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# The sequence of trifurcating Fibonacci numbers

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**Abstract**

One of the remarkable generalizations of Fibonacci sequence is a $k$-Fibonacci sequence and subsequently generalized into the 'Bifurcating Fibonacci sequence'. In this paper, further generalization as the sequence of 'trifurcating Fibonacci numbers' is studied and Binet-like formula for these numbers is obtained. The analogous of Cassini's identity, Catalan's identity, d'Ocagne's identity and some fundamental identities for the terms of this sequence has also been investigated.

**Keywords:** Fibonacci sequence, bifurcating Fibonacci sequence, generalization of Fibonacci sequence, Binet formula, identities related with the Fibonacci sequence.

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# 1. Introduction

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is a sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … , where each term is the sum of two preceding terms. The corresponding recurrence relation is $F_n = F_{n-1} + F_{n-2}$ ; $n \geq 2$. Various generalizations of this sequence have appeared in recent years [5, 7, 8]. Related to this work (i) some work alters the first two terms of the sequence from 0,1 to any arbitrary integers $a, b$ while maintaining the recurrence relation (ii) some more work preserves the first two terms of the sequence but alters the recurrence relation (iii) even the combined approach of altering the initial terms as well as recurrence relation was considered by several authors. For further details about this sequence, one can refer Koshy [6], Patel, Shah [7], Singh, Sikhwal, Bhatnagar [8] and related papers available in the literature.

One interesting generalization depending on exactly one real parameter $k$ is the sequence of $k$-Fibonacci numbers $\{F_{k,n}\}$ which is defined using a linear recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ ; $n \geq 2$ where $F_{k,0} = 0$ and $F_{k,1} = 1$. For $k = 1$, we get the standard Fibonacci sequence and for $k = 2$, we get the sequence of Pell numbers. This sequence was studied by Arvadia and Shah [1]. Edson and Yayenie [4] generalized this sequence to a sequence which depends on two real parameters $a, b$. They defined the bifurcating sequence $\{F_n^{(a,b)}\}_{n\geq 0}$ by the recurrence relation

$$F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}; \text{if } n \text{ is even} \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}; \text{if } n \text{ is odd} \end{cases} ; n \geq 2$$

where $F_0^{(a,b)} = 0, F_1^{(a,b)} = 1$.

Diwan and Shah [2, 3], Yayenie [10], Verma and Bala [9] studied this sequence extensively and obtained significant results. It is easy to observe that (i) by considering $a = b = 1$, we get standard Fibonacci sequence (ii) by considering $a = b = 2$, we get the sequence of Pell numbers and (iii) by considering $a = b = k$, we get the sequence of $k$-Fibonacci numbers. In this paper, we further generalize this sequence to a sequence of trifurcating Fibonacci numbers, which depends on the three real parameters $a, b, c$.

**Definition:** For any three nonzero positive integers $a, b$ and $c$, the trifurcating Fibonacci sequence $\left\{F_n^{(a,b,c)}\right\}_{n\geq 0}$ is defined recursively by $F_0^{(a,b,c)} = 0, F_1^{(a,b,c)} = 1$ and the recurrence relation

$$F_n^{(a,b,c)} = \begin{cases} aF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}; \text{if } n \equiv 0(mod\ 3) \\ bF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}; \text{if } n \equiv 1(mod\ 3). \\ cF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}; \text{if } n \equiv 2(mod\ 3) \end{cases}$$

To avoid cumbersome notation, we denote $F_n^{(a,b,c)}$ by $P_n$. Few terms of this trifurcating generalized Fibonacci sequence are shown in the Table 1. In this paper we obtain various interesting results for this sequence.

| $\boldsymbol{n}$ | $\boldsymbol{P_n}$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | $c$ |
| 3 | $ac + 1$ |
| 4 | $abc + b + c$ |
| 5 | $abc^2 + bc + c^2 + ac + 1$ |
| 6 | $a^2bc^2 + 2abc + ac^2 + a^2c + a + b + c$ |
| 7 | $a^2b^2c^2 + 2ab^2c + 2abc^2 + a^2bc + ab + 2bc + b^2 + c^2 + ac + 1$ |

Table 1

# 2. Fundamental identities for the trifurcating sequence $\{\boldsymbol{P_n}\}_{\boldsymbol{n\geq 0}}$

In this section, we derive some interesting identities for the terms of the sequence $\{P_n\}_{n\geq 0}$. We first show that any two consecutive terms of $\{P_n\}_{n\geq 0}$ are always relatively prime.

**Theorem 2.1.** $\gcd(P_n, P_{n-1}) = 1$; for all $n = 1, 2, \ldots$ .

**Proof.** We prove the result by considering three cases when $n = 3k, 3k + 1$ or $3k + 2$. We present the proof only for the case $n = 3k$ and other cases follows accordingly. Now Euclidean algorithm leads to the following system of equations:

$$\begin{aligned} P_{3k} &= aP_{3k-1} + P_{3k-2} \\ P_{3k-1} &= cP_{3k-2} + P_{3k-3} \\ P_{3k-2} &= bP_{3k-3} + P_{3k-4} \\ &\vdots \end{aligned}$$

$$P_4 = bP_3 + P_2$$
$$P_3 = aP_2 + P_1$$
$$P_2 = cP_1 + 0$$

It now easily follows from Euclidean algorithm that $\gcd(P_n, P_{n-1}) = P_1 = 1$.

We now prove certain summation formulae for the terms of $\{P_n\}_{n\geq 0}$.

**Lemma 2.2.**
a) $P_{3n+2} = (bc+1)(P_3 + P_6 + \cdots + P_{3n}) + (P_2 + P_5 + \cdots + P_{3n-2}) + c$
b) $P_{3n+1} = (ab+1)(P_2 + P_5 + P_8 + \cdots + P_{3n-1})$
$\quad +(b-1)(P_1 + P_4 + \ldots + P_{3n-2}) + 1$
c) $P_{3n} = (ac+1)(P_1 + P_4 + \ldots + P_{3n-2}) + (a-1)(P_0 + P_3 + \cdots + P_{3n-3})$.

