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To some structural properties of ∞-languages

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Abstract

Properties of catenation of sequences of finite (words) and infinite (ω-words) lengths are largely studied in formal language theory. These operations are derived from the mechanism how they are accepted or generated by the corresponding devices. Finite automata accept structures containing only words, ω-automata accept only ω-words. Structures containing both words and ω-words (∞-words) are mostly generated by various types of ∞-automata (∞-machines). The aim of the paper is to investigate algebraic properties of operations on ∞-words generated by *IGk*-automata, where *k* is to model the depth of memory. It has importance in many applications (shift registers, discrete systems with memory…). It is shown that resulting algebraic structures are of „pure“ groupoid or partial groupoid type.

Keywords: ∞-words; ∞-language; $\rho_{n,p,r}$-catenation; closure of an ∞-language;
ρ-operation.

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1. Introduction

The notion of an ∞-language was introduced by Nivat ([10]) as a free-monoid structure containing words finite („clasical“ languages) and infinite (ω-languages) lengths. The theory of ω-languages has been intensively developed so far, mostly as a generalization of acceptance conditions of various types of automata ([1], [8], [9], [11], [12], [13], [14], [15] among others). Devices capable to accept (or generate) simultaneously words or finite or infinite (ω-words) lengths were described and investigated in [2], [3]. Such devices (k-machines, IGk-machines) also provide to implement the depth of memory and possess various applications (shift registers, modelling of phenomena working in a discrete time scale). They also make a lot of properties. In [3] lattice structures were described. The way how they generate words and ω-words motivates to study various types of catenation of words, which is the aim of the paper. The paper is organized as follows. In Section 2 we introduce basic concepts. In Sections 3 and 4 we examine properties of ρ-closure and ρ-operation, respectively.

2. Preliminaries

An *alphabet* is any finite set (including the empty set) and is denoted by Σ; its elements are called *letters*. Let ω be the least infinite ordinal and n, $0 \leq n \leq \omega$ be an ordinal. For $\Sigma \neq \emptyset$ the set of all finite sequences of elements of Σ including the empty sequence λ is denoted by Σ^*, the set of all infinite sequences of elements of Σ by Σ^ω and the set $\Sigma^* \cup \Sigma^\omega$ by Σ^∞. For $\Sigma = \emptyset$, by definition, $\Sigma^* = \Sigma^\omega = \Sigma^\infty = \emptyset$. The elements of Σ^* are *words*, the elements of Σ^ω are ω-*words*, the elements of Σ^∞ are ∞-*words*. Instead of $(a_0, a_1, \ldots, a_{n-1}) \in \Sigma^*, n \geq 1$ and $(a_0, a_1, \ldots) \in \Sigma^\omega$ we write simply $a_0 a_1 \ldots a_{n-1}$ and $a_0 a_1 \ldots$. For $w \in \Sigma^\infty$, the *length of* w, denoted by $|w|$ is defined as follows: if $w = a_0 a_1 \ldots a_{n-1} \in \Sigma^*$ then $|w| = n$, if $w = a_0 a_1 \ldots \in \Sigma^\omega$ then $|w| = \omega$ and if $w = \lambda$ then $|w| = 0$. A subset of Σ^* and Σ^ω and Σ^∞ is referred to as a *language* and an ω-*language* and an ∞-*language* (*over* Σ) respectively. For $L \subseteq \Sigma^\infty$ we define $m(l) = inf\{|w|; w \in L\}$. Let $w \in \Sigma^\infty - \{\lambda\}$, $1 \leq m \leq n < 1 + |w|$; by $w(n)$ we denote the n-th letter of w, *by* $w[m,n]$ the word $w(m) \ldots w(n)$ and if $|w| = \omega$ by $w[m,\omega]$ the word $w(m)w(m+1) \ldots$. Instead of $w[1,n]$ we write only $w[n]$. In case $m > n$ we formally put $w[m,n] = \lambda$. The usual operation of *catenation* on Σ^* may be extended to Σ^∞ as a partial operation as follows. Let $w \in \Sigma^*$, $|w| = n$, $w' \in \Sigma^\infty$; if $w' \in \Sigma^*$, then ww' is defined by catenation on Σ^* and if $w' \in \Sigma^\omega$, then $ww' = w(1)w(2) \ldots w(n)w'(1)w'(2) \ldots$. For $w \in \Sigma^*$, $k \geq 1$, the symbol w^k denotes the result of k catenations of w and the symbol w^ω denotes the result of infinite number of catenations of w; by definition, $w^0 = \lambda$.

