Classification of Hyper MV-algebras of Order 3

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Abstract

In this paper, we investigated the number of hyper MV-algebras of order 3. In fact, we prove that there are 33 hyper MV-algebras of order 3, up to isomorphism.

Key words: hyper MV-algebra

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1 Introduction

The concept of MV-algebras was introduced by Chang in [1] in order to show Lukasiewicz logic to be standard complete, i.e. complete with respect to evaluations of propositional variables in the real unit interval $[0, 1]$. In [6], Mundici showed that any MV-algebra is an interval of an Abelian lattice ordered group with a strong unit. Also, he introduced the concept of state on MV-algebra. Georgescu and Iorgulescu [2] introduced a new non-commutative algebraic structures, which were called pseudo MV-algebras. It can be obtained by dropping commutative axioms in MV-algebras, which are a generalization of MV-algebras. The hyper structure theory was introduced by F. Marty [5] at the 8th congress of Scandinavian Mathematicians in 1934. Since then many researches have worked in these areas. Recently in [4], Sh. Ghorbani, A. Hasankhni and E. Eslami applied the hyper structure to MV-algebras and introduced the concept of a hyper MV-algebra which is a generalization of an MV-algebra and investigated some related results. Now, in this paper we find all hyper MV-algebras of order 3.

2 Preliminary

Definition 2.1. [1] An MV-algebra $(X, \oplus, ^*, 0)$ is a set X equipped with a binary operation $\oplus$, a unary operation $*$ and a constant 0 satisfying the following equations:

(MV_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(MV_2) $x \oplus y = y \oplus x$,

(MV_3) $x \oplus 0 = x$,

(MV_4) $(x^*)^* = x$,

(MV_5) $x \oplus 0^* = 0^*$,

(MV_6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,

for all $x, y, z \in X$.

Definition 2.2. [3]

A hyperalgebra $(M, \oplus, ^*, 0)$ with a hyperoperation $\oplus : M \times M \longrightarrow \mathcal{P}^*(M)$, a unary operation $^* : M \longrightarrow M$ and a constant 0, is said to be a hyper MV-algebra if and only if satisfies the following axioms, for all $x, y, z \in M$:

(HMV_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(HMV_2) $x \oplus y = y \oplus x$,
(HMV_3) $(x^*)^* = x$,
(HMV_4) $0^* \in x \oplus 0^*$,
(HMV_5) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(HMV_6) $0^* \in x \oplus x^*$,
(HMV_7) If $x \leqslant y$ and $y \leqslant x$, then $x = y$,

where $x \leqslant y$ is defined by $0^* \in x^* \oplus y$. For every $X, Y \subseteq M$, $X \leqslant Y$ if there exist $x \in X$ and $y \in Y$ such that $x \leqslant y$. We define $1 = 0^*$

Theorem 2.3. *[3] Let $(M, \oplus, ^*, 0)$ be a hyper-MV algebra. Then for all $x, y, z \in M$ and for all non-empty subsets A, B and C of M the following hold:*

(i) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
(ii) $0 \leqslant x \leqslant 1$, $x \leqslant x$ *and* $A \leqslant A$,
(iii) If $x \leqslant y$ *then* $y^* \leqslant x^*$ *and* $A \leqslant B$ *implies* $B^* \leqslant A^*$,
(iv) If $x \leqslant 0$ *or* $1 \leqslant x$, *then* $x = 0$ *or* $x = 1$, *respectively,*
(v) $0 \oplus 0 = \{0\}$,
(vi) $x \in x \oplus 0$,
(vii) If $x \oplus 0 = y \oplus 0$, *then* $x = y$.

3 Classification of hyper MV-algebras of order 3

In this section we try to find all hyper MV-algebras of order 3, up to isomorphism.

Theorem 3.1. *Let M be a hyper MV-algebra and x be an element of M such that $0 \oplus x = \{x\}$ and $x^* = x$. Then the following statements hold:*
(i) $(1 \oplus x)^* \oplus x = \{x\}$,
(ii) $(1 \oplus x)^* \oplus 1 = x \oplus x$,
(iii) $x \notin 1 \oplus x$ *and* $0 \notin 1 \oplus x$.

Proof. Since $0^* = 1$, then by hypothesis and $(HMV5)$;

$$(1 \oplus x)^* \oplus x = (0^* \oplus x)^* \oplus x = (x^* \oplus 0)^* \oplus 0 = (x \oplus 0)^* \oplus 0 = x^* \oplus 0 = x \oplus 0 = \{x\}$$

$$\begin{aligned}(1 \oplus x)^* \oplus 1 &= (x \oplus 1)^* \oplus 1 = ((x^*)^* \oplus 1)^* \oplus 1 = \\ &= (1^* \oplus x^*)^* \oplus x^* = (0 \oplus x)^* \oplus x^* = x^* \oplus x^* = x \oplus x\end{aligned}$$

and so (i) and (ii) hold.

