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The sum of the reduced harmonic series generated by four primes determined analytically and computed by using CAS Maple

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Abstract

The paper deals with the reduced harmonic series generated by four primes. A formula for the sum of these convergent reduced harmonic series is derived. These sums (concretely 42 from all 12650 sums generated by four different primes smaller than 100) are computed by using the computer algebra system Maple 15 and its programming language, although the formula is valid not only for four arbitrary primes, but also for four integers. We can say that the reduced harmonic series generated by four primes (or by four integers) belong to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically by means of a simple formula.

Key words: reduced harmonic series, sum of convergent infinite series, computer algebra system Maple.

MSC2010: 40A05, 65B10.

1 Introduction

This paper is inspired by a small study material from the *Berkeley Math Circle* (see [5]) and it is a free continuation of the papers [2], [3], and [4].

In two last mentioned papers the sum $S(a,b)$ of the convergent reduced harmonic series

$$\begin{aligned}G(a,b)=\frac{1}{a}+\frac{1}{b}+\left(\frac{1}{a^2}+\frac{1}{ab}+\frac{1}{b^2}\right)+\left(\frac{1}{a^3}+\frac{1}{a^2b}+\frac{1}{ab^2}+\frac{1}{b^3}\right)+\\+\left(\frac{1}{a^4}+\frac{1}{a^3b}+\frac{1}{a^2b^2}+\frac{1}{ab^3}+\frac{1}{b^4}\right)+\left(\frac{1}{a^5}+\frac{1}{a^4b}+\cdots+\frac{1}{b^5}\right)+\cdots,\end{aligned}\tag{1}$$

generated by two primes a and b, and the sum $S(a,b,c)$ of the convergent reduced harmonic series

$$\begin{aligned}G(a,b,c)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)+\\+\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{a^2b}+\frac{1}{a^2c}+\frac{1}{b^2a}+\frac{1}{b^2c}+\frac{1}{c^2a}+\frac{1}{c^2b}+\frac{1}{abc}\right)+\\+\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{a^3b}+\frac{1}{a^3c}+\frac{1}{b^3a}+\frac{1}{b^3c}+\frac{1}{c^3a}+\frac{1}{c^3b}+\right.\\\left.+\frac{1}{a^2b^2}+\frac{1}{a^2c^2}+\frac{1}{b^2c^2}+\frac{1}{a^2bc}+\frac{1}{b^2ac}+\frac{1}{c^2ab}\right)+\cdots,\end{aligned}\tag{2}$$

generated by three primes a, b, and c, were derived and also computed for primes less than 100. It was shown that for arbitrary two primes (or integers) a and b it holds the formula

$$S(a,b)=\frac{a+b-1}{(a-1)(b-1)}\tag{3}$$

and for arbitrary three primes (or integers) a, b, and c it holds the formula

$$S(a,b,c)=\frac{(a+b-1)(c-1)+ab}{(a-1)(b-1)(c-1)}.\tag{4}$$

In the paper [2] the sum S of all the unit fractions that have denominators with only factors from the set $\{2,7,11,13\}$ was determined. This sum was calculated by using numeric method based on the programming language in the computer algebra system Maple 15 and also by analytical method. By these both attempts was obtained the same result: $S=1.780\overline{5}$.

In this paper we shall deal with a certain variant of these two problems – the determination of the sum of the reduced harmonic series generated by four primes.

Let us recall the basic terms and notions. The *harmonic series* is the sum of reciprocals of all natural numbers (except zero), so this is the series

in the form $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$.
The divergence of this series can be easily proved e.g. by using the integral test or the comparison test of convergence.

The *reduced harmonic series* is defined as the subseries of the harmonic series, which arises by omitting some its terms. As an example of the reduced harmonic series we can take the series formed by reciprocals of primes and number one $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots$.
This reduced harmonic series is divergent. The first proof of its divergence was made by Leonhard Euler (15.4.1707–18.9.1783) in 1737 (see e.g. [1]).

