathfrak{F}_1 = \mathfrak{F}_2$.

To prove: The mapping is continuous.

Let $\mathfrak{F}_n \rightarrow 0$ in$\mathfrak{S}_+(\mathbb{R})$as $n \rightarrow \infty$. Then we have$\left[\frac{\mathfrak{F}_n\star\eta_m}{\eta_m}\right] \xrightarrow{\delta} 0$as $n \rightarrow \infty$.

From the Theorem 3.5, $\left[\frac{\mathfrak{F}_n\star\eta_m}{\eta_m}\right] \star \eta_m = \mathfrak{F}_n \star \eta_m \rightarrow 0$as $n \rightarrow \infty$. ☖

Theorem3.9: Let $\mathfrak{d} \in \mathcal{C}_{0+}^{\infty}(\mathbb{R})$ and $\mathfrak{F} \in \mathfrak{S}_+(\mathbb{R}) \Rightarrow \mathfrak{M}(\mathfrak{F} \star \mathfrak{d})(\xi) = \frac{1}{\xi}\mathfrak{F}^{\mathfrak{M}}(\xi)\mathfrak{d}^{\mathfrak{M}}(\xi)$.

4. TheBoehmian Space $\mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$

We delineate Boehmian space as ensues. Let $\mathfrak{S}_+(\mathbb{R})$ be the space's immediately decreasing function [3]. We have

$$\mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R}) = \left\{\mathfrak{d}^{\mathfrak{M}} : \forall \mathfrak{d} \in \mathcal{C}_{0+}^{\infty}(\mathbb{R})\right\} \tag{4.1}$$

here $\mathfrak{d}^{\mathfrak{M}}$ express the Mahgoub transform of $\mathfrak{d}$ and also characterize $\mathfrak{F}\blacksquare\mathfrak{d}^{\mathfrak{M}}$ by

$$\left(\mathfrak{F}\blacksquare\mathfrak{d}^{\mathfrak{M}}\right)(\xi) = \frac{1}{\xi}\mathfrak{F}(\xi)\mathfrak{d}^{\mathfrak{M}}(\xi) \tag{4.2}$$

Lemma 4.1 Let $\mathfrak{F} \in \mathfrak{S}_+(\mathbb{R}), \mathfrak{d}^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R}) \Rightarrow \mathfrak{F}\blacksquare\mathfrak{d}^{\mathfrak{M}} \in \mathfrak{S}_+(\mathbb{R})$.

Proof. Let $\mathfrak{F} \in \mathfrak{S}_+(\mathbb{R}), \mathfrak{d}^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$, by Leibnitz' Theorem and applying the definition of $\mathfrak{S}_+(\mathbb{R})$, we found

$$\left|\xi^k D_\xi^m(\mathfrak{F}\blacksquare\mathfrak{d}^{\mathfrak{M}})(\xi)\right| \leq \left|\xi^k \sum_{j=1}^{m} D^{m-j}\left(\frac{1}{\xi}\mathfrak{F}(\xi)\right) D^j\mathfrak{d}^{\mathfrak{M}}(\xi)\right|$$

$$\leq \sum_{j=1}^{m}\left|\xi^k D^{m-j}\left(\frac{1}{\xi}\mathfrak{F}(\xi)\right)\right| \left|D^j\mathfrak{d}^{\mathfrak{M}}(\xi)\right|$$

$$= \textstyle\sum_{j=1}^{m}\left|\xi^k D^{m-j}\mathfrak{F}_1(\xi)\right| \left|\vartheta \int_K \mathfrak{d}(\tau)e^{-\frac{\tau}{\vartheta}}d\tau\right|$$

Where $\mathfrak{F}_1(\xi) = \frac{1}{\xi}\mathfrak{F}(\xi) \in \mathfrak{S}_+(\mathbb{R})$ and K is a compact support accommodating the $suppu(\tau)$.

$$\left|\xi^k D_\xi^m (\mathfrak{F} \blacksquare \mathfrak{d}^{\mathfrak{M}})(\xi)\right| \leq M\gamma_{k,m-j}(\mathfrak{F}_1) < \infty,$$

for some positive constant M. ☖

Lemma 4.2 A mapping $\mathfrak{S}_+ \times \mathcal{C}_{0+}^{\infty\mathfrak{M}} \to \mathfrak{S}_+$ is defined by

$$(\mathfrak{F}, \mathfrak{d}^{\mathfrak{M}}) \to \mathfrak{F} \blacksquare \mathfrak{d}^{\mathfrak{M}}$$

Satisfying the following axioms:

(1) If $\mathfrak{d}_1^{\mathfrak{M}}, \mathfrak{d}_2^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$, then $\mathfrak{d}_1^{\mathfrak{M}} \blacksquare \mathfrak{d}_2^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$.

(2) If $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathfrak{S}_+(\mathbb{R}), \mathfrak{d}^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$, then $(\mathfrak{F}_1 + \mathfrak{F}_2) \blacksquare \mathfrak{d}^{\mathfrak{M}} = \mathfrak{F}_1 \blacksquare \mathfrak{d}^{\mathfrak{M}} + \mathfrak{F}_2 \blacksquare \mathfrak{d}^{\mathfrak{M}}$.

(3) For $\mathfrak{d}_1^{\mathfrak{M}}, \mathfrak{d}_2^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R}), \mathfrak{d}_1^{\mathfrak{M}} \blacksquare \mathfrak{d}_2^{\mathfrak{M}} = \mathfrak{d}_2^{\mathfrak{M}} \blacksquare \mathfrak{d}_1^{\mathfrak{M}}$.

(4) For $\mathfrak{F} \in \mathfrak{S}_+(\mathbb{R}), \mathfrak{d}_1^{\mathfrak{M}}, \mathfrak{d}_2^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$ then $\left(\mathfrak{F} \blacksquare \mathfrak{d}_1^{\mathfrak{M}}\right) \blacksquare \mathfrak{d}_2^{\mathfrak{M}} = \mathfrak{F} \blacksquare \left(\mathfrak{d}_1^{\mathfrak{M}} \blacksquare \mathfrak{d}_2^{\mathfrak{M}}\right)$.

Proof .Axioms of above lemma as follows:

(1) Let $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathcal{C}_{0+}^{\infty}(\mathbb{R})$ then $\mathfrak{d}_1 \blacksquare \mathfrak{d}_2 \in \mathcal{C}_{0+}^{\infty}(\mathbb{R})$.

$$\Rightarrow (\mathfrak{d}_1 \blacksquare \mathfrak{d}_2)^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$$

By using Theorem (3.9) implies $\mathfrak{d}_1^{\mathfrak{M}} \blacksquare \mathfrak{d}_2^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$.

(2) Proof is straightforward.

(3) We have

$$\begin{aligned}\left(\mathfrak{d}_1^{\mathfrak{M}} \blacksquare \mathfrak{d}_2^{\mathfrak{M}}\right)(\xi) &= \frac{1}{\xi}\mathfrak{d}_1^{\mathfrak{M}}(\xi)\mathfrak{d}_2^{\mathfrak{M}}(\xi) \\ &= \frac{1}{\xi}\mathfrak{d}_2^{\mathfrak{M}}(\xi)\mathfrak{d}_1^{\mathfrak{M}}(\xi) \\ &= \left(\mathfrak{d}_2^{\mathfrak{M}} \blacksquare \mathfrak{d}_1^{\mathfrak{M}}\right)(\xi).\end{aligned}$$

$$\left(\mathfrak{d}_1^{\mathfrak{M}} \blacksquare \mathfrak{d}_2^{\mathfrak{M}}\right) = \left(\mathfrak{d}_2^{\mathfrak{M}} \blacksquare \mathfrak{d}_1^{\mathfrak{M}}\right)$$

(4)Let $\mathfrak{F} \in \mathfrak{S}_+(\mathbb{R})$ and $\mathfrak{d}_1^{\mathfrak{M}}, \mathfrak{d}_2^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$, then

$$\begin{aligned}\left(\left(\mathfrak{F} \blacksquare \mathfrak{d}_1^{\mathfrak{M}}\right) \blacksquare \mathfrak{d}_2^{\mathfrak{M}}\right)(\xi) &= \frac{1}{\xi}\left(\mathfrak{F} \blacksquare \mathfrak{d}_1^{\mathfrak{M}}\right)(\xi)\mathfrak{d}_2^{\mathfrak{M}}(\xi) \\ &= \frac{1}{\xi}\left\{\frac{1}{\xi}\mathfrak{F}(\xi)\mathfrak{d}_1^{\mathfrak{M}}(\xi)\right\}\mathfrak{d}_2^{\mathfrak{M}}(\xi) \\ &= \frac{1}{\xi}\mathfrak{F}(\xi)\frac{1}{\xi}\mathfrak{d}_1^{\mathfrak{M}}(\xi)\,\mathfrak{d}_2^{\mathfrak{M}}(\xi)\end{aligned}$$

$$= \frac{1}{\xi}\mathfrak{F}(\xi)\left(\mathfrak{d}_1^{\mathfrak{M}} \blacksquare \mathfrak{d}_2^{\mathfrak{M}}\right)(\xi)$$
$$= \left(\mathfrak{F} \blacksquare \left(\mathfrak{d}_1^{\mathfrak{M}} \blacksquare \mathfrak{d}_2^{\mathfrak{M}}\right)\right)(\xi),$$
$$\left(\mathfrak{F} \blacksquare \mathfrak{d}_1^{\mathfrak{M}}\right) \blacksquare \mathfrak{d}_2^{\mathfrak{M}} = \mathfrak{F} \blacksquare \left(\mathfrak{d}_1^{\mathfrak{M}} \blacksquare \mathfrak{d}_2^{\mathfrak{M}}\right) \qquad \square$$

Denote by$\Delta_+^{\mathfrak{M}}$ where$\Delta_+^{\mathfrak{M}}$ is the collection of all Mahgoub transform's delta sequence inΔ_+. i.e.,

$$\Delta_+^{\mathfrak{M}} = \left\{\left(\eta_n^{\mathfrak{M}}\right) : (\eta_n) \in \Delta_+, \forall\, n \in N\right\}. \tag{4.3}$$

Lemma 4.3 Let $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathfrak{S}_+(\mathbb{R}), \left(\eta_n^{\mathfrak{M}}\right) \in \Delta_+^{\mathfrak{M}}$ such that $\mathfrak{F}_1 \blacksquare \eta_n^{\mathfrak{M}} = \mathfrak{F}_2 \blacksquare \eta_n^{\mathfrak{M}}$, $\forall n$, then $\mathfrak{F}_1 = \mathfrak{F}_2$ in $\mathfrak{S}_+(\mathbb{R})$.

Proof. Let$\mathfrak{F}_1, \mathfrak{F}_2 \in \mathfrak{S}_+(\mathbb{R}), \left(\eta_n^{\mathfrak{M}}\right) \in \Delta_+^{\mathfrak{M}}$. Since $\mathfrak{F}_1 \blacksquare \eta_n^{\mathfrak{M}} = \mathfrak{F}_2 \blacksquare \eta_n^{\mathfrak{M}}$, using Eq.(4.2)

$\Rightarrow \frac{1}{\xi}\mathfrak{F}_1(\xi)\eta_n^{\mathfrak{M}}(\xi) = \frac{1}{\xi}\mathfrak{F}_2(\xi)\eta_n^{\mathfrak{M}}(\xi)$

Hence $\mathfrak{F}_1(\xi) = \mathfrak{F}_2(\xi)$ for allξ. $\square$

Lemma 4.4 For all $(\tau_n), (\eta_n) \in \Delta_+$, $\left(\eta_n^{\mathfrak{M}} \blacksquare {\tau_n}^{\mathfrak{M}}\right) \in \Delta_+^{\mathfrak{M}}$.

Proof. Since$(\tau_n), (\eta_n) \in \Delta_+$,$\eta_n \star \tau_n \in \Delta_+$, $\forall n$ hence from Theorem 3.9, we get

$\mathfrak{M}(\eta_n \star \tau_n)(\xi) = \frac{1}{\xi}\eta_n^{\mathfrak{M}}(\xi){\tau_n}^{\mathfrak{M}}(\xi) = \eta_n^{\mathfrak{M}} \blacksquare {\tau_n}^{\mathfrak{M}} \in \Delta_+^{\mathfrak{M}}$, for each n. $\square$

Lemma 4.5 Let $\lim_{n\to\infty} \mathfrak{F}_n = \mathfrak{F}$in $\mathfrak{S}_+(\mathbb{R})$, $\mathfrak{d}^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$ then $\mathfrak{F}_n \blacksquare \mathfrak{d}^{\mathfrak{M}} \to \mathfrak{F} \blacksquare \mathfrak{d}^{\mathfrak{M}}$in $\mathfrak{S}_+(\mathbb{R})$.

Proof.we know that $\mathfrak{d}^{\mathfrak{M}}$ is bounded in $\mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R})$ we have

$$\left(\mathfrak{F}_n \blacksquare \mathfrak{d}^{\mathfrak{M}}\right)(\xi) \to \frac{1}{\xi}\mathfrak{F}(\xi)\mathfrak{d}^{\mathfrak{M}}(\xi)$$

$\to \left(\mathfrak{F} \blacksquare \mathfrak{d}^{\mathfrak{M}}\right)(\xi)$.

Hence $\mathfrak{F}_n \blacksquare \mathfrak{d}^{\mathfrak{M}} \to \mathfrak{F} \blacksquare \mathfrak{d}^{\mathfrak{M}}$. $\square$

Lemma 4.6 Let $\lim_{n\to\infty} \mathfrak{F}_n = \mathfrak{F}$ in $\mathfrak{S}_+(\mathbb{R})$, $\left(\eta_n^{\mathfrak{M}}\right) \in \Delta_+^{\mathfrak{M}}$ then $\mathfrak{F}_n \blacksquare \eta_n^{\mathfrak{M}} \to \mathfrak{F}$ in $\mathfrak{S}_+(\mathbb{R})$.

Proof.Let $(\eta_n) \in \Delta_+$, $\eta_n^{\mathfrak{M}}(\xi) \to \xi$ is uniformly on compact subsets of $\mathbb{R}_+$. Hence

$$\left|\xi^k D_\xi^m\left(\mathfrak{F}_n \blacksquare \eta_n^{\mathfrak{M}} - \mathfrak{F}\right)(\xi)\right| = \left|\xi^k D_\xi^m\left(\frac{1}{\xi}\mathfrak{F}_n(\xi)\eta_n^{\mathfrak{M}}(\xi)\right) - \mathfrak{F}(\xi)\right|$$

$\to \left|\xi^k D_\xi^m(\mathfrak{F}_n - \mathfrak{F})(\xi)\right|$ as $n \to \infty$

Thus $\left|\xi^k D_\xi^m\left(\mathfrak{F}_n \blacksquare \eta_n^{\mathfrak{M}} - \mathfrak{F}\right)(\xi)\right| \to 0$ as $n \to \infty$.