(iii) If $x \in 1 \oplus x$, then $x = x^* \in (1 \oplus x)^*$ and so $x \oplus x = x^* \oplus x \subseteq (1 \oplus x)^* \oplus x$. By (i), $x \oplus x \subseteq \{x\}$. Hence $x \oplus x = \{x\}$. Now, since by (HMV_6), $1 = 0^* \in x \oplus x^* = x \oplus x = \{x\}$, then $x = 1$ and so $0 = 1^* = x^* = x = 1$, which is a contradiction. Hence $x \notin 1 \oplus x$. Now, let $0 \in 1 \oplus x$. Then $1 = 0^* \in (1 \oplus x)^*$ and so $1 \oplus x \subseteq (1 \oplus x)^* \oplus x$. By (i), $1 \oplus x \subseteq \{x\}$. Thus $1 \oplus x = \{x\}$, which is a contradiction. Hence $0 \notin 1 \oplus x$.

Note. From now one in this paper, we let $M = \{0, a, 1\}$ be a hyper MV-algebra of order 3.

Theorem 3.2. *(i)* $1 \leq 1$, $0 \leq 0$, $a \leq a$, $0 \leq 1$ *and* $0 \leq a$,
(ii) $a \not\leq 0$,
(iii) $a^* = a$,
(iv) $1 \in 1 \oplus a$.

Proof. (i). By Theorem 2.3(ii), the proof is clear.

(ii). By Theorem 2.3(iv), the proof is clear.

(iii). By Definition 2.2, $0^* = 1$ and by (HMV_3), $0 = (0^*)^* = 1^*$. Now, if $a^* = 1$, then $0 = 1^* = (a^*)^* = a$, which is a contradiction. By similar way, if $a^* = 0$, then $1 = 0^* = (a^*)^* = a$, which is a contradiction. Hence, $a^* = a$.

(iv). By (HMV_4), $1 = 0^* \in 0^* \oplus a = 1 \oplus a$. □

Theorem 3.3. *If $0 \oplus a = \{a\}$ or $1 \oplus a = \{1\}$, then M is an MV-algebra.*

Proof. Let $0 \oplus a = \{a\}$. Since $a^* = a$, then by Theorem 3.1(iii), $a \notin 1 \oplus a$ and $0 \notin 1 \oplus a$ and so $1 \oplus a = \{1\}$.

Moreover, By Theorem 3.1(iii) and (i), $0 \notin 1 \oplus 0$ and $(1 \oplus 0)^* \oplus 0 = \{0\}$. Since $0 \notin \{a\} = 0 \oplus a$ and $0 \notin 1 \oplus 0$, then $(1 \oplus 0)^* = \{0\}$ and so $1 \oplus 0 = \{1\}$. By Theorem 3.1(i) and (ii), $0 \oplus 1 = \{1\} = (1 \oplus a)^* \oplus 1 = a \oplus a$. Hence $a \oplus a = \{1\}$. Now, by (HMV_1),

$$1 \oplus 1 = (a \oplus a) \oplus 1 = a \oplus (1 \oplus a) = a \oplus 1 = \{1\}.$$

Therefore, $x \oplus y$ is singleton for all $x, y \in M$ and so M is an MV-algebra. □

Now, if $1 \oplus a = \{1\}$, then $\{0\} = \{1^*\} = (1 \oplus a)^*$ and so $0 \oplus a = (1 \oplus a)^* \oplus a$. By (HMV_5),

$$0 \oplus a = (1 \oplus a)^* \oplus a = 0 \oplus (0 \oplus a)^*.$$

By Theorem 3.2, $a \not< 0$, $1 \notin 0 \oplus a$. If $0 \in 0 \oplus a$, then $0 \oplus a = \{0, a\}$ and

$$\begin{aligned}\{0, a\} = 0 \oplus a = 0 \oplus (0 \oplus a)^* = 0 \oplus \{0, a\}^* = \\ = 0 \oplus \{1, a\} = (0 \oplus 1) \cup (0 \oplus a) = (0 \oplus 1) \cup \{0, a\}.\end{aligned}$$

Hence $0 \oplus 1 \subseteq \{0, a\}$. By $(HMV4)$, $1 \in 0 \oplus 1$. Thus $1 \in \{0, a\}$, which is a contradiction. Thus $0 \notin 0 \oplus a$ and so $0 \oplus a = \{a\}$. Therefore, M is a same MV-algebra, which is as follows:

$\oplus_1$	0	a	1
0	$\{0\}$	$\{a\}$	$\{1\}$
a	$\{a\}$	$\{1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

Definition 3.4. We call a hyper MV-algebra is proper, if it is not an MV-algebra.

