

On the Roots and Stability of Vertex connectivity Polynomial

Priya Krishnan*
Anil Kumar Vasu[†]

Abstract

The introduction of the concept of vertex connectivity polynomial of graphs in [Priya and Anil Kumar, 2021b] triggered the need to study the nature of roots as well as the stability properties of the same for various graph classes. This paper mainly deals with results about the nature of roots, stability and schur stability of the vertex connectivity polynomial.

Keywords: Vertex connectivity polynomial, Vertex-connected vertices, Schur stability.

2020 AMS subject classifications:05C31, 05C40.¹

*Department of Mathematics, University of Calicut, Thenhipalam, Kerala-673635, India; priyakrishna27.clt@gmail.com.

[†]Department of Mathematics, University of Calicut, Thenhipalam, Kerala-673635, India; anil@uoc.ac.in.

¹Received on March 17th. Accepted on June 25th. Published on June 30th. doi: 10.23755/rm.v42i0.731. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

The adjacency property of vertices in a graph is not sufficient to characterize a graph in terms of its connectedness as the deletion of the corresponding edge may or may not disconnect the graph. More generally, the connectivity property of vertices in a graph may not be preserved after the deletion of edges of the geodesics linking them. This motivated the authors to introduce the concept of “vertex-connected vertices”, which shares geodesics preserving the connectivity of a graph. The study of vertex-connected vertices in [Priya and Anil Kumar, 2021a] triggered the need to define vertex connectivity polynomial discussed in [Priya and Anil Kumar, 2021b] for simple finite connected graphs to explicitly reveal the number of vertex pairs that disconnect a graph.

A polynomial $p(z)$ is said to be *stable*, or a *Hurwitz polynomial*, if all its zeros lie in the open left half-plane [Fuhrmann, 2011] and is *schur stable* if all its zeros belong to the open unit disk. Hurwitz polynomials are important in control systems theory, because they represent the characteristic equations of stable linear systems [Gajender and Sharma, 2014]. Thus the study of a graph polynomial is worthy only if it succeeds in predicting the behavior of some stable physical systems. This motivated the authors to study about the nature of roots and stability of the vertex connectivity polynomial of a graph. Throughout this paper, G denotes a finite simple connected graph with vertex set and edge set denoted by $V(G)$ and $E(G)$ respectively. All the graph theoretic terminology and notations used in the paper are as in [Harary, 1969].

2 Main Results

Definition 2.1. *Let G be a graph of order n . Then, the vertex connectivity polynomial of G , denoted by $V[G; x]$, is defined in [Priya and Anil Kumar, 2021b] as:*

$$V[G; x] = \sum_{i=1}^{\text{diam}(G)+1} |D_i| x^i,$$

where D_i is the set of all vertex pairs which disconnects G into i components.

Definition 2.2. *Let $G = (V, E)$ be a graph and let $u, v \in V$. Then, the vertices u and v are said to be vertex-connected if the deletion of the edges of none of the geodesics connecting u and v disconnects G into multiple components. The graph G is vertex-connected if each of its vertex pairs are vertex-connected. [Priya and Anil Kumar, 2021a]*

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Definition 2.3. Let $G = (V, E)$ be a graph. Then, a pair of vertices $(u, v) \in V \times V$ at a distance $d > 0$ are said to be d -metrically connective if it disconnects the graph G into d components.

The graph G is metrically connective if every pair of its vertices are d -metrically connective for some $d > 0$. [Priya and Anil Kumar, 2021a]

Theorem 2.4. (Routh-Hurwitz Criteria [Fuhrmann, 2011]) Given a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n,$$

where the coefficients a_i 's are real constants, the n Hurwitz matrices using the coefficients a_i of the above polynomial are defined as

$$H_1 = [a_1], H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix}, H_3 = \begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \dots,$$

$$H_n = \begin{bmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_n \end{bmatrix},$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(x)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive: $\det(H_j) > 0, j = 1, 2, \dots, n$.

Definition 2.5. Let G be a graph. Then, the roots of the vertex connectivity polynomial of G are called the vertex connectivity roots of G .