This yield $\mathfrak{F}_n \blacksquare \eta_n^{\mathfrak{M}} \to \mathfrak{F}$ in$\mathfrak{S}_+(\mathbb{R})$. ☖

Lemma 4.7 Define$\mathfrak{F} \to \left[\frac{\mathfrak{F}\blacksquare\eta_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right]$ is a continuous mapping which is embedding from $\mathfrak{S}_+(\mathbb{R})$ into $\mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$. (4.4)

Proof. Let $\frac{\mathfrak{F}\blacksquare\eta_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}$is a quotient of sequences where$\mathfrak{F} \in \mathfrak{S}_+(\mathbb{R}), \eta_n^{\mathfrak{M}} \in \Delta_+^{\mathfrak{M}}$. We have $(\mathfrak{F}\blacksquare\eta_n^{\mathfrak{M}})\blacksquare\eta_m^{\mathfrak{M}} = \mathfrak{F}\blacksquare(\eta_m^{\mathfrak{M}}\blacksquare\eta_n^{\mathfrak{M}})$.We show that the map (4.3) is

one - to - one.

Let$\left[\frac{\mathfrak{F}_1\blacksquare\eta_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right] = \left[\frac{\mathfrak{F}_2\blacksquare\tau_n^{\mathfrak{M}}}{\tau_n^{\mathfrak{M}}}\right]$, then$(\mathfrak{F}_1\blacksquare\eta_n^{\mathfrak{M}})\blacksquare\tau_m^{\mathfrak{M}} = (\mathfrak{F}_2\blacksquare\tau_m^{\mathfrak{M}})\blacksquare\eta_n^{\mathfrak{M}}, m, n \in N$.
Now using of Lemma 4.2& 4.3, we get $\mathfrak{F}_1 = \mathfrak{F}_2$.

To establish the continuity of Eq.(4.4), let $\mathfrak{F}_n \to 0$as $n \to \infty$in $\mathfrak{S}_+(\mathbb{R})$. Then $\mathfrak{F}_n\blacksquare\eta_n^{\mathfrak{M}} \to 0$ as $n \to \infty$by Lemma 4.6, and hence

$\left[\frac{\mathfrak{F}\blacksquare\eta_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right] \to 0$,as $n \to \infty$ in $\mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$. ☖

5. The Mahgoub transform of Bohemians

Let $\mathfrak{H} = \left[\frac{\mathfrak{F}_n}{\eta_n}\right] \in \mathbb{B}(\mathfrak{X})$, then we delineate the Mahgoub transform of $\mathfrak{H}$ by the relation

$$\mathfrak{H}_1^{\mathfrak{M}} = \left[\frac{\mathfrak{F}_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right] \text{in } \mathbb{B}(\mathfrak{X}^{\mathfrak{M}}). \quad (5.1)$$

Theorem 5.1 $\mathfrak{H}_1^{\mathfrak{M}}: \mathbb{B}(\mathfrak{X}) \to \mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$ is well defined.

Proof. Let $\mathfrak{H}_1 = \mathfrak{H}_2 \in \mathbb{B}(\mathfrak{X})$, where $\mathfrak{H}_1 = \left[\frac{\mathfrak{F}_n}{\eta_n}\right], \mathfrak{H}_2 = \left[\frac{g_n}{\tau_n}\right]$Then the concept of quotients yields $\mathfrak{F}_n \star \tau_m = g_m \star \eta_n$. Applying Theorem 3.9, we get $\frac{1}{\xi}\mathfrak{F}_n^{\mathfrak{M}}(\xi)\tau_m^{\mathfrak{M}}(\xi) = \frac{1}{\xi}g_m^{\mathfrak{M}}(\xi)\eta_n^{\mathfrak{M}}(\xi)$,

$i.e. \mathfrak{F}_n^{\mathfrak{M}}\blacksquare\tau_m^{\mathfrak{M}} = g_m^{\mathfrak{M}}\blacksquare\eta_n^{\mathfrak{M}} \Rightarrow \frac{f_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}} \sim \frac{g_n^{\mathfrak{M}}}{\tau_n^{\mathfrak{M}}}$. Thus $\mathfrak{H}_1^{\mathfrak{M}} = \mathfrak{H}_2^{\mathfrak{M}}$. ☖

Theorem 5.2 $\mathfrak{H}^{\mathfrak{M}}: \mathbb{B}(\mathfrak{X}) \to \mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$ is continuous regards to δ-convergence.

Proof. Let $\mathfrak{H}_n \to 0$ in $\mathbb{B}(\mathfrak{X})$as $n \to \infty$. using [4] we get,$\mathfrak{H}_n = \left[\frac{\mathfrak{F}_{n,k}}{\eta_k}\right]$ and $\mathfrak{F}_{n,k} \to 0$ in $\mathfrak{S}_+(\mathbb{R})$ as $n \to \infty$ in $\mathfrak{S}_+(\mathbb{R})$. Now we apply the Mahgoub transform to both sides revenue $\mathfrak{F}_{n,k}^{\mathfrak{M}} \to 0$ as $n \to \infty$. Hence

$\mathfrak{H}_n^{\mathfrak{M}} = \left[\frac{\mathfrak{F}_{n,k}^{\mathfrak{M}}}{\eta_k^{\mathfrak{M}}}\right] \to 0$ as $n \to \infty$ in $\mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$. ☖

Theorem 5.3 $\mathfrak{H}^{\mathfrak{M}} : \mathbb{B}(\mathfrak{X}) \rightarrow \mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$is one-to-one mapping.

Proof. Let $\mathfrak{H}_1^{\mathfrak{M}} = \left[\frac{\mathfrak{F}_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right] = \left[\frac{g_n^{\mathfrak{M}}}{\tau_n^{\mathfrak{M}}}\right] = \mathfrak{H}_2^{\mathfrak{M}}$,then $\mathfrak{F}_n^{\mathfrak{M}} \blacksquare \tau_m^{\mathfrak{M}} = g_m^{\mathfrak{M}} \blacksquare \eta_n^{\mathfrak{M}}$.

Hence

$$(\mathfrak{F}_n \star \tau_m)^{\mathfrak{M}} = (g_m \star \eta_n)^{\mathfrak{M}}.$$

Since the Mahgoub transform is one to one, we get $\mathfrak{F}_n \star \tau_m = g_m \star \eta_n$.Thus

$$\frac{\mathfrak{F}_n}{\eta_n} \sim \frac{g_n}{\tau_n}.$$

Hence $\left[\frac{\mathfrak{F}_n}{\eta_n}\right] = \mathfrak{H}_1 = \left[\frac{g_n}{\tau_n}\right] = \mathfrak{H}_2$. ☖

Theorem 5.4 Let $\mathfrak{H}_1, \mathfrak{H}_2 \in \mathbb{B}(\mathfrak{X})$, then

(1) $(\mathfrak{H}_1 + \mathfrak{H}_2)^{\mathfrak{M}} = \mathfrak{H}_1^{\mathfrak{M}} + \mathfrak{H}_2^{\mathfrak{M}}$;

(2) $(\mathscr{k}\mathfrak{H})^{\mathfrak{M}} = \mathscr{k}\, \mathfrak{H}^{\mathfrak{M}}, \mathscr{k} \in \mathbb{C}$.

Theorem5.5 $\mathfrak{H}^{\mathfrak{M}} : \mathbb{B}(\mathfrak{X}) \rightarrow \mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$ is continuous regards to Δ_+ -convergence.

Proof. Let $\mathfrak{H}_n \overset{\Delta}{\rightarrow} \mathfrak{H}$ in $\mathbb{B}(\mathfrak{X})$ as $n \rightarrow \infty$ Then $\exists \mathfrak{F}_n \rightarrow 0\ \mathfrak{S}_+(\mathbb{R})$ and $(\eta_n) \in \Delta_+$ such that $(\mathfrak{H}_n - \mathfrak{H}) * \eta_n = \left[\frac{\mathfrak{F}_n * \eta_{\mathscr{k}}}{\eta_{\mathscr{k}}}\right]$ and $\mathfrak{F}_n \rightarrow 0$ as $n \rightarrow \infty$.Applying in Eq.(5.1) , we get

$$\mathfrak{M}\big((\mathfrak{H}_n - \mathfrak{H}) * \eta_n\big) = \left[\frac{\mathfrak{M}(\mathfrak{F}_n * \eta_{\mathscr{k}})}{\eta_{\mathscr{k}}^{\mathfrak{M}}}\right].$$

Hence we have $\mathfrak{M}\big((\mathfrak{H}_n - \mathfrak{H}) * \eta_n\big) = \left[\frac{\mathfrak{F}_n^{\mathfrak{M}} \eta_{\mathscr{k}}^{\mathfrak{M}}}{\xi \eta_{\mathscr{k}}^{\mathfrak{M}}}\right] \rightarrow 0$ as $n \rightarrow \infty$ in $\mathbb{B}\big(\mathfrak{X}^{\mathfrak{M}}\big)$. therefore

$\mathfrak{M}\big((\mathfrak{H}_n - \mathfrak{H}) * \eta_n\big) = \frac{1}{\xi}\big(\mathfrak{H}_n^{\mathfrak{M}} - \mathfrak{H}^{\mathfrak{M}}\big)\eta_n^{\mathfrak{M}} \rightarrow 0$ as $n \rightarrow \infty$. ☖

Theorem5.6 Let $\mathfrak{H}^{\mathfrak{M}} : \mathbb{B}(\mathfrak{X}) \rightarrow \mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$ is onto.

Proof. Let $\left[\frac{\mathfrak{F}_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right] \in \mathbb{B}\big(\mathfrak{X}^{\mathfrak{M}}\big)$ be arbitrary then $\mathfrak{F}_n^{\mathfrak{M}} \blacksquare \eta_m^{\mathfrak{M}} = \mathfrak{F}_m^{\mathfrak{M}} \blacksquare \eta_n^{\mathfrak{M}}$ for each $m, n \in N$.Then

$\mathfrak{F}_n \star \eta_m = \mathfrak{F}_m \star \eta_n$.That is, $\frac{\mathfrak{F}_n}{\eta_n}$is the corresponding quotient of sequences of $\frac{\mathfrak{F}_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}$. Thus $\frac{\mathfrak{F}_n}{\eta_n} \in \mathbb{B}(\mathfrak{X})$ is such that $\mathfrak{M}\left[\frac{\mathfrak{F}_n}{\eta_n}\right] = \left[\frac{\mathfrak{F}_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right]$ in $\mathbb{B}\big(\mathfrak{X}^{\mathfrak{M}}\big)$.

Let $\mathfrak{H}^{\mathfrak{M}} = \left[\frac{\mathfrak{F}_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right] \in \mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$, then we express the inverse of Mahgoub transform of $\mathfrak{H}^{\mathfrak{M}}$ given by

$\mathfrak{H}^{\mathfrak{M}^{-1}} = \left[\frac{\mathfrak{F}_n}{\eta_n}\right]$ in the space $\mathbb{B}(\mathfrak{X})$. ☖

Theorem5.7 Let $\left[\frac{\mathfrak{F}_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right] \in \mathbb{B}(\mathfrak{X}^{\mathfrak{M}})$ and $\mathfrak{d}^{\mathfrak{M}} \in \mathcal{C}_{0+}^{\infty\mathfrak{M}}(\mathbb{R}), \mathfrak{d} \in \mathcal{C}_{0+}^{\infty}(\mathbb{R})$.

$\mathfrak{H}\left(\left[\frac{\mathfrak{F}_n}{\eta_n}\right] \star \mathfrak{d}\right) = \left[\frac{\mathfrak{F}_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right] \blacksquare \mathfrak{d}^{\mathfrak{M}}$ and $\mathfrak{H}^{\mathfrak{M}^{-1}}\left(\left[\frac{\mathfrak{F}_n^{\mathfrak{M}}}{\eta_n^{\mathfrak{M}}}\right] \blacksquare \mathfrak{d}^{\mathfrak{M}}\right) = \left[\frac{\mathfrak{F}_n}{\eta_n}\right] \star \mathfrak{d}$.

We can easily proof from the definitions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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On cyclic multigroup family

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Abstract

In this paper, the concept of cyclic multigroup is studied from the preliminary knowledge of cyclic group which is a well-known concept in crisp environment. By using cyclic multigroups, we then delineate a cyclic multigroup family and investigate its structural properties. It is observed that the union of class of cyclic multigroups generated by $\boldsymbol{\mathcal{A}}$ is a cyclic multigroup. However, the intersection is an identity cyclic multigroup. In particular, we obtain a series of class of cyclic multigroups generated by $\boldsymbol{\mathcal{A}}$.

Keywords: Multiset, Multigroup, Cyclic multigroup, Cyclic multigroup family.

2010 AMS subject classification: $08A72$, $03E72$, $94D05$.†

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1 Introduction

In set theory, repetition of objects are not allowed in a collection. This perspective rendered set almost irrelevant because many real life problems admit repetition. To remedy the inadequacy in the idea of sets, the concept of multisets was introduced in [6] as a generalization of sets by relaxing the restriction of distinctness on the nature of the objects forming a set. Multiset is very promising in mathematics, computer science, website design, etc. See [4, 5] for details.

Generalization of algebraic structures is playing a prominent role in the sphere of mathematics. One of such generalization of algebraic structures is the notion of multigroups. Multigroups are actually a generalization of groups and have come into the centre of interest. In [1], the multigroup proposed is analogous to fuzzy group [2] in that the underlying structure is a multiset. Although multigroup concept was earlier used in [9, 12] as an extension of group theory, however the recent definition of multigroup in [1] is adopted in this paper because it shows a strong analogy in the behaviour of group and makes it possible to extend some of the major notions and results of groups to that of multigroups. Some of the related works can be found in [3], [7], [8], [10], [11] etc.

The aim of this paper is to promote research and the development of multiset knowledge by studying cyclic multigroup family based on the sufficient condition for a multiset to be a cyclic multigroup.

2 Preliminaries

In this section, we give the preliminary definitions and results that will be required in this paper from $[1, 8]$.

Definition 2.1 Let $\mho$ be a non-empty set. A multiset A drawn from $\mho$ is characterized by a count function C_A defined as $C_A : \mho \longrightarrow \boldsymbol{\mathcal{D}}$, where $\mathcal{D}$ represents the set of non-negative integers.

For each $x \in \mho$, $C_A(x)$ is the characteristics value of x in A and indicates the number of occurrences of the element x in A. An expedient notation of A drawn from $\mho = \{x_1, x_2, \ldots, x_n\}$ is $[x_1, x_2, \ldots, x_n]_{C_A(x_1), C_A(x_2), \ldots, C_A(x_n)}$ such that $C_A(x_i)$ is the number of times x_i occurs in A, $(i = 1, 2, \ldots, n)$.

The class of all multisets over $\mho$ is denoted by $MS(\mho)$.

Definition 2.2 Let $A, B \in \mho$. Then A is a submultiset of B written as $A \subseteq B$ or $B \supseteq A$ if $C_A(x) \leq C_B(x), \forall\, x \in \mho$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$.

Definition 2.3 Let $A, B \in MS(\mho)$. Then the union and intersection denoted by $A \cup B$ and $A \cap B$ are respectively defined as follows:

$$C_{A \cup B}(x) = C_A(x) \vee C_B(x) = max\{C_A(x), C_B(x)\} \text{ and}$$

$$C_{A \cap B}(x) = C_A(x) \wedge C_B(x) = min\{C_A(x), C_B(x)\}, \forall\, x \in \mho.$$

Definition 2.4 Let $\{A_i\}_{i \in \Lambda}$ be an arbitrary family of multisets over $\mho$. Then for each $i \in \Lambda$, $\bigcup_{i \in \Lambda} A_i = \bigvee_{i \in \Lambda} C_{A_i}(x)$ and $\bigcap_{i \in \Lambda} A_i = \bigwedge_{i \in \Lambda} C_{A_i}(x)$.