Lemma 3.5. *Let $M = \{0, a, 1\}$ be a proper hyper MV-algebra of order 3. Then*

(i) $0 \oplus a = \{0, a\}$,
(ii) $0 \oplus 1 = \{1\}$, $\{0, 1\}$ *or* M,
(iii) $a \oplus a = \{1\}$, $\{0, 1\}$, $\{1, a\}$ *or* M,
(iv) $1 \oplus a = \{0, 1\}$, $\{1, a\}$ *or* M,
(v) $1 \oplus 1 = \{1\}$, $\{0, 1\}$ $\{1, a\}$ *or* M,
(vi) *If* $a \oplus a = \{1\}$, *then* $0 \oplus 1 = M$.

Proof. (i). Since $a \not< 0$, then $1 \notin 0 \oplus a$. By Theorem 2.3 (vi), $a \in 0 \oplus a$. Thus $0 \oplus a = \{a\}$ or $\{0, a\}$. If $0 \oplus a = \{a\}$, then by Theorem 3.3, M is not proper. Thus $0 \oplus a = \{0, a\}$

(ii). Since $0 \leq 0$, then $1 = 0^* \in 0^* \oplus 0 = 1 \oplus 0 = 0 \oplus 1$. Hence it is sufficient to show that $0 \oplus 1 \neq \{1, a\}$. Let $0 \oplus 1 = \{1, a\}$, by the contrary. Then by (HMV_1),

$$\{1, a\} = 0 \oplus 1 = (0 \oplus 0) \oplus 1 = (0 \oplus 1) \oplus 0 = \{1, a\} \oplus 0 = \{0, a, 1\},$$

which is impossible. Therefore, $0 \oplus 1 \neq \{1, a\}$ and so $0 \oplus 1 = \{1\}, \ \{0, 1\}$ or M.

(iii), (v). Since $a \leq a$ and $0 \leq 1$, then $1 \in a \oplus a$ and $1 \in 1 \oplus 1$ and so (v) and (iii) are hold.

(iv). Since $0 \leq a$, then $1 \in 1 \oplus a$. By Theorem 3.3, if $a \oplus 1 = \{1\}$, then M is an MV algebra which is impossible. Hence $1 \oplus a = \{0, 1\}, \ \{1, a\}$ or M.

(vi). Let $a \oplus a = \{1\}$. Then by (HMV_1),

$$0 \oplus 1 = 0 \oplus (a \oplus a) = (0 \oplus a) \oplus a = \{0, a\} \oplus a = (0 \oplus a) \cup (a \oplus a) = M.$$

By Lemma 3.5 (ii), we know that $0 \oplus 1 = \{1\}$, $\{0, 1\}$ or M. So, for the classification of all hyper MV-algebras of order 3, we consider the following three cases.

Case 1: $0 \oplus 1 = \{1\}$

Lemma 3.6. *Let $M = \{0, a, 1\}$ be a proper hyper MV-algebra of order 3 and $0 \oplus 1 = \{1\}$. Then*

(i) $a \oplus a = \{1, a\}$ or M,

(ii) $1 \oplus 1 = \{1\}$,

(iii) $1 \oplus a = M$.

Proof. (i). By Lemma 3.5 (i) and (iii), $0 \oplus a = \{0, a\}$ and $1 \in a \oplus a$. Hence

$$(0 \oplus a) \oplus a = \{0, a\} \oplus a = (0 \oplus a) \cup (a \oplus a) = \{0, a\} \cup (a \oplus a) = M.$$

Since by (HMV_1), $(0 \oplus a) \oplus a = 0 \oplus (a \oplus a)$, then $0 \oplus (a \oplus a) = M$. By Lemma 3.5$(iii)$, $a \oplus a = \{1\}, \ \{0, 1\}, \ \{1, a\}$ or M. If $a \oplus a = \{1\}$, then $0 \oplus (a \oplus a) = 0 \oplus 1 = \{1\}$, which is a contradiction.

If $a \oplus a = \{0, 1\}$, then by Theorem 2.3(v), $0 \oplus (a \oplus a) = 0 \oplus \{0, 1\} = (0 \oplus 0) \cup (0 \oplus 1) = \{0, 1\}$, which is a contradiction. Hence, $a \oplus a = \{1, a\}$ or M.

