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On cyclic multigroup family

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Abstract

In this paper, the concept of cyclic multigroup is studied from the preliminary knowledge of cyclic group which is a well-known concept in crisp environment. By using cyclic multigroups, we then delineate a cyclic multigroup family and investigate its structural properties. It is observed that the union of class of cyclic multigroups generated by $\mathcal{A}$ is a cyclic multigroup. However, the intersection is an identity cyclic multigroup. In particular, we obtain a series of class of cyclic multigroups generated by $\mathcal{A}$.

Keywords: Multiset, Multigroup, Cyclic multigroup, Cyclic multigroup family.

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1 Introduction

In set theory, repetition of objects are not allowed in a collection. This perspective rendered set almost irrelevant because many real life problems admit repetition. To remedy the inadequacy in the idea of sets, the concept of multisets was introduced in [6] as a generalization of sets by relaxing the restriction of distinctness on the nature of the objects forming a set. Multiset is very promising in mathematics, computer science, website design, etc. See [4, 5] for details.

Generalization of algebraic structures is playing a prominent role in the sphere of mathematics. One of such generalization of algebraic structures is the notion of multigroups. Multigroups are actually a generalization of groups and have come into the centre of interest. In [1], the multigroup proposed is analogous to fuzzy group [2] in that the underlying structure is a multiset. Although multigroup concept was earlier used in [9, 12] as an extension of group theory, however the recent definition of multigroup in [1] is adopted in this paper because it shows a strong analogy in the behaviour of group and makes it possible to extend some of the major notions and results of groups to that of multigroups. Some of the related works can be found in [3], [7], [8], [10], [11] etc.

The aim of this paper is to promote research and the development of multiset knowledge by studying cyclic multigroup family based on the sufficient condition for a multiset to be a cyclic multigroup.

2 Preliminaries

In this section, we give the preliminary definitions and results that will be required in this paper from $[1, 8]$.

Definition 2.1 Let $\mho$ be a non-empty set. A multiset A drawn from $\mho$ is characterized by a count function C_A defined as $C_A : \mho \longrightarrow \boldsymbol{\mathcal{D}}$, where $\mathcal{D}$ represents the set of non-negative integers.

For each $x \in \mho$, $C_A(x)$ is the characteristics value of x in A and indicates the number of occurrences of the element x in A. An expedient notation of A drawn from $\mho = \{x_1, x_2, \dots, x_n\}$ is $[x_1, x_2, \dots, x_n]_{C_A(x_1), C_A(x_2), \dots, C_A(x_n)}$ such that $C_A(x_i)$ is the number of times x_i occurs in A, $(i = 1, 2, \dots, n)$.

The class of all multisets over $\mho$ is denoted by $MS(\mho)$.

Definition 2.2 Let $A, B \in \mho$. Then A is a submultiset of B written as $A \subseteq B$ or $B \supseteq A$ if $C_A(x) \leq C_B(x), \forall\, x \in \mho$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$.

Definition 2.3 Let $A, B \in MS(\mho)$. Then the union and intersection denoted by $A \cup B$ and $A \cap B$ are respectively defined as follows:

$$C_{A \,\cup\, B}(x) = C_A(x) \vee C_B(x) = max\{C_A(x), C_B(x)\} \text{ and}$$

$$C_{A \,\cap\, B}(x) = C_A(x) \wedge C_B(x) = min\{C_A(x), C_B(x)\}, \forall\, x \in \mho.$$

Definition 2.4 Let $\{A_i\}_{i\in\Lambda}$ be an arbitrary family of multisets over $\mho$. Then for each $i \in \Lambda$, $\bigcup_{i\in\Lambda} A_i = \bigvee_{i\in\Lambda} C_{A_i}(x)$ and $\bigcap_{i\in\Lambda} A_i = \bigwedge_{i\in\Lambda} C_{A_i}(x)$.

Definition 2.5 The direct product of multisets A and B is defined as

$$A \times B \;=\; \left\{[x, y]_{C_{A\times B}}[(x, y)] \;\mid\; C_{A\times B}[(x, y)] = C_A(x)C_A(y)\right\}.$$

Definition 2.6 Let $\mho$ be a non-empty set. The sets of the form

$A_n = \{x \in \mho \mid C_A(x) \geq n,\;\; n \in \mathbb{Z}^+\}$ are called the n – level sets of A.