Definition 2.5 The direct product of multisets A and B is defined as

$$A \times B = \left\{[x, y]_{C_{A \times B}}[(x, y)] \mid C_{A \times B}[(x, y)] = C_A(x) C_A(y)\right\}.$$

Definition 2.6 Let $\mho$ be a non-empty set. The sets of the form

$A_n = \{x \in \mho \mid C_A(x) \geq n,\ \ n \in \mathbb{Z}^+\}$ are called the n – level sets of A.

Definition 2.7 Let $\mho$ and ξ be two non-empty sets and $f : \mho \longrightarrow \xi$ be a mapping. Then the image $f(A)$ of a multiset $A \in MS(\mho)$ is defined as

$$C_{f(A)}(y) = \begin{cases} \bigvee_{f(x)=y} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & f^{-1}(y) = \emptyset \end{cases}$$

Definition 2.8 Let $\mathcal{X}$ be a group. By a multigroup over $\mathcal{X}$ we mean a count function $C_A : \mathcal{X} \longrightarrow \mathcal{D}$ such that

$C_A(xy) \geq C_A(x) \wedge C_A(y), \forall\, x, y \in \mathcal{X}$ and $C_A(x^{-1}) \geq C_A(x), \forall\, x \in \mathcal{X}$.

Moreover, an abelian multigroup over $\mathcal{X}$ is defined as a multigroup satisfying the condition $C_A(xy) \geq C_A(yx), \forall\, x, y \in \mathcal{X}$.

Let e be the identity element of $\mathcal{X}$. It can be easily verified that if A is a multigroup over a group $\mathcal{X}$, then $C_A(e) \geq C_A(x)$ and $C_A(x^{-1}) \geq C_A(x), \forall\, x \in \mathcal{X}$.

We denote the class of all multigroups over $\mathcal{X}$ by $MG(\mathcal{X})$.

Proposition 2.1 Let $A \in MS(\mho)$. Then $A \in MG(\mathcal{X})$ if and only if $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y), \forall\, x, y \in \mathcal{X}$.

Proposition 2.2 Let $A \in MG(\mathcal{X})$. Then $A_n,\ n \in \mathbb{Z}^+$ are subgroups of $\mathcal{X}$.

Proposition 2.3 Let $\mathcal{X}, \mathcal{Y}$ be groups and $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a homomorphism. If $A \in MG(\mathcal{X})$, then $f(A) \in MG(\mathcal{Y})$.

3 Cyclic Multigroup Family

Definition 3.1 Let $\mathcal{X} = \langle a \rangle$ be a cyclic group. If $\mathcal{A} = \left\{[a^n]_{C_{\mathcal{A}}(a^n)} \mid n \in \mathbb{Z}\right\}$ is a multigroup, then $\mathcal{A}$ is called a cyclic multigroup generated by $[a]_{C_{\mathcal{A}}(a)}$ and denoted by $\langle [a]_{C_{\mathcal{A}}(a)} \rangle$.

Proposition 3.1 If $\mathcal{A}$ is a cyclic multigroup and $m \in \mathbb{Z}^+$, then $\mathcal{A}^m = \left\{\left([a^n]_{C_{\mathcal{A}}(a^n)}\right)^m \mid n \in \mathbb{Z}\right\}$ is also a cyclic multigroup.

Proof. Let us show that $\mathcal{A}^m$ satisfies the two conditions in Definition 2.8. We can consider only its count function because the $m-th$ power of $\mathcal{A}$ effects just only the count function of $\mathcal{A}^m$.

Since $\mathcal{A}$ is a multigroup and $C_{\mathcal{A}}(a) \in \mathcal{D}$, we have $(C_{\mathcal{A}}(a^{n_1}a^{n_2}))^m \geq \left(C_{\mathcal{A}}(a^{n_1}) \wedge C_{\mathcal{A}}(a^{n_2})\right)^m = (C_{\mathcal{A}}(a^{n_1}))^m \wedge (C_{\mathcal{A}}(a^{n_2}))^m$ and consequently, $\left(C_{\mathcal{A}}(a^{-n})\right)^m \geq \left(C_{\mathcal{A}}(a^n)\right)^m$.

This completes the proof of the proposition.

Example 3.1 Let $\mathcal{X} = \langle a \rangle$ be a cyclic group of order 12 such that $C_{\mathcal{A}}(a^0) = t_0$, $C_{\mathcal{A}}(a^4) = C_{\mathcal{A}}(a^8) = t_1,\ C_{\mathcal{A}}(a^2) = C_{\mathcal{A}}(a^6) = C_{\mathcal{A}}(a^{10}) = t_2,\ C_{\mathcal{A}}(x) = t_3$ for other elements $x \in \mathcal{X}$, where $t_i \in \mathcal{D},\ 0 \leq i \leq 3$ with $t_1 > t_1 > t_2 > t_3$. It is clear that $\mathcal{A}$ is a multigroup over $\mathcal{X}$. Thus, $\mathcal{A} = \left\{[a^n]_{C_{\mathcal{A}}(a^n)} \mid n \in \mathbb{Z}\right\}$ is a cyclic multigroup generated by $[a]_{C_{\mathcal{A}}(a)}$.

Definition 3.2 Let e be the identity element of the group $\mathcal{X}$. We define the identity cyclic multigroup $\mathcal{E}$ by $\mathcal{E} = \left\{[e]_{C_{\mathcal{A}}}(e) \mid C_{\mathcal{A}}(e) \geq C_{\mathcal{A}}(a^n),\ n \in \mathbb{Z}\right\}$.

Proposition 3.2 If $m \leq n$, then the multigroup $\mathcal{A}^n$ is a submultigroup of $\mathcal{A}^m$.

Proof. Clearly $\mathcal{A}^n$ and $\mathcal{A}^m$ are multigroups by Definition 2.8. For every $a \in \mathcal{D}$, $a^m \leq a^n$ implies $\mathcal{A}^m \subseteq \mathcal{A}^n$ (since $C_{\mathcal{A}^m}(a) \leq C_{\mathcal{A}^n}(a)\ \forall\ a \in \mathcal{X}$).

Proposition 3.3 If $\mathcal{A}^i$ and $\mathcal{A}^j$ are cyclic multigroups, and $i < j$, then $\mathcal{A}^i \cup \mathcal{A}^j$ is also a cyclic multigroup for any $i, j \in \mathbb{Z}^+$.

Proof. It is sufficient to consider only count functions. Without loss of generality, let $i \leq j$. Since $\mathcal{A}^i \subseteq \mathcal{A}^j$, we have

$$\begin{aligned} C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^n a^m) = C_{\mathcal{A}^i}(a^n a^m) \vee C_{\mathcal{A}^j}(a^n a^m) &= C_{\mathcal{A}^j}(a^n a^m) \\ &\geq C_{\mathcal{A}^j}(a^n) \wedge C_{\mathcal{A}^j}(a^m) \\ &= C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^n) \wedge C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^m) \end{aligned}$$

and

$$\begin{aligned} C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^{-n}) &= C_{\mathcal{A}^i}(a^{-n}) \vee C_{\mathcal{A}^j}(a^{-n}) \\ &= C_{\mathcal{A}^i}(a^n) \vee C_{\mathcal{A}^j}(a^n) = C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^n) \end{aligned}$$

Hence, $\mathcal{A}^i \cup \mathcal{A}^j$ is a cyclic multigroup.

Proposition 3.4 If $\mathcal{A}^i$ and $\mathcal{A}^j$ are cyclic multigroups, then $\mathcal{A}^i \cap \mathcal{A}^j$ is also a cyclic multigroup**.**

Proof. Similar to Proposition 3.3.

Remark 3.1 Since a cyclic group is an abelian group, it is obvious by Definition 2.8 that the cyclic multigroups $\mathcal{A}^m$, $\mathcal{A}^i \cup \mathcal{A}^j$ and $\mathcal{A}^i \cap \mathcal{A}^j$ are also abelian multigroups.

Definition 3.3 Let $\mathcal{A}$ be a cyclic multigroup, then the following class of cyclic multigroups $\{\mathcal{A}, \mathcal{A}^2, \mathcal{A}^3, \dots, \mathcal{A}^m, \dots, \mathcal{E}\}$ is called the cyclic multigroup family generated by $\mathcal{A}$ and denoted by $\langle\mathcal{A}\rangle$.

Proposition 3.5 Let $\langle\mathcal{A}\rangle = \{\mathcal{A}, \mathcal{A}^2, \mathcal{A}^3, \dots, \mathcal{A}^m, \dots, \mathcal{E}\}$. Then $\bigcup_{n=1}^{\infty} \mathcal{A}^n = \mathcal{A}$ and $\bigcap_{n=1}^{\infty} \mathcal{A}^n = \mathcal{E}$.

Proof. The proof is immediate from Propositions 3.3 and 3.4.

Proposition 3.6 Let $\mathcal{A}$ be a cyclic multigroup. Then $\mathcal{A} \subseteq \mathcal{A}^2 \subseteq \mathcal{A}^3 \subseteq \cdots \subseteq \mathcal{A}^n \subseteq \cdots \subseteq \mathcal{E}$.

Proof. It is known that $C_{\mathcal{A}}(a) \in \mathcal{D}$. Hence,
$C_{\mathcal{A}}(a) \leq \left(C_{\mathcal{A}^2}(a)\right)^2$, $C_{\mathcal{A}}(a^2) \leq \left(C_{\mathcal{A}^2}(a^2)\right)^2, \dots,$ $C_{\mathcal{A}}(a^n) \leq \left(C_{\mathcal{A}^2}(a^n)\right)^2$.

By Definition 2.2, we have $\mathcal{A} \subseteq \mathcal{A}^2$. By generalizing it for any $i, j \in \mathbb{Z}^+$ with $i \leq j$, we then obtain $\left(C_{\mathcal{A}^i}(a)\right)^i \leq \left(C_{\mathcal{A}^j}(a)\right)^j, \left(C_{\mathcal{A}^i}(a^2)\right)^i \leq \left(C_{\mathcal{A}^j}(a^2)\right)^j, \dots, \left(C_{\mathcal{A}^i}(a^n)\right)^i \leq \left(C_{\mathcal{A}^j}(a^n)\right)^j$. So $\mathcal{A}^i \subseteq \mathcal{A}^j$ for any $i, j \in \mathbb{Z}^+$ with $i \leq j$, which means that $\mathcal{A} \subseteq \mathcal{A}^2 \subseteq \mathcal{A}^3 \subseteq \cdots \subseteq \mathcal{A}^n \subseteq \cdots$.

Finally, we have $\mathcal{E} = \bigcap_{n=1}^{\infty} \mathcal{A}^n$, which is immediate from Proposition 3.5 since

$$\lim_{n \to \infty} C_{\mathcal{A}}(a^n) = \begin{cases} t_0, & if \ a = e, \\ 0, & if \ a \neq e. \end{cases}$$

This completes the proof for the required relations.

Corollary 3.1 Let $\langle \mathcal{A} \rangle = \{\mathcal{A}, \mathcal{A}^2, \mathcal{A}^3, \dots, \mathcal{A}^m, \dots, \mathcal{E}\}$. Then
$$\mathcal{A} < \mathcal{A}^2 < \mathcal{A}^3 < \cdots < \mathcal{A}^m < \cdots < \mathcal{E}.$$

Proof. The proof is similar to Proposition 3.6.

Proposition 3.7 Let φ be a group homomorphism of a cyclic multigroup $\mathcal{A}$. Then the image of $\mathcal{A}$ under φ is a cyclic multigroup.

Proof. It is well known that in the theory of classical cyclic groups, the image of any cyclic group is a cyclic group and the homomorphic image of a multigroup is a multigroup (from Proposition 2.3). From these two results and Definition 2.8, it is clearly seen that the image of $\mathcal{A}$ under φ is a cyclic multigroup.

Proposition 3.8 Let $\mathcal{X}_n$ be the $n-$ level set of the cyclic group $\mathcal{X}$. If $i, j \in \mathbb{Z}^+$ such that $i < j$, then $\mathcal{A}_n^i$ is a subgroup of $\mathcal{A}_n^j$.

Proof. It is obvious that sets $\mathcal{X}_n$ and $\mathcal{X}_n^m$ are cyclic subgroups of $\mathcal{X}_n$ in crisp sense. Since $i < j$, then $\mathcal{A}_n^j(a) \geq \mathcal{A}_n^i(a) \geq n$, $\forall\, a \in \mathcal{X}_n^j$. Thus, $\mathcal{X}_n^i \subseteq \mathcal{X}_n^j$. Therefore, $\mathcal{X}_n^i$ is a subgroup of $\mathcal{X}_n^j$.

Remark 3.2 From Propositions 3.6 and 3.8, we have that a normal series of $\mathcal{X}$ is a finite sequence $\mathcal{X}_n^m, \mathcal{X}_n^{m-1}, \dots, \mathcal{X}_n$ of normal level subgroups of $\mathcal{X}$ such that $\mathcal{X}_n^m > \mathcal{X}_n^{m-1} > \cdots > \mathcal{X}_n$ since $\mathcal{X}$ is a cyclic group.

Proposition 3.9 Let $\{\mathcal{A}^m, \mathcal{A}^{m-1}, \dots, \mathcal{A}\}$ be a finite cyclic multigroup family. Then $\mathcal{A}^m \times \mathcal{A}^{m-1} \times \dots \times \mathcal{A} = \mathcal{A}^m$.

Proof. It is easily verified using the definition of product of multigroups and Proposition 3.6.

4 Conclusion

The paper introduced the concept of cyclic multigroup family and investigated its related structure properties. For future studies, one can extend this idea to other non-classical algebraic structures such as soft group, rough group, neutrosophic group and smooth group.

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A conceptual proposal on the *undecidability* of the distribution law of prime numbers and theoretical consequences

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In music and in mathematics
there is the same perfume of eternity

Abstract†

Within the conceptual framework of number theory, we consider prime numbers and the classic still unsolved problem to find a complete law of their distribution. We ask ourselves if such persisting difficulties could be understood as due to theoretical incompatibilities. We consider the problem in the conceptual framework of computational theory. This article is a contribution to the philosophy of mathematics proposing different possible understandings of the supposed *theoretical* unavailability and indemonstrability of the existence of a law of distribution of prime numbers. Tentatively, we conceptually consider demonstrability as computability, in our case the conceptual availability of an algorithm able to compute the general properties of the presumed primes' distribution law *without computing such distribution*. The link between the conceptual availability of a distribution law of primes and decidability is given by considering how to decide if a number is prime without computing. The supposed distribution law should allow for any given prime knowing the next prime without factorial computing. *Factorial properties of numbers, such as their property of primality, require their factorisation (or equivalent, e.g., the sieves), i.e., effective computing. However, we have factorisation techniques available, but there are no (non-quantum) known algorithms which can effectively factor arbitrary large integers. Then factorisation is undecidable. We consider the theoretical unavailability of a distribution law for factorial properties, as being prime, equivalent to its non-computability, undecidability.*

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The availability and demonstrability of a hypothetical law of distribution of primes are inconsistent with its undecidability. The perspective is to transform this conjecture into a theorem.

