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Legendre Wavelet expansion of functions and their Approximations

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Abstract

In this paper, nine new Legendre wavelet estimators of functions having bounded third and fourth derivatives have been obtained.These estimators are new and best approximation in wavelet analysis. Legendre wavelet estimator of a function f of bounded higher order derivatives is better and sharper than the estimator of a function f of bounded less order derivative.

Keywords : Legendre Wavelet, Legendre Wavelet Expansion, Orthonormal basis,Legendre Wavelet Approximation .
Mathematics Subject Classification:42C40, 65T60, 65L10, 65L60, 65R20. [1]

1 Introduction

Several researchers have determined the approximation of a functions by trigonometric polynomials in Fourier analysis. In Fourier analysis, a function can be represented generally in one Fourier series. In wavelet analysis, a function can be expanded in many wavelet series corresponding to different wavelets. This is an advantage of wavelet analysis. There is no such advantage in Fourier analysis.Thus a signal can be represented by several wavelet series. Hence Wavelet Analysis is superior to Fourier analysis and has so many applications in Engineering and Technology. The Wavelet approximation of a functions by its Haar wavelet series and related approximations have been studied by Devore[7], Debnath[5], Meyer[9] , Morlet[3], Mhaskar[2], Sablonnière[6] and Lal & Kumar[8]. The purpose of this paper is to discuss the Legendre wavelet series of function having bounded third and fourth derivatives, i.e. $0 \leq |f^{'''}(x)| < \infty \;\; \forall x \in [0,1]$ and $0 \leq |f^{iv}(x)| < \infty \quad \forall x \in [0,1]$ and to obtain Legendre wavelet estimators of these functions. This is a significant observation of this research paper that estimate of a function is better and the sharper than the estimate having less order bounded derivative.Therefore comparison of estimated approximations has very importance in Wavelet analysis.

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2 Definitions and Preliminaries

2.1 Legendre Wavelet

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\psi \in L^2(R)$, called mother wavelet. We write

$$\psi_{b,a}(x) = |a|^{\frac{-1}{2}} \psi \left(\frac{x-b}{a} \right), \quad a \neq 0.$$

If we restrict the values of dilation and translation parameter to $a = a_0^{-n}$,
$b = m b_0 a_0{}^{-n}, a_0 > 1, b_0 > 0$ respectively, the following family of discrete wavelets are constructed:

$$\psi_{n,m}(x) = |a_0|^{\frac{n}{2}} \psi(a_0^n x - m b_0)$$

The Legendre wavelet over the interval [0,1) is defined as

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m+\frac{1}{2}}\ 2^{\frac{k}{2}} P_m(2^k x - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq x < \frac{\hat{n}+1}{2^k} \\ 0 & \text{, otherwise,} \end{cases}$$

where $n = 1, 2, ..., 2^{k-1}$ and $m = 0, 1, 2, 3, ...,$ $\hat{n} = 2n - 1$ and k is the positive integer. In this definition,the polynomials P_m are Legendre Polynomials of degree m over the interval [-1,1] defined as follows:

$$P_0(x) = 1, P_1(x) = x$$

$$(m+1)P_{m+1}(x) = (2m+1)xP_m(x) - mP_{m-1}(x) \ , \quad m = 1, 2, 3, ...$$

The set of $\{P_m(x) : m = 1, 2, 3, ...\}$ in the Hilbert space $L^2[-1, 1]$ is a complete orthogonal set. Orthogonality of Legendre polynomial on the interval [-1,1] implies that

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x)\overline{P_n(x)}dx = \begin{cases} \frac{2}{2m+1} & , m = n \\ 0 & \text{, otherwise.} \end{cases}$$

$$for\ m, n = 0, 1, 2, 3...$$

Furthermore, the set of wavelets $\psi_{n,m}$ makes an orthonormal basis in $L^2[0, 1)$,$i.e.$

$$\int_0^1 \psi_{n,m}(x)\psi_{n'm'}(x)dx = \delta_{n,n'}\delta_{m,m'}$$

in which δ denotes Kronecker delta function defined by

$$\delta_{n,m} = \begin{cases} 1, & \text{n=m} \\ 0, & \text{otherwise.} \end{cases}$$

The function $f(x) \in L^2[0,1)$ is expressed in the Legendre wavelet series as :

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x)$$

where $c_{n,m} = \langle f, \psi_{n,m}\rangle$. The $(2^{k-1}, M)^{th}$ partial sums of above series are given by $S_{2^{k-1},M}(f)(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) = C^T\psi(x)$ in which C and $\psi(x)$ are $2^{k-1}(M+1)$ vectors of the form

$$C^T = [c_{1,0}, c_{1,1}, ...c_{1,M}, c_{2,0}, c_{2,1}, ...c_{2,M}, ..., c_{2^{k-1},0}, ...c_{2^{k-1},M}]$$

and

$$\psi(x) = [\psi_{1,0}, \psi_{1,1}, ...\psi_{1,M}, \psi_{2,0}, \psi_{2,1}, ...\psi_{2,M}, ..., \psi_{2^{k-1},0}, ...\psi_{2^{k-1},M}]^T$$

2.2 Legendre Wavelet Approximation

Let $S_{2^{k-1},M}(f)(x)$ denote the $(2^{k-1}, M)^{th}$ partial sums of the series $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x)$ i.e.

$$S_{2^{k-1},M}(f)(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x)$$

The Legendre wavelet approximation $E_{2^{k-1},M}(f)$ of a function $f \in L^2[0,1)$ by $(2^{k-1}, M)^{th}$ partial sums $S_{2^{k-1},M}(f)$ of its Legendre Wavelet series is given by

$$E_{2^{k-1},M}(f) = min\|f - S_{2^{k-1},M}(f)\|_2, (Zygmund[1], pp.115)$$

where

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}}.$$

If $E_{2^{k-1},M}(f) \to 0$ as $k \to \infty$, $M \to \infty$. then $E_{2^{k-1},M}(f)$ is called the best approximation of f of order $(2^{k-1}, M)$ (Zygmund[1],pp.115)

3 Example

Express the following function in the Legendre wavelet series :
$f(t) = t^3 \;\; \forall t \in [0,1)$

Proof:

$$
\begin{aligned}
f(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t) \\
c_{n,m} &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(t)\psi_{n,m}(t)dt \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} t^3 \left(\frac{2m+1}{2}\right)^{\frac{1}{2}} 2^{\frac{k}{2}} P_m(2^k t-\hat{n})dt \\
&= \left(\frac{2m+1}{2}\right)^{\frac{1}{2}} 2^{\frac{k}{2}} \int_{-1}^{1} \left(\frac{v+\hat{n}}{2^k}\right)^3 P_m(v)\frac{dv}{2^k},\ v=2^k t-\hat{n} \\
c_{n,m} &= \left(\frac{2m+1}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3+v^3+3\hat{n}^2 v+3\hat{n}v^2)P_m(v)dv
\end{aligned}
$$