Definition 2.7 Let $\mho$ and ξ be two non-empty sets and $f : \mho \longrightarrow \xi$ be a mapping. Then the image $f(A)$ of a multiset $A \in MS(\mho)$ is defined as

$$C_{f(A)}(y) = \begin{cases} \bigvee_{f(x)=y} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & f^{-1}(y) = \emptyset \end{cases}$$

Definition 2.8 Let $\mathcal{X}$ be a group. By a multigroup over $\mathcal{X}$ we mean a count function $C_A : \mathcal{X} \longrightarrow \mathcal{D}$ such that

$$C_A(xy) \geq C_A(x) \wedge C_A(y), \forall\, x, y \in \mathcal{X} \text{ and } C_A(x^{-1}) \geq C_A(x), \forall\, x \in \mathcal{X}.$$

Moreover, an abelian multigroup over $\mathcal{X}$ is defined as a multigroup satisfying the condition $C_A(xy) \geq C_A(yx), \forall\, x, y \in \mathcal{X}$.

Let e be the identity element of $\mathcal{X}$. It can be easily verified that if A is a multigroup over a group $\mathcal{X}$, then $C_A(e) \geq C_A(x)$ and $C_A(x^{-1}) \geq C_A(x), \forall\, x \in \mathcal{X}$.

We denote the class of all multigroups over $\mathcal{X}$ by $MG(\mathcal{X})$.

Proposition 2.1 Let $A \in MS(\mho)$. Then $A \in MG(\mathcal{X})$ if and only if $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y), \forall\, x, y \in \mathcal{X}$.

Proposition 2.2 Let $A \in MG(\mathcal{X})$. Then $A_n,\ n \in \mathbb{Z}^+$ are subgroups of $\mathcal{X}$.

Proposition 2.3 Let $\mathcal{X}, \mathcal{Y}$ be groups and $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a homomorphism. If $A \in MG(\mathcal{X})$, then $f(A) \in MG(\mathcal{Y})$.

3 Cyclic Multigroup Family

Definition 3.1 Let $\mathcal{X} = \langle a \rangle$ be a cyclic group. If $\mathcal{A} = \left\{[a^n]_{C_{\mathcal{A}}(a^n)} \mid n \in \mathbb{Z}\right\}$ is a multigroup, then $\mathcal{A}$ is called a cyclic multigroup generated by $[a]_{C_{\mathcal{A}}(a)}$ and denoted by $\langle [a]_{C_{\mathcal{A}}(a)} \rangle$.

Proposition 3.1 If $\mathcal{A}$ is a cyclic multigroup and $m \in \mathbb{Z}^+$, then $\mathcal{A}^m = \left\{\left([a^n]_{C_{\mathcal{A}}(a^n)}\right)^m \mid n \in \mathbb{Z}\right\}$ is also a cyclic multigroup.

Proof. Let us show that $\mathcal{A}^m$ satisfies the two conditions in Definition 2.8. We can consider only its count function because the $m-th$ power of $\mathcal{A}$ effects just only the count function of $\mathcal{A}^m$.

Since $\mathcal{A}$ is a multigroup and $C_{\mathcal{A}}(a) \in \mathcal{D}$, we have $(C_{\mathcal{A}}(a^{n_1}a^{n_2}))^m \geq \left(C_{\mathcal{A}}(a^{n_1}) \wedge C_{\mathcal{A}}(a^{n_2})\right)^m = (C_{\mathcal{A}}(a^{n_1}))^m \wedge (C_{\mathcal{A}}(a^{n_2}))^m$ and consequently, $\left(C_{\mathcal{A}}(a^{-n})\right)^m \geq \left(C_{\mathcal{A}}(a^n)\right)^m$.

This completes the proof of the proposition.

Example 3.1 Let $\mathcal{X} = \langle a \rangle$ be a cyclic group of order 12 such that $C_{\mathcal{A}}(a^0) = t_0$, $C_{\mathcal{A}}(a^4) = C_{\mathcal{A}}(a^8) = t_1,\ C_{\mathcal{A}}(a^2) = C_{\mathcal{A}}(a^6) = C_{\mathcal{A}}(a^{10}) = t_2,\ C_{\mathcal{A}}(x) = t_3$ for other elements $x \in \mathcal{X}$, where $t_i \in \mathcal{D},\ 0 \leq i \leq 3$ with $t_1 > t_1 > t_2 > t_3$. It is clear that $\mathcal{A}$ is a multigroup over $\mathcal{X}$. Thus, $\mathcal{A} = \left\{[a^n]_{C_{\mathcal{A}}(a^n)} \mid n \in \mathbb{Z}\right\}$ is a cyclic multigroup generated by $[a]_{C_{\mathcal{A}}(a)}$.