**Proof.** Since, $P_{3n+2} = cP_{3n+1} + P_{3n}$ we get

$$P_2 = cP_1 + P_0$$
$$P_5 = cP_4 + P_3$$
$$P_8 = cP_7 + P_6$$
$$\vdots$$
$$P_{3n-1} = cP_{3n-2} + P_{3n-3}$$
$$P_{3n+2} = cP_{3n+1} + P_{3n}.$$

Adding all these equations we get

$$P_2 + P_5 + P_8 + \ldots + P_{3n+2} = c(P_1 + P_4 + \ldots + P_{3n+1}) + (P_0 + P_3 + \cdots + P_{3n}) \tag{2.1}$$

Again, $P_{3n} = aP_{3n-1} + P_{3n-2}$ gives

$$P_3 = aP_2 + P_1$$
$$P_6 = aP_5 + P_4$$
$$P_9 = aP_8 + P_7$$
$$\vdots$$
$$P_{3n-3} = aP_{3n-4} + P_{3n-5}$$
$$P_{3n} = aP_{3n-1} + P_{3n-2}$$

Adding these equations, we get

$$\begin{aligned} P_3 + P_6 + P_9 + \cdots + P_{3n} &= a(P_2 + P_5 + P_8 + \cdots + P_{3n-1}) \\ &\quad + (P_1 + P_4 + \ldots + P_{3n-2}) \end{aligned} \tag{2.2}$$

Also, since $P_{3n+1} = bP_{3n} + P_{3n-1}$, we have

$$P_4 = bP_3 + P_2$$
$$P_7 = bP_6 + P_5$$
$$P_{10} = bP_9 + P_8$$
$$\vdots$$

$P_{3n-2} = bP_{3n-3} + P_{3n-4}$
$P_{3n+1} = bP_{3n} + P_{3n-1}$

Adding all these equations we get

$$P_4 + P_7 + \cdots + P_{3n-2} + P_{3n+1} = b(P_3 + P_6 + \cdots + P_{3n}) + (P_2 + P_5 + \ldots + P_{3n-1}) \tag{2.3}$$

Using (2.3) in (2.1) we get

$$\begin{aligned} P_2 + P_5 + P_8 + \cdots + P_{3n+2} &= c\big(1 + b(P_3 + P_6 + \cdots + P_{3n}) + (P_2 + P_5 + \cdots + P_{3n-1})\big) \\ &\quad +(P_0 + P_3 + \cdots + P_{3n}) \\ &= c + (bc+1)(P_0 + P_3 + P_6 + \cdots + P_{3n}) + c(P_2 + P_5 + \ldots + P_{3n-1}) \end{aligned}$$

This finally gives

$$\begin{aligned} P_{3n+2} &= (bc+1)(P_0 + P_3 + P_6 + \cdots + P_{3n}) \\ &\quad +(c-1)(P_2 + P_5 + \ldots + P_{3n-1}) + c \end{aligned} \tag{2.4}$$

Other results can be proved accordingly by considering the pair of equations (2.1), (2.2) and further (2.1), (2.3) together.

**Lemma 2.3.**

a) $\sum_{i=0}^{n} P_{3i} = \frac{aP_{3n}(b-1)(c-1)-(P_{3n+1}-1)(ac+1)(c-1)+(P_{3n+2}-c)(ab+1)(ac+1)}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$.

b) $\sum_{i=0}^{n} P_{3i+1} = \frac{(P_{3n+1}-1)(a-1)(c-1)-(P_{3n+2}-c)(ab+1)(a-1)+aP_{3n}(ab+1)(bc+1)}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$.

c) $\sum_{i=0}^{n} P_{3i+2} = \frac{(P_{3n+2}-c)(a-1)(b-1)-aP_{3n}(bc+1)(b-1)+(P_{3n+1}-1)(ac+1)(bc+1)}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$.

**Proof.** We only prove result (a) here and other two results can be proved in a similar way. Using (b) and (c) of lemma 2.2, we get

$$\begin{aligned} aP_{3n} &= \frac{(ac+1)}{(b-1)}\{(P_{3n+1}-1) - (ab+1)(P_2 + P_5 + P_8 + \cdots + P_{3n-1})\} \\ &\quad +(a-1)(P_0 + P_3 + \cdots + P_{3n}) \end{aligned}$$

Then $aP_{3n} - \frac{(ac+1)}{(b-1)}(P_{3n+1}-1) - (a-1)(P_0 + P_3 + \cdots + P_{3n})$

$$= -\frac{(ac+1)(ab+1)}{(b-1)}(P_2 + P_5 + P_8 + \cdots + P_{3n-1})$$

Using (2.4) we get

$$\begin{aligned} &aP_{3n} - \frac{(ac+1)}{(b-1)}(P_{3n+1}-1) - (a-1)(P_0 + P_3 + \cdots + P_{3n}) \\ &= -\frac{(ab+1)(ac+1)}{(b-1)(c-1)}\big((P_{3n+2}-c) - (bc+1)(P_0 + P_3 + P_6 + \cdots + P_{3n})\big) \end{aligned}$$

Then, $\big((ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)\big)(P_0 + P_3 + \cdots + P_{3n})$

$$= a(b-1)(c-1)P_{3n} - (ac+1)(c-1)(P_{3n+1}-1)$$

$$+(ab+1)(ac+1)(P_{3n+2}-c).$$

Hence, $P_0 + P_3 + \cdots + P_{3n}$

$$= \frac{aP_{3n}(b-1)(c-1)-(P_{3n+1}-1)(ac+1)(c-1)+(P_{3n+2}-c)(ab+1)(ac+1)}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}.$$

We now obtain the sum of first $k$ trifurcating Fibonacci numbers.

**Theorem 2.4.**

$$\sum_{n=1}^{k} P_n = \frac{\begin{array}{c} \left\{(b-1)(c-1)-\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor(bc+1)(b-1)\right\}\left\lfloor\frac{4-\chi(k)}{3}\right\rfloor aP_{3\left\lfloor\frac{k}{3}\right\rfloor-3} \\ +\left\{\left\lfloor\frac{1+\chi(k)}{3}\right\rfloor(b-1)(c-1)+(ab+1)(bc+1)-\left\lfloor\frac{2+\chi(k)}{3}\right\rfloor(bc+1)(b-1)\right\}aP_{3\left\lfloor\frac{k}{3}\right\rfloor} \\ +\left\{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor[(a-1)(c-1)+(ac+1)(bc+1)]-(ac+1)(c-1)\right\}\left\lfloor\frac{4-\chi(k)}{3}\right\rfloor\left(P_{3\left\lfloor\frac{k}{3}\right\rfloor-2}-1\right) \\ +\left\lfloor\frac{2+\chi(k)}{3}\right\rfloor\left\{(a-1)(c-1)+(ac+1)(bc+1)-\left\lfloor\frac{1+\chi(k)}{3}\right\rfloor(ac+1)(c-1)\right\} \\ +\left\lfloor\frac{4-\chi(k)}{3}\right\rfloor\left\{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor(a-1)(b-1)+(ab+1)(ac+1)\right\}\left(P_{3\left\lfloor\frac{k}{3}\right\rfloor-1}-c\right) \\ +\left\{\left\lfloor\frac{2+\chi(k)}{3}\right\rfloor(a-1)(b-1)+\left\lfloor\frac{1+\chi(k)}{3}\right\rfloor(ab+1)(ac+1)-(ab+1)(a-1)\right\}\left(P_{3\left\lfloor\frac{k}{3}\right\rfloor+2}-c\right) \end{array}}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}.$$

**Proof.** We first obtain the value of $\sum_{n=1}^{k} P_n$ in three cases when $k$ is of the form $3m-2, 3m-1$ and $3m$ and then combine the results to obtain a single result.