The following are some of the observations about the vertex connectivity roots of graphs.

- (i) $(0, \infty)$ is a zero-free interval of the vertex connectivity polynomial $V[G; x]$ for any graph G .
- (ii) If G is metrically connective, then it has at most $diam(G)$ vertex connectivity roots.
- (iii) If G is a tree, then it has $diam(G) + 1$ vertex connectivity roots counting multiplicities.

Theorem 2.6. Let G be a graph. Then, zero is a vertex connectivity root of G of multiplicity greater than one iff G is a tree.

Proof. Assume that zero is a vertex connectivity root of G of multiplicity $k > 1$. Then, $V[G; x] = x^k g(x)$, a polynomial of degree greater than or equal to 2. This means that every vertex pair of G disconnects the graph into at least two components so that none of its edges belongs to a cycle. Hence it follows that G is a tree. Conversely, if G is a tree, all its edges being cut edges, none of the vertex pairs are vertex-connected. That is, $V[G; x]$ is a polynomial free of linear coefficient.

This completes the proof. \square

Theorem 2.7. *Let G be a graph of order n . Then, zero is the only vertex connectivity root of G iff either G is vertex-connected or every vertex pair of G disconnects the graph into exactly two components.*

Proof. Assume that zero is the only vertex connectivity root of G . Then, $V[G; x] = rx^k$, where $r, k > 0$. But, $V[G; 1] = \sum_{i=1}^{\text{diam}(G)+1} |D_i| = \binom{n}{2}$ so that $r = \binom{n}{2}$. That is, $V[G; x] = \binom{n}{2}x^k$.

Case (i): If $k = 1$, then $V[G; x] = \binom{n}{2}x$ so that G is vertex-connected.

Case (ii): If $k > 1$, then all the $\binom{n}{2}$ vertex pairs disconnects the graph into k components. But if a vertex pair disconnects G into k components, then corresponding to that there exists a vertex pair disconnecting the graph into two components. Thus it follows that $V[G; x] = \binom{n}{2}x^2$.

Conversely, if either G is vertex-connected or every vertex pair of G disconnects the graph into exactly two components, then we get $V[G; x]$ as $\binom{n}{2}x$ and $\binom{n}{2}x^2$ respectively. In both the cases, it follows that zero is the only vertex connectivity root of G .

This completes the proof. \square

Corollary 2.8. *Let G be a non vertex-connected graph of order n . Then, zero is the only vertex connectivity root of G iff $G = P_2$.*

Proof. Let zero be the only vertex connectivity root of G . Since G is not vertex-connected, from theorem 2.7 it follows that G is a tree and $V[G; x] = \binom{n}{2}x^2$. This means that all the $\binom{n}{2}$ vertex pairs disconnects G into two components. Now if possible assume $n > 2$. Then, since every edge of G is a cut edge, we can find a vertex pair at 2 units distance which disconnects G into three components. Therefore $n = 2$ and $G = P_2$.

This completes the proof. \square

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Theorem 2.9. *Let G be a graph and n_1, n_2 are respectively be the number of vertex-connected vertex pairs and the number of vertex pairs which disconnects G , where $n_1 > n_2$. If n_1 is prime, then G lacks non-zero real vertex connectivity roots.*

Proof. Let $V[G; x] = xg(x)$, where $g(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + n_1$, $\sum_{i=1}^n a_i = n_2$, for some n . Clearly all the zeros of $g(x)$ are non-zero. Let β be a zero of $g(x)$ such that $|\beta| \leq 1$. Then,

$$g(\beta) = a_n \beta^n + a_{n-1} \beta^{n-1} + \dots a_1 \beta + n_1 = 0.$$

That is,

$$|n_1| = |a_n \beta^n + a_{n-1} \beta^{n-1} + \dots a_1 \beta| \leq |a_n| + |a_{n-1}| + \dots |a_1| = n_2,$$

a contradiction. Therefore, $|\beta| > 1$.