Definition 3.2 Let e be the identity element of the group $\mathcal{X}$. We define the identity cyclic multigroup $\mathcal{E}$ by $\mathcal{E} = \left\{[e]_{C_{\mathcal{A}}}(e) \mid C_{\mathcal{A}}(e) \geq C_{\mathcal{A}}(a^n),\ n \in \mathbb{Z}\right\}$.

Proposition 3.2 If $m \leq n$, then the multigroup $\mathcal{A}^n$ is a submultigroup of $\mathcal{A}^m$.

Proof. Clearly $\mathcal{A}^n$ and $\mathcal{A}^m$ are multigroups by Definition 2.8. For every $a \in \mathcal{D}$, $a^m \leq a^n$ implies $\mathcal{A}^m \subseteq \mathcal{A}^n$ (since $C_{\mathcal{A}^m}(a) \leq C_{\mathcal{A}^n}(a)\ \forall\ a \in \mathcal{X}$).

Proposition 3.3 If $\mathcal{A}^i$ and $\mathcal{A}^j$ are cyclic multigroups, and $i < j$, then $\mathcal{A}^i \cup \mathcal{A}^j$ is also a cyclic multigroup for any $i, j \in \mathbb{Z}^+$.

Proof. It is sufficient to consider only count functions. Without loss of generality, let $i \leq j$. Since $\mathcal{A}^i \subseteq \mathcal{A}^j$, we have

$$\begin{aligned} C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^n a^m) = C_{\mathcal{A}^i}(a^n a^m) \vee C_{\mathcal{A}^j}(a^n a^m) &= C_{\mathcal{A}^j}(a^n a^m) \\ &\geq C_{\mathcal{A}^j}(a^n) \wedge C_{\mathcal{A}^j}(a^m) \\ &= C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^n) \wedge C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^m) \end{aligned}$$

and

$$\begin{aligned} C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^{-n}) &= C_{\mathcal{A}^i}(a^{-n}) \vee C_{\mathcal{A}^j}(a^{-n}) \\ &= C_{\mathcal{A}^i}(a^n) \vee C_{\mathcal{A}^j}(a^n) = C_{\mathcal{A}^i \cup \mathcal{A}^j}(a^n) \end{aligned}$$

Hence, $\mathcal{A}^i \cup \mathcal{A}^j$ is a cyclic multigroup.

Proposition 3.4 If $\mathcal{A}^i$ and $\mathcal{A}^j$ are cyclic multigroups, then $\mathcal{A}^i \cap \mathcal{A}^j$ is also a cyclic multigroup**.**

Proof. Similar to Proposition 3.3.

Remark 3.1 Since a cyclic group is an abelian group, it is obvious by Definition 2.8 that the cyclic multigroups $\mathcal{A}^m$, $\mathcal{A}^i \cup \mathcal{A}^j$ and $\mathcal{A}^i \cap \mathcal{A}^j$ are also abelian multigroups.

Definition 3.3 Let $\mathcal{A}$ be a cyclic multigroup, then the following class of cyclic multigroups $\{\mathcal{A}, \mathcal{A}^2, \mathcal{A}^3, \ldots, \mathcal{A}^m, \ldots, \mathcal{E}\}$ is called the cyclic multigroup family generated by $\mathcal{A}$ and denoted by $\langle \mathcal{A} \rangle$.

Proposition 3.5 Let $\langle \mathcal{A} \rangle = \{\mathcal{A}, \mathcal{A}^2, \mathcal{A}^3, \ldots, \mathcal{A}^m, \ldots, \mathcal{E}\}$. Then $\bigcup_{n=1}^{\infty} \mathcal{A}^n = \mathcal{A}$ and $\bigcap_{n=1}^{\infty} \mathcal{A}^n = \mathcal{E}$.

Proof. The proof is immediate from Propositions 3.3 and 3.4.

Proposition 3.6 Let $\mathcal{A}$ be a cyclic multigroup. Then $\mathcal{A} \subseteq \mathcal{A}^2 \subseteq \mathcal{A}^3 \subseteq \cdots \subseteq \mathcal{A}^n \subseteq \cdots \subseteq \mathcal{E}$.

Proof. It is known that $C_{\mathcal{A}}(a) \in \mathcal{D}$. Hence,
$C_{\mathcal{A}}(a) \leq \left(C_{\mathcal{A}^2}(a)\right)^2$, $C_{\mathcal{A}}(a^2) \leq \left(C_{\mathcal{A}^2}(a^2)\right)^2, \ldots,$ $C_{\mathcal{A}}(a^n) \leq \left(C_{\mathcal{A}^2}(a^n)\right)^2$.