For $k = 3m-2$, using the above lemma we get

$$\sum_{n=1}^{k} P_n = (P_3 + \cdots + P_{3m-3}) + (P_1 + P_4 + \cdots + P_{3m-2}) + (P_2 + P_5 + \ldots + P_{3m-4})$$

$$= \frac{\left\{\begin{array}{c} aP_{3m-3}(b-1)(c-1)-(P_{3m-2}-1)(ac+1)(c-1)+(P_{3m-1}-c)(ab+1)(ac+1)+ \\ (P_{3m-2}-1)(a-1)(c-1)-(P_{3m+2}-c)(ab+1)(a-1)+aP_{3m}(ab+1)(bc+1)+ \\ (P_{3m-1}-c)(a-1)(b-1)-aP_{3m-3}(bc+1)(b-1)+(P_{3m-2}-1)(ac+1)(bc+1) \end{array}\right\}}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$$

On simplification, we get,

$$\sum_{n=1}^{k} P_n = \frac{\left\{\begin{array}{c} aP_{3m-3}\{(b-1)(c-1)-(bc+1)(b-1)\}+(P_{3m-2}-1)\{(a-1)(c-1)+ \\ (ac+1)(bc+1)-(ac+1)(c-1)\}+(P_{3m-1}-c)\{(a-1)(b-1)+(ab+1)(ac+1)\} \\ -(P_{3m+2}-c)(ab+1)(a-1)+aP_{3m}(ab+1)(bc+1) \end{array}\right\}}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$$

Next, for the case $k = 3m-1$, we get

$$\sum_{n=1}^{k} P_n = (P_3 + \cdots + P_{3m-3}) + (P_1 + P_4 + \cdots + P_{3m-2}) + (P_2 + P_5 + \ldots + P_{3m-1})$$

$$=\frac{\begin{Bmatrix} aP_{3m-3}(b-1)(c-1)-(P_{3m-2}-1)(ac+1)(c-1)+(P_{3m-1}-c)(ab+1)(ac+1)+ \\ (P_{3m+1}-1)(a-1)(c-1)-(P_{3m+2}-c)(ab+1)(a-1)+aP_{3m}(ab+1)(bc+1)+ \\ (P_{3m+2}-c)(a-1)(b-1)-aP_{3m}(bc+1)(b-1)+(P_{3m+1}-1)(ac+1)(bc+1) \end{Bmatrix}}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$$

Simplifying this we get

$$\sum_{n=1}^{k} P_n = \frac{\begin{matrix} aP_{3m-3}(b-1)(c-1)+(P_{3m-1}-c)(ab+1)(ac+1)-(P_{3m-2}-1)(ac+1)(c-1)+ \\ \{(ac+1)(bc+1)+(a-1)(c-1)\}(P_{3m+1}-1)+(P_{3m+2}-c)\{(a-1)(b-1) \\ -(ab+1)(a-1)\}+aP_{3m}\{(ab+1)(bc+1)-(bc+1)(b-1)\} \end{matrix}}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$$

Finally, for $k = 3m$, we get

$$\sum_{n=1}^{k} P_n = (P_3 + \cdots + P_{3m}) + (P_1 + P_4 + \cdots + P_{3m-2})$$
$$+(P_2 + P_5 + \ldots + P_{3m-1})$$
$$=\frac{\begin{Bmatrix} aP_{3m}(b-1)(c-1)-(P_{3m+1}-1)(ac+1)(c-1)+(P_{3m+2}-c)(ab+1)(ac+1)+ \\ (P_{3m+1}-1)(a-1)(c-1)-(P_{mn+2}-c)(ab+1)(a-1)+aP_{3m}(ab+1)(bc+1)+ \\ (P_{3m+2}-c)(a-1)(b-1)-aP_{3m}(bc+1)(b-1)+(P_{3m+1}-1)(ac+1)(bc+1) \end{Bmatrix}}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$$

This on simplification gives

$$\sum_{n=1}^{k} P_n = \frac{\begin{matrix} aP_{3m}\{(b-1)(c-1)+(ab+1)(bc+1)-(bc+1)(b-1)\}+(P_{3m+1}-1)\{(a-1)(c-1)+(ac+1) \\ (bc+1)-(ac+1)(c-1)\}+(P_{3m+2}-c)\{(a-1)(b-1)+(ab+1)(ac+1)-(ab+1)(a-1)\} \end{matrix}}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$$

Combining all these three results, we finally get the required result.

The following are the interesting identities related with the summation of trifurcating Fibonacci numbers as well as its squares.

**Theorem 2.5.** $\sum_{k=0}^{n} a^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} P_k = P_n + P_{n+1} - 1.$

**Proof.** Using the definition of trifurcating Fibonacci numbers, we have

$$P_{3n} = aP_{3n-1} + P_{3n-2}\ ;\ P_{3n+1} = bP_{3n} + P_{3n-1};\ P_{3n+2} = cP_{3n+1} + P_{3n}$$

This can be written as

$$aP_{3n-1} = P_{3n} - P_{3n-2}\ ;\ bP_{3n} = P_{3n+1} - P_{3n-1}\ ;\ cP_{3n+1} = P_{3n+2} - P_{3n}$$

Thus, we have the following system of equations:

$cP_1 = P_2 - P_0$
$aP_2 = P_3 - P_1$
$bP_3 = P_4 - P_2$
$\vdots$
$aP_n = P_{n+1} - P_{n-1}$; if $n \equiv 0 (mod\ 3)$
$bP_n = P_{n+1} - P_{n-1}$; if $n \equiv 1 (mod\ 3)$
$P_n = P_{n+1} - P_{n-1}$; if $n \equiv 2 (mod\ 3)$

Adding all the above equations and using the fact that $P_0 = 0$ and $P_1 = 1$, we get $\sum_{k=0}^{n} a^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} P_k = P_n + P_{n+1} - 1.$

**Theorem 2.6.** $\sum_{k=0}^{n} a^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} P_k^2 = P_n P_{n+1}$.

**Proof.** We prove this result only for the case $n \equiv 0(mod\ 3)$ and remaining cases can be proved accordingly. We let $n = 3m$ and apply induction on $m$.