Now, if possible, assume that β is real. Then, $g(x)$ is reducible over the set of all real numbers. That is,

$$g(x) = (x - \beta)h(x),$$

where $h(x)$ is a nonconstant integer polynomial. Now,

$$n_1 = g(0) = -\beta h(0).$$

Since n_1 is prime and since $|\beta| > 1$, $|h(0)|$ must be 1. Let $\beta_1, \dots, \beta_{n-1}$ be the zeros of $h(x)$ and a be its leading coefficient. Then,

$$|\beta_1, \dots, \beta_{n-1}| = \frac{1}{|a|} \leq 1.$$

But $\beta_1, \dots, \beta_{n-1}$ are the zeros of $g(x)$ also so that $|\beta_i| > 1 \forall i = 1, \dots, n-1$, a contradiction. Therefore $g(x)$ has no real zeros and hence G lacks non-zero real vertex connectivity roots.

This completes the proof. □

Corollary 2.10. *Let G be a graph and n_1, n_2 are respectively be the number of vertex-connected vertex pairs and the number of vertex pairs which disconnects G , where $n_1 > n_2$. Then, all the non-zero vertex connectivity roots of G lie strictly outside the unit circle.*

Theorem 2.11. *Let G be a graph with $|D_i| = m \forall i \leq n$, where $2 \leq n \leq \text{diam}(G) + 1$. Then, all the non-zero vertex connectivity roots of G are distinct and are of unit modulus. In addition, if G is a tree of diameter p , where p is an odd prime, then G lacks non-zero real vertex connectivity roots.*

Proof. Without loss of generality, let G be a graph which is not a tree. Since $|D_i| = m, \forall i = 1, 2, \dots, n$,

$$V[G, x] = m \sum_{i=1}^n x^i = mx[1 + x + \dots + x^{n-1}].$$

The zeros of the polynomial $1 + x + \dots + x^{n-1}$ are the n^{th} roots of unity except 1 and hence it follows that the vertex connectivity roots of G are distinct. Now, if G is a tree, then $V[G; x]$ is a polynomial of degree $\text{diam}(G) + 1$. That is,

$$V[G, x] = m \sum_{i=2}^{\text{diam}(G)+1} x^i = mx^2[1 + x + \dots + x^{\text{diam}(G)-1}].$$

Since $\text{diam}(G) = p$, the polynomial $1 + x + \dots + x^{p-1}$ is irreducible over the set of all rationals and hence it has no rational roots. But the only possible real $p - 1^{\text{th}}$ root of unity is -1 , which is rational. Therefore, it can be concluded that G is free of non-zero real vertex connectivity roots.

This completes the proof. □

Corollary 2.12. *If n is odd, then all the non-zero vertex connectivity roots of C_n lies on the unit circle. In addition, if $\frac{n-1}{2}$ is even, then C_n has exactly two real vertex connectivity roots.*

Proof. This follows from the fact that $V[C_n; x] = \sum_{i=1}^{\frac{n-1}{2}} nx^i$ for odd n and -1 is a k^{th} root of unity if k is even. □

Theorem 2.13. (i) *Let G_1 and G_2 be two disjoint graphs. If a is a vertex connectivity root of both G_1 and G_2 , then a is also a vertex connectivity root of $G_1 + G_2$.*

(ii) *Let G be a vertex-connected graph of order n and let G' be the graph obtained by adjoining one pendent vertex to m distinct vertices of $G, m \leq n$. Then, the vertex connectivity roots of G' are the roots of the polynomial $V[G; x] + nm x^2 + \binom{m}{2} x^3$.*

(iii) *Let G be metrically connective and let $d = \text{diam}(G)$. Then, G has exactly d vertex connectivity roots counting multiplicities.*

Proof. (i) This follows from the fact that $V[G_1 + G_2; x] = V[G_1; x] + V[G_2; x]$.

(ii) Since G is vertex-connected, the newly adjoined vertices along with every vertex of G disconnects G' into exactly two components and mutually disconnects G' into three components.

Therefore, $V[G'; x] = V[G; x] + nm x^2 + \binom{m}{2} x^3$.

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(iii) This follows from the fact that the degree of $V[G; x]$ is d .

