Ratio Mathematica Volume 42, 2022

On the approximation of conjugate of functions belonging to the generalized Lipschitz class by Euler-matrix product summability method of conjugate series of Fourier series

Jitendra Kumar Kushwaha*
Krishna Kumar †

Abstract

In this paper, a new theorem on the approximation of conjugate of functions belonging to the generalized Lipschitz class by Euler-Matrix product summability method of conjugate series of Fourier series has been obtained. Sometimes a series is not summable by any individual summability method. But it becomes summable by taking product summability means of given series. So working in this direction we have used Euler-Matrix product summability method. Since $Lip\alpha$ $Lip\,(\alpha, p)$ classes are the particular cases of generalized Lipschitz class. Therefore, many of the known results may become particular cases of our result. On the bases of above facts we can say that our result may be useful for the coming researchers in future.

Keywords: generalized Lipschitz class, conjugate series of Fourier series, product summability method, Euler mean, matrix mean.
2022 AMS subject classifications: 42B05, 42B08. [1]

---

*Department of Math and Stat, DDU Gorakhpur University; kjitendrakumar@yahoo.com.
†Department of Math and Stat, DDU Gorakhpur University; kkmaths1986@gmail.com.
[1] Received on May 17th, 2022. Accepted on June 27th, 2022. Published on June 30th, 2022. doi: 10.23755/rm.v42i0.788. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

The degree of approximation of functions belonging to various classes by using Cesáro, Nörlund and generalized Nörlund summability methods has been obtained by a number of researchers like Chandra [1], Holland [3], Lal et al ([5],[6]) and Kushwaha [4], Qureshi [7]. Later on Tiwary et al [9] has discussed the degree of approximation of functions by using $(E, q)A$ product summability means of Fourier series. No work seems to have been done so for to find the degree of approximation of conjugate of functions belonging to generalized Lipschitz class by using Euler-Matrix product summability means. Now, in this paper, we are presenting a new theorem on the degree of approximation of conjugate functions belonging to the generalized Lipschitz class by Euler-Matrix product summability method. This new result may become the generalization of many of the known results.

2 Definitions

In this section,we have given following definitions:

Definition 2.1. *A function $f \in Lip\alpha$ if*

$$|f(x+t) - f(x-t)| = O(|t|^{\alpha}) \ \textit{for} \ 0 \leq \alpha < 1.$$

Definition 2.2. *A function $f \in Lip(\alpha, p)$ if*

$$\left(\int_0^{2\pi} |f(x+t) - f(x-t)|^p \, dx\right)^{1/p} = O(|t^{\alpha}|), 0 \leq \alpha < 1, p \geq 1.$$

Given a positive increasing function $\xi(t)$ and integer $p \geq 1$,

$f \in Lip\,(\xi(t), p)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x-t)|^p \, dx\right)^{1/p} = O\left(\xi(t)\right).$$

If $\xi(t) = t^{\alpha}$, then $Lip\,(\xi(t), p)$ coincides to $Lip(\alpha, p)$.

Definition 2.3. *L_∞-norm of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by*

$$\|f\|_\infty = sup\left\{|f(x)|/f : \mathbb{R} \to \mathbb{R}\right\}.$$

L_p-norm is defined by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p\right)^{1/p}, p \geq 1.$$

The degree of approximation of a function $f : \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial t_n(Zygmund [11]) is defined by

$$\|t_n - f\|_\infty = sup\left\{|t_n - f| : x \in \mathbb{R}\right\} or \|t_n - f\|_p = min\|t_n - f\|.$$

Let f be 2π periodic and integrable over $(-\pi, \pi)$ in Lebesgue sense and $f \in Lip\left(\xi(t), p\right)$. Let its Fourier series be given by

$$\begin{aligned} f(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x). \end{aligned} \tag{1}$$

The conjugate series of the Fourier series (1) is given by

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) = -\sum_{n=1}^{\infty} B_n(x). \tag{2}$$

If f is Lebesgue integrable, then

$$\overline{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi(t)\cot(t/2)dt = -\frac{1}{2\pi}\lim_{\epsilon \to 0}\int_0^{\pi} \psi(t)\cot(t/2)dt$$

exists for almost all x (Zygmund [11]).