For $m = 1$, we have

$$\sum_{k=0}^{3} a^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} P_k^2 = bP_0^2 + cP_1^2 + aP_2^2 + bP_3^2$$

Since $P_0 = 0, P_1 = 1, P_2 = c$ and $P_3 = (1 + ac)$, we get

$$\sum_{k=0}^{3} a^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} = c(1 + ac) + b(1 + ac)^2 = P_3 P_4$$

We next assume that the result holds for some positive integer $m = l > 1$. That is let $\sum_{k=0}^{3l} a^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} P_k^2 = P_{3l} P_{3l+1}$ holds.

Now, $\sum_{k=0}^{3(l+1)} a^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} P_k^2$

$$\begin{aligned}
&= \sum_{k=0}^{3l} a^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} P_k^2 + cP_{3l+1}^2 + aP_{3l+2}^2 + bP_{3l+3}^2. \\
&= P_{3l}P_{3l+1} + cP_{3l+1}^2 + aP_{3l+2}^2 + bP_{3l+3}^2 \\
&= P_{3l+1}P_{3l+2} + aP_{3l+2}^2 + bP_{3l+3}^2 \\
&= P_{3l+2}P_{3l+3} + bP_{3l+3}^2 = P_{3(l+1)}P_{3(l+1)+1}
\end{aligned}$$

Thus, by induction, the result to be proved holds for every positive integer $n$.

# 3. Binet-like formula for the trifurcating Fibonacci sequence:

Generating function is used to solve the linear homogeneous recurrence relations. In this section, the generating function for the trifurcating Fibonacci sequence is derived and it is used to obtain Binet-like formula for these numbers. We first prove a result which will be needed to obtain the generating function of $P_n$.

**Lemma 3.1.** $P_{n+3} - (abc + a + b + c)P_n + P_{n-3} = 0$.

**Proof.** We prove the result by considering the three cases when $n = 3k, 3k + 1$ or $3k + 2$. We present the proof only for the case $n = 3k$ and other cases follows accordingly.

Using the definition of $P_n$, we get

$$P_{3k+3} = aP_{3k+2} + P_{3k+1} = a(cP_{3k+1} + P_{3k}) + bP_{3k} + P_{3k-1}$$

$$= acP_{3k+1} + (a+b)P_{3k} + cP_{3k-2} + P_{3k-3}.$$

Now, by definition we have $P_{3k+1} = bP_{3k} + P_{3k-1}$. Multiplying this by $ac$ we get $acP_{3k+1} = abcP_{3k} + acP_{3k-1} = abcP_{3k} + c(P_{3k} - P_{3k-2})$ Substituting this value in above equation we get

$$P_{3k+3} = (abc + a + b + c)P_{3k} - P_{3k-3}$$

Hence, $P_{3k+3} - (abc + a + b + c)P_{3k} + P_{3k-3} = 0$.

**Lemma 3.2.** The generating function of the subsequence $\{P_m\}_{m\geq 0}$ of $\{P_n\}_{n=0}^{\infty}$ is

(i) $f(x) = \frac{cx^2+x^5}{(1-(abc+a+b+c)x^3-x^6)}$ ; when $m \equiv 2(mod\ 3)$

(ii) $g(x) = \frac{x-ax^4}{(1-(abc+a+b+c)x^3-x^6)}$ ; when $m \equiv 1(mod\ 3)$

(iii) $h(x) = \frac{(ac+1)x^3}{(1-(abc+a+b+c)x^3-x^6)}$ ; when $m \equiv 0(mod\ 3)$.

**Proof**. We present the proof only for the case $m \equiv 2(mod\ 3)$. The other cases can be proved accordingly. We first let $f(x) = \sum_{n=1}^{\infty} P_{3n-1}x^{3n-1} = P_2x^2 + P_5x^5 + P_8x^8 + \cdots$. Then

$(abc + a + b + c)x^3\ f(x)$

$$= (abc + a + b + c)P_2x^5 + (abc + a + b + c)P_5x^8 + (abc + a + b + c)P_8x^{11} \ldots$$

Using lemma 3.1, we get

$(1 - (abc + a + b + c)x^3 - x^6)f(x)$

$= cx^2 + x^5 - \sum_{m=2}^{\infty}(p_{3k+2} - (abc + a + b + c)p_{3k-1} - p_{3k-4}) = cx^2 + x^5$.

Thus, $f(x) = \frac{cx^2+x^5}{(1-(abc+a+b+c)x^3-x^6)}$, as required.

The following result gives the generating function for $P_n$.

**Theorem 3.3.** The generating function for the trifurcating Fibonacci sequence $\{P_n\}$ is $F(x) = \frac{x(1+cx+x^2+acx^2-ax^3+x^4)}{1-(abc+a+b+c)x^3-x^6}$.

**Proof.** We begin with the formal power series representation of the generating function for $\{P_n\}$. Let

$$F(x) = P_0 + P_1x + P_2x^2 + \cdots + P_kx^k + \cdots = \sum_{m=0}^{\infty} P_mx^m$$

Then, $cxF(x) = cP_0x + cP_1x^2 + cP_2x^3 + \cdots + cP_kx^{k+1} + \cdots$

$$= \sum_{m=0}^{\infty} cP_mx^{m+1} = \sum_{m=1}^{\infty} cP_{m-1}x^m.$$

Also, $x^2F(x) = P_0x^2 + P_1x^3 + P_2x^2 + \cdots + P_kx^k + \cdots = \sum_{m=0}^{\infty} P_mx^m$

Since $P_{3k+2} = cP_{3k+1} + P_{3k}$, we get

$(1 - cx - x^2)F(x) = x + \sum_{n=1}^{\infty}(P_{3n} - cP_{3n-1} - P_{3n-2})x^{3n}$

$$+\sum_{n=1}^{\infty}(P_{3n+1}-cP_{3n}-P_{3n-1})x^{3n+1}$$

Since $P_{3k}=aP_{3k-1}+P_{3k-2}$ and $P_{3k+1}=bP_{3k}+P_{3k-1}$ we get
$(1-cx-x^2)F(x)=x+(a-c)\sum_{n=1}^{\infty}P_{3n-1}x^{3n}+(b-c)\sum_{n=1}^{\infty}P_{3n}x^{3n+1}$.
For convenience, we let $f(x)=\sum_{n=1}^{\infty}P_{3n-1}x^{3n}$ and $g(x)=\sum_{n=1}^{\infty}P_{3n}x^{3n}$
Using lemma 3.2 (a) and 3.2 (b) we get

$$(1-cx-x^2)F(x)=x+(a-c)\frac{(ac+1)x^3}{(1-(abc+a+b+c)x^3-x^6)}$$
$$+(b-c)\frac{cx^2+x^5}{(1-(abc+a+b+c)x^3-x^6)}$$

On simplification, we get the required result.