3. Closure of an ∞-language

In this section we define the notion of a ρ-catenation. Subsequently the concept of a ρ-closure is introduced and its closure characterization is derived.

2.1 Definition. Let n, p, r be positive integers and $u \in \Sigma^\infty, v \in \Sigma^\infty$. For u,v satisfying the property

$$u[p, p+n-1] = v[r, r+n-1]$$

we define the *operation* $\rho_{n,p,r}$ by

$$\rho_{n,p,r}(u, v) = u[p+n-1]v[r+n, |v|] \tag{2.1}$$

called a $\rho_{n,p,r}$-*catenation* or only ρ-*catenation* if n,p,r are given by the context.

Apparently, each $\rho_{n,p,r}$-*catenation* is a partial operation in Σ^∞. In other words, each $\rho_{n,p,r}$-*catenation* defines a partial groupoid in Σ^∞. In this manner (regarding the given n,p,r) the set of partial operations (groupoids) in Σ^∞ is given. Instead of $\rho_{n,p,r}(u, v)$ we write as customary $u\rho_{n,p,r}v$ or only $u\rho v$, if n, p, r are clear from the context. To simplify the text, by stating $u\rho_{n,p,r}v$ it is supposed that $(u, v) \in Dom(\rho_{n,p,r})$.

2.2 Lemma. Suppose $u \in \Sigma^\infty$ and $u\rho_{n,p,r}u$ for some n, p, r . Then it holds $u\rho_{n,p,r}u = u$.

Proof. It is an immediate consequence of Definition 2.1

Remarks. 1° Suppose $u \in \Sigma^\infty$ and $n, p, r \leq |u|$. Then there is obviously an infinite number of words $x \in \Sigma^\infty$ such that $u\rho_{n,p,r}x = u$ playing the role of the „identity“ element of $\rho_{n,p,r}$-catenation.

2° Operation $\rho_{n,p,r}$ is in general not commutative. For example consider words u,v over $\Sigma = \{a, b\}$, $u = (ab)^3, v = a^3$. Applying the previous definition we get $u\rho_{1,3,1}v = aba^3$, $v\rho_{1,3,1}u = a^2(ab)^3$, $u\rho_{1,3,1}v \neq v\rho_{1,3,1}u$.

3° Operation $\rho_{n,p,r}$ is in general not associative. For example consider $\rho_{1,3,2} - catenation$ in $\{a, b\}^\infty$ and let $u = (ab)^4, v = a^5, w = a^7$. Construct $(u\rho_{1,3,2}v)\rho_{1,3,2}w$, $u\rho_{1,3,2}(v\rho_{1,3,2}w)$. Due to Definition 2.1 we get $(u\rho_{1,3,2}v)\rho_{1,3,2}w = aba^5$, whereas $u\rho_{1,3,2}(v\rho_{1,3,2}w) = aba^7$. Of course, some of catenations in the given expressions need not be defined.

2.3 Theorem Let $u \in \Sigma^\infty, v \in \Sigma^\infty$ and suppose $u\rho_{n,p,r}v$, $v\rho_{n,r,s}w$ for some $n, p, r, s \geq 1$. Then $u\rho_{n,p,s}w$ and there holds

$$u\rho_{n,p,s}w = u\rho_{n,p,r}(v\rho_{n,r,s}w). \tag{2.2}$$

Proof. Suppose that $u\rho_{n,p,r}v, v\rho_{n,r,s}w$ hold for given $n, p, r, s \geq 1$. From Definition 2.1 it follows that $u[p, p+n-1] = v[r, r+n-1]$ and $v[r, r+n-1] = w[s, s+n-1]$. Then evidently $u[p, p+n-1] = w[s, s+n-1]$, applying Definition 2.1 we get $u\rho_{n,p,s}w = u[p+n-1]w[s+n, |w|]$ and the first part of the statement is verified.