By above expression

$$
\begin{aligned}
c_{n,0} &= \left(\frac{1}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3+v^3+3\hat{n}^2 v+3\hat{n}v^2)P_0(v)dv \\
&= \left(\frac{1}{2^{7k+1}}\right)^{\frac{1}{2}} (2\hat{n}^3+2\hat{n}) \\
c_{n,1} &= \left(\frac{\sqrt{3}}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3+v^3+3\hat{n}^2 v+3\hat{n}v^2)P_1(v)dv \\
&= \left(\frac{\sqrt{3}}{2^{7k+1}}\right)^{\frac{1}{2}} \left(\frac{2}{5}+2\hat{n}^2\right) \\
c_{n,2} &= \left(\frac{\sqrt{5}}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3+v^3+3\hat{n}^2 v+3\hat{n}v^2)P_2(v)dv \\
&= \left(\frac{\sqrt{5}}{2^{7k+1}}\right)^{\frac{1}{2}} \left(\frac{4\hat{n}}{5}\right) \\
c_{n,3} &= \left(\frac{\sqrt{7}}{2^{7k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} (\hat{n}^3+v^3+3\hat{n}^2 v+3\hat{n}v^2)P_3(v)dv
\end{aligned}
$$

$$
\begin{aligned}
c_{n,3} &= \left(\frac{4}{35}\right)\left(\frac{\sqrt{7}}{2^{7k+1}}\right)^{\frac{1}{2}} \\
c_{n,m} &= 0,\ for\ m \geq 4
\end{aligned}
$$

Then,

$$
f(t) = \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}(t) + \sum_{n=1}^{2^{k-1}} c_{n,1}\psi_{n,1}(t) + \sum_{n=1}^{2^{k-1}} c_{n,2}\psi_{n,2}(t) + \sum_{n=1}^{2^{k-1}} c_{n,3}\psi_{n,3}(t)
$$

Now,

$$
\begin{aligned}
||f||_2^2 &= \frac{1}{7} = \sum_{n=1}^{2^{k-1}} c_{n,0}^2||\psi_{n,0}||_2^2 + \sum_{n=1}^{2^{k-1}} c_{n,1}^2||\psi_{n,1}||_2^2 + \sum_{n=1}^{2^{k-1}} c_{n,2}^2||\psi_{n,2}||_2^2 + \sum_{n=1}^{2^{k-1}} c_{n,3}^2||\psi_{n,3}||_2^2 \\
&= \sum_{n=1}^{2^{k-1}} c_{n,0}^2 + \sum_{n=1}^{2^{k-1}} c_{n,1}^2 + \sum_{n=1}^{2^{k-1}} c_{n,2}^2 + \sum_{n=1}^{2^{k-1}} c_{n,3}^2 \\
&= \sum_{n=1}^{2^{k-1}} \left[\left(\frac{1}{2^{7k+1}}\right)^{\frac{1}{2}}(2\hat{n}^3 + 2\hat{n})\right]^2 + \sum_{n=1}^{2^{k-1}} \left[\left(\frac{\sqrt{3}}{2^{7k+1}}\right)^{\frac{1}{2}}\left(\frac{2}{5} + 2\hat{n}^2\right)\right]^2 + \sum_{n=1}^{2^{k-1}} \left[\left(\frac{\sqrt{5}}{2^{7k+1}}\right)^{\frac{1}{2}}\left(\frac{4\hat{n}}{5}\right)\right]^2 \\
&+ \sum_{n=1}^{2^{k-1}} \left[\left(\frac{4}{35}\right)\left(\frac{\sqrt{7}}{2^{7k+1}}\right)^{\frac{1}{2}}\right]^2 \\
&= \frac{1}{7}.
\end{aligned}
$$

4 Theorems

In this paper, we prove following new theorems:

Theorem (4.1)

Let a function $f \in L^2[0,1)$ such that its third derivative be bounded ,i.e. $0 \leq |f'''(x)| < \infty \; \forall \; x \in [0,1)$. Then the Legendre wavelet approximations of f satisfy :

$(i) E^{(1)}_{2^{k-1},0}(f) = ||f - \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}||_2 = O\left(\frac{1}{2^k}\right)$

$(ii) E^{(2)}_{2^{k-1},1}(f) = ||f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{1} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{2k}}\right)$

$(iii) E^{(3)}_{2^{k-1},2}(f) = ||f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{2} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{3k}}\right)$

(iv)For $f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}$,

$E^{(4)}_{2^{k-1},M}(f) = ||f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}||_2$

$$= \left(\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2 \right)^{\frac{1}{2}} = O\left(\frac{1}{(2M-3)^{\frac{5}{2}}} \frac{1}{2^{3k}} \right), \forall M \geq 2.$$

Theorem (4.2)
If a function $f \in L^2[0,1)$ having bounded fourth derivative ,i.e. $0 \leq |f^{iv}(x)| < \infty \; \forall \; x \in [0,1)$. Then its Legendre wavelet approximations are given by

$(i) E^{(5)}_{2^{k-1},0}(f) = ||f - \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}||_2 = O\left(\frac{1}{2^k}\right)$

$(ii) E^{(6)}_{2^{k-1},1}(f) = ||f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{1} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{2k}}\right)$

$(iii) E^{(7)}_{2^{k-1},2}(f) = ||f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{2} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{3k}}\right)$

$(iv) E^{(8)}_{2^{k-1},3}(f) = ||f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{3} c_{n,m}\psi_{n,m}||_2 = O\left(\frac{1}{2^{4k}}\right)$

(v)For $f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}$,

$E^{(9)}_{2^{k-1},M}(f) = ||f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}||_2$

$$= \left(\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2 \right)^{\frac{1}{2}} = O\left(\frac{1}{(2M-5)^{\frac{7}{2}}} \frac{1}{2^{4k}} \right), \; \forall \; M \geq 3.$$

5 Proofs

5.1 Proof of the Theorem (4.1)

(i) The error $e^{(0)}_n(x)$ between $f(x)$ and its expression over any subinterval is defined as
$e^{(0)}_n(x) = c_{n,0}\psi_{n,0}(x) - f(x)$, $x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$, $n = 1, 2, 3, ...2^{k-1}$

$$\begin{aligned} ||e^{(0)}_n||_2^2 &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (e^{(0)}_n(x))^2 dx \\ &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (c_{n,0}^2\psi_{n,m}^2(x) + (f(x))^2 - 2c_{n,0}\psi_{n,0}(x)f(x))dx \end{aligned}$$