By Definition 2.2, we have $\mathcal{A} \subseteq \mathcal{A}^2$. By generalizing it for any $i, j \in \mathbb{Z}^+$ with $i \leq j$, we then obtain $\left(C_{\mathcal{A}^i}(a)\right)^i \leq \left(C_{\mathcal{A}^j}(a)\right)^j, \left(C_{\mathcal{A}^i}(a^2)\right)^i \leq \left(C_{\mathcal{A}^j}(a^2)\right)^j, \dots, \left(C_{\mathcal{A}^i}(a^n)\right)^i \leq \left(C_{\mathcal{A}^j}(a^n)\right)^j$. So $\mathcal{A}^i \subseteq \mathcal{A}^j$ for any $i, j \in \mathbb{Z}^+$ with $i \leq j$, which means that $\mathcal{A} \subseteq \mathcal{A}^2 \subseteq \mathcal{A}^3 \subseteq \cdots \subseteq \mathcal{A}^n \subseteq \cdots$.

Finally, we have $\mathcal{E} = \bigcap_{n=1}^{\infty} \mathcal{A}^n$, which is immediate from Proposition 3.5 since

$$\operatorname*{Lim}_{n\to\infty} C_{\mathcal{A}}(a^n) = \begin{cases} t_0, & if \;\; a = e, \\ 0, & if \;\; a \neq e. \end{cases}$$

This completes the proof for the required relations.

Corollary 3.1 Let $\langle \mathcal{A} \rangle = \{\mathcal{A}, \mathcal{A}^2, \mathcal{A}^3, \dots, \mathcal{A}^m, \dots, \mathcal{E}\}$. Then
$$\mathcal{A} < \mathcal{A}^2 < \mathcal{A}^3 < \cdots < \mathcal{A}^m < \cdots < \mathcal{E}.$$

Proof. The proof is similar to Proposition 3.6.

Proposition 3.7 Let φ be a group homomorphism of a cyclic multigroup $\mathcal{A}$. Then the image of $\mathcal{A}$ under φ is a cyclic multigroup.

Proof. It is well known that in the theory of classical cyclic groups, the image of any cyclic group is a cyclic group and the homomorphic image of a multigroup is a multigroup (from Proposition 2.3). From these two results and Definition 2.8, it is clearly seen that the image of $\mathcal{A}$ under φ is a cyclic multigroup.

Proposition 3.8 Let $\mathcal{X}_n$ be the $n-$ level set of the cyclic group $\mathcal{X}$. If $i, j \in \mathbb{Z}^+$ such that $i < j$, then $\mathcal{A}_n^i$ is a subgroup of $\mathcal{A}_n^j$.

Proof. It is obvious that sets $\mathcal{X}_n$ and $\mathcal{X}_n^m$ are cyclic subgroups of $\mathcal{X}_n$ in crisp sense. Since $i < j$, then $\mathcal{A}_n^j(a) \geq \mathcal{A}_n^i(a) \geq n$, $\forall\, a \in \mathcal{X}_n^j$. Thus, $\mathcal{X}_n^i \subseteq \mathcal{X}_n^j$. Therefore, $\mathcal{X}_n^i$ is a subgroup of $\mathcal{X}_n^j$.

Remark 3.2 From Propositions 3.6 and 3.8, we have that a normal series of $\mathcal{X}$ is a finite sequence $\mathcal{X}_n^m, \mathcal{X}_n^{m-1}, \dots, \mathcal{X}_n$ of normal level subgroups of $\mathcal{X}$ such that $\mathcal{X}_n^m > \mathcal{X}_n^{m-1} > \cdots > \mathcal{X}_n$ since $\mathcal{X}$ is a cyclic group.

Proposition 3.9 Let $\{\mathcal{A}^m, \mathcal{A}^{m-1}, \dots, \mathcal{A}\}$ be a finite cyclic multigroup family. Then $\mathcal{A}^m \times \mathcal{A}^{m-1} \times \dots \times \mathcal{A} = \mathcal{A}^m$.

Proof. It is easily verified using the definition of product of multigroups and Proposition 3.6.

4 Conclusion

The paper introduced the concept of cyclic multigroup family and investigated its related structure properties. For future studies, one can extend this idea to other non-classical algebraic structures such as soft group, rough group, neutrosophic group and smooth group.

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