Rewriting this expression we obtain

$$u\rho_{n,p,s}w = u[1] \dots u[p+n-1]w[s+n]w[s+n+1] \dots w[|w|]. \tag{2.3}$$

Now we costruct the right part of (2.2). By Definition 2.1 we have

$$v\rho_{n,r,s}w = v[1] \dots v[r] \dots v[r+n-1]w[s+n] \dots w[|w|],$$

where $v[r, r+n-1] = v[r] \dots v[r+n-1] = w[s] \dots w[s+n-1] = w[s, s+n-1]$ and

(2.4) $$u\rho_{n,p,r}\left(v\rho_{n,r,s}w\right) = u[1] \dots u[p] \dots u[p+n-1]w[s+n] \dots w[|w|].$$

From (2.3) and (2.4) the statement (2.2) holds and the proof is completed.

2.4 Theorem Let $u, v \in \Sigma^{\infty}$ and suppose $u\rho_{n,p,r}v$ for fixed $n > 1, p \geq 1, r \geq 1$.

Then $u\rho_{m,p,r}v$ for any $m < n$ and it holds

(2.5) $$u\rho_{n,p,r}v = u\rho_{m,p,r}v \ .$$

Proof. Let $u\rho_{n,p,r}v$ for the given n, p, r. From Definition 2.1 it follows that $u[p, p+n-1] = v[r, r+n-1]$. Since $m < n$, then apparently $u[p, p+m-1] = v[r, r+m-1]$ holds for any $m < n$ as well and thus $u\rho_{m,p,r}v$. Due to (2.1) $u\rho_{n,p,r}v = u[p+n-1]v[r+n, |v|]$. In a detailed version we have

(2.6) $$u\rho_{n,p,r}v = u[1] \dots u[p+n-1]v[r+n] \dots v[|v|].$$

With a view to $m < n$, (2.6) may be rewritten as

(2.7) $$u\rho_{n,p,r}v = u[1] \dots u[p+m-1]u[p+m] \dots u[p+n-1]v[r+n]v[r+n+1] \dots v[|v|].$$

Now, we construct $u\rho_{m,p,r}v$ for $m < n$. It holds $u[p, p+m-1] = v[r, r+m-1]$ and by (2.1)

(2.8) $$u\rho_{m,p,r}v = u[p+m-1]v[r+m, |v|].$$

In detail

(2.9) $$u\rho_{m,p,r}v = u[1] \dots u[p+m-1]v[r+m] \dots v[|v|].$$

With a view to $m < n$, (2.9) may be rewritten as

(2.10) $$u\rho_{m,p,r}v = u[1] \dots u[p] \dots u[p+m-1]v[r+m] \dots v[r+n-1]v[r+n] \dots v[|v|].$$

Due to (2.7) and (2.10) $u[1] \dots u[p+m-1]$ and $v[r+n] \dots v[|v|]$ are common parts. It remains to verify that $v[r+m] \dots v[r+n-1] = u[p+m] \dots u[p+n-1]$. Using assumptions of Definition 2.1 we have $u[p, p+n-1] = v[r, r+n-1]$ and thus also $v[r+m] \dots v[r+n-1] = u[p+m] \dots u[p+n-1]$. Hence (2.7) and (2.10) are identical words and the proof is completed.