We now obtain the Binet-like formula for the sequence of trifurcating Fibonacci numbers.

**Theorem 3.4.** The terms of the trifurcating Fibonacci sequence $\{P_n\}$ are given by $P_n=\frac{\gamma(n)\alpha^{\lfloor\frac{n}{3}\rfloor}-\delta(n)\beta^{\lfloor\frac{n}{3}\rfloor}}{\alpha-\beta}$
where $\gamma(n)=\left(\left\lfloor\frac{\chi(n)+2}{3}\right\rfloor c^{\left\lfloor\frac{\chi(n)+1}{3}\right\rfloor}\alpha+(-1)^{\chi(n)}a^{\left\lfloor\frac{4-\chi(n)}{3}\right\rfloor}c^{\left\lfloor1-\frac{\chi(n)}{3}\right\rfloor}+\left\lfloor1-\frac{\chi(n)}{3}\right\rfloor\right)$
and $\delta(n)=\left(\left\lfloor\frac{\chi(n)+2}{3}\right\rfloor c^{\left\lfloor\frac{\chi(n)+1}{3}\right\rfloor}\beta+(-1)^{\chi(n)}a^{\left\lfloor\frac{4-\chi(n)}{3}\right\rfloor}c^{\left\lfloor1-\frac{\chi(n)}{3}\right\rfloor}+\left\lfloor1-\frac{\chi(n)}{3}\right\rfloor\right)$
with $\alpha=\frac{u+\sqrt{u^2+4}}{2},\beta=\frac{u-\sqrt{u^2+4}}{2},u=a+b+c+abc$ and

$$\chi(n)=\begin{cases}0 & if\ n\equiv 0(mod\ 3)\\ 1 & if\ n\equiv 1(mod\ 3)\\ 2 & if\ n\equiv 2(mod\ 3)\end{cases}.$$

**Proof.** From the generating function of $\{P_n\}$, we have

$$F(x)=-\frac{x(1+cx+(1+ac)x^2-ax^3+x^4)}{(x^3+\alpha)(x^3+\beta)}$$

This can be rewritten as

$$F(x)=-\frac{1}{\alpha-\beta}\left[\frac{(1+ac)\alpha-(a\alpha+1)x+(\alpha-c)x^2}{(x^3+\alpha)}-\frac{(1+ac)\beta-(a\beta+1)x+(\beta-c)x^2}{(x^3+\beta)}\right]$$

Using McLaurin series expansion, we get

$$F(x)=-\frac{1}{\alpha-\beta}\begin{bmatrix}\sum_{n=0}^{\infty}\frac{(-1)^n(1+ac)\alpha}{\alpha^{n+1}}x^{3n}-\sum_{n=0}^{\infty}\frac{(-1)^n(a\alpha+1)}{\alpha^{n+1}}x^{3n+1}\\ +\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha-c)}{\alpha^{n+1}}x^{3n+2}-\sum_{n=0}^{\infty}\frac{(-1)^n(1+ac)\beta}{\beta^{n+1}}x^{3n}\\ +\sum_{n=0}^{\infty}\frac{(-1)^n(a\beta+1)}{\beta^{n+1}}x^{3n+1}-\sum_{n=0}^{\infty}\frac{(-1)^n(\beta-c)}{\beta^{n+1}}x^{3n+2}\end{bmatrix}$$

$$= -\frac{1}{\alpha-\beta}\left[\begin{array}{c} \sum_{n=0}^{\infty}(-1)^n(1+ac)\left(\frac{\beta^n-\alpha^n}{(\alpha\beta)^n}\right)x^{3n} \\ -\sum_{n=0}^{\infty}(-1)^n\left(\frac{(a\alpha+1)\beta^{n+1}-(a\beta+1)\alpha^{n+1}}{(\alpha\beta)^{n+1}}\right)x^{3n+1} \\ +\sum_{n=0}^{\infty}(-1)^n\left(\frac{(\alpha-c)\beta^{n+1}-(\beta-c)\alpha^{n+1}}{(\alpha\beta)^{n+1}}\right)x^{3n+2} \end{array}\right]$$

Now, if $\alpha,\beta$ are the roots of $1-(a+b+c+abc)x-x^2=0$ then

$$\alpha=\frac{u+\sqrt{u^2+4}}{2},\beta=\frac{u-\sqrt{u^2+4}}{2}.$$

If we let $u=a+b+c+abc$, then it is easy to observe that $\alpha\beta=-1,\alpha+\beta=u$ and $\alpha-\beta=\sqrt{u^2+4}$ .

Then

$$F(x)=\left[\begin{array}{c} \sum_{n=0}^{\infty}(1+ac)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)x^{3n}-\sum_{n=0}^{\infty}\left(\frac{(a\beta+1)\alpha^{n+1}-(a\alpha+1)\beta^{n+1}}{\alpha-\beta}\right)x^{3n+1} \\ +\sum_{n=0}^{\infty}\left(\frac{(\beta-c)\alpha^{n+1}-(\alpha-c)\beta^{n+1}}{\alpha-\beta}\right)x^{3n+2} \end{array}\right]$$

Thus,

$$F(x)=\sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(\begin{array}{c} \left(\begin{array}{c}\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor+(-1)^{\chi(n)}a^{\left\lfloor\frac{4-\chi(n)}{3}\right\rfloor}c^{\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor} \\ +\left\lfloor\frac{\chi(n)+2}{3}\right\rfloor c^{\left\lfloor\frac{\chi(n)+1}{3}\right\rfloor}\alpha\end{array}\right)\alpha^{\left\lfloor\frac{n}{3}\right\rfloor} \\ -\left(\begin{array}{c}\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor+(-1)^{\chi(n)}a^{\left\lfloor\frac{4-\chi(n)}{3}\right\rfloor}c^{\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor} \\ +\left\lfloor\frac{\chi(n)+2}{3}\right\rfloor c^{\left\lfloor\frac{\chi(n)+1}{3}\right\rfloor}\beta\end{array}\right)\beta^{\left\lfloor\frac{n}{3}\right\rfloor}\end{array}\right)x^n$$

For convenience if we write $\chi(n)=\begin{cases}0 & if\ n\equiv 0(mod\ 3)\\ 1 & if\ n\equiv 1(mod\ 3)\\ 2 & if\ n\equiv 2(mod\ 3)\end{cases}$ and

$\gamma(n)=\left(\left\lfloor\frac{\chi(n)+2}{3}\right\rfloor c^{\left\lfloor\frac{\chi(n)+1}{3}\right\rfloor}\alpha+(-1)^{\chi(n)}a^{\left\lfloor\frac{4-\chi(n)}{3}\right\rfloor}c^{\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor}+\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor\right)$,

$\delta(n)=\left(\left\lfloor\frac{\chi(n)+2}{3}\right\rfloor c^{\left\lfloor\frac{\chi(n)+1}{3}\right\rfloor}\beta+(-1)^{\chi(n)}a^{\left\lfloor\frac{4-\chi(n)}{3}\right\rfloor}c^{\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor}+\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor\right)$ then

$F(x)$ can be written as $F(x)=\sum_{n=0}^{\infty}\frac{\gamma(n)\alpha^{\left\lfloor\frac{n}{3}\right\rfloor}-\delta(n)\beta^{\left\lfloor\frac{n}{3}\right\rfloor}}{\alpha-\beta}x^n$

This gives $P_n=\frac{\gamma(n)\alpha^{\left\lfloor\frac{n}{3}\right\rfloor}-\delta(n)\beta^{\left\lfloor\frac{n}{3}\right\rfloor}}{\alpha-\beta}$ , as desired.