Let $T = (a_{n,k})$ be an infinite lower triangular matrix satisfying Töeplitz (p. 131) condition of regularity i.e. $a_{n,k} \to 1$ as $n \to \infty$, $a_{n,k} = 0$ for $k > n$ and $\sum_{k=0}^{n} |a_{n,k}| \leq M$ a finite constant. Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose k^{th} partial sums is $s_k = \sum_{n=0}^{k} u_n$. The sequence to sequence transformation $t_n = \sum_{k=0}^{n} a_{a,k} s_k$ defines the sequence $\{t_n\}$ of lower triangular matrix summability means of sequence $\{s_n\}$ generated by the sequence of coefficients $(a_{n,k})$. The series $\sum_{n=0}^{\infty} u_n$ is said to summable to sum s by lower triangular matrix method if $\lim_{n\to\infty} t_n$ exists and is equal to s (Zygmund; p.74) and we write $t_n \to s(T)$, as $n \to \infty$.

The (E, q) means of $\{s_n\}$ is defined by -

$$W_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k.$$

The (E, q) transform of Matrix transform A of s_n is defined by

$$\begin{aligned} \overline{\eta}_n &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \overline{t}_k \\ &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} a_{\nu,k} \overline{s}_k \right\}. \end{aligned}$$

If $\overline{\eta}_n \to \infty$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be $(E, q)A$-summable to sum s.

We use the following notations :

(i) $\psi(x, t) = f(x + t) - f(x - t)$.

(ii) $\overline{M}_n(t) = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} a_{\nu,k} \frac{\cos(\nu+1/2)t}{\sin(t/2)} \right\}$.

3 Lemmas:-

For the proof of our theorem, we have required following lemmas:

Lemma 3.1. $\overline{M}_n(t) = O\left(\frac{1}{t}\right)$ *for* $0 \le t \le (n+1)^{-1}$.

Proof. For $0 \le t \le (n+1)^{-1}$, we have

$$\begin{aligned} \left|\overline{M}_n(t)\right| &= \frac{1}{2\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\nu=0}^{k}a_{\nu,k}\frac{\cos(\nu+1/2)t}{\sin(t/2)}\right\}\right]\right| \\ &\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\nu=0}^{k}a_{\nu,k}\frac{|\cos(\nu+1/2)t|}{(t/\pi)}\right\}\right] \\ &\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\nu=0}^{k}a_{\nu,k}\right\}\right] \\ &= O\left(\frac{1}{t}\right). \end{aligned}$$

Lemma 3.2. $\overline{M}_n(t) = O\left(\frac{A_{n,\tau}}{t}\right)$ *for* $(n+1)^{-1} \le t \le \pi$.

Proof. For $(n+1)^{-1} \le t \le \pi$, we have by Jordan's Lemma, $\sin(t/2) \ge (t/\pi)$, then

$$\begin{aligned} \left|\overline{M}_n(t)\right| &= \frac{1}{2\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\nu=0}^{k}a_{\nu,k}\frac{\cos(\nu+1/2)t}{\sin(t/2)}\right\}\right]\right| \\ &\le \frac{1}{2\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\nu=0}^{k}a_{\nu,k}\frac{\cos(\nu+1/2)t}{(t/\pi)}\right\}\right]\right| \\ &= \frac{1}{2t(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\nu=0}^{k}a_{\nu,k}\cos(\nu+1/2)t\right\}\right]\right| \\ &= \frac{1}{2t(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{O\left(\frac{a_{k,k-\tau-1}}{t}\right)+A_{k,\tau}\right\}\right]\right| \\ &= O\left(\frac{A_{n,\tau}}{t}\right). \end{aligned}$$

4 Theorem

In this section, we have proved the theorem:

Theorem 4.1. *Let f be a 2π-periodic, Lebesgue integrable function belonging to $Lip\,(\xi(t),p)$ class and $T=(a_{m,n})$ be an infinite lower triangular matrix. Then the degree of approximation of conjugate function by $(E,q)A$-summability means of its conjugate series of Fourier series is given by*

$$\|\overline{\eta}-\overline{f}(x)\|_p = O\left((n+1)^{1/p}\xi\left(\frac{1}{n+1}\right)\right),$$

provided $\xi(t)$ satisfies following conditions:

$$\left\{\int_0^{1/(n+1)}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^p\right\}^{1/p} = O\left(\frac{1}{n+1}\right), \tag{3}$$

and

$$\left\{\int_{1/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^q\right\}^{1/q} = O\left((n+1)^{\delta}\right), \tag{4}$$

where δ ia an arbitrary number such that $q(1-\delta)-1>0$, $p^{-1}+q^{-1}=1$ such that $1\le p<\infty$, conditions (3) and (4) holds uniformly in x.