2.5 Definition. Let a $\rho_{n,p,r}-$ catenation be given. Define a relation $R_{n,p,r}$ on Σ^{∞} by

(2.11) $$R_{n,p,r} = \left\{u, v \in \Sigma^{\infty}; (u, v) \in Dom\left(\rho_{n,p,r}\right)\right\} \subseteq \Sigma^{\infty} \times \Sigma^{\infty}.$$

2.6 Lemma. The relation $R_{n,p,r}$ is

(i) reflexive,
(ii) not symmetric,
(iii) not antisymmetric,

(iv) not transitive.

Proof. (i) Reflexivity of $R_{n,p,r}$ follows immediately from Lemma 2.2. (ii) Consider $u = abbb, v = aabbb$. By Definition 2.1 it holds $u = abbb\rho_{2,3,3}aabbb = v$, whereas $v = aabbb\rho_{2,3,3}abbb = u$ does not hold, so the relation $R_{n,p,r}$ is not transitive. (iii) Put $u = abababab = (ab)^4, v = bababababa = (ba)^5$. By Definition 2.1 we have $u\rho_{1,2,5}v, v\rho_{1,2,5}u$, but $u \neq v$ and hence the relation $R_{n,p,r}$ is not antisymmetric. (iv) Let $u = abbb, v = bababa, w = bbbb$. From Definition it follows $u\rho_{1,1,2}v, v\rho_{1,1,2}w$, but $u\rho_{1,1,2}w$ does not hold and the relation $R_{n,p,r}$ is not transitive.

2.5 Definition. Let $L \subseteq \Sigma^\infty$ be an ∞-language, $n \geq 1$ integer and $u, v \in \Sigma^\infty$. Put

$$C_n^\rho(u,v) = \bigcup_{p,r} u\rho_{n,p,r}v, C_n^\rho(L) = \bigcup_{u,v} C_n^\rho(u,v), C^\rho(L) = \bigcup_n C_n^\rho(L).$$

The set $C_n^\rho(L)$ is called the *n-th ρ-closure of L* and the set $C^\rho(L) = \bigcup_n C_n^\rho(L)$ the *ρ-closure of L* respectively.

2.6 Lemma. Let $L \subseteq \Sigma^\infty - \{\lambda\}$ be an ∞-language. Then $L \subseteq C^\rho(L)$ holds true.

Proof. Suppose $w \in L$. Trivially $w(1) = w(1)$ and by Definition 2.1 it holds $w\rho_{1,1,1}w = w[1]w[2,|w|] = w$ and hence by Definition 2.5 $w \in C_1^\rho(L)$ and also $w \in C^\rho(L)$ and the statement holds true.

2.7 Theorem Let $L \subseteq (\Sigma^\infty - \{\lambda\})$ be an ∞-language. Then for every $i \geq 1$ there holds

$$C_{i+1}^\rho(L) \subseteq C_i^\rho(L).$$

Proof. Let $w \in C_{i+1}^\rho(L)$. According to Definition 2.1 there exist $u, v \in \Sigma^\infty$ and $i, p, r \geq 1$ with the property $u[p, p+i] = v[r, r+i]$ for which $u\rho_{i+1,p.r}v = u[p+i]v[r+i+1, |v|] = w$ holds. Obviously if $u[p, p+i] = v[r, r+i]$ then also $u[p, p+i-1] = v[r, r+i-1]$ holds. By Definition 2.5 we get $u\rho_{i,p,r}v = u[p+i-1]v[r+i, |v|] = w' \in C_i^\rho(L)$. But apparently w, w' are identical words. Hence $w \in C_i^\rho(L)$ and the statement is valid.

As a consequence of Definition 2.5 and Theorem 2.7 the following Corollary 2.8 holds:

2.8 Corollary Let $L \subseteq (\Sigma^\infty - \{\lambda\})$ be an ∞-language. Then $C^\rho(L) = C_1^\rho(L)$ holds true.