The following results are the easy consequence from this theorem.

**Corollary 3.5.** (i) $P_{3n}=(1+ac)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$

(ii) $P_{3n+1} = \left(\frac{(a\beta+1)\alpha^{n+1}-(a\alpha+1)\beta^{n+1}}{\alpha-\beta}\right)$

(iii) $P_{3n+2} = \left(\frac{(\beta-c)\alpha^{n+1}-(\alpha-c)\beta^{n+1}}{\alpha-\beta}\right)$.

# 4. Some more identities relating trifurcating Fibonacci numbers:

In this section, we use the above Binet-like formula to derive some interesting properties for the terms of trifurcating Fibonacci sequence. If we let $w = \gamma(n)\delta(n)$, then we observe that

$$w = \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor^2 + 2(-1)^{\chi(n)} \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor a^{\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{\left\lfloor 1-\frac{\chi(n)}{3} \right\rfloor}$$
$$+ \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \left\lfloor \frac{\chi(n)+2}{3} \right\rfloor c^{\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor}(\alpha+\beta) + a^{2\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{2\left\lfloor 1-\frac{\chi(n)}{3} \right\rfloor}$$
$$+(-1)^{\chi(n)} \left\lfloor \frac{\chi(n)+2}{3} \right\rfloor a^{\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor + \left\lfloor 1-\frac{\chi(n)}{3} \right\rfloor}(\alpha+\beta) + \left\lfloor \frac{\chi(n)+2}{3} \right\rfloor^2 c^{2\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor}\alpha\beta$$

This on simplification gives the value of $w = \gamma(n)\delta(n)$ as

$$w = \begin{cases} (1+ac)^2 & ; \text{if } n \equiv 0(mod\ 3) \\ (1+ac)(1+b) & ; \text{if } n \equiv 1(mod\ 3) \\ (1+ac)(1+bc) & ; \text{if } n \equiv 2(mod\ 3) \end{cases}$$

This can be further written as $w = \begin{cases} P_3^2 & ; \text{if } n \equiv 0(mod\ 3) \\ (1+b)P_3 & ; \text{if } n \equiv 1(mod\ 3) \\ (1+bc)P_3 & ; \text{if } n \equiv 2(mod\ 3) \end{cases}$

We first obtain an identity for the terms of $\{P_n\}$, which is analogous to that of Catalan's identity for Fibonacci numbers.

**Theorem 4.1.** For any two nonnegative integers $k$ and $r \left(\le \frac{k}{3}\right)$, we have

$$P_{k-3r}P_{k+3r} - P_k^2 = (-1)^{l-r+1} w \left(\frac{P_{3r}}{P_3}\right)^2.$$

**Proof.** If $k \equiv 0(mod\ 3)$ then taking $k = 3l$ and using corollary 3.5, we get

$P_{3l-3r}P_{3l+3r} - P_{3l}^2$

$$= (1+ac)\left(\frac{\alpha^{l-r}-\beta^{l-r}}{\alpha-\beta}\right)(1+ac)\left(\frac{\alpha^{l+r}-\beta^{l+r}}{\alpha-\beta}\right) - (1+ac)^2\left(\frac{\alpha^l-\beta^l}{\alpha-\beta}\right)^2$$
$$= \frac{(1+ac)^2}{(\alpha-\beta)^2}\{(\alpha^{l-r}-\beta^{l-r})(\alpha^{l+r}-\beta^{l+r}) - (\alpha^l-\beta^l)^2\}$$
$$= \frac{(1+ac)^2}{(\alpha-\beta)^2}\left\{\alpha^{2l} - (-1)^l\left(\frac{\beta}{\alpha}\right)^r - (-1)^l\left(\frac{\alpha}{\beta}\right)^r + \beta^{2l} - (\alpha^{2l} - 2(-1)^l + \beta^{2l})\right\}$$

$$= \frac{(1+ac)^2}{(\alpha-\beta)^2}(-1)^{l-r+1}\{\alpha^{2r} - 2(-1)^r + \beta^{2r}\}$$
$$= \frac{(1+ac)^2}{(\alpha-\beta)^2}(-1)^{l-r+1}(\alpha^r - \beta^r)^2 = (-1)^{l-r+1}\left(\frac{(1+ac)(\alpha^r-\beta^r)}{(\alpha-\beta)}\right)^2$$