$$
\begin{aligned}
&= c_{n,0}^2 \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \psi_{n,0}^2(x)dx + \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - 2c_{n,0} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)\psi_{n,0}(x)dx \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - c_{n,0}^2. \qquad (5.1)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx &= \int_0^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k}+h\right)\right)^2 dh, x = \frac{\hat{n}-1}{2^k}+h \\
&= \int_0^{\frac{1}{2^{k-1}}} \left[f\left(\frac{\hat{n}-1}{2^k}\right) + hf^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^2}{2}f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right]^2, \\
&\quad 0<\theta<1 \;\; by\, Taylor's \;\; expansion \\
&= \int_0^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 dh + \int_0^{\frac{1}{2^{k-1}}} h^2 \left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 dh + \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{4}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 dh \\
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^6}{36}\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right)^2 dh + \int_0^{\frac{1}{2^{k-1}}} 2hf\left(\frac{\hat{n}-1}{2^k}\right)f^{'}\left(\frac{\hat{n}-1}{2^k}\right)dh \\
&+ \int_0^{\frac{1}{2^{k-1}}} h^2 f\left(\frac{\hat{n}-1}{2^k}\right)f^{''}\left(\frac{\hat{n}-1}{2^k}\right)dh + \int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{3}f\left(\frac{\hat{n}-1}{2^k}\right)f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&+ \int_0^{\frac{1}{2^{k-1}}} h^3 f^{'}\left(\frac{\hat{n}-1}{2^k}\right)f^{''}\left(\frac{\hat{n}-1}{2^k}\right)dh + \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{3}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^5}{6}f^{''}\left(\frac{\hat{n}-1}{2^k}\right)f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&= \frac{2}{2^k}\left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{8}{3}\frac{1}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{8}{5}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 \\
&+ \frac{1}{36}\int_0^{\frac{1}{2^{k-1}}} h^6\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right)^2 dh + \frac{4}{2^{2k}}f\left(\frac{\hat{n}-1}{2^k}\right)f^{'}\left(\frac{\hat{n}-1}{2^k}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{8}{3}\frac{1}{2^{3k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{3}\int_0^{\frac{1}{2^{k-1}}} h^3 f\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh \\
&+ \quad \frac{4}{2^{4k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) dh + \frac{1}{3}\int_0^{\frac{1}{2^{k-1}}} h^4 f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh \\
&+ \quad \frac{1}{6}\int_0^{\frac{1}{2^{k-1}}} h^5 f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh. \qquad (5.2)
\end{aligned}
$$

Now,

$$
\begin{aligned}
c_{n,0} &= < f(x), \psi_{n,0}(x) > \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)\psi_{n,0}(x)dx \\
&= 2^{\frac{k-1}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)dx \\
&= 2^{\frac{k-1}{2}} \int_0^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k}+h\right) dh, x = \frac{\hat{n}-1}{2^k}+h \\
&= 2^{\frac{k-1}{2}} \int_0^{\frac{1}{2^{k-1}}} \left[f\left(\frac{\hat{n}-1}{2^k}\right) + hf^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^2}{2} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^3}{6} f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right] dh \\
&= 2^{\frac{k-1}{2}} \left[\frac{2}{2^k} f\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{2^{2k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{3}\frac{1}{2^{3k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{6}\int_0^{\frac{1}{2^{k-1}}} h^3 f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right].
\end{aligned}
$$

Next,

$$
\begin{aligned}
c_{n,0}^2 &= \frac{2}{2^k}\left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{8}{9}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 \\
&+ \quad \frac{2^k}{2}\left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6} f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right)^2 + \frac{4}{2^{2k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{'}\left(\frac{\hat{n}-1}{2^k}\right) \\
&+ \quad \frac{8}{3}\frac{1}{2^{3k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{3}\int_0^{\frac{1}{2^{k-1}}} h^3 f\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh
\end{aligned}
$$

$$+ \quad \frac{8}{3}\frac{1}{2^{4k}}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)f^{''}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{2}{2^k}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+ \quad \frac{4}{3}\frac{1}{2^{2k}}f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh. \tag{5.3}$$

Now, by using equations (5.1), (5.2) and (5.3) we have

$$\begin{aligned}
||e_n^{(0)}||_2^2 &= \frac{2}{3}\frac{1}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2+\frac{32}{45}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2+\frac{1}{36}\int\limits_0^{\frac{1}{2^{k-1}}}h^6\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right)^2dh \\
&- \frac{2^k}{2}\left(\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\right)^2+\frac{4}{3}\frac{1}{2^{4k}}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)f^{''}\left(\frac{\hat{n}-1}{2^k}\right) \\
&+ \frac{1}{3}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}h^4f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh-\frac{2}{2^k}f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&+ \frac{1}{6}f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}h^5f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh-\frac{4}{3}\frac{1}{2^{2k}}f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\int\limits_0^{\frac{1}{2^{k-1}}}\frac{h^3}{6}f^{'''}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \\
&= I_1+I_2+I_3-I_4+I_5+I_6-I_7+I_8-I_9,\ say.
\end{aligned}$$

Since $|f^{'}(x)|\le M_1, |f^{''}(x)|\le M_2, |f^{'''}(x)|\le M_3, \forall x\in[0,1),$

therefore

$$\begin{aligned}
|I_1| &\le \frac{2}{3}\frac{1}{2^{3k}}M_1^2 \\
|I_2| &\le \frac{32}{45}\frac{1}{2^{5k}}M_2^2 \\
|I_3| &\le \frac{32}{63}\frac{1}{2^{7k}}M_3^2 \\
|I_4| &\le \frac{2}{9}\frac{1}{2^{7k}}M_3^2 \\
|I_5| &\le \frac{4}{3}\frac{1}{2^{4k}}M_1M_2 \\
|I_6| &\le \frac{32}{15}\frac{1}{2^{5k}}M_1M_3 \\
|I_7| &\le \frac{4}{3}\frac{1}{2^{5k}}M_1M_3 \\
|I_8| &\le \frac{16}{9}\frac{1}{2^{6k}}M_2M_3 \\
|I_9| &\le \frac{8}{9}\frac{1}{2^{6k}}M_2M_3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
||e_n^{(0)}||_2^2 &\leq |I_1|+|I_2|+|I_3|+|I_4|+|I_5|+|I_6|+|I_7|+|I_8|+|I_9| \\
&\leq \frac{2}{3}\frac{1}{2^{3k}}M_1^2+\frac{32}{45}\frac{1}{2^{5k}}M_2^2+\frac{32}{63}\frac{1}{2^{7k}}M_3^2+\frac{2}{9}\frac{1}{2^{7k}}M_3^2+\frac{4}{3}\frac{1}{2^{4k}}M_1M_2 \\
&+ \frac{32}{15}\frac{1}{2^{5k}}M_1M_3+\frac{4}{3}\frac{1}{2^{5k}}M_1M_3+\frac{16}{9}\frac{1}{2^{6k}}M_2M_3+\frac{8}{9}\frac{1}{2^{6k}}M_2M_3 \\
&= \frac{2}{3}\frac{1}{2^{3k}}M_1^2+\frac{32}{45}\frac{1}{2^{5k}}M_2^2+\frac{56}{63}\frac{1}{2^{7k}}M_3^2+\frac{4}{3}\frac{1}{2^{4k}}M_1M_2+\frac{52}{15}\frac{1}{2^{5k}}M_1M_3+\frac{24}{9}\frac{1}{2^{6k}}M_2M_3 \\
&< \frac{2}{2^{3k}}\left[M_1^2+\left(\frac{M_2}{2^k}\right)^2+\left(\frac{M_3}{2^{2k}}\right)^2+\frac{2M_1M_2}{2^k}+\frac{2M_1M_3}{2^{2k}}+\frac{2M_2M_3}{2^{3k}}\right] \\
&= \frac{2}{2^{3k}}\left(M_1+\frac{M_2}{2^k}+\frac{M_3}{2^{2k}}\right)^2 \\
&= \frac{2M^2}{2^{3k}}\left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}\right)^2, M=max[M_1,M_2,M_3].
\end{aligned}$$