2.9 Example Let $L = \{ab, ba^k, a^\omega; k \geq 1\} \subseteq \{a,b\}^\infty$ be an ∞ - language. To find $C_n^\rho(L)$ and $C^\rho(L)$ applying Definition 2.5 we get the results as follows.
(i) $C_1^\rho(L)$: $C_1^\rho(ab, ab) = \{ab\}, C_1^\rho(ab, ba^k) = \{a^k, aba^k; k \geq 1\}, C_1^\rho(ba^k, ab) = \{b, ba^kb; k \geq 1\}, C_1^\rho(ba^k, ba^k) = \{ba^k; k \geq 1\}, C_1^\rho(ab, a^\omega) = \{a^\omega\}, C_1^\rho(a^\omega, ab) = \{a^kb; k \geq 1\}, C_1^\rho(a^\omega, a^\omega) = \{a^\omega\}, C_1^\rho(ba^k, a^\omega) = \{ba^\omega\}, C_1^\rho(a^\omega, ba^k) = \{a^k; k \geq 1\}$; therefore $C_1^\rho(L) = \{ab, a^k, aba^k, ba^kb, ba^k, a^\omega, a^kb, ba^\omega; k \geq 1\}$.

(ii) $C_2^\rho(L)$: $C_2^\rho(ab,ab)=\{ab\}, C_2^\rho(ab,ba^k)=\emptyset, C_2^\rho(ba^k,ab)=\emptyset, C_2^\rho(a^\omega,a^\omega)=\{a^\omega\}, C_2^\rho(ba^k,a^\omega)=\{ba^\omega\}$ for $k\geq 2, C_2^\rho(a^\omega,ba^k)=\{a^k;k\geq 2\}, C_2^\rho(ba^k,ba^k)=\{ba^k;k\geq 1\}, C_2^\rho(ab,a^\omega)=\emptyset, C_2^\rho(a^\omega,ab)=\emptyset$; therefore $C_2^\rho(L)=\{ab,ba^k,a^\omega,ba^\omega,a^{k+1};k\geq 1\}$.
(iii) $C_3^\rho(L)$: $C_3^\rho(ab,ab)$=$C_3^\rho(ab,ba^k)=C_3^\rho(ba^k,ab)=C_3^\rho(ab,a^\omega)=C_3^\rho(a^\omega,ab)=\emptyset, C_3^\rho(ba^k,ba^k)=\{ba^k;k\geq 2\}, C_3^\rho(a^\omega,a^\omega)=\{a^\omega\}, C_3^\rho(ba^k,a^\omega)=\{ba^\omega\}$ for $k\geq 3, C_3^\rho(a^\omega,ba^k)=\{a^k;k\geq 3\}$; therefore $C_3^\rho(L)=\{ba^k,a^\omega,ba^\omega,a^{k+1};k\geq 2\}$.
(iv) $C_n^\rho(L)$ for $n\geq 4$: $C_n^\rho(ab,ab)$=$C_n^\rho(ab,ba^k)=C_n^\rho(ba^k,ab)=C_n^\rho(ab,a^\omega)=C_n^\rho(a^\omega,ab)=\emptyset, C_n^\rho(ba^k,ba^k)=\{ba^k;k\geq n-1\}, C_n^\rho(a^\omega,a^\omega)=\{a^\omega\}, C_n^\rho(ba^k,a^\omega)=\{ba^\omega\}$ for $k\geq n-1, C_n^\rho(a^\omega,ba^k)=\{a^k;k\geq n-1\}$; therefore $C_n^\rho(L)=\{ba^k,a^\omega,ba^\omega,a^{k+1};k\geq n-1\}$.
Conclusion: $C^\rho(L)=\{ab,a^k,aba^k,ba^kb,ba^k,a^\omega,a^kb,ba^\omega;k\geq 1\}=C_1^\rho(L)$.

