Term Functions and Fundamental Relation of Fuzzy Hyperalgebras

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Abstract

We introduce and study term functions over fuzzy hyperalgebras. We start from this idea that the set of nonzero fuzzy subsets of a fuzzy hyperalgebra can be organized naturally as a universal algebra, and constructing the term functions over this algebra. We present the form of generated subfuzzy hyperalgebra of a given fuzzy hyperalgebra as a generalization of universal algebras and multialgebras. Finally, we characterize the form of the fundamental relation of a fuzzy hyperalgebra.

Keywords: Hyperalgebra, Fuzzy hyperalgebra, Equivalence relation, Term function, Fundamental relation, Quotient set.

1 Introduction

Hyperstructure theory was born in 1934 when Marty defined hypergroups, began to analysis their properties and applied them to groups, relational algebraic functions (see [15]). Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties ([7]). As it is well known, in 1965 Zadeh ([28]) introduced the notion of a set μ on a nonempty set X as a function from X to the unite real interval $\mathbb{I} = [0, 1]$ as a fuzzy set. In 1971, Rosenfeld ([25]) introduced fuzzy sets in the context of group theory and formulated the concept of a fuzzy subgroup of a group. Since then, many researchers are engaged in extending the concepts of abstract algebra to the framework of the fuzzy setting (for instance see [23]).

The study of fuzzy hyperstructure is an interesting research topic of fuzzy sets and applied to the theory of algebraic hyperstructure. As it is known a hyperoperation assigns to every pair of elements of H a nonempty subset of H, while a fuzzy hyperoperation assigns to every pair of elements of H a nonzero fuzzy set on H. Recently, Sen, Ameri and Chowdhury introduced and analyzed fuzzy semihypergroups in [21]. This idea was followed by other researchers and extended to other branches of algebraic hyperstructures, for instance Leoreanu and Davvaz introduced and studied fuzzy hyperring notion in [13], Chowdhury in [5] studied fuzzy transposition hypergroups and Leoreanu studied fuzzy hypermodules in [15].

In this paper we follow the idea in [20] and introduced fuzzy hyperalgebras, as the largest class of fuzzy algebraic system. We introduce and study term functions over algebra of all nonzero fuzzy subsets of a fuzzy hyperalgebra, as an important tool to introduce fundamental relation on fuzzy hyperalgebra. Finally, we construct fundamental relation of fuzzy algebras and investigate its basic properties.

This paper is organized in four sections. In section 2 we gather the definitions and basic properties of hyperalgebras and fuzzy sets that we need to develop our paper. In section 3 we introduce term functions over the algebra of nonzero fuzzy subsets of a fuzzy hyperalgebra and we obtained some basic results on fuzzy hyperalgebras, in section 4 we will present the form of the fundamental relation of a fuzzy hyperalgebra.

2 Preliminaries

In this section we present some definitions and simple properties of hyperalgebras from [2] and [3], which will be used in the next sections. In the sequel H is a fixed nonvoid set, $P^*(H)$ is the family of all nonvoid subsets of H, and for a positive integer n we denote for H^n the set of n-tuples over H (for more see [6] and [7]).

For a positive integer n a n-ary *hyperoperation* β on H is a function $\beta : H^n \rightarrow P^*(H)$. We say that n is the *arity* of β. A subset S of H is *closed* under the n-ary hyperoperation β if $(x_1, \ldots, x_n) \in S^n$ implies that $\beta(x_1, \ldots, x_n) \subseteq S$. A *nullary hyperoperation* on H is just an element of $P^*(H)$; i.e. a nonvoid subset of H.

A *hyperalgebraic system* or a *hyperalgebra* $\langle H, (\beta_i : i \in I)\rangle$ is the set H with together a collection $(\beta_i \mid i \in I)$ of hyperoperations on H.

A subset S of a hyperalgebra $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ is a *subhyperalgebra* of $\mathbb{H}$ if S is closed under each hyperoperation β_i, for all $i \in I$, that is $\beta_i(a_1, ..., a_{n_i}) \subseteq S$, whenever $(a_1, ..., a_{n_i}) \in S^{n_i}$. The *type* of $\mathbb{H}$ is the map from I into the set $\mathbb{N}^*$ of nonnegative integers assigning to each $i \in I$ the arity of β_i. In this paper we will assume that for every $i \in I$, the arity of β_i is n_i.

For $n > 0$ we extend an n-ary hyperoperation β on H to an n-ary operation $\overline{\beta}$ on $P^*(H)$ by setting for all $A_1, ..., A_n \in P^*(H)$

$$\overline{\beta}(A_1, ..., A_n) = \bigcup\{\beta(a_1, ..., a_n) | a_i \in A_i (i = 1, ..., n)\}$$

It is easy to see that $\langle P^*(H), (\overline{\beta}_i : i \in I)\rangle$ is an algebra of the same type of $\mathbb{H}$.

Definition 2.1. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\overline{\mathbb{H}}$=$\langle \overline{H}, (\overline{\beta}_i : i \in I)\rangle$ be two similar hyperalgebras. A map h from $\mathbb{H}$ into $\overline{\mathbb{H}}$ is called a

(i) A *homomorphism* if for every $i \in I$ and all $(a_1, ..., a_{n_i}) \in H^{n_i}$ we have that

$$h(\beta_i((a_1, ..., a_{n_i})) \subseteq \overline{\beta}_i(h(a_1), ..., h(a_{n_i}));$$

(ii) a *good homomorphism* if for every $i \in I$ and all $(a_1, ..., a_{n_i}) \in H^{n_i}$ we have that

$$h(\beta_i((a_1, ..., a_{n_i})) = \overline{\beta}_i(h(a_1), ..., h(a_{n_i})),$$

for more details about homomorphism of hyperalgebras see [12]. Let ρ be an equivalence relation on H. We can extend ρ on $P^*(H)$ in the following ways:

(i) Let $\{A, B\} \subseteq P^*(H)$. We write $A\overline{\rho}B$ iff

$$\forall a \in A, \exists b \in B, \text{ such that } a\rho b \quad \text{and} \quad \forall b \in B, \exists a \in A, \text{ such that } a\rho b.$$

(ii) we write $A\overline{\overline{\rho}}B$ iff $\forall a \in A, \forall b \in B$ we have $a\rho b$.