Next,

$$\begin{aligned}
(E^{(1)}_{2^{k-1},0}(f))^2 &= \int_0^1\left(\sum_{n=1}^{2^{k-1}}e_n^{(0)}(x)\right)^2dx \\
&= \int_0^1\sum_{n=1}^{2^{k-1}}(e_n^{(0)}(x))^2dx+2\sum_{n=1}^{2^{k-1}}\sum_{n\neq n'}^{2^{k-1}}\int_0^1 e_n^{(0)}(x)e_n^{(0')}(x)dx \\
&= \sum_{n=1}^{2^{k-1}}\int_0^1(e_n(x))^2dx,\ \text{due to disjoint supports of } e_n \text{ and } e'_n \\
&= \sum_{n=1}^{2^{k-1}}||e_n^{(0)}||_2^2 \\
&\leq (2^{k-1})\frac{2M^2}{2^{3k}}\left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}\right)^2 \\
&= \frac{M^2}{2^{2k}}\left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}\right)^2.
\end{aligned}$$

Then,

$$\begin{aligned}
E^{(1)}_{2^{k-1},0}(f) &\leq \frac{M}{2^k}\left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}\right) \\
&\leq M\left(\frac{1}{2^k}+\frac{1}{2^k}+\frac{1}{2^k}\right) \\
&= 3M\left(\frac{1}{2^k}\right) \\
&= O\left(\frac{1}{2^k}\right).
\end{aligned}$$

$$
\begin{aligned}
(ii)e_n^{(1)}(x) &= c_{n,0}\psi_{n,0}(x) + c_{n,1}\psi_{n,1}(x) - f(x) \;\;, \quad x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right) \\
||e_n^{(1)}||_2^2 &= \int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - c_{n,0}^2 - c_{n,1}^2. \qquad (5.4)
\end{aligned}
$$

Now, consider

$$
\begin{aligned}
c_{n,1} &= < f(x), \psi_{n,1}(x) > \\
&= \int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)\psi_{n,1}(x)dx \\
&= \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)P_1(2^k x - \hat{n})dx \\
&= \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \int\limits_{0}^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k} + h\right) P_1(2^k h - 1)dh, \;\; x = \frac{\hat{n}-1}{2^k} + h \\
&= \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \int\limits_{0}^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k} + h\right) (2^k h - 1)dh \\
&= \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \int\limits_{0}^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k}\right) (2^k h - 1)dh + \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \int\limits_{0}^{\frac{1}{2^{k-1}}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) h(2^k h - 1)dh \\
&+ \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \int\limits_{0}^{\frac{1}{2^{k-1}}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) \frac{h^2}{2}(2^k h - 1)dh + \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \int\limits_{0}^{\frac{1}{2^{k-1}}} f^{'''}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \frac{h^3}{6}(2^k h - 1)dh \\
c_{n,1} &= \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \left[\frac{2}{3}\frac{1}{2^{2k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{3}\frac{1}{2^{3k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right] \\
&+ \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \int\limits_{0}^{\frac{1}{2^{k-1}}} f^{'''}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \frac{h^3}{6}(2^k h - 1)dh.
\end{aligned}
$$

Now,

$$
\begin{aligned}
c_{n,1}^2 &= \frac{2}{3}\frac{1}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{3}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 \\
&+ \frac{3}{2}2^k \left(\int\limits_{0}^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h - 1) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) dh\right)^2 + \frac{4}{3}\frac{1}{2^{4k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right)
\end{aligned}
$$

$$+\quad \frac{2}{2^k} f'\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h-1) f'''\left(\frac{\hat{n}-1}{2^k}\right) dh$$

$$+\quad \frac{2}{2^{2k}} f''\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h-1) f'''\left(\frac{\hat{n}-1}{2^k}\right) dh. \qquad (5.5)$$

By using equations (5.2), (5.3), (5.4) and (5.5), we have

$$\begin{aligned}
||e_n^{(1)}||_2^2 &= \frac{2}{45}\frac{1}{2^{5k}}\left(f''\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{1}{36}\int_0^{\frac{1}{2^{k-1}}} h^6\left(f'''\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right)^2 dh \\
&- \frac{4}{3}\frac{1}{2^{2k}} f''\left(\frac{\hat{n}-1}{2^k}\right)\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6} f'''\left(\frac{\hat{n}-1}{2^k}\right) dh - \frac{2^k}{2}\left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6} f'''\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right)^2 \\
&+ \frac{1}{6} f''\left(\frac{\hat{n}-1}{2^k}\right)\int_0^{\frac{1}{2^{k-1}}} h^5 f'''\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh - \frac{3}{2}2^k\left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h-1) f'''\left(\frac{\hat{n}-1}{2^k}\right) dh\right)^2 \\
&- \frac{2}{2^{2k}} f''\left(\frac{\hat{n}-1}{2^k}\right)\int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{6}(2^k h-1) f'''\left(\frac{\hat{n}-1}{2^k}\right) dh \\
&= I_1+I_2+I_3+I_4+I_5+I_6+I_7, say.
\end{aligned}$$

Therefore

$$\begin{aligned}
|I_1| &\le \frac{2}{45}\frac{1}{2^{5k}} M_2^2 \\
|I_2| &\le \frac{32}{63}\frac{1}{2^{7k}} M_3^2 \\
|I_3| &\le \frac{8}{9}\frac{1}{2^{6k}} M_2 M_3 \\
|I_4| &\le \frac{2}{9}\frac{1}{2^{7k}} M_3^2 \\
|I_5| &\le \frac{16}{9}\frac{1}{2^{6k}} M_2 M_3 \\
|I_6| &\le \frac{18}{75}\frac{1}{2^{7k}} M_3^2 \\
|I_7| &\le \frac{12}{15}\frac{1}{2^{6k}} M_2 M_3 \\
||e_n^{(1)}||_2^2 &\le |I_1|+|I_2|+|I_3|+|I_4|+|I_5|+|I_6|+|I_7| \\
&\le \frac{2}{45}\frac{1}{2^{5k}} M_2^2 + \frac{32}{63}\frac{1}{2^{7k}} M_3^2 + \frac{8}{9}\frac{1}{2^{6k}} M_2 M_3 + \frac{2}{9}\frac{1}{2^{7k}} M_3^2 + \frac{16}{9}\frac{1}{2^{6k}} M_2 M_3 + \frac{18}{75}\frac{1}{2^{7k}} M_3^2 \\
&+ \frac{12}{15}\frac{1}{2^{6k}} M_2 M_3
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{45}\frac{1}{2^{5k}}M_2^2 + \frac{1528}{1575}\frac{1}{2^{7k}}M_3^2 + \frac{144}{45}\frac{1}{2^{6k}}M_2M_3 \\
&< \frac{2}{2^{5k}}\left(M_2^2 + \left(\frac{M_3}{2^k}\right)^2 + \frac{2M_2M_3}{2^k}\right) \\
&= \frac{2}{2^{5k}}M^2\left(1+\frac{1}{2^k}\right)^2, M = max[M_2, M_3].
\end{aligned}
$$