2 Reduced harmonic series generated by four primes

Now, let us consider the reduced harmonic series $G(a,b,c,d)$ below, generated by four primes a, b, c, d:

$$\begin{aligned} G(a,b,c,d) = {} & \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \right. \\ & \left. + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd}\right) + \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} + \frac{1}{a^2b} + \frac{1}{a^2c} + \frac{1}{a^2d} + \right. \\ & + \frac{1}{b^2a} + \frac{1}{b^2c} + \frac{1}{b^2d} + \frac{1}{c^2a} + \frac{1}{c^2b} + \frac{1}{c^2d} + \frac{1}{d^2a} + \frac{1}{d^2b} + \frac{1}{d^2c} + \\ & \left. + \frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}\right) + \left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} + \right. \\ & + \frac{1}{a^3b} + \frac{1}{a^3c} + \frac{1}{a^3d} + \frac{1}{b^3a} + \frac{1}{b^3c} + \frac{1}{b^3d} + \frac{1}{c^3a} + \frac{1}{c^3b} + \frac{1}{c^3d} + \\ & + \frac{1}{d^3a} + \frac{1}{d^3b} + \frac{1}{d^3c} + \frac{1}{a^2bc} + \frac{1}{a^2bd} + \frac{1}{a^2cd} + \frac{1}{b^2ac} + \frac{1}{b^2ad} + \\ & + \frac{1}{b^2cd} + \frac{1}{c^2ab} + \frac{1}{c^2ad} + \frac{1}{c^2bd} + \frac{1}{d^2ab} + \frac{1}{d^2ac} + \frac{1}{d^2bc} + \\ & \left. + \frac{1}{a^2b^2} + \frac{1}{a^2c^2} + \frac{1}{a^2d^2} + \frac{1}{b^2c^2} + \frac{1}{b^2d^2} + \frac{1}{c^2d^2} + \frac{1}{abcd}\right) + \cdots . \end{aligned} \tag{5}$$

Analogously as in the cases of the reduced harmonic series generated by two and three primes, we assume that its sum $S(a,b,c,d)$ is finite, so the series (5) converges. Because all its terms are positive, then the series (5) converges absolutely and so we can rearrange it.

For easier determining the sum $S(a,b,c,d)$ of the series $G(a,b,c,d)$ it is necessary to rearrange it and divide it into ten subseries $G(a)$, $G(b)$, $G(c)$, $G(d)$, $G(ab)$, $G(ac)$, $G(ad)$, $G(bc)$, $G(bd)$, and $G(cd)$, where

$$G(a) = \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \cdots = \frac{1}{a}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \cdots\right), \tag{6}$$

$$G(b) = \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \frac{1}{b^4} + \cdots = \frac{1}{b}\left(1 + \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \cdots\right), \tag{7}$$

$$G(c) = \frac{1}{c} + \frac{1}{c^2} + \frac{1}{c^3} + \frac{1}{c^4} + \cdots = \frac{1}{c}\left(1 + \frac{1}{c} + \frac{1}{c^2} + \frac{1}{c^3} + \cdots\right), \tag{8}$$

$$G(d) = \frac{1}{d} + \frac{1}{d^2} + \frac{1}{d^3} + \frac{1}{d^4} + \cdots = \frac{1}{d}\left(1 + \frac{1}{d} + \frac{1}{d^2} + \frac{1}{d^3} + \cdots\right), \tag{9}$$

$$\begin{aligned} G(ab) = {} & \frac{1}{ab} + \frac{1}{a^2b} + \frac{1}{b^2a} + \frac{1}{abc} + \frac{1}{abd} + \frac{1}{a^3b} + \frac{1}{b^3a} + \frac{1}{a^2bc} + \frac{1}{a^2bd} + \\ & + \frac{1}{b^2ac} + \frac{1}{b^2ad} + \frac{1}{c^2ab} + \frac{1}{d^2ab} + \frac{1}{a^2b^2} + \frac{1}{abcd} + \cdots = \\ = {} & \frac{1}{ab}\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \right. \\ & \left. + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd} + \cdots\right), \end{aligned} \tag{10}$$

$$\begin{aligned} G(ac) = {} & \frac{1}{ac} + \frac{1}{a^2c} + \frac{1}{c^2a} + \frac{1}{acd} + \frac{1}{a^3c} + \\ & + \frac{1}{c^3a} + \frac{1}{a^2cd} + \frac{1}{c^2ad} + \frac{1}{d^2ac} + \frac{1}{a^2c^2} + \cdots = \\ = {} & \frac{1}{ac}\left(1 + \frac{1}{a} + \frac{1}{c} + \frac{1}{d} + \frac{1}{a^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{ac} + \frac{1}{ad} + \frac{1}{cd} + \cdots\right), \end{aligned} \tag{11}$$