Definition 2.2. If $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a hyperalgebra and ρ be an equivalence relation on H. Then ρ is called *regular* (resp. *strongly regular*) if for every $i \in I$, and for all $a_1, ..., a_{n_i}, b_1, ..., b_{n_i} \in H$ the following implication holds:

$$a_1\rho b_1, ..., a_{n_i}\rho b_{n_i} \Rightarrow \beta_i(a_1, ..., a_{n_i})\overline{\rho}\beta_i(b_1, ..., b_{n_i})$$

$$(resp.\ \ a_1\rho b_1, ..., a_{n_i}\rho b_{n_i} \Rightarrow \beta_i(a_1, ..., a_{n_i})\overline{\overline{\rho}}\beta_i(b_1, ..., b_{n_i})).$$

Definition 2.3. Recall that for a nonempty set H, a fuzzy subset μ of H is a function

$$\mu : H \rightarrow [0, 1].$$

If μ_i is a collection of fuzzy subsets of H, then we define the fuzzy subset $\bigcap_{i\in I}\mu_i$ by:

$$(\bigcap_{i\in I}\mu_i)(x) = \bigwedge_{i\in I}\{\mu_i(x)\}, \quad \forall x \in H.$$

Definition 2.4. Let ρ be an equivalence relation on a hyperalgebra $\langle H, (\beta_i : i \in I)\rangle$ and μ and υ be two fuzzy subsets on H. We say that $\mu\rho\upsilon$ if the following two conditions hold:

(i) $\mu(a) > 0 \Rightarrow \exists b \in H : \upsilon(b) > 0$, and $a\rho b$

(ii) $\upsilon(x) > 0 \Rightarrow \exists y \in H : \mu(y) > 0,$ and $x\rho y.$

3 Fuzzy Hyperalgebra and Term Functions

Definition 3.1. A *fuzzy n-ary hyperoperation* f^n on S is a map $f^n : S \times ... \times S \longrightarrow F^*(S)$, which associated a nonzero fuzzy subset $f^n(a_1, ..., a_n)$ with any n-tuple $(a_1, ..., a_n)$ of elements of S. The couple $\langle S, f^n \rangle$ is called a *fuzzy n-ary hypergroupoid*. A *fuzzy nullary hyperoperation* on S is just an element of $F^*(S)$; i.e. a nonzero fuzzy subset of S.

Definition 3.2. Let H be a nonempty set and for every $i \in I$, β_i be a fuzzy n_i-ary hyperoperation on H. Then $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$ is called *fuzzy hyperalgebra*, where $(n_i : i \in I)$ is the type of this fuzzy hyperalgebra.

Definition 3.3. If $\mu_1, ..., \mu_{n_i}$ be n_i nonzero fuzzy subsets of a fuzzy hyperalgebra $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$, we define for all $t \in H$

$$\beta_i(\mu_1, ..., \mu_{n_i})(t) = \bigvee_{(x_1,...,x_{n_i}) \in H^{n_i}} (\mu_1(x_1) \bigwedge ... \bigwedge \mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, ..., x_{n_i})(t))$$

Finally, if $A_1, ..., A_{n_k}$ are nonempty subsets of H, for all $t \in H$

$$\beta_k(A_1, ..., A_{n_k})(t) = \bigvee_{(a_1,...,a_{n_k}) \in H^{n_k}} (\beta_k(a_1, ..., a_{n_k})(t)).$$

If A is a nonempty subset of H, then we denote the characteristic function of A by χ_A. Note that, if $f : H_1 \longrightarrow H_2$ is a map and $a \in H_1$, then $f(\chi_a) = \chi_{f(a)}$.

Example 3.4.

(i) A *fuzzy hypergroupoid* is a fuzzy hyperalgebra of type (2), that is a set H together with a fuzzy hyperoperation $\circ$. A fuzzy hypergroupoid $\langle H, \circ \rangle$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in H$ is called *fuzzy hypersemigroup*[22]. In this case for any $\mu \in F^*(H)$, we define $(a \circ \mu)(r) = \bigvee_{t \in H} ((a \circ t)(r) \wedge \mu(t))$ and $(\mu \circ a)(r) = \bigvee_{t \in H} (\mu(t) \wedge (t \circ a)(r))$ for all $r \in H$.

(ii) A *fuzzy hypergroup* is a fuzzy hypersemigroup such that for all $x \in H$ we have $x \circ H = H \circ x = \chi_H$ (fuzzy reproduction axiom)(for more details see [22]).

(iii) A *fuzzy hyperring* $\mathbb{R}$=$\langle R, \oplus, \odot \rangle$ ([13]) is a fuzzy hyperalgebra of type $(2, 2)$, which in that the following axioms hold:

1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in R$;

2) $x \oplus R = R \oplus x = \chi_R$ for all $x \in R$;

3) $a \oplus b = b \oplus a$ for all $a, b \in R$;

4) $a \odot (b \odot c) = (a \odot b) \odot c$ for all $a, b, c \in R$;

5) $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ and $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$ for all $a, b, c \in R$.

Example 3.5. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$ be a hyperalgebra and μ be a nonzero fuzzy subset of H. Define the following fuzzy n-ary hyperoperations on H, for every $i \in I$ and for all $(a_1, ..., a_{n_i}) \in H^{n_i}$;

$$\beta_i^{\diamond}(a_1, ..., a_{n_i})(t) = \begin{cases} \mu(a_1) \bigwedge ... \bigwedge \mu(a_{n_i}) & t \in \beta(a_1, ..., a_{n_i}) \\ 0 & otherwise \end{cases}$$

and letting

$$\beta_i^{\circ}(a_1, ..., a_{n_i}) = \chi_{\{a_1, ..., a_{n_i}\}}.$$

Evidently $\mathbb{H}^{\diamond}$=$\langle H, (\beta_i^{\diamond} : i \in I) \rangle$, $\mathbb{H}^{\circ}$=$\langle H, (\beta_i^{\circ} : i \in I) \rangle$ are fuzzy hyperalgebras.

Theorem 3.6. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$ be a fuzzy hyperalgebra, then for every $i \in I$ and every $a_1, ..., a_{n_i} \in H$ we have $\beta_i(\chi_{a_1}, ..., \chi_{a_{n_i}}) = \beta_i(a_1, ..., a_{n_i})$.

Definition 3.7. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$ be a fuzzy hyperalgebra . A nonempty subset S of H is called a *subfuzzy hyperalgebra* if for $\forall i \in I, \forall a_1, ..., a_{n_i} \in S$, the following condition hold:

$$\beta_i(a_1, ..., a_{n_i})(x) > 0 \text{ then } x \in S.$$

We denote by $\mathcal{S}(\mathbb{U})$ the set of the subfuzzy hyperalgebras of $\mathbb{H}$.

Definition 3.8. Consider the fuzzy hyperalgebra $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\phi \neq X \subseteq H$ be nonempty. Clearly, $\langle X\rangle = \bigcap\{B : B \in \mathcal{S}(\mathbb{H})|\ X \subseteq B\}$ with the fuzzy hyperoperations of $\mathbb{H}$ form a subfuzzy hyperalgebra of $\mathbb{H}$ called the *subfuzzy hyperalgebra of* $\mathbb{H}$ *generated* by the subset X . Evidently if X is a subfuzzy hyperalgebra for $\mathbb{H}$ then $\langle X\rangle = X$.

Theorem 3.9. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and $\phi \neq X \subseteq H$. We consider $X_0 = X$ and for any $k \in \mathbb{N}$,

$$X_{k+1} = X_k \cup \{a \in H \mid \exists i \in I, n_i \in \mathbb{N}, x_1, ..., x_{n_i} \in X_k; \beta_i(x_1, ..., x_{n_i})(a) > 0\}.$$

Then $\langle X\rangle = \bigcup_{k\in\mathbb{N}} X_k$.