Next,

$$
\begin{aligned}
(E^{(2)}_{2^{k-1},1}(f))^2 &= \sum_{n=1}^{2^{k-1}} ||e_n^{(1)}||_2^2 \\
&\leq (2^{k-1})\frac{2}{2^{5k}}M^2\left(1+\frac{1}{2^k}\right)^2 \\
&= \frac{M^2}{2^{4k}}\left(1+\frac{1}{2^k}\right)^2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
E^{(2)}_{2^{k-1},1}(f) &\leq \frac{M}{2^{2k}}\left(1+\frac{1}{2^k}\right) \\
&= O\left(\frac{1}{2^{2k}}\right).
\end{aligned}
$$

(iii) $e_n^{(2)}(x) = c_{n,0}\psi_{n,0}(x) + c_{n,1}\psi_{n,1}(x) + c_{n,2}\psi_{n,2}(x) - f(x) \; , \; x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$
Similarly, it can be proved that

$$
E^{(3)}_{2^{k-1},2}(f) = O\left(\frac{1}{2^{3k}}\right).
$$

$(iv)\; 0 \leq |f^{'''}(x)| < M_1 \; , \forall x \in [0,1)$

$$
\begin{aligned}
c_{n,m} &= \int_0^1 f(x)\psi_{n,m}(x)dx \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)\sqrt{\frac{2m+1}{2}}2^{\frac{k}{2}}P_m(2^kx-\hat{n})dx \\
&= \sqrt{\frac{2m+1}{2^{k+1}}}\int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k}\right)P_m(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{\frac{2m+1}{2^{k+1}}}\int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+1}(t)-P_{m-1}(t))}{2m+1} \\
&= \left(\frac{1}{2^{k+1}(2m+1)}\right)^{\frac{1}{2}} \\
&\times \left[\left\{f\left(\frac{\hat{n}+t}{2^k}\right)(P_{m+1}(t)-P_{m-1}(t))\right\}_{-1}^{1} - \int_{-1}^{1}\frac{1}{2^k}f'\left(\frac{\hat{n}+t}{2^k}\right)(P_{m+1}(t)-P_{m-1}(t))dt\right] \\
&= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right)(P_{m-1}(t)-P_{m+1}(t))dt\right] \\
&= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right)(P_{m-1}(t))dt - \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right)(P_{m+1}(t))dt\right] \\
&= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}\left[f'\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m}(t)-P_{m-2}(t))}{(2m-1)} - f'\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+2}(t)-P_{m}(t))}{(2m+3)}\right] \\
&= \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}\left[f''\left(\frac{\hat{n}+t}{2^k}\right)\frac{(P_{m+2}(t)-P_{m}(t))}{(2m+3)} - f''\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m}(t)-P_{m-2}(t))}{(2m-1)}\right] \\
&= \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)}\int_{-1}^{1}\left[f''\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+3}(t)-P_{m+1}(t))}{(2m+5)}\right] \\
&- \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)}\int_{-1}^{1}\left[f''\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+1}(t)-P_{m-1}(t))}{(2m+1)}\right] \\
&+ \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)}\int_{-1}^{1}\left[f''\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m-1}(t)-P_{m-3}(t))}{(2m-3)}\right] \\
&- \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)}\int_{-1}^{1}\left[f''\left(\frac{\hat{n}+t}{2^k}\right)\frac{d(P_{m+1}(t)-P_{m-1}(t))}{(2m+1)}\right] \\
&= \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)}\int_{-1}^{1} f'''\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_{m+1}(t)-P_{m-1}(t))}{(2m+1)} - \frac{(P_{m+3}(t)-P_{m+1}(t))}{(2m+5)}\right]dt \\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)}\int_{-1}^{1} f'''\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_{m-1}(t)-P_{m-3}(t))}{(2m-3)} - \frac{(P_{m+1}(t)-P_{m-1}(t))}{(2m+1)}\right]dt
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \\
&\times \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{2(2m+3)P_{m+1}(t)-(2m+5)P_{m-1}(t)-(2m+1)P_{m+3}(t)}{(2m+1)(2m+5)(2m+3)}\right]dt \\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \\
&\times \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{2(2m-1)P_{m-1}(t)-(2m+1)P_{m-3}(t)-(2m-3)P_{m+1}(t)}{(2m+1)(2m-1)(2m-3)}\right]dt.
\end{aligned}
$$

Let

$$
\begin{aligned}
\tau_1(t) &= 2(2m+3)P_{m+1}(t)-(2m+5)P_{m-1}(t)-(2m+1)P_{m+3}(t) \\
\tau_2(t) &= 2(2m-1)P_{m-1}(t)-(2m+1)P_{m-3}(t)-(2m-3)P_{m+1}(t)
\end{aligned}
$$

Then,

$$
\begin{aligned}
c_{n,m} &= \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+1)(2m+3)(2m+5)}\left[\int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\tau_1(t)dt\right] \\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+1)(2m-1)(2m-3)}\left[\int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\tau_2(t)dt\right] \\
|c_{n,m}| &\leq \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+1)(2m+3)(2m+5)}\left[\int_{-1}^{1} \left|f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\right||\tau_1(t)|dt\right] \\
&+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+1)(2m-1)(2m-3)}\left[\int_{-1}^{1} \left|f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\right||\tau_2(t)dt|\right] \\
&\leq M_1\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+1)(2m+3)(2m+5)}\int_{-1}^{1}|\tau_1(t)|dt \\
&+ M_1\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+1)(2m-1)(2m-3)}\int_{-1}^{1}|\tau_2(t)|dt. \qquad (5.6)
\end{aligned}
$$

