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Approximation of functions by (C,2)(E,1) product summability method of Fourier series

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Abstract

Various investigators such as Leindler [10], Chandra [1], Mishra et al. [7], Khan [11], Kushwaha [6] have determined the degree of approximation of 2π *-periodic* functions belonging to classes $Lip\alpha$, $Lip(\alpha,r)$, $Lip(\xi(t),r)$ of functions through trigonometric Fourier approximation using different summability means. Recently Nigam [12] has determined that the Fourier series is summable under the summability means (C,2)(E,1) but he did not find the degree of approximation of function belonging to various classes. In this paper a theorem concerning the degree of approximation of function f belonging to $Lip(\xi(t),r)$ class by (C,2)(E,1) product summability method of Fourier series has been established which in turn generalizes the result of H. K. Nigam [12].

Keywords: Degree of approximation; Fourier series; Pruduct summability methods.
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1. Introduction

The study of the theory of trigonometric approximation is of great mathematical interest and of great practical importance. Broadly speaking, signals are treated as function of single variable and images are represented by function of two variables. The study of these concepts is directly related to the emerging area of information technology. Studies on trigonometric approximation of functions in L_p-norm using different linear operators such Hölder,Nörlund, Euler, Riesz, Borel etc. were made by several researchers like Mohapatra & Chandra [9], Holland, Mohapatra & sahney [8], Chandra [2]. The degree o approximation of a function belonging to different class of functions by product summability methods were made by Lal & Singh [5], Lal & Kushwaha [6]. The aim of this paper is to study Fourier series and conjugate series by product operators. The advantage of considering product operators over linear operators can be understood with the observation that the infinite series, which is neither summable by left linear operators nor by right linear operators individually, is summable to some number by the product operators obtained from the same linear operators placed in the same sequential order. Moreover , in studies of error estimates $E_n(f)$ through Trigonometric Fourier Approximation, product operators give better approximation than individual linear operators. Generalizing the result of Nigam [12], the degree of approximation of function f belonging to $Lip(\xi(t),r)$ class by (C,2)(E,1) product summability method of Fourier series has been established.

Therefore, in this paper, (C,2)(E,1) product summability method is introduced and a theorem on the approximation of functions belonging to $L(\xi(t),r)$ class has been established.

Let $\sum_{n=0}^{\infty} u_n$ be given infinite series with s_n for its n^{th} partial sum.

Let $\{t_n^{E_1}\}$ denote the sequence of (E,1) mean of the sequence $\{s_n\}$. If the (E,1) transform of s_n is defined

as $$t_n^{E_1}(f;x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k(f;x) \to s \text{ as } n \to \infty \tag{1.1}$$

The series $\sum_{n=0}^{\infty} u_n$ is said to summable to the number s by the (E,1) method (Hardy [14]).

Let $\{t_n^{C_2}\}$ denote the sequence of (C, 2) mean of the sequence $\{s_n\}$. If the (C, 2) transform of s_n is defined as

$$t_n^{C_2}(f;x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1)\, s_k(f;x) \to s \text{ as } n \to \infty \tag{1.2}$$

the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by (C, 2) method (Cesàro method).

Thus if

$$t_n^{C_2 \cdot E_1}(f;x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) \frac{1}{2^k} \sum_{v=0}^{n} \binom{n}{v} s_v(f;x) \to s \text{ as } n \to \infty \tag{1.3}$$

Where $t_n^{C_2 \cdot E_1}$ denotes the sequence of (C,2)(E,1) product mean of the sequence s_n. The series $\sum_{n=0}^{\infty} u_n$ is said to summable to the number s by (C,2)(E,1) method. We observe that (C,2)(E,1) method is regular.

Let f be 2π *-periodic* and Lebesgue integrable function. The Fourier series associated with f at a point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.4}$$

with partial sum $s_n(f;x)$.

Throughout this paper, we use following notations:

$$\phi(t) = \phi(x, t) = f(x+t) + f(x-t) - f(x)$$

$$M_n(t) = \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^{n} \left[\frac{n-k+1}{2^k} \sum_{v=0}^{k} \left\{ \binom{k}{v} \frac{\sin(v+1/2)t}{\sin(t/2)} \right\} \right].$$

2. Main Theorem

We prove the following theorem

Theorem . If $f : R \to R$ is 2π -periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to $Lip(\xi(t), r)$ class then the degree of approximation of f by the (C,2)(E,1) product means $t_n^{C_2 \cdot E_1}(f;x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) \frac{1}{2^k} \sum_{v=0}^{n} \binom{n}{v} s_v(f;x)$ of its Fourier series (1.4) is given by

$$\left\| t_n^{C_2.E_1} - f \right\|_r = O\left(\xi\left(\frac{1}{n+1}\right)\right).$$

3. Lemmas

3.1 Lemma 1

For $0 < t < 1/(n+1)$, $|K_n(t)| = O(n+1)$.