Proof. Let $M = \bigcup_{k\in N} X_k$, and $\forall i \in I$, consider $t_1, ..., t_{n_i} \in M$ and $\beta_i(t_1, ..., t_{n_i})(x) > 0$. From $X_0 \subseteq X_1 \subseteq ... \subseteq X_k \subseteq ...$ it follows the existence of $m \in \mathbb{N}$ such that $t_1, ..., t_{n_i} \in X_m$, which implies, according to the definition of X_{m+1} that $x \in X_{m+1}$. Thus $x \in M$ and $M = \bigcup_{k\in\mathbb{N}} X_k$ is a subfuzzy hyperalgebra. From $X = X_0 \subseteq M$, by definition of the generated subfuzzy hyperalgebra, it results $\langle X\rangle \subseteq \langle M\rangle = M$. To prove the inverse inclusion we will show by induction on $k \in \mathbb{N}$ that $X_k \subseteq \langle X\rangle$ for any $k \in \mathbb{N}$, and we have $X_0 = X \subseteq \langle X\rangle$. We suppose that $X_k \subseteq \langle X\rangle$. From $\langle X\rangle \in \mathcal{S}(\mathbb{H})$ and the definition X_{k+1} we can deduce that $X_{k+1} \subseteq \langle X\rangle$. Hence $M \subseteq \langle X\rangle$. The two inclusion lead us to $M = \langle X\rangle$.$\Box$

Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra then, the set of the nonzero fuzzy subsets of H denoted by $F^*(H)$, can be organized as a universal algebra with the operations;

$$\beta_i(\mu_1, ..., \mu_{n_i})(t) = \bigvee_{(x_1,...,x_{n_i})\in H^{n_i}} (\mu_1(x_1) \bigwedge ... \bigwedge \mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, ..., x_{n_i})(t))$$

for every $i \in I$, $\mu_1, ..., \mu_{n_i} \in F^*(H)$ and $t \in H$. We denote this algebra by $\mathcal{F}^*(\mathbb{H})$.

In [13] Gratzer presents the algebra of the term functions of a universal algebra. If we consider an algebra $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ we call $n-$ary term functions on $\mathbb{B}$ $(n \in \mathbb{N})$ those and only those functions from B^n into B, which can be obtained by applying (i) and (ii) from bellow for finitely many times:

(i) the functions $e_i^n : B^n \to B$, $e_i^n(x_1, ..., x_n) = x_i$, $i = 1, ..., n$ are $n-$ary term functions on $\mathbb{B}$;

(ii) if $p_1, ..., p_{n_i}$ are $n-$ary term functions on $\mathbb{B}$, then $\beta_i(p_1, ..., p_{n_i}) : B^n \to B$, $\beta_i(p_1, ..., p_{n_i})(x_1, ..., x_n) = \beta_i(p_1(x_1, ..., x_n), ..., p_{n_i}(x_1, ..., x_n))$ is also a $n-$ary term function on $\mathbb{B}$.

We can observe that (ii) organize the set of $n-$ary term functions over $\mathbb{B}$ $(P^{(n)}(\mathbb{B}))$ as a universal algebra, denoted by $\mathcal{B}^{(n)}(\mathbb{B})$.

If $\mathbb{H}$ is a fuzzy hyperalgebra then for any $n \in \mathbb{N}$, we can construct the algebra of $n-$ary term functions on $\mathcal{F}^*(\mathbb{H})$, denoted by $\mathcal{B}^{(n)}(\mathcal{F}^*(\mathbb{H})) = \langle P^{(n)}(\mathcal{F}^*(\mathbb{H})), (\beta_i : i \in I)\rangle$.

Theorem 3.10. A necessary and sufficient condition for $\mathcal{F}^*(\mathbb{B})$ to be a subalgebra of $\mathcal{F}^*(\mathbb{U})$ is that $\mathbb{B}$ is to be a subfuzzy hyperalgebra for $\mathbb{U}$.

Proof. Obvious.□

The next result immediately follows from Theorem 3.10.

Corollary 3.11. (i) Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and $\mathbb{B}$ a subfuzzy hyperalgebra of $\mathbb{H}$, and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$,$(n \in \mathbb{N})$. If $\mu_1, ..., \mu_n \in F^*(B)$, then $p(\mu_1, ..., \mu_n) \in F^*(B)$.

(ii) Let $\mathbb{H}$= $\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and $\mathbb{B}$ a subfuzzy hyperalgebra of $\mathbb{H}$, and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$,$(n \in \mathbb{N})$. If $x_1, ..., x_n \in B$, then $p(\chi_{x_1}, ..., \chi_{x_n}) \in F^*(B)$.□

Theorem 3.12. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and $\phi \neq X \subseteq H$. Then $a \in \langle X\rangle$ if and only if $\exists n \in \mathbb{N}, \exists p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$, and $\exists x_1, ..., x_n \in X$, such that

$p(\chi_{x_1},...,\chi_{x_n})(a) > 0.$

Proof. We denote

$$M = \{a \in H \mid \exists n \in \mathbb{N}, \exists p \in P^{(n)}(\mathcal{F}^*(\mathbb{H})), \quad \exists x_1,...,x_n \in X : p(\chi_{x_1},...\chi_{x_n})(a) > 0\}.$$

For any $x \in X$ we have $e_1^1(\chi_x)(x) = \chi_x(x) = 1$, thus $x \in X$ and hence $X \subseteq M$. Also from Corollary 3.11 (ii), it follows that $p(\chi_{x_1},...,\chi_{x_n}) \in \mathcal{F}^*(\langle X \rangle)$, therefore $M \subseteq \langle X \rangle$. We will prove now that M is subfuzzy hyperalgebra of $\mathbb{H}$. For any $i \in I$, if $c_1,...,c_{n_i} \in M$ and $\beta_i(c_1,...,c_{n_i})(x) > 0$, we must show that $x \in M$. For $c_1,...,c_{n_i} \in M$, it means that there exist $m_k \in \mathbb{N}, p_k \in P^{m_k}(\mathcal{F}^*(\mathbb{H})), x_1^k,...,x_{m_k}^k \in X, k \in \{1,...,n_i\}$, such that $p_k(\chi_{x_1^k},...,\chi_{x_{m_k}^k})(c_k) > 0$, $\forall k \in \{1,...,n_i\}$. According to the Corollary 8.2 from [12], for any $n-$ary term function p over $\mathcal{F}^*(\mathbb{H})$ and for $m \geq n$ there exists an $m-$ary term function q over $\mathcal{F}^*(\mathbb{H})$, such that $p(\mu_1,...,\mu_n) = q(\mu_1,...,\mu_m)$, for all $\mu_1,...,\mu_m \in F^*(H)$; this allows us to consider instead of $p_1,...,p_{n_i}$ the term functions $q_1,...,q_{n_i}$ all with the same arity $m = m_1 + ... + m_{n_i}$ and the elements $y_1,...,y_m \in X$ (which are the elements $x_1^1,...,x_{m_1}^1,...,x_1^{n_i},...,x_{m_{n_i}}^{n_i}$), such that $q_k(\chi_{y_1},...,\chi_{y_m})(c_k) > 0$,$\forall k \in \{1,...,n_i\}$. But we have $\beta_i(q_1(\chi_{y_1},...,\chi_{y_m}),...,q_{n_i}(\chi_{y_1},...,\chi_{y_m}))(x) =$

$$\bigvee_{(a_1,...,a_{n_i}) \in H^{n_i}} (q_1(\chi_{y_1},...,\chi_{y_m})(a_1) \wedge ... \wedge q_{n_i}(\chi_{y_1},...,\chi_{y_m})(a_{n_i}) \wedge \beta_i(a_1,...,a_{n_i})(x)),$$

and for $(a_1,...,a_{n_i}) = (c_1,...,c_{n_i})$ we have $(\beta_i(q_1,...,q_{n_i})(\chi_{y_1},...,\chi_{y_m}))(x) > 0$. On the other hands we have $\beta_i(q_1,...,q_{n_i}) \in P^{(m)}(\mathcal{F}^*(\mathbb{H})), (m \in \mathbb{N})$, $y_1,...,y_m \in X$ which implies that $x \in M$. Therefore, $M = \langle X \rangle$ and this complete the proof.□