Keywords: algorithm; computation; decidability; incompleteness; indemonstrability; law of distribution; prime numbers; symbolic; undecidability.

2010 AMS subject classification: 11N05; 00A30; 11A51.

1 Introduction

Number theory is an antique and fascinating discipline. Number theory considers endless properties of numbers such as perfect numbers, golden ratios, and Fibonacci numbers.

An endless list of approaches, problems, properties, and results added one to the other over time deal with prime numbers and the possibility to find a suitable law of their distribution.

With regards to prime numbers, mathematicians introduced several conjectures, and not definitive, proven partial results.

To name a few, we recall properties and results relating to prime number generation such as the *Fundamental theorem of Arithmetic* (by Gauss in the 1801), the Goldbach's conjecture (approximately 1742), the classic sieve of Eratosthenes (275–194 B.C.), the sieve of Sundaram (approximately 1934), the sieve of Atkin (approximately 2003), and the Mersenne prime (1536) - of the form $M_n = 2^n - 1$ - for pseudorandom number generators, all used for applications such as cryptography.

Throughout history, several important mathematicians have tentatively contributed to the identification of the asymptotic law of distribution of prime numbers and its proof. We just mention Legendre (approximately 1808), Dirichlet (approximately 1837), Gauss (approximately in 1849 reported the connection between prime numbers and logarithms), Riemann (in 1859) wrote his very famous article (Riemann, 1859), Euler's theorem (approximately 1763) as a generalisation of Fermat's little theorem, Chebyshev (approximately 1850), and Yitang Zhang's contributions to the twin-prime conjecture (approximately 2013).

However, since providing a complete review of the literature is beyond the scope of this article, we leave it to the reader to familiarize themselves with the literature on this subject.

The contribution to the philosophy of mathematics of the present article is to propose different possible understandings of the unavailability and *indemonstrability of the existence of the law of distribution of prime numbers. Further research is expected to allow suitable formalisations.*

In Section 2 we consider generic indemonstrability as a fact of incompleteness and *platonic consistency of knowledge.* This is further explored in Section 4, in a constructivist understanding, where we propose indemonstrability to prevent inevitably implicit inconsistencies because a paradigm shift is required instead.

In Section 3 we propose to consider demonstrability as having symbolic nature and as decidability. Indemonstrability cannot be demonstrated and it can be intended as a fact of incompleteness, case of undecidability. The link between the conceptual availability of a distribution law of primes and decidability is given by considering how to *decide* if a number is prime *without* computing. The supposed distribution law should allow for any given prime knowing what the next prime with *without computing such sequences*.

However, factorial properties of numbers, such as their property of being prime, require their factorisation (or equivalent, e.g., the sieves), i.e., effective computing.

Because of that it is not possible to know in advance the properties of the factorisation, in the same way as it is not possible to solve the alt of a Turing Machine (TM) -the *halting problem* consists on determining if an arbitrary computer program and its input will finish running or continue to run forever (such as being *in loop*). A general algorithm to solve the halting problem for all possible program-input couples cannot exist-**,** it is not possible to know the result of the processing of a Neural Network without performing the entire processing, and to know the patterns generated by a Cellular Automata without performing the entire processing.

In Section 4, regarding the research relating to a Prime's Distribution Law (PDL), we present, for the general reader, a short, partial overview of the situation as it currently consists mainly of a list of conjectures. Such conjectures have been not falsified but, rather, computationally confirmed by considering numerically large cases.

In Section 5 we tentatively conceptually consider demonstrability as computability, i.e., in our case the *conceptual existence* of an algorithm able to compute the general properties of the presumed primes' distribution law *without computing such distribution*. We tentatively consider generic indemonstrability, unavailability as undecidability of the law of distribution and the probabilistic nature of the Prime Number Theorem (PNT) as an aspect of its undecidability. We consider then the usability of such undecidability, in the historical conceptual framework of the very effective usability of imaginary numbers. We ask ourselves if the non-demonstrability of existence of the PDL and its non-discovery can be intended as a prototype of the non-distribution and of possible different non-equivalent non-distributions. Besides, such non-demonstrability of existence and the persisting non-availability of the PDL may be considered as a prototype of the generic non-demonstrability, of theoretical incompleteness, and theoretical incomprehensibility.

We conclude by mentioning how from the issues considered above it is possible to use such incompleteness in order to introduce paradigm shifts and non-equivalent, mutually irreducible, incommensurable approaches.

2 Indemonstrability as a fact of consistency

We consider here a kind of *platonic consistency of the knowledge*, as *theoretical incompleteness* [1, 2] which manifests when dealing with incomplete problems or indemonstrability of incomplete or wrong theses. In a *constructivist understanding* it is a kind of *experiment having no reaction as a result, stating that the experiment is inadmissible, inconsistent, wrong*.

As a classic example, consider the unsuccessful attempts to demonstrate the fifth postulate in Euclidian geometry. The history of the attempts to demonstrate the fifth postulate reveals how the conclusion was obtained by appealing to a new proposition that was equivalent to the fifth postulate itself.

The Italian mathematician Eugenio Beltrami discovered the Giovanni Girolamo Saccheri's article *Euclides ab omni naevo vindicatus* (*Euclid Freed of Every Flaw*), published in 1733 in which he tried to prove the Euclid's postulate of parallel lines. By using a similar approach, Beltrami, among others, inadvertently introduced a paradigm shift towards the non-Euclidean geometries by reasoning per *reductio ad absurdum*, i.e., as a result of the impossibility of proving the absurdity of the negation of the fifth postulate [3, 4].

An example of a relationship between theoretical incompleteness and indemonstrability is given by the two celebrated Gödel's syntactic incompleteness theorems [5].

The meaning of the first theorem states that within any mathematical theory, having at least the power of arithmetic, there exists a formula that, neither the formula nor its negation is syntactically provable. In other words, it is possible to construct a formally correct proposition that, however, cannot be proven or disproved. This is logically equivalent to the construction of a logical formula that denies its provability.

The meaning of the second theorem is that no coherent system is able to demonstrate its own syntactic coherence. The two theorems can be intended to *prove* the inexhaustibility *in principle* of pure mathematics [6-8]. "In other words, infinite-state logical theories when sufficiently complex are necessarily incomplete. Whether this result implies a sort of incompleteness of other kinds of theories (for instance, those of physics) is still an open question [9, p. 7].

As for incompleteness in physics, it is closely related to the *uncertainty principles*. It relates to the well-known uncertainty principle, first introduced by Werner Heisenberg [10]. Furthermore, there is the *principle of complementarity* introduced by Neils Bohr [11] stating that the corpuscular and undulatory aspects of a physical phenomenon cannot be observed simultaneously.

This is the case of the measurement of *homologous components* such as position and momentum.

From now on we consider a tentative relationship among some generic concepts such as indemonstrability, incompleteness and undecidability:

- theoretical incompleteness and indemonstrability; indemonstrability as a fact of incompleteness;
- demonstrability of incompleteness;
- the other issue is that of *indemonstrability and (as?) undecidability*.
-

3 Indemonstrability and undecidability

A problem is considered as "undecidable" when there is no algorithm that produces the corresponding solution in a finite time for each instance of the input data. A typical example is the *classic halting problem* for the Turing Machine [12]. The set of decidable problems is incomplete. In this regard, Turing himself introduced an issue of 'completion' by inserting the concept of Oracle [13], representing another logic, possibly incommensurable, that, however, combines, interferes, superimposes, and acts on that in use. All this in the framework of a general theory of truth, e.g., Tarskian semantics, see, for instance [1].

However, even in case of availability of effective computational algorithms, the finite precision or finite memory (in case for symbolic manipulation) implies theoretical incompleteness [14-16].

Moreover, another example is given by the non-explicit, non-symbolic computation, for instance, of Artificial Neural Networks (ANNs), see, for example [17, 18].

The computational processing is represented and performed in a non-analytical, non-symbolic way through weighted connections and levels. If we look instant per instant at the calculation performed, it is incomprehensible and we have to wait for the final result. This also applies to other computational processes such as Cellular Automata. *The computation acquires properties not formally prescribed like learning* [19, 20].

Particular classes of ANNs, such as those with non-Turing computable weights, and Recurrent-ANNs [21, 22] show a non-Turing behaviour for which the principles of *hypercomputation* [23-25] and *naturally-inspired computation* [26] *apply*.

Indemonstrability cannot be symbolically demonstrated and is intended to be a fact of incompleteness, case of undecidability. Furthermore, it is possible to conceptually consider symbolic demonstrability as having logical equivalences with decidability.

4 Prime numbers

Please download the latex template and see the .pdf file to see how to format editing definitions, theorems, corollaries.

At this point we may ask ourselves how to interpret the non-comprehension, the non-availability of the PDL, which is used in areas such as cryptography [27]? As incompleteness of the theory of numbers, undecidability, and indemonstrability [28]?

The problem has been frequented by mathematicians for centuries, with important, but not definitive results.

At this point we may consider two questions:

- In a constructivist understanding, can we intend such barrier to prevent an inevitably, implicitly inconsistent demonstration because a paradigm shift is required instead?
- In a platonic understanding, can we intend such a barrier to protect from an inevitably wrong demonstration contrasting with the general consistency and requiring different entry points?

4.1 A brief summary of the current situation

Attention to prime numbers first focused on the question whether they were infinite or not, and then turned to the understanding how they are distributed within natural numbers. It dates back to the 3rd century BC and to the Euclid's first proof that infinitely many primes exist (see the *Elements*, Book IX, Proposition 20), see the Polignac's conjecture below. In modern times Euler gave an alternative proof of this result by using, for the first time, concepts coming from infinitesimal mathematical analysis. Gauss understood the still fundamental key to the understanding of a crucial characteristic of the prime numbers: their density.

Riemann introduced his conjecture, listed below, which concerns the distribution of the zeros of a particular complex function, known as the zeta function, which has a very close connection with the distribution of primes. In particular, the distribution of the zeros of the zeta function is linked to the possibility of accurately counting the prime numbers.

In what follows, we propose a very short overview on the very large world of attempts to deal with the still unsolved problem of finding a PDL. This world includes mainly conjectures and few theorems. We give approximate reference dates for the convenience of the general reader.

4.2 An overview

The overview [29-31] includes the following subjects.

1) Goldbach's conjecture 1742: every even integer greater than 2 can be expressed as the sum of two primes.
2) Cramér's conjecture, 1936: it gives an asymptotic estimate for the size of gaps between consecutive prime numbers

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{(\ln p_n)^2} = 1,$$

where:

- p_n denotes the nth prime;
- *ln* is the natural logarithm.

This is based on a probabilistic model assuming that the probability that a natural number x is prime is $1/\ln x$, from which it can be shown that the conjecture is true with probability 1. In other words, if the prime numbers follow a "random" distribution, it is very likely that the conjecture is true. In short, the Cramér's Conjecture states that the difference between two consecutive prime numbers always remains less than the square of the natural logarithm of the smaller of the first two.

This conjecture implies the following:

3) Opperman's conjecture, approximately 1882: the conjecture states that, for every integer $x > 1$, there is at least one prime number between $x(x - 1)$ and x^2.

This conjecture in turn implies the next conjecture:

4) Legendre's (1752 – 1833) conjecture: it states that there exists at least one prime number between n^2 and $(n + 1)^2$ for all natural numbers.

The previous conjectures are all more restrictive than the Bertrand Postulate (which has been proven and is now a theorem):

5) Bertrand Postulate, approximately 1845: in its less restrictive formulation it states that for every $n>1$ there is always at least one prime p such that $n<p<2n$.
6) Polignac's Twin prime conjecture (approximately 1846 and previously considered by Euclid): it states that there are infinitely many twin primes, or pairs of primes that differ by *2*. As numbers get larger, primes become less frequent and twin primes become rarer as well. In this regard in 1919 Brun's Theorem showed that the sum of the reciprocals of the twin primes converges to a sum, now known as Brun's constant. In 2010, the value of Brun's constant was approximately *1.902160583209* ±

0.000000000781 based on all twin primes less than 2×10^{16}. Conversely, the sum of the reciprocals of all primes diverges to infinity [32].

7) Riemann Hypothesis, 1859: it deals with the distribution of the zeros of a particular complex function, now called "Riemann zeta function", which has a very close connection with the distribution of primes. In particular, the distribution of its zeros is linked to the possibility of accurately counting the prime numbers. The Riemann Hypothesis can be described geometrically by saying that the zeros of the Riemann zeta function are confined to two lines in the complex plane [33, 34].

8) The *Prime Number Theorem* (PNT) describes the *asymptotic distribution of prime numbers*: it states a general view of how primes are distributed among positive integers and also states that the primes become less common as they become larger. Let $\pi(x)$ be the prime-counting function that gives the number of primes less than or equal to x, for any real number x. The PNT then states that $x / \log x$ is a good approximation to $\pi(x)$, *that is* $\pi(x) \sim x \log x$. This notation means only that the quotient limit of the two functions π (x) and $x / \ln(x)$ for x which tends to infinity is *1*, but not that the limit of the difference of the two functions, as x tends to infinity, is *0*. This means that for large enough N, the probability that a random integer not greater than N is prime is very close to $1 / \log(N)$. The PNT is based on several previous and subsequent, increasingly specifying contributions, such as Legendre’s conjecture stating that $\pi(a)$ is approximated by the function $a / (A \log a + B)$, where A and B are unspecified constants; Gauss studied the problem; Dirichlet introduced a logarithmic integral $li(x)$ as approximating function; the connection between the prime number theorem and the Riemann zeta function is very deep and allowed by the *Euler product*.

The plausibility of such conjectures and approaches is supported by a large number of computational simulations which did not lead to falsifying cases.

5 Indemonstrability as undecidability of the distribution?

The main conclusions of the study may be presented in a short Conclusions section, which may stand alone or form a subsection of a Discussion or Results.

We are tentatively proposing to consider here the incomplete, probabilistic or approximate nature of PNT not as much as a limit to be solved by more appropriated approaches, but as an unavoidable theoretical aspect, *price* to be paid for consistency within the *theory of computation* rather than within *number* *theory* itself. We consider here that number theory and its properties and theorems may be not incompatible with the availability of regularities in the distribution of prime numbers, while such availability can be considered *incompatible* with other properties and theories, such as general forms of theoretical, structural incompleteness, such as the *halting problem* for the Turing Machine in theory of computation.

We consider here, reasoning in proof by contradiction, that the computability of such distribution is possible.

5.1 Demonstrability as computability

The question relates to the *conceptual availability* of an algorithm able to compute general properties of the primes' distribution.

Such properties are supposed to allow to know for any number the properties of the following sequence of prime numbers *without computing each item of such sequences*.

We just mention that the case of the knowledge of properties of a function, e.g., continuity, differentiability, minimum and maximum points, asymptotes, is different. Properties of a function are known from its *formal definition* and not from the knowledge of the properties of the distribution of all the values assumed in its domain of validity, i.e., law of distribution.

We may know the analytical properties of an exponential function without computing its values in any points on the abscissa axis.

The same holds for sequences of numbers such as the Fibonacci sequence defined as $F_n=F_{n-1}+F_{n-2}$, with $F_1=F_2=1$ (two successive Fibonacci numbers are relatively prime).

How do we *decide* if a number is prime *without* computing?