Proof

For $0 < t < 1/(n+1)$, $\sin nt \le n \sin t$

$$|M_n(t)| \le \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\sin(\nu+1/2)}{\sin(t/2)}\right]\right|$$

$$|M_n(t)| \le \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{(2\nu+1)\sin(t/2)}{\sin(t/2)}\right]\right|$$

$$\le \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}(2k+1)\sum_{\nu=0}^{k}\binom{k}{\nu}\right]\right|$$

$$= \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}[(n-k+1)(2k+1)]$$

$$= \frac{n+1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}(2k+1) - \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}[k(2k+1)]$$

$$= \frac{1}{\pi(n+2)}\sum_{k=0}^{n}(2k+1) - \frac{1}{\pi(n+1)(n+2)}\left[2\sum_{k=0}^{n}k^2 + \sum_{k=0}^{n}k\right]$$

$$= \frac{(n+1)^2}{\pi(n+2)} - \frac{1}{\pi(n+1)(n+2)}\left[\frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}\right]$$

$$= O(n+1).$$

3.2 Lemma 2

For $1/(n+1) \le t \le \pi$, $|K_n(t)| = O(1/t)$.

Proof

For $1/(n+1) \le t \le \pi$, applying Jordan's lemma, $\sin(t/2) \ge t/\pi$ and $\sin nt \le 1$.

$$|M_n(t)| \le \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\sin(\nu+1/2)}{\sin(t/2)}\right]\right|$$

$$\le \frac{(n+1)}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{1}{(t/\pi)}\right]\right|$$

$$-\frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{k}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{1}{(t/\pi)}\right]\right|$$
$$=\frac{1}{t(n+2)}\sum_{k=0}^{n}1-\frac{1}{t(n+1)(n+2)}\sum_{k=0}^{n}k$$
$$=O(1/t).$$

4. Proof of the Theorem

Following Titchmarsh [13] and using Riemann Lebesgue theorem, $s_n(f;x)$ of the series (1.4) is given by

$$s_n(f;x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(n+1/2)t}{\sin(t/2)}dt$$

Using (1), the (E,1) transforms of $s_n(f;x)$ is given by

$$t_n^{(E,1)}-f(x)=\frac{1}{\pi 2^{n+1}}\int_0^{\pi}\phi(t)\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\sin(n+1/2)t}{\sin(t/2)}\right)dt$$

The (C,2) (E,1) transform of $s_n(f;x)$ is given by

$$t_n^{C_2.E_1}-f(x)=\frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi(t)}{\sin(t/2)}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\sin(\nu+1/2)t\right\}dt\right]$$

$$=\int_0^{\pi}\phi(t)K_n(t)\,dt$$
$$=\int_0^{1/(n+1)}\phi(t)K_n(t)\,dt+\int_{1/(n+1)}^{\pi}\phi(t)\,K_n(t)\,dt$$
$$=I_1+I_2 \tag{4.1}$$

Now , $I_1=\int_0^{1/(n+1)}\xi(t)\left|K_n(t)\right|dt$

$$=O\left[\int_0^{1/(n+1)}\xi(t)(n+1)dt\right], \text{ by Lemma (1)}$$
$$=O(n+1)\left[\int_0^{1/(n+1)}\xi(t)\,dt\right]$$

$$= \left(O(n+1)\xi\left(\frac{1}{n+1}\right)\right)\int_{\varepsilon}^{1/(n+1)} dt \text{ , where } 0<\varepsilon<1/(n+1)$$

by first mean value theorem of calculus

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right). \tag{4.2}$$

Lastly, $I_2 = O\left[\int_{1/(n+1)}^{\pi} \phi(t)\ K_n(t)\ dt \right]$

$$= O\left[\int_{1/(n+1)}^{\pi} \frac{\xi(t)}{(n+1)t}\ dt \right] \text{, by lemma (2)}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right). \tag{4.3}$$

Combining (4.1)-(4.3), we get

$$\left\| t_n^{C_2 . E_1} - f \right\|_r = O\left(\xi\left(\frac{1}{n+1}\right)\right).$$

This completes the proof of the theorem.

5. Conclusions

The result of main theorem is

$$\left\| t_n^{C_2 . E_1} - f \right\|_r = O\left(\xi\left(\frac{1}{n+1}\right)\right).$$

from which the results of H.K. Nigam [12] can be derived directly.

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