Remark 3.13. If $\mathbb{H}$ has a fuzzy nullary hyperoperation then

$$< \phi >= \{a \in H \mid \exists \mu \in P^0(\mathcal{F}^*(\mathbb{H})), \text{ such that } \mu(a) > 0\}.$$

Recall that if $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$ and $\mathbb{B}$=$\langle B, (\beta_i : i \in I) \rangle$ are fuzzy hyperalgebras with the same type, then a map $h : H \to B$ is called a *good homomorphism* if for any $i \in I$ we have ;

$$h(\beta_i(a_1, ..., a_{n_i})) = \beta_i(h(a_1), ..., h(a_{n_i})), \forall a_1, ..., a_{n_i} \in H.$$

An equivalence relation on H φ is said to be an *ideal* if for any $i \in I$ we have:

$$\beta_i(x_1, ..., x_{n_i})(a) > 0 \quad \text{and} \quad x_k \varphi y_k (k \in \{1, ..., n_i\}) \;\; \Rightarrow \;\; \exists b \in H : \beta_i(y_1, ..., y_{n_i})(b) > 0 \quad \text{and} \quad a\varphi b.$$

For example the fuzzy regular relations on a fuzzy hypersemigroup are ideal equivalence. (for more details see [13, 21])

Definition 3.14. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and φ an equivalence relation on H. Then H/φ can be described as a fuzzy hyperalgebra $\mathbb{H}/\varphi$ with the fuzzy hyperoperations:

$$\beta_i(\varphi(x_1), ..., \varphi(x_{n_i}))(\varphi(x_{n_i+1})) = \bigvee_{x_k \varphi y_k} \beta_i(y_1, ..., y_{n_i})(y_{n_i+1}).$$

Theorem 3.15. Let $h : H \to B$ be a good homomorphism of fuzzy hyperalgebras $\mathbb{H}$ and $\mathbb{B}$. Then the relation $\varphi = \{(x, y) \in H | h(x) = h(y)\}$ is an ideal relation on $\mathbb{H}$. Conversely, if φ is an ideal relation on $\mathbb{H}$, then $p = p_\varphi : H \to H/\varphi$ is homomorphism (which is not strong).

Proof. Straightforward.□

Remark 3.16. Let $\mathbb{H}$ and $\mathbb{B}$ be fuzzy hyperalgebras of the same type and h be a homomorphism from $\mathbb{H}$ into $\mathbb{B}$. We will construct the algebras $\mathcal{F}^*(\mathbb{H})$ and $\mathcal{F}^*(\mathbb{B})$. The homomorphism h induces a mapping $h' : \mathcal{F}^*(\mathbb{H}) \to \mathcal{F}^*(\mathbb{B})$ by $h'(\mu) = h(\mu)$, for any $\mu \in F^*(H)$.

Let us consider H a set and $F^*(H)$ the set of its nonzero fuzzy subsets. Let φ be an equivalence on H and let us consider the relation $\overline{\varphi}$ on $F^*(H)$ as follows:

$$\mu\overline{\varphi}\nu \Leftrightarrow \quad \forall a \in H : \mu(a) > 0 \quad \Rightarrow \quad \exists b \in H : \nu(b) > 0 \quad \text{and} \quad a\varphi b \text{ and}$$

$$\forall b \in H : \quad \nu(b) > 0 \quad \Rightarrow \quad \exists a \in H : \mu(a) > 0 \quad \text{and} \quad a\varphi b.$$

It is immediate that $\overline{\varphi}$ is an equivalence relation on $F^*(H)$. The next result immediately follows.

Theorem 3.17. An equivalence relation φ on a fuzzy hyperalgebra $\mathbb{H}$ is ideal if and only if $\overline{\varphi}$ is a congruence relation on $\mathcal{F}^*(\mathbb{H})$.

Proof. Let us suppose that φ is an ideal equivalence on $\mathbb{H}$ and let us consider $i \in I$ and $\mu_k, \nu_k \in F^*(H)$ nonzero and $\mu_k\overline{\varphi}\nu_k$, $k \in \{1, ..., n_i\}$. Then for any $a \in H$ such that $\beta_i(\mu_1, ..., \mu_{n_i})(a) > 0$, we have

$\beta_i(\mu_1, ..., \mu_{n_i})(a) = \bigvee\limits_{(x_1,...,x_{n_i})\in H^{n_i}} \mu_1(x_1) \wedge ... \wedge \mu_{n_i}(x_{n_i}) \wedge \beta_i(x_1, ..., x_{n_i})(a).$

Thus there exists $(x_1, ..., x_{n_i}) \in H^{n_i}$, such that $\mu_k(x_k) > 0$ for $k \in \{1, ..., n_i\}$ and $\beta_i(x_1, ..., x_{n_i})(a) > 0$. From the definition $\overline{\varphi}$ and hence there exists $(y_1, ..., y_{n_i}) \in H^{n_i}$, such that $\nu_k(y_k) > 0$ for $k \in \{1, ..., n_i\}$ and $x_k\varphi y_k$, and sice φ is an ideal and $\beta_i(x_1, ..., x_{n_i})(a) > 0$, there exists $b \in H$, such that $\beta_i(y_1, ..., y_{n_i})(b) > 0$ and $a\varphi b$. Analogously, it can be proved that for all $b \in H$, such that $\beta_i(y_1, ..., y_{n_i})(b) > 0$, there exists $a \in H$, such that $\beta_i(x_1, ..., x_{n_i})(a) > 0$ and $a\varphi b$. Hence, it is proved that $\overline{\varphi}$ is a congruence on $\mathcal{F}^*(\mathbb{H})$.

Conversely, let us take $i \in I$ and $a, x_k, y_k \in H$, $k \in \{1, ..., n_i\}$ such that $x_k\varphi y_k$ and $\beta_i(x_1, ..., x_{n_i})(a) > 0$. Obviously, $\chi_{x_k}\overline{\varphi}\chi_{y_k}$, $\forall k \in \{1, ..., n_i\}$, and because $\overline{\varphi}$ is a congruence on $\mathcal{F}^*(\mathbb{H})$ We can write $\beta_i(\chi_{x_1}, ..., \chi_{x_{n_i}})\overline{\varphi}\beta_i(\chi_{y_1}, ..., \chi_{y_{n_i}})$, hence $\beta_i(x_1, ..., x_{n_i})\overline{\varphi}\beta_i(y_1, ..., y_{n_i})$, which leads us to the existence $b \in H$, such that $\beta_i(y_1, ..., y_{n_i})(b) > 0$ and $a\varphi b$. This complete the proof.$\square$

Corollary 3.18. (i) If $\mathbb{H}=\langle H, (\beta_i : i \in I)\rangle$ is a fuzzy hyperalgebra, φ is an ideal equivalence relation on $\mathbb{H}$ and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$ If for any nonzero, μ_k, ν_k, such that $\mu_k\overline{\varphi}\nu_k$

$k \in \{1, ..., n\}$, then $p(\mu_1, ..., \mu_n)\overline{\varphi}p(\nu_1, ..., \nu_n)$.