When considering a number, we may take into account, for instance, some of its properties 1) will not require its factorisation –we consider here the case of factorization of an integer. We do not consider here the cases related to polynomial factorization and rings- or 2) will consider its factorisation.

As stated by the fundamental theorem of arithmetic *every integer > 1 either is prime itself or is the product of prime numbers. This product is unique regardless of the order of the factors*. The first explicit proof of the theorem of arithmetic, namely that the set of integer numbers has a unique factorization, is due to Carl Friederich Gauss, who inserted it in the *Disquisitiones Arithmeticae*, published in 1798, but already introduced by Euclid.

Examples of properties of the first kind (not requiring factorisation) are generic properties such as considering if a number is greater or less than another, the number of its digits, and if it is even or odd. Similarly, properties of values of a function are known from its formal definition and do not require the effective computation of values. *Positions within the sequence of natural numbers correspond to properties*.

Examples of properties of the second kind (requiring the factorisation of the number) relate the identification of the number as given by the exponentiation of a base and prime numbers.

In the first case, it is possible to detect a property *without computing and factorise*.

However, *factorial properties of numbers,* as their factorial breakdown, exponential factor values and their being prime, i.e., non-decomposability, require their factorisation (or equivalent, e.g., the sieves), i.e., effective computing.

Properties of a distribution law, e.g., the graph of a function, its continuity, regularity, domains, and values of its derivatives, allow to know subsequent values moving along the graph without computing each value corresponding to the punctual abscissas.

In the case of factorisations, each of them must be computed since not made available by any property of a distribution law.

In the second case, factorisation is then *necessary*.

For instance, each value of the function $f=x^n$ is available on its graph. Rather, each factorisation of an integer (factorisation is different from "combinatorial calculus" when factors are known) is in principle unknown and must be computed case by case, being not available from sequences or any graph.

In the first case, we have available the complete *computational procedure*, i.e., an algorithm.

In the second case, we have *factorisation techniques* available, but there are no known algorithms (can integer factorization be solved in polynomial time on a non-quantum computer [35]?) which can *effectively* factor arbitrary large integers, see, for instance, [36] and [37].

The adjective *effectively* refers to the definition of TM for which the *algorithm* should produce the solution in a *finite* time for each instance of the input data [12]. This also refers to *tractable* problems that can be solved by algorithms in *polynomial time*, i.e., for a problem of size *n*, the time or number of steps needed to find the solution is a polynomial function of *n*. Conversely, algorithms for solving *intractable* problems require times that are exponential functions of the problem size *n*.

Then factorisation is undecidable.

Furthermore, we mention that sieves, such as the Eratosthenes, Legendre (it is an extension of Eratosthenes' idea), Brun, Selberg, and Turán sieves [38], have an *exponential time complexity* with regard to input size, making them pseudo-polynomial algorithms.

We consider the theoretical unavailability of a distribution law for factorial properties, as being prime, equivalent to its non-computability, undecidability.

The availability and demonstrability of a hypothetical PDL are inconsistent with its undecidability.

In the second case, factorisation is then *necessary*. *Because of that it is not possible to know in advance the properties of the factorisations, in the same way as it is not possible to solve the alt of a TM* (see the Introduction), *it is not possible to know the result of the processing of a Neural Network without performing the entire processing, and to know the patterns generated by a Cellular Automata without performing the entire processing*. *Positions within the sequence of natural numbers do not correspond to the distributed property of being prime number.*

In light of that, we tentatively propose the *speculative conjecture* that the complete knowledge of the PDL, that allows the availability of a rule, is not possible since it would disprove the Alt Problem for a TM. We conclude that the PDL is *undecidable*. We may conclude the indemonstrability of the *Riemann Hypothesis (Millennium Problem)*, the Riemann hypothesis is undecidable in arithmetic.

Conceptual non-availability of an algorithm defines all undecidable problems as correspondent to the Alt Problem for a Turing Machine.

The probabilistic nature of PNT should be considered an aspect of its undecidability.

This will *theoretically* provide reassurance about the usage of prime numbers for a large variety of applications such as cryptography and pseudorandom number generation.

5.2 Using the indemonstrability

A theoretical incompletable list of non-equivalent models and approaches are necessary to deal with the endless acquisitions and modality of acquisition of properties in complexity and emergent phenomena. This is the case for uncertainty principles and theoretical incompleteness such as that of mathematics, of the Turing machines, and of the so-called *Logical Openness* in the Dynamic Usage of Models -DYSAM [39, pp. 64-88], based on established approaches in the literature, such as *Ensemble learning* [40, 41] and *Evolutionary Game Theory* [42, 43]. Other cases relate to the undecidability and irreducibility of emergence [17, 44], the usage of the *non-computable* and *unknowable* imaginary numbers, however very effective and used, and the non-symbolic computation of ANN and CA.

The non-demonstrability of the PDL primes' distribution law is well used in cryptography in the same way as some pharmaceutical products are used for their *side-effects*.

This relates to the usage of the *theoretically incomprehensible* [45] which is suitable for introducing paradigm-shifts and non-equivalent, incommensurable, mutually irreducible approaches.

Can the non-demonstrability of the primes' distribution law become the prototype of the non-distribution(s) having some possible different levels of equivalence; the prototype of the non-demonstrability, of theoretical incompleteness, and of theoretical incomprehensibility?

Conclusions

We shortly considered the research about primes in mathematics and the theoretical, still elusive, results looking for a PDL.

We considered as these endless difficulties may be interpreted as logical consistency, since the availability of such distribution law could be theoretically incompatible with other consolidated theories and properties.

This is the case for the theoretical incompleteness of mathematics, the Turing machines, and of the so-called Logical Openness in the use of Dynamic Usage of Models (DYSAM).

We considered the conceptual incompatibility of the availability of a PDL and the Alt Problem for a TM, i.e., implying that the PDL is *undecidable*.

The link between the conceptual availability of a PDL and decidability is given by considering how to *decide* if a number is prime *without* its computation. The supposed PDL should allow to know the sequence of primes *without their computation*, but considering only their sequential positions *which coincide, however, with the numbers in question*.

However, factorial properties of numbers, such as their primality, require their factorisation (or equivalent, e.g., the sieves), i.e., effective computing.

Because of that it is not possible to know in advance the properties of the factorisation, in the same way as it is not possible to solve the alt of a TM, it is not possible to know the result of the processing of a Neural Network without performing the entire processing, and to know the patterns generated by a Cellular Automata without performing the entire processing. *Positions within the sequence of natural numbers do not correspond to the distributed property of a prime number.*

We may conclude that the availability and demonstrability of a hypothetical PDL are inconsistent with its undecidability.

The perspective is to transform this conjecture into a theorem.

Furthermore, we considered the unavailability of a PDL as corresponding, representing incompleteness in mathematics and physics. However, such incompleteness can be *used*, e.g., for cryptography, imaginary numbers, and non-symbolic computation, in order to introduce paradigm-shifts and non-equivalent, mutually irreducible approaches.

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Legendre Wavelet expansion of functions and their Approximations

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Abstract

In this paper, nine new Legendre wavelet estimators of functions having bounded third and fourth derivatives have been obtained.These estimators are new and best approximation in wavelet analysis. Legendre wavelet estimator of a function f of bounded higher order derivatives is better and sharper than the estimator of a function f of bounded less order derivative.

Keywords : Legendre Wavelet, Legendre Wavelet Expansion, Orthonormal basis,Legendre Wavelet Approximation .

Mathematics Subject Classification:42C40, 65T60, 65L10, 65L60, 65R20. [1]

1 Introduction

Several researchers have determined the approximation of a functions by trigonometric polynomials in Fourier analysis. In Fourier analysis, a function can be represented generally in one Fourier series. In wavelet analysis, a function can be expanded in many wavelet series corresponding to different wavelets. This is an advantage of wavelet analysis. There is no such advantage in Fourier analysis.Thus a signal can be represented by several wavelet series. Hence Wavelet Analysis is superior to Fourier analysis and has so many applications in Engineering and Technology. The Wavelet approximation of a functions by its Haar wavelet series and related approximations have been studied by Devore[7], Debnath[5], Meyer[9] , Morlet[3], Mhaskar[2], Sablonnière[6] and Lal & Kumar[8]. The purpose of this paper is to discuss the Legendre wavelet series of function having bounded third and fourth derivatives, i.e. $0 \leq |f^{'''}(x)| < \infty \ \ \forall x \in [0,1]$ and $0 \leq |f^{iv}(x)| < \infty \quad \forall x \in [0,1]$ and to obtain Legendre wavelet estimators of these functions. This is a significant observation of this research paper that estimate of a function is better and the sharper than the estimate having less order bounded derivative.Therefore comparison of estimated approximations has very importance in Wavelet analysis.

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2 Definitions and Preliminaries

2.1 Legendre Wavelet

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\psi \in L^2(R)$, called mother wavelet. We write

$$\psi_{b,a}(x) = |a|^{\frac{-1}{2}} \psi \left(\frac{x-b}{a} \right), \ \ a \neq 0.$$

If we restrict the values of dilation and translation parameter to $a = a_0^{-n}$,
$b = mb_0 {a_0}^{-n}, a_0 > 1, b_0 > 0$ respectively, the following family of discrete wavelets are constructed:

$$\psi_{n,m}(x) = |a_0|^{\frac{n}{2}} \psi(a_0^n x - mb_0)$$

The Legendre wavelet over the interval [0,1) is defined as

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m+\frac{1}{2}} \ \ 2^{\frac{k}{2}} P_m(2^k x - \hat{n}), & \dfrac{\hat{n}-1}{2^k} \leq x < \dfrac{\hat{n}+1}{2^k} \\ 0 & \text{, otherwise,} \end{cases}$$

where $n = 1, 2, ..., 2^{k-1}$ and $m = 0, 1, 2, 3, ...,$ $\hat{n} = 2n - 1$ and k is the positive integer. In this definition,the polynomials P_m are Legendre Polynomials of degree m over the interval [-1,1] defined as follows:

$$P_0(x) = 1, P_1(x) = x$$

$$(m+1)P_{m+1}(x) = (2m+1)xP_m(x) - mP_{m-1}(x) \ , \quad m = 1, 2, 3, ...$$

The set of $\{P_m(x) : m = 1, 2, 3, ...\}$ in the Hilbert space $L^2[-1, 1]$ is a complete orthogonal set. Orthogonality of Legendre polynomial on the interval [-1,1] implies that

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x)\overline{P_n(x)}dx = \begin{cases} \dfrac{2}{2m+1} & , m = n \\ 0 & \text{, otherwise.} \end{cases}$$

$$for \ m, n = 0, 1, 2, 3...$$

Furthermore, the set of wavelets $\psi_{n,m}$ makes an orthonormal basis in $L^2[0, 1)$,$i.e.$

$$\int_0^1 \psi_{n,m}(x)\psi_{n'm'}(x)dx = \delta_{n,n'}\delta_{m,m'}$$

in which δ denotes Kronecker delta function defined by

$$\delta_{n,m} = \begin{cases} 1, & \text{n=m} \\ 0, & \text{otherwise.} \end{cases}$$

The function $f(x) \in L^2[0,1)$ is expressed in the Legendre wavelet series as :

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x)$$

where $c_{n,m} = \langle f, \psi_{n,m} \rangle$. The $(2^{k-1}, M)^{th}$ partial sums of above series are given by $S_{2^{k-1},M}(f)(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) = C^T\psi(x)$ in which C and $\psi(x)$ are $2^{k-1}(M+1)$ vectors of the form

$$C^T = [c_{1,0}, c_{1,1}, ...c_{1,M}, c_{2,0}, c_{2,1}, ...c_{2,M}, ..., c_{2^{k-1},0}, ...c_{2^{k-1},M}]$$

and

$$\psi(x) = [\psi_{1,0}, \psi_{1,1}, ...\psi_{1,M}, \psi_{2,0}, \psi_{2,1}, ...\psi_{2,M}, ..., \psi_{2^{k-1},0}, ...\psi_{2^{k-1},M}]^T$$

2.2 Legendre Wavelet Approximation

Let $S_{2^{k-1},M}(f)(x)$ denote the $(2^{k-1}, M)^{th}$ partial sums of the series $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x)$ i.e.

$$S_{2^{k-1},M}(f)(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x)$$

The Legendre wavelet approximation $E_{2^{k-1},M}(f)$ of a function $f \in L^2[0,1)$ by $(2^{k-1}, M)^{th}$ partial sums $S_{2^{k-1},M}(f)$ of its Legendre Wavelet series is given by

$$E_{2^{k-1},M}(f) = min\|f - S_{2^{k-1},M}(f)\|_2, (Zygmund[1], pp.115)$$

where

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

If $E_{2^{k-1},M}(f) \to 0$ as $k \to \infty,\ M \to \infty.$ then $E_{2^{k-1},M}(f)$ is called the best approximation of f of order $(2^{k-1}, M)$ (Zygmund[1],pp.115)

3 Example

Express the following function in the Legendre wavelet series :
$f(t) = t^3 \ \forall t \ \in [0,1)$

Proof:

$$
\begin{aligned}
f(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t) \\
c_{n,m} &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(t)\psi_{n,m}(t)dt \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} t^3 \left(\frac{2m+1}{2}\right)^{\frac{1}{2}} 2^{\frac{k}{2}} P_m(2^k t - \hat{n})dt \\
&= \left(\frac{2m+1}{2}\right)^{\frac{1}{2}} 2^{\frac{k}{2}} \int_{-1}^{1} \left(\frac{v+\hat{n}}{2^k}\right)^3 P_m(v)\frac{dv}{2^k},\ v = 2^k t - \hat{n} \\
c_{n,m} &= \left(\frac{2m+1}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3 + v^3 + 3\hat{n}^2 v + 3\hat{n}v^2)P_m(v)dv \\
& \text{By above expression} \\
c_{n,0} &= \left(\frac{1}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3 + v^3 + 3\hat{n}^2 v + 3\hat{n}v^2)P_0(v)dv \\
&= \left(\frac{1}{2^{7k+1}}\right)^{\frac{1}{2}} (2\hat{n}^3 + 2\hat{n}) \\
c_{n,1} &= \left(\frac{\sqrt{3}}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3 + v^3 + 3\hat{n}^2 v + 3\hat{n}v^2)P_1(v)dv \\
&= \left(\frac{\sqrt{3}}{2^{7k+1}}\right)^{\frac{1}{2}} \left(\frac{2}{5} + 2\hat{n}^2\right) \\
c_{n,2} &= \left(\frac{\sqrt{5}}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3 + v^3 + 3\hat{n}^2 v + 3\hat{n}v^2)P_2(v)dv \\
&= \left(\frac{\sqrt{5}}{2^{7k+1}}\right)^{\frac{1}{2}} \left(\frac{4\hat{n}}{5}\right) \\
c_{n,3} &= \left(\frac{\sqrt{7}}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3 + v^3 + 3\hat{n}^2 v + 3\hat{n}v^2)P_3(v)dv
\end{aligned}
$$

$$
\begin{aligned}
c_{n,3} &= \left(\frac{4}{35}\right)\left(\frac{\sqrt{7}}{2^{7k+1}}\right)^{\frac{1}{2}} \\
c_{n,m} &= 0,\ for\ m \geq 4
\end{aligned}
$$