$$\begin{aligned} G(ad) = {} & \frac{1}{ad} + \frac{1}{a^2d} + \frac{1}{d^2a} + \frac{1}{a^3d} + \frac{1}{d^3a} + \frac{1}{a^2d^2} + \cdots = \\ = {} & \frac{1}{ad}\left(1 + \frac{1}{a} + \frac{1}{d} + \frac{1}{a^2} + \frac{1}{d^2} + \frac{1}{ad} + \cdots\right), \end{aligned} \tag{12}$$

$$\begin{aligned} G(bc) &= \frac{1}{bc} + \frac{1}{b^2c} + \frac{1}{c^2b} + \frac{1}{bcd} + \frac{1}{b^3c} + \\ &+ \frac{1}{c^3b} + \frac{1}{b^2cd} + \frac{1}{c^2bd} + \frac{1}{d^2bc} + \frac{1}{b^2c^2} + \cdots = \\ &= \frac{1}{bc}\left(1 + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd} + \cdots\right), \end{aligned} \tag{13}$$

$$\begin{aligned} G(bd) &= \frac{1}{bd} + \frac{1}{b^2d} + \frac{1}{d^2b} + \frac{1}{b^3d} + \frac{1}{d^3b} + \frac{1}{b^2d^2} + \cdots = \\ &= \frac{1}{bd}\left(1 + \frac{1}{b} + \frac{1}{d} + \frac{1}{b^2} + \frac{1}{d^2} + \frac{1}{bd} + \cdots\right), \end{aligned} \tag{14}$$

$$\begin{aligned} G(cd) &= \frac{1}{cd} + \frac{1}{c^2d} + \frac{1}{d^2c} + \frac{1}{c^3d} + \frac{1}{d^3c} + \frac{1}{c^2d^2} + \cdots = \\ &= \frac{1}{cd}\left(1 + \frac{1}{c} + \frac{1}{d} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{cd} + \cdots\right). \end{aligned} \tag{15}$$

3 Analytic solution

Now, we determine by the analytic way the unknown sum $S(a, b, c, d)$ by means of the sums of the series (6) – (15). By the formula

$$s = \frac{a_1}{1-q},$$

for the sum s of the convergent infinite geometric series with the first term a_1 and with the ratio q, $|q| < 1$, we get the sums $S(a)$, $S(b)$, $S(c)$, and $S(d)$ of the series (6) – (9):

$$S(a) = \frac{1}{a} \cdot \frac{1}{1 - 1/a} = \frac{1}{a} \cdot \frac{a}{a-1} = \frac{1}{a-1} \tag{16}$$

and, analogously

$$S(b) = \frac{1}{b-1}, \qquad S(c) = \frac{1}{c-1}, \qquad S(d) = \frac{1}{d-1}. \tag{17}$$

It is clear that the sum $S(ab)$ of the series (10) we can write in the form

$$S(ab) = \frac{1}{ab}\big[1 + S(a, b, c, d)\big]. \tag{18}$$

The sum $S(ac)$ of the series (11) is the product of the fraction $1/(ac)$ and the sum of number one and the reduced harmonic series generated by three primes a, c, and d. So, by the formula (4) above, we can write

$$\begin{aligned} S(ac) &= \frac{1}{ac}\left(1+\frac{(a+c-1)(d-1)+ac}{(a-1)(c-1)(d-1)}\right) = \\ &= \frac{(d-1)\big[(a-1)(c-1)+(a+c-1)\big]+ac}{ac(a-1)(c-1)(d-1)} = \\ &= \frac{(d-1)(ac-a-c+1+a+c-1)+ac}{ac(a-1)(c-1)(d-1)} = \\ &= \frac{(d-1)ac+ac}{ac(a-1)(c-1)(d-1)} = \frac{acd}{ac(a-1)(c-1)(d-1)} = \\ &= \frac{d}{(a-1)(c-1)(d-1)}. \end{aligned} \tag{19}$$