This completes the proof. □

Theorem 2.14. *Let G be a graph of order n and diameter d . Then, any vertex connectivity root β of G either has nonpositive real part or satisfies*

$$|\beta| < \frac{1 + \sqrt{1 + 2n(n-1)}}{2}.$$

Proof. Let β be a non-zero vertex connectivity root of G . If $|\beta| \leq 1$, then the result holds trivially since $\frac{1 + \sqrt{1 + 2n(n-1)}}{2} > 1$.

Case(i) : Let G be a tree. Then, the degree of $V[G; x]$ is $d + 1$.

We have, $V[G; x] = x^2g(x)$, where $g(x) = a_{d-1}x^{d-1} + \dots + a_1x + a_0$. Since $\sum_{i=0}^{d-1} a_i = \binom{n}{2}$, $|a_i| \leq \binom{n}{2} \forall i = 0, 1, \dots, d-1$. Now, if $|\beta| > 1$ and $Re(\beta) > 0$, then

$$\begin{aligned} \frac{|g(\beta)|}{|\beta^{d-1}|} &= \left| a_{d-1} + \frac{a_{d-2}}{\beta} + \dots + \frac{a_1}{\beta^{d-2}} + \frac{a_0}{\beta^{d-1}} \right| \\ &\geq \left| a_{d-1} + \frac{a_{d-2}}{\beta} \right| - \frac{n(n-1)}{2} \left[\frac{1}{|\beta^2|} + \dots + \frac{1}{|\beta^{d-1}|} \right] \\ &> Re(a_{d-1}) + a_{d-2}Re\left(\frac{1}{\beta}\right) - \frac{n(n-1)}{2} \left[\frac{1}{|\beta^2| - |\beta|} \right] \\ &\geq 1 - \frac{n(n-1)}{2} \left[\frac{1}{|\beta^2| - |\beta|} \right], \text{ since } a_{d-1}, a_{d-2} \geq 1 \text{ and } Re(\beta) > 0. \\ &= \frac{|\beta^2| - |\beta| - \frac{n(n-1)}{2}}{|\beta^2| - |\beta|} \\ &\geq 0, \end{aligned}$$

whenever $|\beta^2| - |\beta| - \frac{n(n-1)}{2} \geq 0$.

That is, if $|\beta| \geq \frac{1 + \sqrt{1 + 4 \frac{n(n-1)}{2}}}{2} = \frac{1 + \sqrt{1 + 2n(n-1)}}{2}$ and $Re(\beta) > 0$, then $\frac{|g(\beta)|}{|\beta^{d-1}|} > 0$ so that β cannot be a zero of $g(x)$.

Case(ii) Let G be a graph which is not a tree. Then at least one edge of G is a part of some cycle so that the linear coefficient of $V[G; x]$ is non-zero. Let $V[G; x] = xh(x)$, where $h(x) = a_{m-1}x^{m-1} + \dots + a_1x + a_0$. Here also $\sum_{i=0}^{m-1} a_i = \binom{n}{2}$, $|a_i| \leq \binom{n}{2} \forall i = 0, 1, \dots, m-1$, so that similar calculation as in case(i) yields that if $|\beta| \geq \frac{1 + \sqrt{1 + 2n(n-1)}}{2}$ and $Re(\beta) > 0$, then β cannot be a zero of $g(x)$.

Therefore, from the above two cases we get that every vertex connectivity root of the graph G either has nonpositive real part or has modulus strictly less than $\frac{1+\sqrt{1+2n(n-1)}}{2}$.

This completes the proof. \square

Theorem 2.15. *Let G be a tree with diameter $d \leq 2$. Then, all the non-zero vertex connectivity roots of G lie in the left half plane. In addition, if the leading coefficient of $V[G; x]$ dominates, then the polynomial is schur stable.*

Proof. Case(i): If $d = 1$, then $G = P_2$ so that $V[G; x] = x^2$. Thus, the result holds trivially.

Case(ii): If $d = 2$, then $V[G; x] = x^2(ax + b)$, a cubic polynomial where $a, b > 0$. Thus the only non-zero zero $x = -\frac{b}{a}$ of $V[G; x]$ lies in the left half plane. Now if the leading coefficient of $V[G; x]$ dominates, then $|\frac{b}{a}| < 1$ and thus all the vertex connectivity roots of G lies inside the unit circle. Hence $V[G; x]$ is schur stable. \square

Corollary 2.16. $V[K_{1,n}; x]$ is schur stable iff $n > 3$.