Then,

$$
f(t) = \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}(t) + \sum_{n=1}^{2^{k-1}} c_{n,1}\psi_{n,1}(t) + \sum_{n=1}^{2^{k-1}} c_{n,2}\psi_{n,2}(t) + \sum_{n=1}^{2^{k-1}} c_{n,3}\psi_{n,3}(t)
$$

Now,

$$
\begin{aligned}
||f||_2^2 &= \frac{1}{7} = \sum_{n=1}^{2^{k-1}} c_{n,0}^2||\psi_{n,0}||_2^2 + \sum_{n=1}^{2^{k-1}} c_{n,1}^2||\psi_{n,1}||_2^2 + \sum_{n=1}^{2^{k-1}} c_{n,2}^2||\psi_{n,2}||_2^2 + \sum_{n=1}^{2^{k-1}} c_{n,3}^2||\psi_{n,3}||_2^2 \\
&= \sum_{n=1}^{2^{k-1}} c_{n,0}^2 + \sum_{n=1}^{2^{k-1}} c_{n,1}^2 + \sum_{n=1}^{2^{k-1}} c_{n,2}^2 + \sum_{n=1}^{2^{k-1}} c_{n,3}^2 \\
&= \sum_{n=1}^{2^{k-1}} \left[\left(\frac{1}{2^{7k+1}}\right)^{\frac{1}{2}}(2\hat{n}^3 + 2\hat{n})\right]^2 + \sum_{n=1}^{2^{k-1}} \left[\left(\frac{\sqrt{3}}{2^{7k+1}}\right)^{\frac{1}{2}}\left(\frac{2}{5} + 2\hat{n}^2\right)\right]^2 + \sum_{n=1}^{2^{k-1}} \left[\left(\frac{\sqrt{5}}{2^{7k+1}}\right)^{\frac{1}{2}}\left(\frac{4\hat{n}}{5}\right)\right]^2 \\
&+ \sum_{n=1}^{2^{k-1}} \left[\left(\frac{4}{35}\right)\left(\frac{\sqrt{7}}{2^{7k+1}}\right)^{\frac{1}{2}}\right]^2 \\
&= \frac{1}{7}.
\end{aligned}
$$

4 Theorems

In this paper, we prove following new theorems:

Theorem (4.1)

Let a function $f \in L^2[0,1)$ such that its third derivative be bounded ,i.e. $0 \leq |f^{'''}(x)| < \infty\ \forall\ x \in [0,1)$. Then the Legendre wavelet approximations of f satisfy :

$(i)E^{(1)}_{2^{k-1},0}(f) = ||f - \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}||_2 = O\left(\frac{1}{2^k}\right)$

$(ii)E^{(2)}_{2^{k-1},1}(f) = ||f - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{1} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{2k}}\right)$

$(iii)E^{(3)}_{2^{k-1},2}(f) = ||f - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{2} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{3k}}\right)$

(iv)For $f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}$,

$E^{(4)}_{2^{k-1},M}(f) = ||f - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}||_2$

$$= \left(\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2 \right)^{\frac{1}{2}} = O\left(\frac{1}{(2M-3)^{\frac{5}{2}}} \frac{1}{2^{3k}} \right), \forall M \geq 2.$$

Theorem (4.2)
If a function $f \in L^2[0,1)$ having bounded fourth derivative ,i.e. $0 \leq |f^{iv}(x)| < \infty \ \forall \ x \in [0,1)$. Then its Legendre wavelet approximations are given by
$(i) E^{(5)}_{2^{k-1},0}(f) = ||f - \sum\limits_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}||_2 = O\left(\frac{1}{2^k}\right)$
$(ii) E^{(6)}_{2^{k-1},1}(f) = ||f - \sum\limits_{n=1}^{2^{k-1}} \sum\limits_{m=0}^{1} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{2k}}\right)$
$(iii) E^{(7)}_{2^{k-1},2}(f) = ||f - \sum\limits_{n=1}^{2^{k-1}} \sum\limits_{m=0}^{2} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{3k}}\right)$
$(iv) E^{(8)}_{2^{k-1},3}(f) = ||f - \sum\limits_{n=1}^{2^{k-1}} \sum\limits_{m=0}^{3} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{4k}}\right)$
(v)For $f(x) = \sum\limits_{n=1}^{\infty} \sum\limits_{m=0}^{\infty} c_{n,m}\psi_{n,m}$,
$E^{(9)}_{2^{k-1},M}(f) = ||f - \sum\limits_{n=1}^{2^{k-1}} \sum\limits_{m=0}^{M} c_{n,m}\psi_{n,m}||_2$

$$= \left(\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2 \right)^{\frac{1}{2}} = O\left(\frac{1}{(2M-5)^{\frac{7}{2}}} \frac{1}{2^{4k}} \right), \ \forall \ M \geq 3.$$

5 Proofs

5.1 Proof of the Theorem (4.1)

(i) The error $e_n^{(0)}(x)$ between $f(x)$ and its expression over any subinterval is defined as
$e_n^{(0)}(x) = c_{n,0}\psi_{n,0}(x) - f(x)$, $x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$, $n = 1, 2, 3, ...2^{k-1}$

$$\begin{aligned} ||e_n^{(0)}||_2^2 &= \int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (e_n^{(0)}(x))^2 dx \\ &= \int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (c_{n,0}^2\psi_{n,m}^2(x) + (f(x))^2 - 2c_{n,0}\psi_{n,0}(x)f(x))dx \end{aligned}$$

$$
\begin{aligned}
&= c_{n,0}^2 \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \psi_{n,0}^2(x)dx + \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - 2c_{n,0} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)\psi_{n,0}(x)dx \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - c_{n,0}^2. \qquad (5.1)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx &= \int_0^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k}+h\right)\right)^2 dh, x = \frac{\hat{n}-1}{2^k}+h \\
&= \int_0^{\frac{1}{2^{k-1}}} \left[f\left(\frac{\hat{n}-1}{2^k}\right) + hf'\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^2}{2}f''\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^3}{6}f'''\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right]^2, \\
&\quad 0<\theta<1 \ \ by\, Taylor's \ \ expansion \\
&= \int_0^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 dh + \int_0^{\frac{1}{2^{k-1}}} h^2\left(f'\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 dh + \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{4}\left(f''\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 dh \\
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^6}{36}\left(f'''\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right)^2 dh + \int_0^{\frac{1}{2^{k-1}}} 2hf\left(\frac{\hat{n}-1}{2^k}\right)f'\left(\frac{\hat{n}-1}{2^k}\right)dh \\
&+ \int_0^{\frac{1}{2^{k-1}}} h^2 f\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)dh + \int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{3}f\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&+ \int_0^{\frac{1}{2^{k-1}}} h^3 f'\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)dh + \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{3}f'\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^5}{6}f''\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&= \frac{2}{2^k}\left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{8}{3}\frac{1}{2^{3k}}\left(f'\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{8}{5}\frac{1}{2^{5k}}\left(f''\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 \\
&+ \frac{1}{36}\int_0^{\frac{1}{2^{k-1}}} h^6\left(f'''\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right)^2 dh + \frac{4}{2^{2k}}f\left(\frac{\hat{n}-1}{2^k}\right)f'\left(\frac{\hat{n}-1}{2^k}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{8}{3}\frac{1}{2^{3k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{3}\int_0^{\frac{1}{2^{k-1}}} h^3 f\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh \\
&+ \quad \frac{4}{2^{4k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) dh + \frac{1}{3}\int_0^{\frac{1}{2^{k-1}}} h^4 f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh \\
&+ \quad \frac{1}{6}\int_0^{\frac{1}{2^{k-1}}} h^5 f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh. \qquad (5.2)
\end{aligned}
$$

Now,

$$
\begin{aligned}
c_{n,0} &= < f(x), \psi_{n,0}(x) > \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)\psi_{n,0}(x)dx \\
&= 2^{\frac{k-1}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)dx \\
&= 2^{\frac{k-1}{2}} \int_0^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k}+h\right) dh, x = \frac{\hat{n}-1}{2^k}+h \\
&= 2^{\frac{k-1}{2}} \int_0^{\frac{1}{2^{k-1}}} \left[f\left(\frac{\hat{n}-1}{2^k}\right) + hf^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^2}{2} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^3}{6} f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right] dh \\
&= 2^{\frac{k-1}{2}} \left[\frac{2}{2^k} f\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{2^{2k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{3}\frac{1}{2^{3k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{6}\int_0^{\frac{1}{2^{k-1}}} h^3 f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right].
\end{aligned}
$$

Next,

$$
\begin{aligned}
c_{n,0}^2 &= \frac{2}{2^k}\left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{8}{9}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 \\
&+ \quad \frac{2^k}{2}\left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6} f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right)^2 + \frac{4}{2^{2k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{'}\left(\frac{\hat{n}-1}{2^k}\right) \\
&+ \quad \frac{8}{3}\frac{1}{2^{3k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{3}\int_0^{\frac{1}{2^{k-1}}} h^3 f\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh
\end{aligned}
$$

$$+ \quad \frac{8}{3}\frac{1}{2^{4k}}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)f^{''}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{2}{2^k}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+ \quad \frac{4}{3}\frac{1}{2^{2k}}f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh. \tag{5.3}$$

Now, by using equations (5.1), (5.2) and (5.3) we have

$$\begin{aligned}
||e_n^{(0)}||_2^2 \;&=\; \frac{2}{3}\frac{1}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2+\frac{32}{45}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2+\frac{1}{36}\int\limits_0^{\frac{1}{2^{k-1}}}h^6\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right)^2dh \\
&-\; \frac{2^k}{2}\left(\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\right)^2+\frac{4}{3}\frac{1}{2^{4k}}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)f^{''}\left(\frac{\hat{n}-1}{2^k}\right) \\
&+\; \frac{1}{3}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}h^4f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh-\frac{2}{2^k}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&+\; \frac{1}{6}f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}h^5f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh-\frac{4}{3}\frac{1}{2^{2k}}f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&=\; I_1+I_2+I_3-I_4+I_5+I_6-I_7+I_8-I_9,\ say.
\end{aligned}$$

Since $|f^{'}(x)|\le M_1, |f^{''}(x)|\le M_2, |f^{'''}(x)|\le M_3, \forall x\in[0,1),$

therefore

$$\begin{aligned}
|I_1| &\le \frac{2}{3}\frac{1}{2^{3k}}M_1^2 \\
|I_2| &\le \frac{32}{45}\frac{1}{2^{5k}}M_2^2 \\
|I_3| &\le \frac{32}{63}\frac{1}{2^{7k}}M_3^2 \\
|I_4| &\le \frac{2}{9}\frac{1}{2^{7k}}M_3^2 \\
|I_5| &\le \frac{4}{3}\frac{1}{2^{4k}}M_1M_2 \\
|I_6| &\le \frac{32}{15}\frac{1}{2^{5k}}M_1M_3 \\
|I_7| &\le \frac{4}{3}\frac{1}{2^{5k}}M_1M_3 \\
|I_8| &\le \frac{16}{9}\frac{1}{2^{6k}}M_2M_3 \\
|I_9| &\le \frac{8}{9}\frac{1}{2^{6k}}M_2M_3.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
||e_n^{(0)}||_2^2 &\leq |I_1|+|I_2|+|I_3|+|I_4|+|I_5|+|I_6|+|I_7|+|I_8|+|I_9| \\
&\leq \frac{2}{3}\frac{1}{2^{3k}}M_1^2+\frac{32}{45}\frac{1}{2^{5k}}M_2^2+\frac{32}{63}\frac{1}{2^{7k}}M_3^2+\frac{2}{9}\frac{1}{2^{7k}}M_3^2+\frac{4}{3}\frac{1}{2^{4k}}M_1M_2 \\
&+ \frac{32}{15}\frac{1}{2^{5k}}M_1M_3+\frac{4}{3}\frac{1}{2^{5k}}M_1M_3+\frac{16}{9}\frac{1}{2^{6k}}M_2M_3+\frac{8}{9}\frac{1}{2^{6k}}M_2M_3 \\
&= \frac{2}{3}\frac{1}{2^{3k}}M_1^2+\frac{32}{45}\frac{1}{2^{5k}}M_2^2+\frac{56}{63}\frac{1}{2^{7k}}M_3^2+\frac{4}{3}\frac{1}{2^{4k}}M_1M_2+\frac{52}{15}\frac{1}{2^{5k}}M_1M_3+\frac{24}{9}\frac{1}{2^{6k}}M_2M_3 \\
&< \frac{2}{2^{3k}}\left[M_1^2+\left(\frac{M_2}{2^k}\right)^2+\left(\frac{M_3}{2^{2k}}\right)^2+\frac{2M_1M_2}{2^k}+\frac{2M_1M_3}{2^{2k}}+\frac{2M_2M_3}{2^{3k}}\right] \\
&= \frac{2}{2^{3k}}\left(M_1+\frac{M_2}{2^k}+\frac{M_3}{2^{2k}}\right)^2 \\
&= \frac{2M^2}{2^{3k}}\left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}\right)^2, M = max[M_1,M_2,M_3].
\end{aligned}
$$

Next,

$$
\begin{aligned}
(E^{(1)}_{2^{k-1},0}(f))^2 &= \int_0^1\left(\sum_{n=1}^{2^{k-1}} e_n^{(0)}(x)\right)^2 dx \\
&= \int_0^1\sum_{n=1}^{2^{k-1}}(e_n^{(0)}(x))^2dx + 2\sum_{n=1}^{2^{k-1}}\sum_{n\neq n'}^{2^{k-1}}\int_0^1 e_n^{(0)}(x)e_n^{(0')}(x)dx \\
&= \sum_{n=1}^{2^{k-1}}\int_0^1(e_n(x))^2dx, \text{ due to disjoint supports of } e_n \text{ and } e_n' \\
&= \sum_{n=1}^{2^{k-1}}||e_n^{(0)}||_2^2 \\
&\leq (2^{k-1})\frac{2M^2}{2^{3k}}\left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}\right)^2 \\
&= \frac{M^2}{2^{2k}}\left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}\right)^2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
E^{(1)}_{2^{k-1},0}(f) &\leq \frac{M}{2^k}\left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}\right) \\
&\leq M\left(\frac{1}{2^k}+\frac{1}{2^k}+\frac{1}{2^k}\right) \\
&= 3M\left(\frac{1}{2^k}\right) \\
&= O\left(\frac{1}{2^k}\right).
\end{aligned}
$$

$$
\begin{aligned}
(ii)e_n^{(1)}(x) &= c_{n,0}\psi_{n,0}(x) + c_{n,1}\psi_{n,1}(x) - f(x) \;, \quad x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right) \\
||e_n^{(1)}||_2^2 &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - c_{n,0}^2 - c_{n,1}^2. \qquad (5.4)
\end{aligned}
$$

Now, consider

$$
\begin{aligned}
c_{n,1} &= < f(x), \psi_{n,1}(x) > \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)\psi_{n,1}(x) dx \\
&= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x) P_1(2^k x - \hat{n}) dx \\
&= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_0^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k} + h\right) P_1(2^k h - 1) dh, \;\; x = \frac{\hat{n}-1}{2^k} + h \\
&= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_0^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k} + h\right) (2^k h - 1) dh \\
&= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_0^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k}\right) (2^k h - 1) dh + \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_0^{\frac{1}{2^{k-1}}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) h(2^k h - 1) dh \\
&+ \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_0^{\frac{1}{2^{k-1}}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) \frac{h^2}{2}(2^k h - 1) dh + \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_0^{\frac{1}{2^{k-1}}} f^{'''}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \frac{h^3}{6}(2^k h - 1) dh \\
c_{n,1} &= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \left[\frac{2}{3}\frac{1}{2^{2k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{3}\frac{1}{2^{3k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right] \\
&+ \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_0^{\frac{1}{2^{k-1}}} f^{'''}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \frac{h^3}{6}(2^k h - 1) dh.
\end{aligned}
$$