Consider,

$$
\begin{aligned}
\int_{-1}^{1}|\tau_1(t)|dt &= \int_{-1}^{1} 1.|\tau_1(t)|dt \\
&\leq \left(\int_{-1}^{1} 1^2.dt\right)^{\frac{1}{2}}\left(\int_{-1}^{1}|\tau_1(t)|^2dt\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{2}\left(\int_{-1}^{1}\left(2(2m+3)P_{m+1}(t)-(2m+5)P_{m-1}(t)-(2m+1)P_{m+3}(t)\right)^2 dt\right)^{\frac{1}{2}}\\
&= \sqrt{2}\left(\int_{-1}^{1}\left[4(2m+3)^2P_{m+1}^2(t)+(2m+5)^2P_{m-1}^2(t)+(2m+1)^2P_{m+3}^2(t)\right] dt\right)^{\frac{1}{2}}\\
&= \sqrt{2}\left[4(2m+3)^2\frac{2}{2m+3}+(2m+5)^2\frac{2}{2m-1}+(2m+1)^2\frac{2}{2m+7}\right]^{\frac{1}{2}}\\
&\qquad \text{by orthogonality condition on } P_m\\
&= 2\left[4(2m+3)+\frac{(2m+5)^2}{2m-1}+\frac{(2m+1)^2}{2m+7}\right]^{\frac{1}{2}}\\
&\leq 2\left[\frac{4(2m+3)(2m-1)+(2m+5)^2+(2m+1)^2}{(2m-1)}\right]^{\frac{1}{2}}\\
&= 2\left[\frac{24m^2+40m+14}{2m-1}\right]^{\frac{1}{2}}\\
&= 2\sqrt{2}\left[\frac{(2m+1)(6m+7)}{2m-1}\right]^{\frac{1}{2}}\\
&\leq 2\sqrt{6}\left[\frac{(2m+1)(2m+3)}{(2m-1)}\right]^{\frac{1}{2}}. \qquad (5.7)
\end{aligned}
$$

Now ,

$$
\begin{aligned}
\int_{-1}^{1}|\tau_2(t)|dt &= \int_{-1}^{1}1.|\tau_2(t)|dt\\
&= \sqrt{2}\left(\int_{-1}^{1}[2(2m-1)P_{m-1}(t)-(2m+1)P_{m-3}(t)-(2m-3)P_{m+1}(t)]^2dt\right)^{\frac{1}{2}}\\
&= \sqrt{2}\left[\int_{-1}^{1}\left[(2m-3)^2P_{m+1}(t)+(2m+1)^2P_{m-3}^2(t)+4(2m-1)^2P_{m-1}^2(t)\right] dt\right]^{\frac{1}{2}}\\
&= \sqrt{2}\left[(2m-3)^2\frac{2}{(2m+3)}+(2m+1)^2\frac{2}{2m-5}+4(2m-1)^2\frac{2}{2m-1}\right]^{\frac{1}{2}}\\
&\qquad \text{by orthogonality condition on } P_m\\
&= 2\left[\frac{(2m-3)^2}{(2m+3)}+\frac{(2m+1)}{(2m-5)}+4(2m-1)\right]^{\frac{1}{2}}\\
&\leq 2\left[\frac{(2m-3)^2+(2m+1)^2+4(2m-1)(2m-5)}{2m-5}\right]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\left[\frac{24m^2-56m+30}{2m-5}\right]^{\frac{1}{2}} \\
&= 2\sqrt{2}\left[\frac{(2m-3)(6m-5)}{2m-5}\right]^{\frac{1}{2}} \\
&\leq 2\sqrt{6}\left[\frac{(2m-3)(2m-1)}{(2m-5)}\right]^{\frac{1}{2}}. \qquad (5.8)
\end{aligned}
$$

Now , by using equations (5.6), (5.7) and (5.8) we have

$$
\begin{aligned}
|C_{n,m}| &\leq M_1\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\frac{2\sqrt{6}}{(2m-3)^{\frac{5}{2}}}+\frac{2\sqrt{6}}{(2m-5)^{\frac{5}{2}}}\right] \\
&\leq M_1\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\frac{4\sqrt{6}}{(2m-5)^{\frac{5}{2}}}\right] \\
&\leq \frac{4\sqrt{6}M_1}{2^{\frac{7k+1}{2}}}\frac{1}{(2m-5)^3}.
\end{aligned}
$$

Therefore,

$$
|C_{n,m}| \leq \frac{4\sqrt{6}M_1}{2^{\frac{7k+1}{2}}}\frac{1}{(2m-5)^3}, \forall m \geq 3. \qquad (5.9)
$$

$$
\begin{aligned}
S_{2^{k-1},M}(f)(x) &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) \\
f(x)-S_{2^{k-1},M}(f)(x) &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) \\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) + \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(x) \\
&- \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) \\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(x).
\end{aligned}
$$

Then,

$$
\begin{aligned}
||f-S_{2^{k-1},M}(f)||_2^2 &= \int_0^1\left(\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(x)\right)^2 dx \\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}^2, \text{ by orthogonality property of } \psi_{n,m}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \left(\frac{4\sqrt{6}M_1}{2^{\frac{7k+1}{2}}} \frac{1}{(2m-5)^3} \right)^2 , \ by\ (5.9) \\
&= 96M_1^2 \sum_{n=1}^{2^{k-1}} \frac{1}{2^{7k+1}} \sum_{m=M+1}^{\infty} \frac{1}{(2m-5)^6} \\
&= \frac{96M_1^2}{4} \frac{1}{2^{6k}} \int_{M+1}^{\infty} \frac{1}{(2m-5)^6} dm \\
&= \frac{12M_1^2}{5} \frac{1}{2^{6k}} \frac{1}{(2M-3)^5} \\
\therefore E^{(4)}_{2^{k-1},M}(f) &\leq \frac{2\sqrt{3}M_1}{\sqrt{5}} \frac{1}{2^{3k}(2M-3)^{\frac{5}{2}}} \\
&= O\left(\frac{1}{(2M-3)^{\frac{5}{2}} 2^{3k}} \right), \ \ M \geq 2.
\end{aligned}
$$

5.2 Proof of the Theorem(4.2)

(i) The error $e^{*}{}^{(0)}_{n}(x)$ between $f(x)$ and its expression over any subinterval is defined as $e^{*}{}^{(0)}_{n}(x) = c_{n,0}\psi_{n,0}(x) - f(x)$, $x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$, $n = 1, 2, 3, ...2^{k-1}$
Now consider,