Because the sum $S(bc)$ of the series (13) is the product of the fraction $1/(bc)$ and the sum of number one and the reduced harmonic series generated by three primes b, c, and d, we can analogously write

$$S(bc) = \frac{d}{(b-1)(c-1)(d-1)}. \tag{20}$$

Obviously, the sum $S(ad)$ of the series (12) is the product of the fraction $1/(ad)$ and the sum of number one and the reduced harmonic series generated by two primes a and d. So, by the formula (3) above, we can write

$$\begin{aligned} S(ad) &= \frac{1}{ad}\left(1+\frac{a+d-1}{(a-1)(d-1)}\right) = \frac{(a-1)(d-1)+a+d-1}{ad(a-1)(d-1)} = \\ &= \frac{ad-a-d+1+a+d-1}{ad(a-1)(d-1)} = \frac{1}{(a-1)(d-1)} \end{aligned} \tag{21}$$

and, analogously for the sums $S(bd)$ and $S(cd)$ of the series (14) and (15), we get

$$S(bd) = \frac{1}{(b-1)(d-1)}, \qquad S(cd) = \frac{1}{(c-1)(d-1)}. \tag{22}$$

By the assumption of the absolute convergence of the series (5) we can write its sum $S(a,b,c,d)$ in the form

$$S(a)+S(b)+S(c)+S(d)+S(ab)+S(ac)+S(ad)+S(bc)+S(bd)+S(cd).$$

According to (16) – (22) we get the equation

$$S(a,b,c,d) = \frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \frac{1}{d-1} + \frac{1+S(a,b,c,d)}{ab} + \\ + \frac{d}{(a-1)(c-1)(d-1)} + \frac{d}{(b-1)(c-1)(d-1)} + \\ + \frac{1}{(a-1)(d-1)} + \frac{1}{(b-1)(d-1)} + \frac{1}{(c-1)(d-1)}.$$

Multiplying both sides of this equation by $ab(a-1)(b-1)(c-1)(d-1)$, we obtain the equation

$$(ab-1)(a-1)(b-1)(c-1)(d-1)S(a,b,c,d) = ab\big[(b-1)(c-1)(d-1) + \\ + (a-1)(c-1)(d-1) + (a-1)(b-1)(d-1) + (a-1)(b-1)(c-1)\big] + \\ + (a-1)(b-1)(c-1)(d-1) + abd\big[(b-1) + (a-1)\big] + \\ + ab\big[(b-1)(c-1) + (a-1)(c-1) + (a-1)(b-1)\big].$$

It is easy to derive (e.g. by means of the computer algebra system Maple 15 and its **simplify** and **factor** statements) that it holds $S(a,b,c,d) =$

$$= \frac{abc + abd + acd + bcd - ab - ac - ad - bc - bd - cd + a + b + c + d - 1}{(a-1)(b-1)(c-1)(d-1)},$$

i.e.

$$S(a,b,c,d) = \frac{\big[(a+b-1)(c-1) + ab\big](d-1) + abc}{(a-1)(b-1)(c-1)(d-1)}. \qquad (23)$$

This formula can be also written in another two equivalent forms:

$$S(a,b,c,d) = \frac{\big[(a+c-1)(b-1) + ac\big](d-1) + abc}{(a-1)(b-1)(c-1)(d-1)}$$

and

$$S(a,b,c,d) = \frac{\big[(b+c-1)(a-1) + bc\big](d-1) + abc}{(a-1)(b-1)(c-1)(d-1)}.$$

4 Numeric solution

For approximate calculation of the sums $S(a,b,c,d)$ for the primes $a,b,c,d < 100$, i.e. for 25 primes $a,b,c,d \in \{2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97\}$, we use the computer algebra system Maple 15. The sums $S(a,b,c,d)$ we calculate for concrete four primes by the following **for** statements and by the procedure `partabcd`:

```
partabcd:=proc(a,b,c,d)
    local s;
    s:=(((a+b-1)*(c-1)+a*b)*(d-1)+a*b*c)/((a-1)*(b-1)*(c-1)*(d-1));
print("S(a,b,c,d) for a=",a,"b=",b,"c=",c,"d=",d,"is",evalf[8](s));
end proc:

P:=[2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,
    83,89,97]:
for i in P do
    for j in P do
        for k in P do
            for l in P do
                if (i < j and j < k and k < l) then
                partabcd(i,j,k,l);
                end if;
            end do;
        end do;
    end do;
end do;
```