Proof. The k^{th} partial sums of conjugate series of Fourier series (2) is given by

$$\overline{s}_k(f;x) = -\frac{1}{2\pi}\int_0^{\pi}\cot(t/2)\psi(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(k+1/2)t}{\sin(t/2)}\psi(t)dt.$$

$$\overline{s}_k(f;x) = -\frac{1}{2\pi}\int_0^{\pi}\cot(t/2)\psi(t)dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(k+1/2)t}{\sin(t/2)}\psi(t)dt.$$

Therefore making (A-transform) of $\overline{s}_k(f;x)$, we get

$$\overline{t}_n-\overline{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}a_{n,k}\frac{\cos(\nu+1/2)t}{\sin(t/2)}dt.$$

Now, making $(E,q)A$-transform of $\overline{s}_k(f;x)$, we get

$$\begin{aligned}
\overline{\eta}_n - \overline{f}(x) &= \frac{1}{2\pi(1+q)^n} \\
&\times \int_0^{\pi} \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} a_{n,k} \frac{\cos(\nu+1/2)t}{\sin(t/2)} \right\} dt \\
&= \int_0^{\pi} \psi(t) \overline{M}_n(t) dt \\
&= \left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi} \right) \psi(t) \overline{M}_n(t) dt \\
&= I_1 + I_2. \qquad (5)
\end{aligned}$$

Clearly,

$$|\psi(x+t) - \psi(x-t)| \leq |f(u+x+t) - f(x+t)| + |f(u-x-t) - f(x-t)|.$$

Now,let

$$\psi(x,t) = \psi(x+t) - \psi(x-t)$$
$$\psi_1(u,x,t) = f(u+x+t) - f(x+t)$$
$$\psi_2(u,x,t) = f(u-x-t) - f(x-t)$$

Hence, by Minkowski's inequality

$$\begin{aligned}
\left\{ \int_0^{2\pi} |\psi(x,t)|^p dx \right\}^{1/p} &\leq \left\{ \int_0^{2\pi} |\psi_1(u,x,t)|^p dx \right\}^{1/p} \\
&+ \left\{ \int_0^{2\pi} |\psi_2(u,x,t)|^p dx \right\}^{1/p} \\
&= O\left(\xi(t)\right). \qquad (6)
\end{aligned}$$

Then

$$f \in Lip\,(\xi(t), p) \Rightarrow \psi \in Lip\,(\xi(t), p)\,.$$

Using the Hölder's inequality, $\psi(t) \in Lip\,(\xi(t), p)$, condition (3), $\sin t \geq (2\pi/t)$, Lemma 1, and second mean value theorem for integrals, we have

$$\begin{aligned}
|I_1| &\leq \left[\int_0^{1/(n+1)} \left(\frac{t\psi(t)}{\xi(t)}\right)^p dt\right]^{1/p} \left[\int_0^{1/(n+1)} \left(\frac{\xi(t)|\overline{M}_n(t)|}{t}\right)^q dt\right]^{1/q} \\
&= O\left((n+1)^{-1}\right) \left[\int_0^{1/(n+1)} \left(\frac{\xi(t)|\overline{M}_n(t)|}{t}\right)^q dt\right]^{1/q} \\
&= O\left((n+1)^{-1}\right) \left[\int_0^{1/(n+1)} \left(\frac{\xi(t)}{t^2}\right)^q dt\right]^{1/q} \\
&= O\left((n+1)^{1/p}\xi\left(\frac{1}{n+1}\right)\right), p^{-1} + q^{-1} = 1. \qquad (7)
\end{aligned}$$

Using the Hölder's inequality, Lemma 2, $|\sin t| \leq 1$, $\sin t \geq (2\pi/t)$, and condition (4), we have

$$\begin{aligned}
|I_2| &\leq \left[\int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta}\psi(t)}{\xi(t)}\right)^p dt\right]^{1/p} \left[\int_{1/(n+1)}^{\pi} \left(\frac{\xi(t)|\overline{M}_n(t)|}{t^{-\delta}}\right)^q dt\right]^{1/q} \\
&\leq O\left((n+1)^{\delta}\right) \left[\int_{1/(n+1)}^{\pi} \left(\frac{\xi(t)|\overline{M}_n(t)|}{t^{-\delta}}\right)^q dt\right]^{1/q} \\
&\leq O\left((n+1)^{\delta}\right) \left[\int_{1/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{-\delta}} O\left(\frac{A_{n,\tau}}{t}\right)\right)^q dt\right]^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&= O\left((n+1)^{\delta}\right)\left[\int_{1/(n+1)}^{\pi}\left(\left(\frac{\xi(t)}{t^{1-\delta}}A_{n,\tau}\right)\right)^{q}dt\right]^{1/q}\\
&= O\left((n+1)^{\delta}\right)\left[\int_{1/\pi}^{(n+1)}\left(\frac{\xi(1/y)}{y^{\delta-1}}A_{n,[y]}\right)^{q}\frac{dy}{y^{2}}\right]^{1/q}\\
&= O\left((n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right)\left[\int_{1/\pi}^{(n+1)}\frac{dy}{y^{\delta q-q+2}}\right]^{1/q}\\
&= \left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\left(O(n+1)^{-q(\delta-1)-1}\right)^{1/q}\right\}\\
&= O\left((n+1)^{1/p}\xi\left(\frac{1}{n+1}\right)\right), p^{-1}+q^{-1}=1. \qquad (8)
\end{aligned}
$$