$$
\begin{aligned}
\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx &= \int_0^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k} + h \right) \right)^2 dh, x = \frac{\hat{n}-1}{2^k} + h \\
&= \int_0^{\frac{1}{2^{k-1}}} \left[f\left(\frac{\hat{n}-1}{2^k} \right) + hf'\left(\frac{\hat{n}-1}{2^k} \right) + \frac{h^2}{2} f''\left(\frac{\hat{n}-1}{2^k} \right) + \frac{h^3}{6} f'''\left(\frac{\hat{n}-1}{2^k} \right) \right. \\
&+ \left. \frac{h^4}{24} f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h \right) \right]^2 dh \\
&= \frac{2}{2^k} \left(f\left(\frac{\hat{n}-1}{2^k} \right) \right)^2 + \frac{8}{3}\frac{1}{2^{3k}} \left(f'\left(\frac{\hat{n}-1}{2^k} \right) \right)^2 + \frac{8}{5}\frac{1}{2^{5k}} \left(f''\left(\frac{\hat{n}-1}{2^k} \right) \right)^2 \\
&+ \frac{32}{63}\frac{1}{2^{7k}} \left(f'''\left(\frac{\hat{n}-1}{2^k} \right) \right)^2 + \int_0^{\frac{1}{2^{k-1}}} \frac{h^8}{576} \left(f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h \right) \right)^2 dh \\
&+ \frac{4}{2^{2k}} f\left(\frac{\hat{n}-1}{2^k} \right) f'\left(\frac{\hat{n}-1}{2^k} \right) + \frac{8}{3}\frac{1}{2^{3k}} f\left(\frac{\hat{n}-1}{2^k} \right) f''\left(\frac{\hat{n}-1}{2^k} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{12} f\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh + \frac{4}{2^{4k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\\
&+ \frac{32}{15}\frac{1}{2^{5k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) + \int_0^{\frac{1}{2^{k-1}}} \frac{h^5}{12} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\\
&+ \frac{16}{9}\frac{1}{2^{6k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) + \int_0^{\frac{1}{2^{k-1}}} \frac{h^6}{24} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\\
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^7}{72} f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh + \frac{4}{3}\frac{1}{2^{4k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
c_{n,0} &= 2^{\frac{k-1}{2}}\left[\frac{2}{2^k} f\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{2^{2k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{3}\frac{1}{2^{3k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{3}\frac{1}{2^{4k}} f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right]\\
&+ 2^{\frac{k-1}{2}}\left[\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24} f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right].
\end{aligned}
$$

Next,

$$
\begin{aligned}
c_{n,0}^2 &= \frac{2}{2^k}\left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{8}{9}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{9}\frac{1}{2^{7k}}\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2\\
&+ \frac{2^k}{2}\left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24} f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right)^2 + \frac{4}{2^{2k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\\
&+ \frac{8}{3}\frac{1}{2^{3k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{3}\frac{1}{2^{4k}} f\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\\
&+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{12} f\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh + \frac{8}{3}\frac{1}{2^{4k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\\
&+ \frac{4}{3}\frac{1}{2^{5k}} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{2^k}\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{12} f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\\
&+ \frac{8}{9}\frac{1}{2^{6k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{3}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\\
&+ \frac{2}{3}\frac{1}{2^{3k}}\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24} f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh.
\end{aligned}
$$

Since

$$|f^{'}(x)| \leq M_1, |f^{''}(x)| \leq M_2, |f^{'''}(x)| \leq M_3 \; and \; |f^{iv}(x)| \leq M_4, \; \forall x \in [0,1)$$

therefore,

$$\begin{aligned} ||e{*}_n^{(0)}||_2^2 &\leq \frac{2}{2^{3k}}\left(M_1 + \frac{M_2}{2^k} + \frac{M_3}{2^{2k}} + \frac{M_4}{2^{3k}}\right)^2 . \\ \therefore E^{(5)}_{2^{k-1},0}(f) &= O\left(\frac{1}{2^k}\right). \end{aligned}$$

(ii) The error $e{*}_n^{(1)}(x)$ between $f(x)$ and its expression over any subinterval is defined as

$e{*}_n^{(1)}(x) = c_{n,0}\psi_{n,0}(x) + c_{n,1}\psi_{n,1}(x) - f(x) \; , x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right), n = 1, 2, 3, ...2^{k-1}$

$||e{*}_n^{(1)}||_2^2 = \int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (e{*}_n^{(1)}(x))^2 dx$

$$= \int\limits_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - c_{n,0}^2 - c_{n,1}^2.$$

Now,

$$\begin{aligned} c_{n,1} &= \sqrt{\frac{3}{2}}2^{\frac{k}{2}}\left[\frac{2}{3}\frac{1}{2^{2k}}f^{'}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{3}\frac{1}{2^{3k}}f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{2}{5}\frac{1}{2^{4k}}f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right] \\ &+ \sqrt{\frac{3}{2}}2^{\frac{k}{2}} \int\limits_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h - 1) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh. \end{aligned}$$

Next,

$$\begin{aligned} c_{n,1}^2 &= \frac{2}{3}\frac{1}{2^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{3}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{6}{25}\frac{1}{2^{7k}}\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 \\ &+ \frac{3}{2}2^k\left(\int\limits_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h - 1) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh\right)^2 \\ &+ \frac{2}{2^k}\int\limits_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h - 1) f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\ &+ \frac{4}{3}\frac{1}{2^{4k}}f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) + \frac{4}{5}\frac{1}{2^{5k}}f^{'}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) \\ &+ \frac{4}{5}\frac{1}{2^{6k}}f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) \end{aligned}$$

$$+ \quad \frac{2}{2^{2k}} \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h-1) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh$$

$$+ \quad \frac{6}{5}\frac{1}{2^{3k}} \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(2^k h-1) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh$$

Therefore,

$$||e^{*(1)}_{n}||_2^2 \quad \leq \quad \frac{2}{2^{5k}}\left(M_2+\frac{M_3}{2^k}+\frac{M_4}{2^{2k}}\right)^2.$$

Then,

$$E^{(6)}_{2^{k-1},1}(f) = O\left(\frac{1}{2^{2k}}\right).$$

(iii) The error $e^{*(2)}_{n}(x)$ between $f(x)$ and its expression over any subinterval is defined as
$e^{*(2)}_{n}(x) = c_{n,0}\psi_{n,0}(x)+c_{n,1}\psi_{n,1}(x)+c_{n,2}\psi_{n,2}(x)-f(x)$, $x\in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$, $n=1,2,3,...2^{k-1}$

$$||e^{*(2)}_{n}||_2^2 \quad = \quad \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (e^{(2)}_n(x))^2 dx$$

$$= \quad \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x))^2 dx - c^2_{n,0} - c^2_{n,1} - c^2_{n,2}.$$

Now,

$$c_{n,2} \quad = \quad < f(x), \psi_{n,2}(x) >$$

$$= \quad \sqrt{\frac{5}{2}}2^{\frac{k}{2}}\frac{2}{15}\left[\frac{1}{2^{3k}}f^{''}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{1}{2^{4k}}f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right]$$

$$+ \quad \sqrt{\frac{5}{2}}2^{\frac{k}{2}}\frac{1}{2}\left[\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(3h^2 2^{2k}-6h2^k+2) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right].$$

Next,

$$c^2_{n,2} \quad = \quad \frac{2}{45}\frac{1}{2^{5k}}\left(f^{''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{2}{45}\frac{1}{2^{7k}}\left(f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)\right)^2$$

$$+ \quad \left(\int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(3h^2 2^{2k}-6h2^k+2) f^{iv}\left(\frac{\hat{n}-1}{2^k}+\theta h\right) dh\right)^2 + \frac{4}{45}\frac{1}{2^{6k}} f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+ \quad \frac{1}{3}\frac{1}{2^{2k}} \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(3h^2 2^{2k} - 6h2^k + 2) f^{''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh$$

$$+ \quad \frac{1}{3}\frac{1}{2^{3k}} \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{24}(3h^2 2^{2k} - 6h2^k + 2) f^{'''}\left(\frac{\hat{n}-1}{2^k}\right) f^{iv}\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh.$$