(ii) Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra, φ an ideal equivalence relation on $\mathbb{H}$. If $x_k \varphi y_k$, $k \in \{1, ..., n\}$, $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$, $x_k, y_k \in H$. Then have $p(\chi_{x_1}, ..., \chi_{x_n})\overline{\varphi}p(\chi_{y_1}, ..., \chi_{y_n})$.

Let h be a homomorphism from $\mathbb{H}$ into $\mathbb{B}$ and take $\varphi = \{(x, y) \in H^2 \mid h(x) = h(y)\}$. Then we have $\overline{\varphi} = \{(\mu, \nu) \in (F^*(H))^2 \mid h'(\mu) = h'(\nu)\}$. Obviously, φ is an ideal of $\mathbb{H}$ if and only if $\overline{\varphi}$ is congruence on $\mathcal{F}^*(\mathbb{H})$.

Theorem 3.19. The map h is a homomorphism ofthe universal algebras $\mathcal{F}^*(\mathbb{H})$ and $\mathcal{F}^*(\mathbb{B})$ if and only if h is a good homomorphism between $\mathbb{H}$ and $\mathbb{B}$.

Proof. Straightforward.$\square$

The next result immediately follows from Theorem 3.12.

Corollary 3.20. (i) Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ be fuzzy hyperalgebras of the same type, $h : H \to B$ a homomorphism and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$. Then for all $\mu_1, ..., \mu_n \in F^*(H)$ we have $h'(p(\mu_1, ..., \mu_n)) = p(h'(\mu_1), ..., h'(\mu_n))$.

(ii) Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ be fuzzy hyperalgebras of the same type, $h : H \to B$ a homomorphism and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$. Then for all $a_1, ..., a_n \in H$, we have $h'(p(\chi_{a_1}, ..., \chi_{a_n})) = p(h'(\chi_{a_1}), ..., h'(\chi_{a_n}))$.$\square$

4 Fundamental Relation of Fuzzy Hyperalgebra

As it is known that if R is an strongly regular equivalence relation on a given hypergroup (resp. hypergroupoid, semihypergroup) H, then we can define a binary operation $\otimes$ on the quotient set H/R, the set of all equivalence classes of H with respect to R, such that $(H/R, \otimes)$ consists a group (resp. groupoid, semigroup). In fact the relation β^* is the

smallest equivalences relation such that the quotient H/β^* is a group (resp. groupoid, semigroup) and it is called *fundamental relation* of H. The equivalence relation β^* was studied on hypergroups by many authors(for more details see [6]). As the fundamental relation plays an important role in the theory of algebraic hyperstructure it extended to other classes of algebraic hyperstructure, such as hyperrings, hypermodules, hypervectorspaces(for more details see [25], [26] and [27]). In [20] Pelea introduced and studied the fundamental relation of a multialgebra based on term functions. In the sequel we extend fundamental relation on fuzzy hyperalgebras and investigate its basic properties.
Let $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ be an universal algebra. If we add to the set of the operations of $\mathbb{B}$ the nullary operations corresponding to the elements of B, the $n-$ary term functions of this new algebra are called the *$n-$ary polynomial functions* of $\mathbb{B}$. The $n-$ary polynomial functions $P^n(\mathbb{B})$ of $\mathbb{B}$ form a universal algebra with the operations $(\beta_i : i \in I)$, denoted by $\mathcal{P}^{(n)}(\mathbb{B})$, $\mathcal{P}^{(n)}(\mathbb{B})$=$\langle P^n(\mathbb{B}), (\beta_i : i \in I)\rangle$.
Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra. For any $n \in \mathbb{N}$, we can construct the algebra $\mathcal{P}^{(n)}(\mathcal{F}^*(\mathbb{H}))$ of $n-$ary polynomial functions on $\mathcal{F}^*(\mathbb{H})$, ($\mathcal{P}^{(n)}(\mathcal{F}^*(\mathbb{H})) = \langle P^n(\mathcal{F}^*(\mathbb{H})), (\beta_i : i \in I)\rangle$) . Consider the subalgebra $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$ of $\mathcal{P}^{(n)}(\mathcal{F}^*(\mathbb{H}))$ obtained by adding to the operations $(\beta_i : i \in I)$ of $\mathcal{F}^*(\mathbb{H})$ only the nullary operations associated to the characteristic functions of the elements of H. Thus the elements of $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$ $(n \in \mathbb{N})$ are those and only those functions from $(F^*(H))^n$ into $F^*(H)$ which can obtained by applying $(i), (ii), (iii)$ from bellow for finitely many times:
(i) the functions $C^n_{\chi_a} : (F^*(H))^n \to F^*(H)$, defined by setting $C^n_{\chi_a}(\mu_1, ..., \mu_n) = \chi_a$, for all $\mu_1, ..., \mu_n \in F^*(H)$ are elements of $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$, for every $a \in H$;
(ii) the functions $e^n_i : (F^*(H))^n \to F^*(H)$, $e^n_i(\mu_1, ..., \mu_n) = \mu_i$, for all $\mu_1, ..., \mu_n \in F^*(H)$, $i = 1, ..., n$ are elements of $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$;
(iii) if $p_1, ..., p_{n_i}$ are elements of $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$, and $i \in I$ then $\beta_i(p_1, ..., p_{n_i}) : (F^*(H))^n \to$

$F^*(H)$, defined by setting for all $\mu_1, ..., \mu_n \in F^*(H)$, $(\beta_i(p_1, ..., p_{n_i}))(\mu_1, ..., \mu_n) =$ $\beta_i(p_1(\mu_1, ..., \mu_n), ..., p_{n_i}(\mu_1, ..., \mu_n))$ is also an element of $\mathcal{P}_H^{(n)}(\mathcal{F}^*(\mathbb{H}))$.

In the continue, we will use only polynomial functions from $\mathcal{P}_H^{(n)}(\mathcal{F}^*(\mathbb{H}))$. Thus we will drop the subscript with no danger of confusion.

Definition 4.1. Let α be the relation defined on H for $x, y \in H$ set $x\alpha y$ follows:

$$x\alpha y \iff p(\chi_{a_1}, ..., \chi_{a_n})(x) > 0 \text{and } p(\chi_{a_1}, ..., \chi_{a_n})(y) > 0, \text{ for some } p \in P^n(\mathcal{F}^*(\mathbb{H})), a_1, ..., a_n \in H.$$

It is clear that α is symmetric. Because for any $a \in H$, $e_1^1(\chi_a)(a) > 0$, the relation α is also reflexive. We take α^* to be the transitive closure of α. Then α^* is an equivalence relation on H.

Lemma 4.2. If $f \in P^1(\mathbb{F}^*(\mathbb{H}))$ and $a, b \in H$ satisfy $a\alpha^*b$ then $f(\chi_a)\overline{\overline{\alpha^*}}f(\chi_b)$.