Theorem 2.17. *Every vertex connectivity root of the path graph P_n lie in the closed left half plane iff $n \leq 5$*

Proof. We have, $V[P_n; x] = \sum_{i=1}^{n-1} (n-i)x^{i+1}$. Thus the result holds trivially for $n = 2$ and $n = 3$.

Now for $n \geq 4$, let $V[P_n; x] = x^2g(x)$.

When $n = 4$, $g(x) = x^2 + 2x + 3$ so that

$$H_1 = [2], H_2 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

are having positive determinants.

If $n = 5$, then $g(x) = x^3 + 2x^2 + 3x + 4$ so that all the Hurwitz matrices

$$H_1 = [2], H_2 = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 0 & 5 & 4 \end{bmatrix}$$

have positive determinant.

For $n = 6$, $g(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$.

$$H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

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so that $\det(H_3) < 0$.

For $n \geq 7$,

$$H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 6 & 5 & 4 \end{bmatrix}$$

so that $\det(H_3) = 0$. Thus for $n \geq 6$, $g(x)$ is not stable so that all the vertex connectivity roots of P_n does not belong to the closed left half plane.

This completes the proof. □

A bistar graph $B_{m,m}$ is the union of two identical star graphs $K_{1,m}$ whose centres joined together to form a new edge.

Theorem 2.18. *The vertex connectivity roots of the bistar graph $B_{m,m}$ always lie in the closed left half plane and are all real iff $m \geq 7$. Also, for all $m \geq 2$, zero is the only vertex connectivity root of multiplicity > 1 .*

Proof. We have, $V[B_{m,m}; x] = x^2[m^2x^2 + m(m+1)x + (2m+1)]$. [Priya and Anil Kumar, 2021b]

The zeros of the polynomial $g(x) = m^2x^2 + m(m+1)x + (2m+1)$ are

$$\begin{aligned} x &= \frac{-m(m+1) \pm \sqrt{m^2(m+1)^2 - 4m^2(2m+1)}}{2m^2} \\ &= \frac{-(m+1) \pm \sqrt{m^2 - 6m - 3}}{2m}. \end{aligned}$$

Since $\sqrt{m^2 - 6m - 3} < \sqrt{m^2} < m + 1$, all the zeros of $g(x)$ lie in the left half plane. The zeros of $g(x)$ are real iff $m^2 - 6m - 3 \geq 0$ iff $m \geq 3 + \sqrt{12}$ or $m \leq 3 - \sqrt{12}$. Since m is a positive integer and since $6 < 3 + \sqrt{12} < 7$, we get that the vertex connectivity roots of $B_{m,m}$ are all real iff $m \geq 7$. Now, since $m^2 - 6m - 3$ is never zero for any integer m , all the zeros of $g(x)$ are distinct.

This completes the proof. □

Corollary 2.19. *For $m \geq 3$, $V[B_{m,m}; x]$ is schur stable.*

Proof. In the proof of theorem 2.18, the zeros of $g(x)$ are $\frac{-(m+1) \pm \sqrt{m^2 - 6m - 3}}{2m}$. It can be observed that $|\frac{-(m+1) \pm \sqrt{m^2 - 6m - 3}}{2m}| < 1$ for $m \geq 3$ so that all the vertex connectivity roots of $B_{m,m}$ lie inside the unit circle.

This completes the proof. □

Theorem 2.20. *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a complex polynomial such that $|a_k| > |a_0| + |a_1| + \dots + |a_{k-1}| + |a_{k+1}| + \dots + |a_n|$ for some $0 \leq k \leq n$. Then exactly k zeros of P lie strictly inside the unit circle, and the other $n - k$ zeros of P lie strictly outside the unit circle.[Zhao, 1947]*

A helm H_n is obtained by adding pendent edges to every vertex on the graph obtained as the join of the cycle C_{n-1} and K_1 . A webgraph WB_n is obtained by joining the pendent vertices of H_n to form a cycle and then adding a single pendent edge to each vertex of the outer cycle. A butterfly graph is a double shell with same apex along with exactly two pendent edges at the apex.