Therefore,

$$||e^{*(2)}_n||_2^2 \quad \leq \quad \frac{2}{2^{7k}}\left(M_3 + \frac{M_4}{2^k}\right)^2.$$

Then,

$$E^{(7)}_{2^{k-1},2}(f) \quad = \quad O\left(\frac{1}{2^{3k}}\right).$$

(iv) The error $e^{*(3)}_n(x)$ between $f(x)$ and its expression over any subinterval is defined as
$e^{*(3)}_n(x) = c_{n,0}\psi_{n,0}(x) + c_{n,1}\psi_{n,1}(x) + c_{n,2}\psi_{n,2}(x) + c_{n,3}\psi_{n,3}(x) - f(x)$,
$x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$, $n = 1, 2, 3, ...2^{k-1}$
Similarly , it can be proved that

$$E^{(8)}_{2^{k-1},3}(f) \quad = \quad O\left(\frac{1}{2^{4k}}\right).$$

(v)

Following the proof of Theorem $(4.1)(iv)$ *we have*

$$\begin{aligned}
c_{n,m} \quad &= \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_{m+1}(t) - P_{m-1}(t))}{(2m+1)} - \frac{(P_{m+3}(t) - P_{m+1}(t))}{(2m+5)}\right] dt \\
&- \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_{m-1}(t) - P_{m-3}(t))}{(2m-3)} - \frac{(P_{m+1}(t) - P_{m-1}(t))}{(2m+1)}\right] dt \\
&= \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+1)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m+2}(t) - P_m(t))}{(2m+3)}\right] \\
&- \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+1)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_m(t) - P_{m-2}(t))}{(2m-1)}\right] \\
&- \quad \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+5)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m+4}(t) - P_{m+2}(t))}{(2m+7)}\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+5)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] \\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m-3)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_m(t)-P_{m-2}(t))}{(2m-1)}\right] \\
&+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m-3)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m-2}(t)-P_{m-4}(t))}{(2m-5)}\right] \\
&+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m+1)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] \\
&- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m+1)} \int_{-1}^{1} f^{'''}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{d(P_m(t)-P_{m-2}(t))}{(2m-1)}\right] \\
&= \left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+1)(2m+3)} \\
&\times \int_{-1}^{1} f^{iv}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_m(t)-P_{m-2}(t))}{(2m-1)} - \frac{(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] dt \\
&+ \left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+5)} \\
&\times \int_{-1}^{1} f^{iv}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_{m+4}(t)-P_{m+2}(t))}{(2m+7)} - \frac{(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] dt \\
&+ \left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m+1)} \\
&\times \int_{-1}^{1} f^{iv}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_m(t)-P_{m-2}(t))}{(2m-1)} - \frac{(P_{m+2}(t)-P_m(t))}{(2m+3)}\right] dt \\
&+ \left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m-3)} \\
&\times \int_{-1}^{1} f^{iv}\left(\frac{\hat{n}+t}{2^k}\right)\left[\frac{(P_m(t)-P_{m-2}(t))}{(2m-1)} - \frac{(P_{m-2}(t)-P_{m-4}(t))}{(2m-5)}\right] dt.
\end{aligned}
$$

$$
|c_{n,m}| \leq \left(\frac{1}{2^{9k}}\right)^{\frac{1}{2}} \frac{8\sqrt{6}M_2}{(2m-7)^4}, \qquad (\because |f^{iv}(x)| \leq M_2 \;\forall\, x \in [0,1)).
$$

Next,

$$
\begin{aligned}
\|f - S_{2^{k-1},M}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} C_{n,m}^2 \\
&\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \left(\left(\frac{1}{2^{9k}} \right)^{\frac{1}{2}} \frac{8\sqrt{6}M_2}{(2m-7)^4} \right)^2 \\
&= \frac{48M_2^2}{7} \frac{1}{2^{8k}} \frac{1}{(2M-5)^7} \\
\therefore E_{2^{k-1},M}^{(9)}(f) &= \sqrt{\frac{48}{7}} \frac{M_2}{2^{4k}} \frac{1}{(2M-5)^{\frac{7}{2}}} \\
&= O\left(\frac{1}{(2M-5)^{\frac{7}{2}}} \frac{1}{2^{4k}} \right), \ \forall\, M \geq 3.
\end{aligned}
$$

6 Conclusions

(1) After discussing the Legendre wavelet approximation of a function f with bounded third and fourth derivatives ,it is trivial to find out the wavelet estimators of a function f of bounded first and second derivatives .
(2)The estimates of the Theorems (4.1) an(4.2) are obtained as following:
$(i)E_{2^{k-1},0}^{(1)}(f) = O\left(\frac{1}{2^k}\right) \to 0 \; as \; k \to \infty$
$(ii)E_{2^{k-1},1}^{(2)}(f) = O\left(\frac{1}{2^{2k}}\right) \to 0 \; as \; k \to \infty$
$(iii)E_{2^{k-1},2}^{(3)}(f) = O\left(\frac{1}{2^{3k}}\right) \to 0 \; as \; k \to \infty$
$(iv)E_{2^{k-1},M}^{(4)}(f) = O\left(\frac{1}{(2M-3)^{\frac{5}{2}}} \frac{1}{2^{3k}}\right) \to 0 \; as \; k \to \infty, M \to \infty$
$(v)E_{2^{k-1},0}^{(5)}(f) = O\left(\frac{1}{2^k}\right) \to 0 \; as \; k \to \infty$
$(vi)E_{2^{k-1},1}^{(6)}(f) = O\left(\frac{1}{2^{2k}}\right) \to 0 \; as \; k \to \infty$
$(vii)E_{2^{k-1},2}^{(7)}(f) = O\left(\frac{1}{2^{3k}}\right) \to 0 \; as \; k \to \infty$
$(viii)E_{2^{k-1},3}^{(8)}(f) = O\left(\frac{1}{2^{4k}}\right) \to 0 \; as \; k \to \infty$
$(ix)E_{2^{k-1},M}^{(9)}(f) = O\left(\frac{1}{(2M-5)^{\frac{7}{2}}} \frac{1}{2^{4k}}\right) \to 0 \; as \; k \to \infty, M \to \infty$
Then
$E_{2^{k-1},0}^{(1)}(f), E_{2^{k-1},1}^{(2)}(f), E_{2^{k-1},2}^{(3)}(f), E_{2^{k-1},M}^{(4)}(f), E_{2^{k-1},1}^{(5)}(f), E_{2^{k-1},1}^{(6)}(f), E_{2^{k-1},2}^{(7)}(f),$ $E_{2^{k-1},3}^{(8)}(f), E_{2^{k-1},M}^{(9)}(f)$ are best possible Legendre wavelet approximation in Wavelet Analysis.
(3)Legendre wavelet estimators of a function f with bounded fourth order derivative is better and sharper than the estimator of a function f of bounded third order derivative.
(4) Legendre wavelet estimator of a function f of bounded higher order derivatives is better and sharper than the estimator of a function f of bounded less order derivatives.

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