2.10 Theorem. The set of ρ-closures is not closed under set union.

Proof. We state an counterexample. Consider $L_1=\{ab\}, L_2=\{a^\omega\}$ over $\{a,b\}^\infty$and put $L=L_1\cup L_2=\{ab,a^\omega\}$. Applying Definition 2.1 and Corollary 2.8 we get $C^\rho(L_1)=C_1^\rho(L_1)=C^\rho(\{ab\})=\{ab\}, C^\rho(L_2)=C_1^\rho(L_2)=C^\rho(\{a^\omega\})=\{a^\omega\}$. Further, $C^\rho(L)=C^\rho(L_1\cup L_2)=C^\rho(\{ab,a^\omega\})=\{a^\omega,a^kb;k\geq 1\}$. Obviously $C^\rho(L_1\cup L_2)\neq C^\rho(L_1)\cup C^\rho L_2)$ and the statement is verified.

2.11 Theorem. Let $L_1,L_2\subseteq\Sigma^\infty$. Then $C^\rho(L_1)\cup C^\rho L_2)\subseteq C^\rho(L_1\cup L_2)$.

Proof. With a view to Corollary 2.6 we may consider C_1^ρ instead of C^ρ. Let $w\in C_1^\rho(L_1)\cup C_1^\rho(L_2)$. According to Definition 2.3 then (a) there exist $u\in L_1, v\in L_1$ and positive integers p,r such that $u\rho_{1,p,r}v=w\in C_1^\rho(L_1)$ or (b) there exist $\overline{u}\in L_2,\overline{v}\in L_2$ and positive integers $\overline{p},\overline{r}$ such that $\overline{u}\rho_{1,\overline{p},\overline{r}}\overline{v}=w\in C_1^\rho(L_2)$. Assuming (a), the statement there exist $u\in L_1\cup L_2, v\in L_1\cup L_2$ and positive integers p,r such that $u\rho_{1,p,r}v=w\in C_1^\rho(L_1\cup L_2)$ is obviously also valid for an arbitrary set L_2. Assuming (b), the statement there exist $\overline{u}\in L_2\cup L_1,\overline{v}\in L_2\cup L_1$ and positive integers $\overline{p},\overline{r}$ such that $\overline{u}\rho_{1,\overline{p},\overline{r}}\overline{v}=w\in C_1^\rho(L_2\cup L_1)$ is valid as well for ab arbitrary set L_1. Thus $w\in C_1^\rho(L_1\cup L_2)$ and the proof is completed.

2.12 Theorem. The set of ρ-closures is not closed under set intersection.

Proof. We state an counterexample. Consider $L_1=\{a^\omega,a^3,b\}, L_2=\{a^3,ab\}$ over $\{a,b\}^\infty$and put $L=L_1\cap L_2=\{a^3\}$. Applying Definition 2.1 and Corollary 2.8 we get $C^\rho(L_1)=C_1^\rho(L_1)=C_1^\rho(\{a^\omega,a^3,b\})=\{a^\omega,b,a^k;k\geq 1\}, C^\rho(L_2)=C_1^\rho(L_2)=C_1^\rho(\{a^3,ab\})=\{a,a^2,a^3,a^4,a^5,b,ab,a^2b,a^3b\}$, $C^\rho(L_1)\cap C^\rho(L_2)=\{a,a^2,a^3,a^4,a^5,b\}$. Further,$C^\rho(L)=C^\rho(L_1\cap L_2)=C^\rho(\{a^3\})=\{a^k\,;1\leq k\leq 5\}$. Obviously $C^\rho(L_1\cap L_2)\neq C^\rho(L_1)\cap C^\rho(L_2)$ and the statement is verified.

2.13 Theorem. Let $L_1,L_2\subseteq\Sigma^\infty$. Then $C^\rho(L_1\cap L_2)\subseteq C^\rho(L_1)\cap C^\rho(L_2)$.

Proof. With a view to Corollary 2.6 we may work with C_1^ρ instead of C^ρ. Let $w \in C_1^\rho(L_1 \cap L_2)$. According to Definition 2.3 there exist $u \in (L_1 \cap L_2), v \in (L_1 \cap L_2)$ and positive integers p, r such that $u\rho_{1,p,r}v = w \in C_1^\rho(L_1 \cap L_2)$. Since $u \in (L_1 \cap L_2), v \in (L_1 \cap L_2)$, then $w \in C_1^\rho(L_1)$ and also $w \in C_1^\rho(L_2)$. Thus $w \in C_1^\rho(L_1) \cap C_1^\rho(L_2)$ and the statement holds.