Proof. By the definition of α^* : $a = y_1\alpha y_2\alpha ... \alpha y_m = b$ for some $m \in \mathbb{N}$ and $y_2, ..., y_{m-1} \in H$. Let $f(\chi_{y_i})(u_i) > 0$, $i = 1, ..., m$. Consider $1 \leq j < m$. Clearly $y_j\alpha y_{j+1}$ means that $p_j(\chi_{a_1}, ..., \chi_{a_n})(y_j) > 0$ and $p_j(\chi_{a_1}, ..., \chi_{a_n})(y_{j+1}) > 0$, for some $n_j \in \mathbb{N}, p_j \in P^{n_j}(\mathcal{F}^*(\mathbb{H}))$, $a_1, ..., a_n \in H$. Define the n_j-ary hyperoperation q_j on $F^*(H)$ by setting

$q_j(\chi_{x_1}, ..., \chi_{x_{n_j}}) = \bigvee\{f(\chi_t) : p_j(\chi_{x_1}, ..., \chi_{x_{n_j}})(t) > 0\}$ for all $x_1, ..., x_{n_j} \in H$. Clearly $q_j \in P^{n_j}(\mathcal{F}^*(\mathbb{H}))$ and for $x \in H$; $q_j(\chi_{a_1}, ..., \chi_{a_n})(x) = \bigvee_{p_j(\chi_{a_1}, ..., \chi_{a_n})(z) > 0} f(\chi_z)(x)$.

From $p_j(\chi_{a_1}, ..., \chi_{a_n})(y_j) > 0$ and $p_j(\chi_{a_1}, ..., \chi_{a_n})(y_{j+1}) > 0$ we get

$$0 < f(\chi_{y_j})(u_j) \leq q_j(\chi_{a_1}, ..., \chi_{a_n})(u_j) \quad \text{and}$$
$$0 < f(\chi_{y_{j+1}})(u_{j+1}) \leq q_j(\chi_{a_1}, ..., \chi_{a_n})(u_{j+1})$$

proving $u_j\alpha u_{j+1}$. Thus $u_1\alpha^*u_m$. Since $f(\chi_a)(u_1) = f(\chi_{y_1})(u_1) > 0$ and $f(\chi_b)(u_m) = f(\chi_{y_m})(u_m) > 0$ were arbitrary, we obtain $f(\chi_a)\overline{\overline{\alpha^*}}f(\chi_b)$.□

Remark 4.3. For a given fuzzy hyperalgebra $\mathbb{H}$ and equivalence relation ρ on H, the set H/ρ can be considered as a hyperalgebra with the hyperoperations

$$\beta_i(\rho(a_1), ..., \rho(a_{n_i})) = \{\rho(z) \mid \beta_i(b_1, ..., b_{n_i})(z) > 0, b_k \in \rho(a_k), \forall k \in \{1, ..., n_i\}\} \quad (1)$$

for all $i \in I$.

Lemma 4.4. Let ρ be an equivalence relation on $\mathbb{H}$ such that $\mathbb{H}/\rho$ be an universal algebra . Then for any $n \in \mathbb{N}$, $p \in P^n(\mathcal{F}^*(\mathbb{H}))$ and $a_1, ..., a_n \in H$ the following gold:

$$p(\chi_{a_1}, ..., \chi_{a_n})(x) > 0 \text{ and } p(\chi_{a_1}, ..., \chi_{a_n})(y) > 0 \implies x\rho y.$$

Proof. We will prove this statement by induction over the steps of construction of an $n-$ary polynomial function(for $n \in \mathbb{N}$ arbitrary).

If $p = C^n_{\chi_a}$, from $C^n_{\chi_a}(\chi_{a_1}, ..., \chi_{a_n})(x) > 0$ and $C^n_{\chi_a}(\chi_{a_1}, ..., \chi_{a_n})(y) > 0$ we deduce that $x = y = a$, thus $x\rho y$.

If $p = e^n_i$ with $i \in \{1, ..., n\}$, from $e^n_i(\chi_{a_1}, ..., \chi_{a_n})(x) > 0$ and $e^n_i(\chi_{a_1}, ..., \chi_{a_n})(y) > 0$ we deduce that $x = y = a_i$, , and hence $x\rho y$.

We suppose that the statement holds for the $n-$ary polynomial functions $p_1, ..., p_{n_k}$ and we will prove it for the $n-$ary polynomial function $\beta_k(p_1, ..., p_{n_k})$. If

$0 < \beta_k(p_1, ..., p_{n_k})(\chi_{a_1}, ..., \chi_{a_n})(x) = \beta_k(p_1(\chi_{a_1}, ..., \chi_{a_n}), ..., p_{n_k}(\chi_{a_1}, ..., \chi_{a_n}))(x) =$

$$\bigvee_{(x_1, ..., x_{n_k}) \in H^{n_k}} (p_1(\chi_{a_1}, ..., \chi_{a_n})(x_1) \wedge ... \wedge p_{n_k}(\chi_{a_1}, ..., \chi_{a_n})(x_{n_k}) \wedge \beta_k(x_1, ..., x_{n_k})(x))$$

and if we set y instead of x, above statement is true. Thus there exist $x_1, ..., x_{n_k}, y_1, ..., y_{n_k} \in H$, such that $p_i(\chi_{a_1}, ..., \chi_{a_n})(x_i) > 0$ and $p_i(\chi_{a_1}, ..., \chi_{a_n})(y_i) > 0$, for $i \in \{1, ..., n_k\}$ and $\beta_k(x_1, ..., x_{n_k})(x) > 0$ and $\beta_k(y_1, ..., y_{n_k})(y) > 0$. Obviously, $x_i\rho y_i$ for all $i \in \{1, ..., n_k\}$ and according to (1) and by the hypothesis we obtain that $\rho(x) = \rho(y)$, i.e., $x\rho y$, as desired.□

The next result immediately follows from previous two lemmas.

Theorem 4.5. The relation α^* is the smallest equivalence relation on fuzzy hyperalgebra $\mathbb{H}$ such that $\mathbb{H}/\rho$ is an universal algebra.

We call $\mathbb{H}/\rho$, *fundamental universal algebra* of fuzzy hyperalgebra $\mathbb{H}$ such that $\mathbb{H}/\rho$.

Proof. At the first, we show that $\mathbb{H}/\rho$ is a universal algebra. For this we take any $x, y \in H$, such that $\alpha^*(x), \alpha^*(y) \in \beta_k(\alpha^*(a_1), ..., \alpha^*(a_{n_k}))$ for $k \in I$ and $a_1, ..., a_{n_k} \in H$. This means that there exist $x_1, ..., x_{n_k}, y_1, ..., y_{n_k} \in H$, such that $\beta_k(x_1, ..., x_{n_k})(x) > 0$ and $\beta_k(y_1, ..., y_{n_k})(y) > 0$ and $x_i \alpha^* a_i \alpha^* y_i$ for all $i \in \{1, ..., n_k\}$.