Thus, $P_{3l-3r}P_{3l+3r} - P_{3l}^2 = (-1)^{l-r+1}P_{3r}^2$

Next, if we let $k \equiv 1(mod\ 3)$ then by considering $k = 3l + 1$, we get

$$P_{3l-3r+1}P_{3l+3r+1} - P_{3l+1}^2$$
$$= \left(\frac{(a\beta+1)\alpha^{l-r+1}-(a\alpha+1)\beta^{l-r+1}}{\alpha-\beta}\right)\left(\frac{(a\beta+1)\alpha^{l+r+1}-(a\alpha+1)\beta^{l+r+1}}{\alpha-\beta}\right)$$
$$-\left(\frac{(a\beta+1)\alpha^{l+1}-(a\alpha+1)\beta^{l+1}}{\alpha-\beta}\right)^2$$
$$= \frac{1}{(\alpha-\beta)^2}\begin{bmatrix} a^2(\alpha^{l-r}-\beta^{l-r})(\alpha^{l+r}-\beta^{l+r}) \\ -a(\alpha^{l-r}-\beta^{l-r})(\alpha^{l+r+1}-\beta^{l+r+1}) \\ -a(\alpha^{l-r+1}-\beta^{l-r+1})(\alpha^{l+r}-\beta^{l+r}) \\ +(\alpha^{l-r+1}-\beta^{l-r+1})(\alpha^{l+r+1}-\beta^{l+r+1}) \\ -\{a^2(\alpha^l-\beta^l)^2 - 2a(\alpha^l-\beta^l)(\alpha^{l+1}-\beta^{l+1}) + (\alpha^{l+1}-\beta^{l+1})^2\} \end{bmatrix}$$
$$= \frac{1}{(\alpha-\beta)^2}\begin{bmatrix} a^2(-1)^{l-r+1}[\beta^{2r}+\alpha^{2r}+2(-1)^r] \\ -(-1)^{l-r+1}[\beta^{2r}+\alpha^{2r}+2(-1)^r] \\ -a(-1)^{l+1}\left[\left(\frac{\beta}{\alpha}\right)^r(\alpha+\beta)+\left(\frac{\alpha}{\beta}\right)^r(\alpha+\beta)+2(\alpha+\beta)\right] \end{bmatrix}$$
$$= \frac{(-1)^{l-r+1}}{(\alpha-\beta)^2}\{a^2(\alpha^r-\beta^r)^2 - au[\beta^{2r}+\alpha^{2r}+2(-1)^r] - (\alpha^r-\beta^r)^2\}$$
$$= \frac{(-1)^{l-r+1}}{(\alpha-\beta)^2}(\alpha^r-\beta^r)^2\{a^2-au-1\} = \frac{(-1)^{l-r+1}}{(1+ac)^2}\{a^2-au-1\}P_{3r}^2$$

Thus, $P_{3l-3r+1}P_{3l+3r+1} - P_{3l+1}^2 = (-1)^{l-r+1}wP_{3r}^2P_3^{-2}$.

Finally, if $k \equiv 2(mod\ 3)$ then by considering $k = 3l + 2$, we get

$$P_{3l-3r+2}P_{3l+3r+2} - P_{3l+2}^2$$
$$= \left(\frac{(\beta-c)\alpha^{l-r+1}-(\alpha-c)\beta^{l-r+1}}{\alpha-\beta}\right)\left(\frac{(\beta-c)\alpha^{l+r+1}-(\alpha-c)\beta^{l+r+1}}{\alpha-\beta}\right) - \left(\frac{(\beta-c)\alpha^{l+1}-(\alpha-c)\beta^{l+1}}{\alpha-\beta}\right)^2.$$
$$= \frac{1}{(\alpha-\beta)^2}\begin{bmatrix} (\alpha^{l-r}-\beta^{l-r})(\alpha^{l+r}-\beta^{l+r}) + c(\alpha^{l-r}-\beta^{l-r})(\alpha^{l+r+1}-\beta^{l+r+1}) \\ +c(\alpha^{l-r+1}-\beta^{l-r+1})(\alpha^{l+r}-\beta^{l+r}) \\ +c^2(\alpha^{l-r+1}-\beta^{l-r+1})(\alpha^{l+r+1}-\beta^{l+r+1}) \\ -(\alpha^{l-r}-\beta^{l-r})^2 - (\alpha^l-\beta^l)(\alpha^{l+1}-\beta^{l+1}) - c^2(\alpha^{l+1}-\beta^{l+1})^2 \end{bmatrix}$$
$$= \frac{1}{(\alpha-\beta)^2}\begin{bmatrix} (-1)^{l-r+1}[\beta^{2r}+\alpha^{2r}+2(-1)^r] \\ -c^2(-1)^{l-r+1}[\beta^{2r}+\alpha^{2r}+2(-1)^r] \\ -c(-1)^{l+1}\left[\left(\frac{\beta}{\alpha}\right)^r(\alpha+\beta)+\left(\frac{\alpha}{\beta}\right)^r(\alpha+\beta)-2(\alpha+\beta)\right] \end{bmatrix}$$
$$= \frac{(-1)^{l-r+1}}{(\alpha-\beta)^2}\{(\alpha^r-\beta^r)^2 - cu[\beta^{2r}+\alpha^{2r}+2(-1)^r] - c^2(\alpha^r-\beta^r)^2\}$$

$= \frac{(-1)^{l-r+1}}{(\alpha-\beta)^2}(\alpha^r - \beta^r)^2\{1 + cu - c^2\} = \frac{(-1)^{l-r+1}}{(1+ac)^2}\{1 + cu - c^2\}P_{3r}^2$

Thus, $P_{3l-3r+2}P_{3l+3r+2} - P_{3l+2}^2 = (-1)^{l-r+1}wP_{3r}^2P_3^{-2}$

Hence, in general we write $P_{k-3r}P_{k+3r} - P_k^2 = (-1)^{l-r+1}wP_{3r}^2P_3^{-2}$.

The following identity is analogous to the Cassini's identity for Fibonacci numbers which follows easily from the above theorem.

**Corollary 4.2.** $P_{k-3}P_{k+3} - P_k^2 = (-1)^n w$ for any integer $k \geq 3$.

The following identity is similar to d'Ocagne's identity of Fibonacci numbers.

**Theorem 4.3.** $P_mP_{n+3} - P_{m+3}P_n = (-1)^n P_{m-n}\left(\frac{w}{P_3}\right)$ where $m, n$ are nonnegative integers such that $m \geq n$ and $m \equiv n(mod\ 3)$.

**Proof.** Since $m \equiv n(mod\ 3)$, we first let both $m, n$ to be of the form $3j, 3k$ respectively, for positive integers $j$ and $k \leq j$. Then

$$\begin{aligned}
P_{3j}P_{3k+3} - P_{3j+3}P_{3k} &= (1 + ac)\left(\frac{\alpha^j-\beta^j}{\alpha-\beta}\right)(1 + ac)\left(\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}\right) \\
&\quad -(1 + ac)\left(\frac{\alpha^{j+1}-\beta^{j+1}}{\alpha-\beta}\right)(1 + ac)\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right) \\
&= \frac{(1+ac)^2}{(\alpha-\beta)^2}\{\alpha^j\beta^k(\alpha - \beta) - \alpha^k\beta^j(\alpha - \beta)\} \\
&= (1 + ac)^2(-1)^k\left(\frac{\alpha^{j-k}-\beta^{j-k}}{\alpha-\beta}\right) = (-1)^{\left\lfloor\frac{n-1}{3}\right\rfloor}wP_{m-n}P_3^{-1}
\end{aligned}$$