Collecting equations from (5) to (8), we get

$$\|\overline{\eta}-\overline{f}(x)\|_{p}=O\left((n+1)^{1/p}\xi\left(\frac{1}{n+1}\right)\right), 1\leq p<\infty.$$

5 Corollaries:-

Corolary 5.1. *If $\xi(t)=t^{\alpha}$ then the degree of approximation of a function $\overline{f}(x)$, conjugate of $f\in Lip(\alpha,p)$, $\frac{1}{p}<\alpha<1$ by $(E,q)A$ means is given by*

$$\|\overline{\eta}-\overline{f}(x)\|_{p}=O\left((n+1)^{-\alpha+1/p}\right), 1\leq p<\infty.$$

Corolary 5.2. *If $p\to\infty$ in case 1, then for $0<\alpha<1$, the degree of approximation of a function $\overline{f}(x)$, conjugate of $f\in Lip\alpha$ by $(E,q)A$ means is given by*

$$\|\overline{\eta}-\overline{f}(x)\|_{p}=O\left((n+1)^{-\alpha}\right).$$

6 Acknowledgement

Auther is highley thankful to **Professeor Shyam lal**, Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India for his encouragement and support to this work.

References

1. P. Chandra, Trigonometric approximation of functions in L_p-norm,*J. Math. Anal. Appl.*, 275, No 1 (2002), 13-26.

2. G. H. Hardy, *Divergent Series*, American Mathematical Society (2000).

3. A. S. B. Holland and B. N. Sahney, On the degree of approximation by (E,q) means, *Studia Sci. Math. Hunger*, 11, (1976), 431-435.

4. J. K. Kushwaha, On the approximation of conjugate function by almost triangular matrix summability means, *Int. J. of Management Tech. and Engi.*, 9, No 3 (2019), 4382-4389; DOI: 16.10089.IJMTE.2019.V913.19.27979.

5. S. Lal and J. K. Kushwaha, Degree of approximation of Lipschitz function by product summability methods, *International Mathematical Forum*, 4, No 43, (2009), 2101-2107.

6. S. Lal and J. K. Kushwaha, Approximation of conjugate of functions belonging to generalized Lipschitz class by lower triangular matrix means, *Int. Journal of Math. Analysis*, 3, No 21, (2009), 1031-1041.

7. K. Qureshi, On the degree of approximation of a function belonging to the weighted class, *Indian Jour. of Pure and Appl. Math*, 4, No 13, (1982), 471-475.

8. E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press (1939).

9. S. K. Tiwary and U. Upadhyay, Degree of approximation of functions belonging to the generalized Lipschitz class by product means of its Fourier series, *Ultra Scientist*, 3, No 25, (2013), 411-416.

10. O. Töeplitz, Über die Lineare mittelbi-dungen Prace, *Mat. Fiz.*, No 22, (1911), 113-119.

11. A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge (1959).

12. C.K.Chui. An introduction to Wavelets (wavelets analysis and it's application), Vol. 1,*Academic Press, USA*,1992.

13. Jitendra Kumar Kushwaha, On the Approximation of Generalized Lipschitz Function by Euler Means of Conjugate Series of Fourier Series.*The Scientific World Journal*. Vol.2013,Article Id 508026.

14. Sandeep Kumar Tiwari and Uttam Upadhyay, Degree of Approximation of Function Belonging to the $W(L^r, \xi(t))$ class by (E, q) A-Product Means of its Fourier series.*IJM* Archive- 4(8),2013,266-272.

15. Xhevat Z. Krasniqi, On the Degree of Approximation of Functions Belonging to the Lipschitz Class by $(E, q)(C, \alpha, \beta)$ Means.*Khayyam J.Math.* 1(2015),no.2 243-252.

16. Jitendra Kumar Kushwaha,Approximation of functions by $(C, 2)(E, 1)$product summability method of Fourier series. *Ratio Mathematica*.Vol 38,2020,pp. 341-348.