Applying Lemma 4.2 to the unary polynomial functions

$\beta_i(z, C^n_{\chi_{x_2}}, ..., C^n_{\chi_{x_{n_k}}})$, $\beta_i(C^n_{\chi_{y_1}}, z, C^n_{\chi_{x_3}}, ..., C^n_{\chi_{x_{n_k}}}), ..., , \beta_i(C^n_{\chi_{y_1}}, ..., C^n_{\chi_{y_{n_k-1}}}, z)$,

we obtain the following relations:

$$\beta_i(\chi_{x_1}, ..., \chi_{x_{n_k}})\overline{\overline{\alpha^*}}\beta(\chi_{y_1}, \chi_{x_2}, ..., \chi_{x_{n_k}})$$
$$\beta_i(\chi_{y_1}, \chi_{x_2}, ..., \chi_{x_{n_k}})\overline{\overline{\alpha^*}}\beta_i(\chi_{y_1}, \chi_{2_2}, \chi_{x_3} ..., \chi_{x_{n_k}})$$
$$\vdots$$
$$\beta_i(\chi_{y_1}, \chi_{y_2}, ..., \chi_{x_{n_k-1}})\overline{\overline{\alpha^*}}\beta_i(\chi_{y_1}, \chi_{y_2}, ..., \chi_{y_{n_k}}),$$

which leads us to $x\alpha^* y$ (from definition α^*), i.e. $\alpha^*(x) = \alpha^*(y)$. Clearly, β_i in (1) is an operation on H/α^*, for any $i \in I$, and $\mathbb{H}/\alpha^*$ is a universal algebra. Now we prove that α^* is smallest. If ρ is an arbitrary equivalence relation on H such that H/ρ is a universal algebra, we can show that $\alpha^* \subseteq \rho$. If $x\alpha y$ then there exist $n \in \mathbb{N}$, $p \in P^n(\mathcal{F}^*(\mathbb{H}))$ and $a_1, ..., a_n \in H$ for which $p(\chi_{a_1}, ..., \chi_{a_n})(x) > 0$ and $p(\chi_{a_1}, ..., \chi_{a_n})(y) > 0$, and hence by Lemma 4.4 we have $x\rho y$, hence $\alpha \subseteq \rho$, which implies $\alpha^* \subseteq \rho$.$\square$

Remark 4.6. For a given fuzzy hyperalgebra $\mathbb{H}$ and equivalence relation α^* on H. Let us define the operations of the universal algebra $\mathbb{H}/\alpha^*$ as follows :

$$\beta_i(\alpha^*(a_1), ..., \alpha^*(a_{n_i})) = \{\alpha^*(b) \mid \beta_i(a_1, ..., a_{n_i})(b) > 0\}.$$

Moreover, we can write

$$\beta_i(\alpha^*(a_1), ..., \alpha^*(a_{n_i})) = \alpha^*(b) \qquad \beta_i(a_1, ..., a_{n_i})(b) > 0.$$

Example 4.7. Let $\mathbb{H}$=$\langle H, \circ\rangle$ be a fuzzy hypersemigroup, i.e. a fuzzy hyperalgebra with one binary fuzzy hyperoperation $\circ$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in H$ (for more details see [21]). Let $\mathcal{F}^*(\mathbb{H})$=$\langle F^*(H), \circ\rangle$ be the universal algebra with one binary operation defined as follows:

$$(\mu \circ \nu)(r) = \bigvee_{x,y \in H} \mu(x) \wedge \nu(y) \wedge (x \circ y)(r) \qquad \forall \;\; \mu, \nu \in F^*(H), r \in H.$$

By distributivity of the lattice $([0,1], \vee, \wedge)$ and associativity of $\circ$ in H, we will prove that the operation $\circ$ in $\mathcal{F}^*(\mathbb{H})$ is associative. So for every $\mu, \nu, \eta \in F^*(H)$ and $r \in H$ we have

$$((\mu \circ \nu) \circ \eta)(r) = \bigvee_{x,y \in H} [(\mu \circ \nu)(x) \wedge \eta(y) \wedge (x \circ y)(r)] =$$

$$\bigvee_{x,y \in H} [(\bigvee_{p,q \in H} \mu(p) \wedge \nu(q) \wedge (p \circ q)(x)) \wedge \eta(y) \wedge (x \circ y)(r)] =$$

$$\bigvee_{p,q,y \in H} [\mu(p) \wedge \nu(q) \wedge \eta(y) \wedge (\bigvee_{x \in H} (p \circ q)(x) \wedge (x \circ y)(r))] =$$

$$\bigvee_{p,q,y \in H} [\mu(p) \wedge \nu(q) \wedge \eta(y) \wedge (\bigvee_{x \in H} (p \circ x)(r) \wedge (q \circ y)(x))] =$$

$$\bigvee_{p,x \in H} [\mu(p) \wedge (p \circ x)(r) \wedge (\bigvee_{q,y \in H} \nu(q) \wedge \eta(y) \wedge (q \circ y)(x))] =$$

$$\bigvee_{p,x \in H} [\mu(p) \wedge (p \circ x)(r) \wedge (\nu \circ \eta)(x)] = (\mu \circ (\nu \circ \eta))(r).$$

Consider now the universal algebra of polynomial functions of $\langle F^*(H), \circ\rangle$. The images of the elements of this algebra are the sums of nonzero fuzzy subsets of $\mathbb{H}$. Thus we can define α on H by:

$a \alpha b \Longleftrightarrow \exists x_1, ..., x_n \in H (n \in \mathbb{N})$: $(\chi_{x_1} \circ ... \circ \chi_{x_n})(a) > 0$ and $(\chi_{x_1} \circ ... \circ \chi_{x_n})(b) > 0$.

Consider the quotient set H/α^* with the hyperoperation

$$\alpha^*(a) \circ \alpha^*(b) = \{\alpha^*(c) \mid (a' \circ b')(c) > 0, \quad a' \alpha^* a, \quad b' \alpha^* b\}.$$

Really $\circ$ is an operation, because α^* is the fundamental relation on $\mathbb{H}$. Also

$$\alpha^*(x) \circ \alpha^*(y) \circ \alpha^*(z)) = \alpha^*(x) \circ \alpha^*(k) = \alpha^*(l), \;\; \text{where} \;\; (y \circ z)(k) > 0 \quad \text{and} \;\; (x \circ k)(l) > 0.$$

Therefore, $0 < (x \circ (y \circ z))(l) = ((x \circ y) \circ z)(l) = \bigvee_{p \in H} [(x \circ y)(p) \wedge (p \circ z)(l)]$. Thus there exists $p \in H$, such that $\alpha^*(l) = \alpha^*(p) \circ \alpha^*(z) = (\alpha^*(x) \circ \alpha^*(y)) \circ \alpha^*(z)$, that the operation $\circ$ in H/α^* is associative. Moreover, if $\mathbb{H}$=$\langle H, \circ\rangle$ be a fuzzy hypergroup, that is $x \circ H = H \circ x = \chi_H$, for every $x \in H$, since for every $\alpha^*(a), \alpha^*(b) \in H/\alpha^*$, there exist $\alpha^*(t), \alpha^*(s) \in H/\alpha^*$, such that, $\alpha^*(a) \circ \alpha^*(t) = \alpha^*(b)$ and $\alpha^*(s) \circ \alpha^*(a) = \alpha^*(b)$, it is concluded that $\mathbb{H}/\alpha^*$=$\langle H/\alpha^*, \circ\rangle$ is a group.