If $m, n$ are of the form $3j + 1$ and $3k + 1$ respectively, then for positive integers $j$ and $k \leq j$, we have

$$\begin{aligned}
&P_{3j+1}P_{3k+3+1} - P_{3j+3+1}P_{3k+1} \\
&= \begin{bmatrix} \left(\frac{(a\beta+1)\alpha^{j+1}-(a\alpha+1)\beta^{j+1}}{\alpha-\beta}\right)\left(\frac{(a\beta+1)\alpha^{k+2}-(a\alpha+1)\beta^{k+2}}{\alpha-\beta}\right) \\ -\left(\frac{(a\beta+1)\alpha^{j+2}-(a\alpha+1)\beta^{j+2}}{\alpha-\beta}\right)\left(\frac{(a\beta+1)\alpha^{k+1}-(a\alpha+1)\beta^{k+1}}{\alpha-\beta}\right) \end{bmatrix} \\
&= \frac{1}{(\alpha-\beta)^2}\begin{bmatrix} a^2\left(\alpha^j\beta^k(\alpha - \beta) - \alpha^k\beta^j(\alpha - \beta)\right) \\ -a\left(\alpha^j\beta^k(\alpha^2 - \beta^2) - \alpha^k\beta^j(\alpha^2 - \beta^2)\right) \\ -(\alpha^j\beta^k(\alpha - \beta) - \alpha^k\beta^j(\alpha - \beta) \end{bmatrix} \\
&= \frac{1}{(\alpha-\beta)}\left[(a^2 - au - 1)\alpha^k\beta^k\left(\alpha^{j-k} - \beta^{j-k}\right)\right] \\
&= (-1)^k(a^2 - au - 1)\left(\frac{\alpha^{j-k}-\beta^{j-k}}{\alpha-\beta}\right) = (-1)^{\left\lfloor\frac{n-1}{3}\right\rfloor}wP_{m-n}P_3^{-1}
\end{aligned}$$

Finally, if $m, n$ are of the form $3j+2$ and $3k+2$ respectively, then for positive integers $j$ and $k \leq j$, we have

$$P_{3j+2}P_{3k+3+2} - P_{3j+3+2}P_{3k+2} = \left(\frac{(\beta-c)\alpha^{j+1}-(\alpha-c)\beta^{j+1}}{\alpha-\beta}\right)\left(\frac{(\beta-c)\alpha^{k+2}-(\alpha-c)\beta^{k+2}}{\alpha-\beta}\right)$$
$$- \left(\frac{(\beta-c)\alpha^{j+2}-(\alpha-c)\beta^{j+2}}{\alpha-\beta}\right)\left(\frac{(\beta-c)\alpha^{k+1}-(\alpha-c)\beta^{k+1}}{\alpha-\beta}\right)$$
$$= \frac{1}{(\alpha-\beta)^2}\begin{bmatrix}\alpha^j\beta^k(\alpha-\beta) - \alpha^k\beta^j(\alpha-\beta) - c^2(\alpha^j\beta^k(\alpha-\beta) - \alpha^k\beta^j(\alpha-\beta)) \\ +c\left(\alpha^j\beta^k(\alpha^2-\beta^2) - \alpha^k\beta^j(\alpha^2-\beta^2)\right)\end{bmatrix}$$
$$= \frac{1}{(\alpha-\beta)}\left[(1+cu-c^2)\alpha^k\beta^k\left(\alpha^{j-k}-\beta^{j-k}\right)\right]$$
$$= (-1)^k(a^2-au-1)\left(\frac{\alpha^{j-k}-\beta^{j-k}}{\alpha-\beta}\right) = (-1)^{\left\lfloor\frac{n-1}{3}\right\rfloor}wP_{m-n}P_3^{-1}$$

Combining all the above cases, we finally get

$$P_mP_{n+3} - P_{m+3}P_n = (-1)^{\left\lfloor\frac{n-1}{3}\right\rfloor}wP_{m-n}P_3^{-1}.$$

We use above Binet-like formula to prove the following identity which combines four consecutive $P_n$'s.

**Theorem 4.4.**

$$a^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor}b^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor}c^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor}P_{k+1}^2 + a^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor}b^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor}c^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor}P_{k+2}^2$$
$$= P_{k+2}P_{k+3} - P_kP_{k+1}.$$

**Proof.** We prove the result only for the case $k = 3n$ and the remaining cases $k = 3n+1$ and $k = 3n+2$ can be handled accordingly.

Now, $cP_{3n+1}^2 + aP_{3n+2}^2$

$$= c\left\{\left(\frac{(a\beta+1)\alpha^{n+1}-(a\alpha+1)\beta^{n+1}}{\alpha-\beta}\right)\right\}^2 + a\left\{\left(\frac{(\beta-c)\alpha^{n+1}-(\alpha-c)\beta^{n+1}}{\alpha-\beta}\right)\right\}^2.$$
$$= \frac{1}{(\alpha-\beta)^2}\begin{bmatrix} a^2c(\alpha^{2n}-2(-1)^n+\beta^{2n}) \\ -2ac(\alpha^{2n+1}-(-1)^n\alpha-(-1)^n\beta+\beta^{2n+1}) \\ +c(\alpha^{2n+2}-2(-1)^{n+1}+\beta^{2n+2}) \\ +2ac(\alpha^{2n+1}-(-1)^n\beta+(-1)^n\alpha+\beta^{2n+1}) \\ +c^2(\alpha^{2n+2}-2(-1)^{n+1}+\beta^{2n+2}) + a(\alpha^{2n}-2(-1)^n+\beta^{2n})\end{bmatrix}$$
$$= \frac{(1+ac)}{(\alpha-\beta)^2}\{a(\alpha^n-\beta^n)^2 + c(\alpha^{n+1}-\beta^{n+1})^2\}$$

Also, $P_{3n+2}P_{3n+3} - P_{3n}P_{3n+1}$

$$= \left(\frac{(\beta-c)\alpha^{n+1}-(\alpha-c)\beta^{n+1}}{\alpha-\beta}\right)(1+ac)\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)$$
$$-(1+ac)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\left(\frac{(a\beta+1)\alpha^{n+1}-(a\alpha+1)\beta^{n+1}}{\alpha-\beta}\right)$$

$$= \frac{(1+ac)}{(\alpha-\beta)^2}\begin{bmatrix} (\alpha^n-\beta^n)(\alpha^{n+1}-\beta^{n+1})+c(\alpha^{n+1}-\beta^{n+1})^2 \\ +a(\alpha^n-\beta^n)^2-(\alpha^n-\beta^n)(\alpha^{n+1}-\beta^{n+1}) \end{bmatrix}$$

$$= \frac{(1+ac)}{(\alpha-\beta)^2}\{a(\alpha^n-\beta^n)^2+c(\alpha^{n+1}-\beta^{n+1})^2\}$$

This proves the required result.

# 4. Conclusions

In this paper, we considered the sequence of ‘trifurcating Fibonacci numbers’ and obtained its Binet-like formula. We also obtained the analogous of Cassini’s identity, Catalan’s identity, d’Ocagne’s identity and some fundamental identities for the terms of this sequence.

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