Now,

$$
\begin{aligned}
c_{n,1}^2 &= \frac{2}{3}\frac{1}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{3}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 \\
&+ \frac{3}{2} 2^k \left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h - 1) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) dh\right)^2 + \frac{4}{3}\frac{1}{2^{4k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{2}{2^k} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h-1) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) dh \\
&+ \quad \frac{2}{2^{2k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h-1) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) dh. \qquad (5.5)
\end{aligned}
$$

By using equations (5.2), (5.3), (5.4) and (5.5), we have

$$
\begin{aligned}
||e_n^{(1)}||_2^2 &= \frac{2}{45}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{1}{36}\int_0^{\frac{1}{2^{k-1}}} h^6\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right)^2 dh \\
&- \frac{4}{3}\frac{1}{2^{2k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6} f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) dh - \frac{2^k}{2}\left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6} f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right)^2 \\
&+ \frac{1}{6} f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\int_0^{\frac{1}{2^{k-1}}} h^5 f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh - \frac{3}{2}2^k\left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h-1) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) dh\right)^2 \\
&- \frac{2}{2^{2k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h-1) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) dh \\
&= I_1+I_2+I_3+I_4+I_5+I_6+I_7, say.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|I_1| &\leq \frac{2}{45}\frac{1}{2^{5k}} M_2^2 \\
|I_2| &\leq \frac{32}{63}\frac{1}{2^{7k}} M_3^2 \\
|I_3| &\leq \frac{8}{9}\frac{1}{2^{6k}} M_2 M_3 \\
|I_4| &\leq \frac{2}{9}\frac{1}{2^{7k}} M_3^2 \\
|I_5| &\leq \frac{16}{9}\frac{1}{2^{6k}} M_2 M_3 \\
|I_6| &\leq \frac{18}{75}\frac{1}{2^{7k}} M_3^2 \\
|I_7| &\leq \frac{12}{15}\frac{1}{2^{6k}} M_2 M_3 \\
||e_n^{(1)}||_2^2 &\leq |I_1|+|I_2|+|I_3|+|I_4|+|I_5|+|I_6|+|I_7| \\
&\leq \frac{2}{45}\frac{1}{2^{5k}} M_2^2 + \frac{32}{63}\frac{1}{2^{7k}} M_3^2 + \frac{8}{9}\frac{1}{2^{6k}} M_2 M_3 + \frac{2}{9}\frac{1}{2^{7k}} M_3^2 + \frac{16}{9}\frac{1}{2^{6k}} M_2 M_3 + \frac{18}{75}\frac{1}{2^{7k}} M_3^2 \\
&+ \frac{12}{15}\frac{1}{2^{6k}} M_2 M_3
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{45}\frac{1}{2^{5k}}M_2^2 + \frac{1528}{1575}\frac{1}{2^{7k}}M_3^2 + \frac{144}{45}\frac{1}{2^{6k}}M_2M_3 \\
&< \frac{2}{2^{5k}}\left(M_2^2 + \left(\frac{M_3}{2^k}\right)^2 + \frac{2M_2M_3}{2^k}\right) \\
&= \frac{2}{2^{5k}}M^2\left(1+\frac{1}{2^k}\right)^2, M = max[M_2, M_3].
\end{aligned}
$$

Next,

$$
\begin{aligned}
(E^{(2)}_{2^{k-1},1}(f))^2 &= \sum_{n=1}^{2^{k-1}} ||e_n^{(1)}||_2^2 \\
&\le (2^{k-1})\frac{2}{2^{5k}}M^2\left(1+\frac{1}{2^k}\right)^2 \\
&= \frac{M^2}{2^{4k}}\left(1+\frac{1}{2^k}\right)^2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
E^{(2)}_{2^{k-1},1}(f) &\le \frac{M}{2^{2k}}\left(1+\frac{1}{2^k}\right) \\
&= O\left(\frac{1}{2^{2k}}\right).
\end{aligned}
$$

(iii) $e_n^{(2)}(x) = c_{n,0}\psi_{n,0}(x) + c_{n,1}\psi_{n,1}(x) + c_{n,2}\psi_{n,2}(x) - f(x)$, $x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$
Similarly, it can be proved that

$$
E^{(3)}_{2^{k-1},2}(f) = O\left(\frac{1}{2^{3k}}\right).
$$

(iv) $0 \le |f^{'''}(x)| < M_1$, $\forall x \in [0,1)$

$$
\begin{aligned}
c_{n,m} &= \int_0^1 f(x)\psi_{n,m}(x)dx \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)\sqrt{\frac{2m+1}{2}}2^{\frac{k}{2}}P_m(2^kx-\hat{n})dx \\
&= \sqrt{\frac{2m+1}{2^{k+1}}}\int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k}\right)P_m(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{\frac{2m+1}{2^{k+1}}}\int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+1}(t)-P_{m-1}(t))}{2m+1}\\
&= \left(\frac{1}{2^{k+1}(2m+1)}\right)^{\frac{1}{2}}\\
&\times \left[\left\{f\left(\frac{\hat{n}+t}{2^k}\right)(P_{m+1}(t)-P_{m-1}(t))\right\}_{-1}^{1} - \int_{-1}^{1}\frac{1}{2^k}f^{'}\left(\frac{\hat{n}+t}{2^k}\right)(P_{m+1}(t)-P_{m-1}(t))dt\right]\\
&= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\int_{-1}^{1} f^{'}\left(\frac{\hat{n}+t}{2^k}\right)(P_{m-1}(t)-P_{m+1}(t))dt\right]\\
&= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\int_{-1}^{1} f^{'}\left(\frac{\hat{n}+t}{2^k}\right)(P_{m-1}(t))dt - \int_{-1}^{1} f^{'}\left(\frac{\hat{n}+t}{2^k}\right)(P_{m+1}(t))dt\right]\\
&= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}\left[f^{'}\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m}(t)-P_{m-2}(t))}{(2m-1)} - f^{'}\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+2}(t)-P_{m}(t))}{(2m+3)}\right]\\
&= \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}\left[f^{''}\left(\frac{\hat{n}+t}{2^k}\right)\frac{(P_{m+2}(t)-P_{m}(t))}{(2m+3)} - f^{''}\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m}(t)-P_{m-2}(t))}{(2m-1)}\right]\\
&= \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)}\int_{-1}^{1}\left[f^{''}\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+3}(t)-P_{m+1}(t))}{(2m+5)}\right]\\
&- \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)}\int_{-1}^{1}\left[f^{''}\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+1}(t)-P_{m-1}(t))}{(2m+1)}\right]\\
&+ \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)}\int_{-1}^{1}\left[f^{''}\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m-1}(t)-P_{m-3}(t))}{(2m-3)}\right]\\
&- \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)}\int_{-1}^{1}\left[f^{''}\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+1}(t)-P_{m-1}(t))}{(2m+1)}\right]\\
&= \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)}\int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_{m+1}(t)-P_{m-1}(t))}{(2m+1)} - \frac{(P_{m+3}(t)-P_{m+1}(t))}{(2m+5)}\right]dt\\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)}\int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_{m-1}(t)-P_{m-3}(t))}{(2m-3)} - \frac{(P_{m+1}(t)-P_{m-1}(t))}{(2m+1)}\right]dt
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \\
&\times \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{2(2m+3)P_{m+1}(t)-(2m+5)P_{m-1}(t)-(2m+1)P_{m+3}(t)}{(2m+1)(2m+5)(2m+3)}\right]dt \\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \\
&\times \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{2(2m-1)P_{m-1}(t)-(2m+1)P_{m-3}(t)-(2m-3)P_{m+1}(t)}{(2m+1)(2m-1)(2m-3)}\right]dt.
\end{aligned}
$$

Let

$$
\begin{aligned}
\tau_1(t) &= 2(2m+3)P_{m+1}(t)-(2m+5)P_{m-1}(t)-(2m+1)P_{m+3}(t) \\
\tau_2(t) &= 2(2m-1)P_{m-1}(t)-(2m+1)P_{m-3}(t)-(2m-3)P_{m+1}(t)
\end{aligned}
$$

Then,

$$
\begin{aligned}
c_{n,m} &= \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+1)(2m+3)(2m+5)}\left[\int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\tau_1(t)dt\right] \\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+1)(2m-1)(2m-3)}\left[\int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\tau_2(t)dt\right] \\
|c_{n,m}| &\leq \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+1)(2m+3)(2m+5)}\left[\int_{-1}^{1} \left|f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\right||\tau_1(t)|dt\right] \\
&+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+1)(2m-1)(2m-3)}\left[\int_{-1}^{1} \left|f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\right||\tau_2(t)dt|\right] \\
&\leq M_1\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+1)(2m+3)(2m+5)}\int_{-1}^{1}|\tau_1(t)|dt \\
&+ M_1\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+1)(2m-1)(2m-3)}\int_{-1}^{1}|\tau_2(t)|dt. \qquad (5.6)
\end{aligned}
$$

Consider,

$$
\begin{aligned}
\int_{-1}^{1}|\tau_1(t)|dt &= \int_{-1}^{1} 1.|\tau_1(t)|dt \\
&\leq \left(\int_{-1}^{1} 1^2.dt\right)^{\frac{1}{2}}\left(\int_{-1}^{1}|\tau_1(t)|^2dt\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{2}\left(\int_{-1}^{1}\left(2(2m+3)P_{m+1}(t)-(2m+5)P_{m-1}(t)-(2m+1)P_{m+3}(t)\right)^{2}dt\right)^{\frac{1}{2}}\\
&= \sqrt{2}\left(\int_{-1}^{1}\left[4(2m+3)^{2}P_{m+1}^{2}(t)+(2m+5)^{2}P_{m-1}^{2}(t)+(2m+1)^{2}P_{m+3}^{2}(t)\right]dt\right)^{\frac{1}{2}}\\
&= \sqrt{2}\left[4(2m+3)^{2}\frac{2}{2m+3}+(2m+5)^{2}\frac{2}{2m-1}+(2m+1)^{2}\frac{2}{2m+7}\right]^{\frac{1}{2}}\\
&\qquad \text{by orthogonality condition on } P_m\\
&= 2\left[4(2m+3)+\frac{(2m+5)^{2}}{2m-1}+\frac{(2m+1)^{2}}{2m+7}\right]^{\frac{1}{2}}\\
&\leq 2\left[\frac{4(2m+3)(2m-1)+(2m+5)^{2}+(2m+1)^{2}}{(2m-1)}\right]^{\frac{1}{2}}\\
&= 2\left[\frac{24m^{2}+40m+14}{2m-1}\right]^{\frac{1}{2}}\\
&= 2\sqrt{2}\left[\frac{(2m+1)(6m+7)}{2m-1}\right]^{\frac{1}{2}}\\
&\leq 2\sqrt{6}\left[\frac{(2m+1)(2m+3)}{(2m-1)}\right]^{\frac{1}{2}}. \qquad (5.7)
\end{aligned}
$$

Now ,

$$
\begin{aligned}
\int_{-1}^{1}|\tau_2(t)|dt &= \int_{-1}^{1}1.|\tau_2(t)|dt\\
&= \sqrt{2}\left(\int_{-1}^{1}[2(2m-1)P_{m-1}(t)-(2m+1)P_{m-3}(t)-(2m-3)P_{m+1}(t)]^{2}dt\right)^{\frac{1}{2}}\\
&= \sqrt{2}\left[\int_{-1}^{1}\left[(2m-3)^{2}P_{m+1}(t)+(2m+1)^{2}P_{m-3}^{2}(t)+4(2m-1)^{2}P_{m-1}^{2}(t)\right]dt\right]^{\frac{1}{2}}\\
&= \sqrt{2}\left[(2m-3)^{2}\frac{2}{(2m+3)}+(2m+1)^{2}\frac{2}{2m-5}+4(2m-1)^{2}\frac{2}{2m-1}\right]^{\frac{1}{2}}\\
&\qquad \text{by orthogonality condition on } P_m\\
&= 2\left[\frac{(2m-3)^{2}}{(2m+3)}+\frac{(2m+1)}{(2m-5)}+4(2m-1)\right]^{\frac{1}{2}}\\
&\leq 2\left[\frac{(2m-3)^{2}+(2m+1)^{2}+4(2m-1)(2m-5)}{2m-5}\right]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\left[\frac{24m^2-56m+30}{2m-5}\right]^{\frac{1}{2}} \\
&= 2\sqrt{2}\left[\frac{(2m-3)(6m-5)}{2m-5}\right]^{\frac{1}{2}} \\
&\leq 2\sqrt{6}\left[\frac{(2m-3)(2m-1)}{(2m-5)}\right]^{\frac{1}{2}}. \qquad (5.8)
\end{aligned}
$$

Now , by using equations (5.6), (5.7) and (5.8) we have

$$
\begin{aligned}
|C_{n,m}| &\leq M_1\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\frac{2\sqrt{6}}{(2m-3)^{\frac{5}{2}}}+\frac{2\sqrt{6}}{(2m-5)^{\frac{5}{2}}}\right] \\
&\leq M_1\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\frac{4\sqrt{6}}{(2m-5)^{\frac{5}{2}}}\right] \\
&\leq \frac{4\sqrt{6}M_1}{2^{\frac{7k+1}{2}}}\frac{1}{(2m-5)^3}.
\end{aligned}
$$

Therefore,

$$
|C_{n,m}| \leq \frac{4\sqrt{6}M_1}{2^{\frac{7k+1}{2}}}\frac{1}{(2m-5)^3}, \forall m \geq 3. \qquad (5.9)
$$

$$
\begin{aligned}
S_{2^{k-1},M}(f)(x) &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) \\
f(x)-S_{2^{k-1},M}(f)(x) &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) \\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x)+\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(x) \\
&- \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) \\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(x).
\end{aligned}
$$

Then,

$$
\begin{aligned}
||f-S_{2^{k-1},M}(f)||_2^2 &= \int_0^1\left(\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(x)\right)^2 dx \\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}^2, \text{ by orthogonality property of } \psi_{n,m}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \left(\frac{4\sqrt{6}M_1}{2^{\frac{7k+1}{2}}} \frac{1}{(2m-5)^3} \right)^2 , \ by\ (5.9) \\
&= 96M_1^2 \sum_{n=1}^{2^{k-1}} \frac{1}{2^{7k+1}} \sum_{m=M+1}^{\infty} \frac{1}{(2m-5)^6} \\
&= \frac{96M_1^2}{4} \frac{1}{2^{6k}} \int\limits_{M+1}^{\infty} \frac{1}{(2m-5)^6} dm \\
&= \frac{12M_1^2}{5} \frac{1}{2^{6k}} \frac{1}{(2M-3)^5} \\
\therefore E^{(4)}_{2^{k-1},M}(f) &\leq \frac{2\sqrt{3}M_1}{\sqrt{5}} \frac{1}{2^{3k}(2M-3)^{\frac{5}{2}}} \\
&= O\left(\frac{1}{(2M-3)^{\frac{5}{2}} 2^{3k}} \right), \ \ M \geq 2.
\end{aligned}
$$