Example 4.8. Let $\mathbb{R}$=$\langle R, \oplus, \odot\rangle$ be a fuzzy hyperring. This means that $\langle R, \oplus\rangle$ is a commutative fuzzy hypergroup, $\langle R, \odot\rangle$ is a fuzzy hypersemigroup and for all $x, y, z \in R$ satisfies: $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ and $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ (for more details see [13]). Let $\mathcal{F}^*(\mathbb{R})$=$\langle F^*(R), \oplus, \odot\rangle$ be the universal algebra with two binary operations defined as follows:

$$(\mu \oplus \nu)(r) = \bigvee_{x,y \in H} [\mu(x) \wedge \nu(y) \wedge (x \oplus y)(r)],$$

$$(\mu \odot \nu)(r) = \bigvee_{x,y \in H} [\mu(x) \wedge \nu(y) \wedge (x \odot y)(r)],$$

for all $\mu, \nu \in F^*(R)$, $r \in R$. Obviously, the operation $\oplus$ in $F^*(R)$ is commutative, and $\oplus$ and $\odot$ in $F^*(R)$ are associative. By distributivity of the lattice [0, 1] and distributivity $\odot$ with respect to $\oplus$ in R, we will prove that the operation $\odot$ in $F^*(R)$ is distributive with respect to the operation $\oplus$, too.

For every $\mu, \nu,\ eta \in F^*(R)$ and $r \in R$ we have:

$$(\mu \odot (\nu \oplus \eta))(r) = \bigvee_{x,y \in R} [\mu(x) \wedge (\nu \oplus \eta)(y) \wedge (x \odot y)(r)] =$$

$$\bigvee_{x,y \in R} [\mu(x) \wedge (\bigvee_{s,t \in R} \nu(s) \wedge \eta(t) \wedge (s \oplus t)(y)) \wedge (x \odot y)(r)] =$$

$$\bigvee_{x,y \in R} [\bigvee_{s,t \in R} (\mu(x) \wedge \nu(s) \wedge \eta(t) \wedge (s \oplus t)(y) \wedge (x \odot y)(r))] =$$

$$\bigvee_{x,s,t \in R} [\mu(x) \wedge \nu(s) \wedge \eta(t) \wedge (\bigvee_{y \in R} (x \odot y)(r) \wedge (s \oplus t)(y))] =$$

$$\bigvee_{x,s,t \in R} [\mu(x) \wedge \nu(s) \wedge \eta(t) \wedge (\bigvee_{p,q \in R} (x \odot s)(p) \wedge (x \odot t)(q) \wedge (p \oplus q)(r))] =$$

$$\bigvee_{x,s,t\in R}[\bigvee_{p,q\in R}(\mu(x)\wedge\eta(t)\wedge(x\odot t)(q)\wedge\mu(x)\wedge\nu(s)\wedge(x\odot s)(p)\wedge(p\oplus q)(r))]=$$
$$\bigvee_{p,q\in R}[(\bigvee_{x,t\in R}\mu(x)\wedge\eta(t)\wedge(x\odot t)(q))\wedge(\bigvee_{x,s\in R}\mu(x)\wedge\nu(s)\wedge(x\odot s)(p))\wedge(p\oplus q)(r)]=$$
$$\bigvee_{p,q\in R}[(\mu\odot\eta)(q)\wedge(\mu\odot\nu)(p)\wedge(p\oplus q)(r)]=((\mu\odot\nu)\oplus(\mu\odot\eta))(r).$$

And analogously, $(\mu\oplus\nu)\odot\eta=(\mu\odot\eta)\oplus(\nu\odot\eta)$. Now we can construct the universal algebra (with two binary operations) of the polynomial functions of $\mathcal{F}^*(\mathbb{R})$ for any $n\in\mathbb{N}$. The images of the elements of this algebra are the sums of products of nonzero fuzzy subsets of $\mathbb{R}$. Thus we can define α on $\mathbb{R}$ by;

$a\alpha b\Longleftrightarrow\exists x_{ij}\in R, i\in\{1,...,k_j\}, j\in\{1,...,l\}, k_j,l\in\mathbb{N}$:

$$(\oplus_{j=1}^{l}(\odot_{i=1}^{k_j}\chi_{x_{ij}}))(a)>0 \text{ and } (\oplus_{j=1}^{l}(\odot_{i=1}^{k_j}\chi_{x_{ij}}))(b)>0.$$

Consider the quotient set R/α^* withe the two following hyperoperations :

$$\alpha^*(a)\oplus\alpha^*(b)=\{\alpha^*(c)\mid(a'\oplus b')(c)>0, a'\alpha^*a, b'\alpha^*b\}, \text{ and}$$
$$\alpha^*(a)\odot\alpha^*(b)=\{\alpha^*(c)\mid(a'\odot b')(c)>0, a'\alpha^*a, b'\alpha^*b\}$$

Actually $\oplus$ and $\odot$ are operations, because α^* is the fundamental relation on $\mathbb{R}$. By considering the previous example, obviously $\langle R/\alpha^*,\oplus\rangle$ is a commutative group. We verify the distributivity of $\odot$ with respect to $\oplus$ for the universal algebra $\mathbb{R}/\alpha^*$=$\langle R/\alpha^*,\oplus,\odot\rangle$. We have

$\alpha^*(a)\odot(\alpha^*(b)\oplus\alpha^*(c))=\alpha^*(a)\odot\alpha^*(d)=\alpha^*(e)$, where $(b\oplus c)(d)>0$ and $(a\odot d)(e)>0$
$0<(a\odot(b\oplus c))(e)=\bigvee_{p\in R}(a\odot p)(e)\wedge(b\oplus c)(p)$. Thus
$0<((a\odot b)\oplus(a\odot c))(e)=\bigvee_{x,y\in R}(a\odot b)(x)\wedge(a\odot c)(y)\wedge(x\oplus y)(e)$. Therefore, there exist $x,y\in R$ such that $\alpha^*(e)=\alpha^*(x)+\alpha^*(y)=(\alpha^*(a)+\alpha^*(b))\oplus(\alpha^*(a)\odot\alpha^*(c))$, and hence it was proved that $\alpha^*(a)\odot(\alpha^*(b)\oplus\alpha^*(c))=(\alpha^*(a)+\alpha^*(b))\oplus(\alpha^*(a)\odot\alpha^*(c))$. Analogously, we can prove that $(\alpha^*(b)\oplus\alpha^*(c))\odot\alpha^*(a))=(\alpha^*(b)\odot\alpha^*(a))\oplus(\alpha^*(c)\odot\alpha^*(a))$. Thus it concluded that $\mathbb{R}/\alpha^*$=$\langle R/\alpha^*,\oplus,\odot\rangle$ is a ring, as desired.$\square$

Conclusion

We introduced and studied term functions over fuzzy hyperalgebras, as the largest class of fuzzy algebraic systems. We use the idea that the set of nonzero fuzzy subsets of a fuzzy hyperalgebra can be organized naturally as a universal algebra, and constructed the term functions over this algebra. We gave the form of generated subfuzzy hyperalgebra of a given fuzzy hyperalgebra as a generalization of universal algebras and multialgebras. Finally, we characterized the form of the fundamental relation of a fuzzy hyperalgebra, to construct the fundamental universal algebra corresponding to a given fuzzy hyperalgebra, and this result guarantee that that fundamental relation on any fuzzy algebraic hyperstructures, such as fuzzy hypergroups, fuzzy hyperrings, fuzzy hypermodules,... exists.

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