2.14 Example. Using the setting of the counterexample from the proof of Theorem 2.12, we have $C^\rho(L_1 \cap L_2) = C^\rho(\{a^3\}) = \{a^k\,; 1 \le k \le 5\} \subseteq C^\rho(L_1) \cap C^\rho(L_2) = \{a, a^2, a^3, a^4, a^5, b\}$ to illustrate Theorem 2.13. Further, we have $C^\rho(L_1) \cup C^\rho L_2) = \{a^\omega, b, ab, a^2b, a^3b, a^k; k \ge 1\} \subseteq C^\rho(L_1 \cup L_2) = \{a^\omega, a^k, a^kb, b; k \ge 1\}$ to illustrate Theorem 2.11.

3. Operation ρ_n

3.1 Definition. Given $L_1, L_2 \subseteq \Sigma^\infty$ and $n \ge 1$, an *operation* ρ_n is defined as follows:

$$\rho_n(L_1, L_2) = \{xuy; there\ exists\ u \in \Sigma^n\ with\ xu \in L_1 and\ uy \in L_2\}.$$

Clearly, for each n, ρ_n is the operation on 2^{Σ^∞}. In this manner the set of operations on 2^{Σ^∞} is given. Instead of $\rho_n(L_1, L_2)$ we also write $L_1\rho_nL_2$.

3.2 Theorem. (i) Let $L_1, L_2 \subseteq \Sigma^\omega$. Then for all $n \ge 1$ there holds $L_1\rho_nL_2 = \emptyset$.

(ii) Given $L_1, L_2 \subseteq \Sigma^*$ and let $L_1 \cup L_2$ be a finite set. Then for all $n > \max\limits_{w \in L_1 \cup L_2} |w|$ here holds $L_1\rho_nL_2 = \emptyset$.

Proof. Both statements (i), (ii) follow immediately from Definition 3.1.

3.3 Example. (a) Let $L_1, L_2 \subseteq \{a, b\}^*$, $L_1 = \{(ab)^k; k \ge 1\}, L_2 = \{a^k, b^k; k \ge 1\}$. Applying Definition 3.1 we have $L_1\rho_1L_2 = \{ab^k, (ab)^k, (ab)^kb^m; k, m \ge 1\}$. Similarly, and with accordance to Theorem 3.2(ii) we get $L_1\rho_nL_2 = \emptyset$ and $L_2\rho_nL_1 = \emptyset$ for any $n \ge 2$. (b) Let $L_1, L_2, L_3 \subseteq \{a, b\}^\infty$, $L_1 = \{a^k, b; k \ge 1\}, L_2 = \{a^3, b^2\}, L_3 = \{a^\omega, ab\}$. Applying Definition 3.1 we have $L_1\rho_1L_2 = \{a^k, b^2; k \ge 3\}, L_2\rho_1L_3 = \{a^\omega, a^3b\}, (L_1\rho_1L_2)\rho_1L_3 = \{a^\omega, a^kb; k \ge 1\}$, $L_1\rho_1(L_2\rho_1L_3) = \{a^\omega, a^3b\}$.

3.4 Theorem. The operation ρ_n is generally
(i) not commutative,
(ii) not associative.
Proof. It follows immediately from the results of Example 3.3.

3.4 Remark. Theorem 3.4 justifies the conclusion that the set 2^{Σ^∞} with the operation ρ_n forms a „pure“ groupoid. Also nonexistence of an identity element may be simply verified.

4. Conclusion

In this paper we examined algebraic properties of operations on ∞-words having direct relation to ∞-languages generated by ∞- automata. It may motivate to consider further types of operations, particularly modeling the depth of memory of such devices.

As a generalization a variant structure of ∞-automata may be considered and the corresponding structures of their ∞-languages studied.

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