Theorem 2.21. (i) *For $n > 3$, the helm H_n is not schur stable.*

- (ii) *For $n > 3$, the web graph WB_n is not schur stable. But all the vertex connectivity roots of WB_n are real and belong to the closed left half plane.*
- (iii) *For $n > 4$, the butterfly graph BF_n is not schur stable. But all the vertex connectivity roots of BF_n are real and belong to the closed left half plane.*

Proof. (i) From [Priya and Anil Kumar, 2021b], we get the vertex connectivity polynomial of H_n as $V[H_n; x] = \binom{n-1}{2} x^3 + n(n-1)x^2 + \binom{n}{2} x$. Here, the quadratic coefficient $n(n-1) > \binom{n-1}{2} + \binom{n}{2}$ so that from theorem 2.20 it follows that only two zeros of $V[H_n; x]$ lie inside the unit circle and one zero outside the unit circle. Hence H_n is not schur stable.

- (ii) We have, $V[WB_n; x] = \binom{n-1}{2} x^3 + (2n-1)(n-1)x^2 + \binom{2n-1}{2} x$. That is, $V[WB_n; x] = \binom{n-1}{2} x [x^2 + \frac{2(2n-1)}{n-2} x + \frac{2(2n-1)}{n-2}]$. Since the discriminant of the quadratic polynomial $p(x) = x^2 + \frac{2(2n-1)}{n-2} x + \frac{2(2n-1)}{n-2}$ is $\frac{12(2n-1)}{(n-2)^2} > 0$, it follows that all the vertex connectivity roots of WB_n are real. The remaining results follows easily from the fact that the zeros of $p(x)$ are exactly $\frac{-2(2n-1) \pm \sqrt{12(2n-1)}}{2(n-2)}$.

- (iii) From [Priya and Anil Kumar, 2021b], it follows that the vertex connectivity polynomial of BF_n is $V[BF_n; x] = x^3 + (2n-4)x^2 + \binom{n-2}{2} x$. For $n = 5, 6, 7$, the non-zero vertex connectivity roots of BF_n are exactly $-3 \pm \sqrt{6}$, $-4 \pm \sqrt{10}$ and $-5 \pm \sqrt{15}$ respectively so that in all those cases the graph is not schur stable. Now for $n > 7$, the linear coefficient $\binom{n-2}{2} > 1 + 2n - 4$ and hence by theorem 2.20, only one zero of $V[BF_n; x]$ lie inside the unit circle and the other two zeros lie outside the unit circle. Hence for $n > 4$, BF_n lacks schur stability. Now, since $2n - 4$ is greater than the magnitude of the square of the discriminant of the quadratic factor in $V[BF_n; x]$ and since the discriminant is positive, it follows that all the

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non-zero vertex connectivity roots of BF_n belong to the left half plane.

This completes the proof. □

3 Conclusions

In this paper, the nature of roots and stability properties of some well known graphs were discussed in detail and inferences were made about its stability. More general properties regarding the graph stability can be inferred only after studying the same for graph operations.

4 Acknowledgements

The first author would like to thank the University Grants Commission of India for funding this research.

References

- Paul A Fuhrmann. *A polynomial approach to linear algebra*. Springer Science & Business Media, 2011.
- Gaurav Gajender and Himanshu Sharma. Hurwitz polynomial. *International journal of innovative research in technology*, 1(7):340–342, 2014.
- F Harary. *Graph Theory*. Addison-Wesley, 1969.
- K Priya and V Anil Kumar. On the vertex connectivity index of graphs. *Chinese Journal of Mathematical Sciences*, 1(1):49–59, 2021a.
- K Priya and V Anil Kumar. On the vertex connectivity polynomial of graphs. *Advances and Applications in Discrete Mathematics*, 26(2):133–147, 2021b.
- Yufei Zhao. Integer polynomials. *MOP 2007 Black Group*, <https://yufeizhao.com>, 69(1):17–20, 1947.