Fourty-two representative sums of these 12650 sums $S(a, b, c, d)$, where 12650 is the number of combinations of size 4 from a collection of size 25, i.e.

$$\binom{25}{4} = \frac{25!}{(25-4)!\,4!} = \frac{25 \cdot 24 \cdot 23 \cdot 22}{4!} = 12650\,,$$

are presented in two following tables. There are 27 sums with a finite decimal expansion (called regular numbers) with at most 4 decimals in the table 1 and another 15 sums, including the sum $S(2, 7, 11, 13) = 1.780\overline{5}$ calculated in the paper [2] and mentioned above, rounded to 6 decimals, in the table 2:

(a,b,c,d)	$S(a,b,c,d)$	(a,b,c,d)	$S(a,b,c,d)$	(a,b,c,d)	$S(a,b,c,d)$
$(2,3,5,7)$	3.375	$(2,3,11,13)$	2.575	$(2,5,11,23)$	1.875
$(2,3,5,11)$	3.125	$(2,3,11,23)$	2.45	$(2,11,23,41)$	1.3575
$(2,3,5,13)$	3.0625	$(2,3,11,31)$	2.41	$(2,11,23,47)$	1.35
$(2,3,5,31)$	2.875	$(2,3,11,61)$	2.355	$(2,11,41,83)$	1.2825
$(2,3,5,61)$	2.8125	$(2,3,11,67)$	2.35	$(2,67,79,89)$	1.0797
$(2,3,7,11)$	2.85	$(2,3,11,89)$	2.3375	$(3,7,11,23)$	1.0125
$(2,3,7,29)$	2.625	$(2,3,13,53)$	2.3125	$(3,7,11,71)$	0.9525
$(2,3,7,41)$	2.5875	$(2,3,31,41)$	2.1775	$(3,11,23,31)$	0.7825
$(2,3,7,71)$	2.55	$(2,3,41,83)$	2.1125	$(3,11,23,43)$	0.7625

Table 1: The table with some values of the sums $S(a, b, c, d)$

(a,b,c,d)	$S(a,b,c,d)$	(a,b,c,d)	$S(a,b,c,d)$	(a,b,c,d)	$S(a,b,c,d)$
$(2,3,5,17)$	2.984375	$(2,5,7,31)$	2.013889	$(2,23,37,43)$	1.200156
$(2,3,5,73)$	2.802083	$(2,5,11,53)$	1.802885	$(3,7,13,19)$	1.001157
$(2,3,7,31)$	2.616667	$(2,7,11,13)$	1.780556	$(3,7,59,89)$	0.800402
$(2,3,11,29)$	2.417857	$(2,5,37,83)$	1.600779	$(3,31,53,79)$	0.600002
$(2,3,19,89)$	2.202652	$(2,7,59,89)$	1.400536	$(79,83,89,97)$	0.047622

Table 2: The table with another values of the sums $S(a,b,c,d)$

5 Conclusion

In this paper the sums $S(a,b,c,d)$ of the convergent reduced harmonic series $G(a,b,c,d)$ generated by four primes a, b, c and d were derived. These sums were computed for $a,b,c,d < 100$, although the formula

$$S(a,b,c,d) = \frac{\big[(a+b-1)(c-1)+ab\big](d-1)+abc}{(a-1)(b-1)(c-1)(d-1)}$$

derived above gives results for arbitrary four different primes a,b,c,d. So that, for example

$$S(101,103,107,109) = \frac{(203\cdot 106 + 101\cdot 103)\cdot 108 + 101\cdot 103\cdot 107}{100\cdot 102\cdot 106\cdot 108} \doteq 0.039056\,.$$

It is clear that this formula is valid not only for four primes, but also for four integers. For example the sum of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{2\cdot 4}+\frac{1}{2\cdot 6}+\frac{1}{2\cdot 8}+\frac{1}{4\cdot 6}+\frac{1}{4\cdot 8}+\frac{1}{6\cdot 8}+\frac{1}{2^3}+\frac{1}{4^3}+\cdots$$

is $S(2,4,6,8) = \dfrac{(5\cdot 5 + 2\cdot 4)\cdot 7 + 2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7} \doteq 2.657143$.

We can say that the reduced harmonic series $G(a,b,c,d)$ generated by four primes (or by four integers) belong to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically by means of a simple formula.

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