5.2 Proof of the Theorem(4.2)

(i) The error $e{*}_n^{(0)}(x)$ between $f(x)$ and its expression over any subinterval is defined as $e{*}_n^{(0)}(x) = c_{n,0}\psi_{n,0}(x) - f(x)$, $x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$, $n = 1, 2, 3, ...2^{k-1}$
Now consider,

$$
\begin{aligned}
\int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx &= \int\limits_{0}^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k} + h \right) \right)^2 dh, x = \frac{\hat{n}-1}{2^k} + h \\
&= \int\limits_{0}^{\frac{1}{2^{k-1}}} \left[f\left(\frac{\hat{n}-1}{2^k} \right) + hf^{'}\left(\frac{\hat{n}-1}{2^k} \right) + \frac{h^2}{2} f^{''}\left(\frac{\hat{n}-1}{2^k} \right) + \frac{h^3}{6} f^{'''}\left(\frac{\hat{n}-1}{2^k} \right) \right. \\
&+ \left. \frac{h^4}{24} f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h \right) \right]^2 dh \\
&= \frac{2}{2^k} \left(f\left(\frac{\hat{n}-1}{2^k} \right) \right)^2 + \frac{8}{3}\frac{1}{2^{3k}} \left(f^{'}\left(\frac{\hat{n}-1}{2^k} \right) \right)^2 + \frac{8}{5}\frac{1}{2^{5k}} \left(f^{''}\left(\frac{\hat{n}-1}{2^k} \right) \right)^2 \\
&+ \frac{32}{63}\frac{1}{2^{7k}} \left(f^{'''}\left(\frac{\hat{n}-1}{2^k} \right) \right)^2 + \int\limits_{0}^{\frac{1}{2^{k-1}}} \frac{h^8}{576} \left(f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h \right) \right)^2 dh \\
&+ \frac{4}{2^{2k}} f\left(\frac{\hat{n}-1}{2^k} \right) f^{'}\left(\frac{\hat{n}-1}{2^k} \right) + \frac{8}{3}\frac{1}{2^{3k}} f\left(\frac{\hat{n}-1}{2^k} \right) f^{''}\left(\frac{\hat{n}-1}{2^k} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{12} f\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh + \frac{4}{2^{4k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) \\
&+ \frac{32}{15}\frac{1}{2^{5k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) + \int_0^{\frac{1}{2^{k-1}}} \frac{h^5}{12} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh \\
&+ \frac{16}{9}\frac{1}{2^{6k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) + \int_0^{\frac{1}{2^{k-1}}} \frac{h^6}{24} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh \\
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^7}{72} f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh + \frac{4}{3}\frac{1}{2^{4k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
c_{n,0} &= 2^{\frac{k-1}{2}}\left[\frac{2}{2^k} f\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{2^{2k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{3}\frac{1}{2^{3k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{3}\frac{1}{2^{4k}} f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right] \\
&+ 2^{\frac{k-1}{2}}\left[\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24} f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right].
\end{aligned}
$$

Next,

$$
\begin{aligned}
c_{n,0}^2 &= \frac{2}{2^k}\left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{8}{9}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{9}\frac{1}{2^{7k}}\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 \\
&+ \frac{2^k}{2}\left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24} f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right)^2 + \frac{4}{2^{2k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{'}\left(\frac{\hat{n}-1}{2^k}\right) \\
&+ \frac{8}{3}\frac{1}{2^{3k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{3}\frac{1}{2^{4k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) \\
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{12} f\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh + \frac{8}{3}\frac{1}{2^{4k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) \\
&+ \frac{4}{3}\frac{1}{2^{5k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{2^k}\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{12} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh \\
&+ \frac{8}{9}\frac{1}{2^{6k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{3}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh \\
&+ \frac{2}{3}\frac{1}{2^{3k}}\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24} f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh.
\end{aligned}
$$

Since

$$|f^{'}(x)| \leq M_1, |f^{''}(x)| \leq M_2, |f^{'''}(x)| \leq M_3 \; and \; |f^{iv}(x)| \leq M_4, \; \forall x \in [0,1)$$

therefore,

$$\begin{aligned} ||e{*}_n^{(0)}||_2^2 &\leq \frac{2}{2^{3k}}\left(M_1 + \frac{M_2}{2^k} + \frac{M_3}{2^{2k}} + \frac{M_4}{2^{3k}}\right)^2. \\ \therefore E^{(5)}_{2^{k-1},0}(f) &= O\left(\frac{1}{2^k}\right). \end{aligned}$$

(ii) The error $e{*}_n^{(1)}(x)$ between $f(x)$ and its expression over any subinterval is defined as

$e{*}_n^{(1)}(x) = c_{n,0}\psi_{n,0}(x) + c_{n,1}\psi_{n,1}(x) - f(x)$, $x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$, $n = 1, 2, 3, ...2^{k-1}$

$||e{*}_n^{(1)}||_2^2 = \int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (e{*}_n^{(1)}(x))^2 dx$

$$= \int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - c^2_{n,0} - c^2_{n,1}.$$

Now,

$$\begin{aligned} c_{n,1} &= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \left[\frac{2}{3}\frac{1}{2^{2k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{3}\frac{1}{2^{3k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{5}\frac{1}{2^{4k}} f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right] \\ &+ \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int\limits_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h - 1) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh. \end{aligned}$$

Next,

$$\begin{aligned} c^2_{n,1} &= \frac{2}{3}\frac{1}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{3}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{6}{25}\frac{1}{2^{7k}}\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 \\ &+ \frac{3}{2} 2^k \left(\int\limits_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h - 1) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh\right)^2 \\ &+ \frac{2}{2^k} \int\limits_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h - 1) f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\ &+ \frac{4}{3}\frac{1}{2^{4k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{5}\frac{1}{2^{5k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) \\ &+ \frac{4}{5}\frac{1}{2^{6k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) \end{aligned}$$

$$+ \quad \frac{2}{2^{2k}} \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h - 1) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh$$

$$+ \quad \frac{6}{5}\frac{1}{2^{3k}} \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h - 1) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh$$

Therefore,

$$||e*_n^{(1)}||_2^2 \quad \leq \quad \frac{2}{2^{5k}}\left(M_2 + \frac{M_3}{2^k} + \frac{M_4}{2^{2k}}\right)^2.$$

Then,

$$E^{(6)}_{2^{k-1},1}(f) = O\left(\frac{1}{2^{2k}}\right).$$

(iii) The error $e*_n^{(2)}(x)$ between $f(x)$ and its expression over any subinterval is defined as
$e*_n^{(2)}(x) = c_{n,0}\psi_{n,0}(x) + c_{n,1}\psi_{n,1}(x) + c_{n,2}\psi_{n,2}(x) - f(x)$, $x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$,
$n = 1, 2, 3, ...2^{k-1}$

$$||e*_n^{(2)}||_2^2 \quad = \quad \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (e_n^{(2)}(x))^2 dx$$

$$= \quad \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - c_{n,0}^2 - c_{n,1}^2 - c_{n,2}^2.$$

Now,

$$c_{n,2} \quad = \quad < f(x), \psi_{n,2}(x) >$$

$$= \quad \sqrt{\frac{5}{2}} 2^{\frac{k}{2}} \frac{2}{15}\left[\frac{1}{2^{3k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{2^{4k}} f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right]$$

$$+ \quad \sqrt{\frac{5}{2}} 2^{\frac{k}{2}} \frac{1}{2}\left[\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(3h^2 2^{2k} - 6h2^k + 2) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh\right].$$

Next,

$$c_{n,2}^2 \quad = \quad \frac{2}{45}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{45}\frac{1}{2^{7k}}\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2$$

$$+ \quad \left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(3h^2 2^{2k} - 6h2^k + 2) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh\right)^2 + \frac{4}{45}\frac{1}{2^{6k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+ \quad \frac{1}{3}\frac{1}{2^{2k}} \int_{0}^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(3h^2 2^{2k} - 6h2^k + 2) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh$$

$$+ \quad \frac{1}{3}\frac{1}{2^{3k}} \int_{0}^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(3h^2 2^{2k} - 6h2^k + 2) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh.$$

Therefore,

$$||e{*}_{n}^{(2)}||_2^2 \quad \leq \quad \frac{2}{2^{7k}}\left(M_3 + \frac{M_4}{2^k}\right)^2.$$

Then,

$$E^{(7)}_{2^{k-1},2}(f) \quad = \quad O\left(\frac{1}{2^{3k}}\right).$$

(iv) The error $e{*}_{n}^{(3)}(x)$ between $f(x)$ and its expression over any subinterval is defined as
$e{*}_{n}^{(3)}(x) \; = \; c_{n,0}\psi_{n,0}(x) + c_{n,1}\psi_{n,1}(x) + c_{n,2}\psi_{n,2}(x) + c_{n,3}\psi_{n,3}(x) - f(x)$,
$x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right), \; n = 1, 2, 3, ...2^{k-1}$
Similarly , it can be proved that

$$E^{(8)}_{2^{k-1},3}(f) \quad = \quad O\left(\frac{1}{2^{4k}}\right).$$

(v)

Following the proof of Theorem $(4.1)(iv)$ *we have*

$$\begin{aligned}
c_{n,m} \quad &= \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right) \left[\frac{(P_{m+1}(t) - P_{m-1}(t))}{(2m+1)} - \frac{(P_{m+3}(t) - P_{m+1}(t))}{(2m+5)}\right] dt \\
&- \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right) \left[\frac{(P_{m-1}(t) - P_{m-3}(t))}{(2m-3)} - \frac{(P_{m+1}(t) - P_{m-1}(t))}{(2m+1)}\right] dt \\
&= \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+1)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right) \left[\frac{d(P_{m+2}(t) - P_m(t))}{(2m+3)}\right] \\
&- \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+1)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right) \left[\frac{d(P_m(t) - P_{m-2}(t))}{(2m-1)}\right] \\
&- \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+5)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right) \left[\frac{d(P_{m+4}(t) - P_{m+2}(t))}{(2m+7)}\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+5)} \int_{-1}^{1} f'''\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] \\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m-3)} \int_{-1}^{1} f'''\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m}(t)-P_{m-2}(t))}{(2m-1)}\right] \\
&+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m-3)} \int_{-1}^{1} f'''\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m-2}(t)-P_{m-4}(t))}{(2m-5)}\right] \\
&+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m+1)} \int_{-1}^{1} f'''\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] \\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m+1)} \int_{-1}^{1} f'''\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m}(t)-P_{m-2}(t))}{(2m-1)}\right] \\
&= \left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+1)(2m+3)} \\
&\times \int_{-1}^{1} f^{iv}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_m(t)-P_{m-2}(t))}{(2m-1)} - \frac{(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] dt \\
&+ \left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+5)} \\
&\times \int_{-1}^{1} f^{iv}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_{m+4}(t)-P_{m+2}(t))}{(2m+7)} - \frac{(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] dt \\
&+ \left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m+1)} \\
&\times \int_{-1}^{1} f^{iv}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_m(t)-P_{m-2}(t))}{(2m-1)} - \frac{(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] dt \\
&+ \left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m-3)} \\
&\times \int_{-1}^{1} f^{iv}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_m(t)-P_{m-2}(t))}{(2m-1)} - \frac{(P_{m-2}(t)-P_{m-4}(t))}{(2m-5)}\right] dt. \\
|c_{n,m}| &\leq \left(\frac{1}{2^{9k}}\right)^{\frac{1}{2}} \frac{8\sqrt{6}M_2}{(2m-7)^4}, \qquad (\because |f^{iv}(x)| \leq M_2 \;\forall\, x \in\ [0,1)).
\end{aligned}
$$

Next,

$$
\begin{aligned}
||f - S_{2^{k-1},M}(f)||_2^2 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} C_{n,m}^2 \\
&\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \left(\left(\frac{1}{2^{9k}} \right)^{\frac{1}{2}} \frac{8\sqrt{6}M_2}{(2m-7)^4} \right)^2 \\
&= \frac{48M_2^2}{7} \frac{1}{2^{8k}} \frac{1}{(2M-5)^7} \\
\therefore E_{2^{k-1},M}^{(9)}(f) &= \sqrt{\frac{48}{7}} \frac{M_2}{2^{4k}} \frac{1}{(2M-5)^{\frac{7}{2}}} \\
&= O\left(\frac{1}{(2M-5)^{\frac{7}{2}}} \frac{1}{2^{4k}} \right), \ \forall\, M \geq 3.
\end{aligned}
$$

6 Conclusions

(1) After discussing the Legendre wavelet approximation of a function f with bounded third and fourth derivatives ,it is trivial to find out the wavelet estimators of a function f of bounded first and second derivatives .
(2)The estimates of the Theorems (4.1) an(4.2) are obtained as following:
$(i)E_{2^{k-1},0}^{(1)}(f) = O\left(\frac{1}{2^k}\right) \to 0 \; as \; k \to \infty$
$(ii)E_{2^{k-1},1}^{(2)}(f) = O\left(\frac{1}{2^{2k}}\right) \to 0 \; as \; k \to \infty$
$(iii)E_{2^{k-1},2}^{(3)}(f) = O\left(\frac{1}{2^{3k}}\right) \to 0 \; as \; k \to \infty$
$(iv)E_{2^{k-1},M}^{(4)}(f) = O\left(\frac{1}{(2M-3)^{\frac{5}{2}}} \frac{1}{2^{3k}}\right) \to 0 \; as \; k \to \infty, M \to \infty$
$(v)E_{2^{k-1},0}^{(5)}(f) = O\left(\frac{1}{2^k}\right) \to 0 \; as \; k \to \infty$
$(vi)E_{2^{k-1},1}^{(6)}(f) = O\left(\frac{1}{2^{2k}}\right) \to 0 \; as \; k \to \infty$
$(vii)E_{2^{k-1},2}^{(7)}(f) = O\left(\frac{1}{2^{3k}}\right) \to 0 \; as \; k \to \infty$
$(viii)E_{2^{k-1},3}^{(8)}(f) = O\left(\frac{1}{2^{4k}}\right) \to 0 \; as \; k \to \infty$
$(ix)E_{2^{k-1},M}^{(9)}(f) = O\left(\frac{1}{(2M-5)^{\frac{7}{2}}} \frac{1}{2^{4k}}\right) \to 0 \; as \; k \to \infty, M \to \infty$
Then
$E_{2^{k-1},0}^{(1)}(f), E_{2^{k-1},1}^{(2)}(f), E_{2^{k-1},2}^{(3)}(f), E_{2^{k-1},M}^{(4)}(f), E_{2^{k-1},1}^{(5)}(f), E_{2^{k-1},1}^{(6)}(f), E_{2^{k-1},2}^{(7)}(f),$
$E_{2^{k-1},3}^{(8)}(f), E_{2^{k-1},M}^{(9)}(f)$ are best possible Legendre wavelet approximation in Wavelet Analysis.
(3)Legendre wavelet estimators of a function f with bounded fourth order derivative is better and sharper than the estimator of a function f of bounded third order derivative.
(4) Legendre wavelet estimator of a function f of bounded higher order derivatives is better and sharper than the estimator of a function f of bounded less order derivatives.

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