(ii). By (HMV_5), and Theorem 2.3(v),

$$(1 \oplus 1)^* \oplus 1 = (0^* \oplus 1)^* \oplus 1 = (1^* \oplus 0)^* \oplus 0 = (0 \oplus 0)^* \oplus 0 = 1 \oplus 0 = \{1\}.$$

If $0 \in 1 \oplus 1$, then $1 \oplus 1 \subseteq (1 \oplus 1)^* \oplus 1 = \{1\}$ and so $0 \notin 1 \oplus 1$, which is a contradiction. If $a \in 1 \oplus 1$, then $a \oplus 1 \subseteq (1 \oplus 1)^* \oplus 1 = \{1\}$. Thus $a \oplus 1 = \{1\}$ and so by Theorem 3.3, M is an MV-algebra, which is a contradiction. Hence, $1 \oplus 1 = \{1\}$.

(iii). By Lemma 3.5, $1 \oplus a = \{0, 1\}$, $\{1, a\}$ or M. If $1 \oplus a = \{0, 1\}$, since by (HMV_1), $1 \oplus (1 \oplus a) = (1 \oplus 1) \oplus a = 1 \oplus a$, then $1 \oplus (1 \oplus a) = \{1\}$, which is a contradiction. If $1 \oplus a = \{1, a\}$, since by (HMV_1), $0 \oplus (1 \oplus a) = (0 \oplus 1) \oplus a = 1 \oplus a$, then $0 \oplus (1 \oplus a) = (0 \oplus 1) \cup (0 \oplus a) = M$, which is a contradiction. Hence, $1 \oplus a = M$.

Theorem 3.7. *There are two non-isomorphic proper hyper MV-algebras of order 3 such that $0 \oplus 1 = \{1\}$.*

Proof. According Theorem 3.6, if M is a proper hyper MV-algebra of order 3 and $0 \oplus 1 = \{1\}$, then we must investigate two following tables, which both of them are non-isomorphic hyper MV-algebras.

$\oplus_2$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{1\}$
a	$\{0, a\}$	$\{1, a\}$	$\{0, a, 1\}$
1	$\{1\}$	$\{0, a, 1\}$	$\{1\}$

$\oplus_3$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{0, a, 1\}$
1	$\{1\}$	$\{0, a, 1\}$	$\{1\}$

Case 2: $0 \oplus 1 = \{0, 1\}$

Lemma 3.8. *Let $M = \{0, a, 1\}$ be a proper hyper MV-algebra of order 3 and $0 \oplus 1 = \{0, 1\}$. Then*
(i) $(a \oplus a) \cup (1 \oplus a) = M$,
(ii) $a \oplus 1 = \{a, 1\}$ or M,
(iii) $a \oplus a = \{a, 1\}$ or M,
(iv) $1 \oplus 1 = \{0, 1\}$ or $\{1\}$.

Proof. (i). Let $0 \oplus 1 = \{0, 1\}$. By Theorem 3.5(iv), since $1 \in 1 \oplus a$, by (HMV_1),

$$(0 \oplus a) \oplus 1 = (0 \oplus 1) \oplus a = \{0, 1\} \oplus a = (0 \oplus a) \cup (1 \oplus a) = \{0, a\} \cup (1 \oplus a) = M.$$

On the other hands

$$(0 \oplus a) \oplus 1 = \{0, a\} \oplus 1 = (0 \oplus 1) \cup (a \oplus 1) = \{0, 1\} \cup (a \oplus 1).$$

Thus $\{0,1\} \cup (a \oplus 1) = M$ and so $a \in a \oplus 1$. New, we consider two cases $0 \in a \oplus 1$ or $0 \neq a \oplus 1$. If $0 \in a \oplus 1$, since by Theorem 3.5, $1 \in a \oplus 1$, then $a \oplus 1 = M$ and so $(a \oplus a) \cup (1 \oplus a) = M$. Now, if $0 \neq a \oplus 1$, then by Theorem 3.5, $a \in a \oplus 1$. Hence by Theorem 3.2(iv), $\{1, a\} \subseteq a \oplus 1$. Thus

$$M = (0 \oplus 1) \cup (a \oplus 1) = \{0, a\} \oplus 1 = \{1, a\}^* \oplus 1 \subseteq (a \oplus 1)^* \oplus 1 \subseteq M$$

and so $(a \oplus 1)^* \oplus 1 = M$. On the other hands, by (HMV_5), $(a \oplus 1)^* \oplus 1 = (0 \oplus a)^* \oplus a$. Hence $(0 \oplus a)^* \oplus a = M$. Since $0 \oplus a = \{0, a\}$, then

$$M = (0 \oplus a)^* \oplus a = \{1, a\} \oplus a = (1 \oplus a) \cup (a \oplus a).$$

(ii). By Lemma 3.5(iv), it is enough to show that $1 \oplus a = \{0, 1\}$. Let $0 \in a \oplus 1$, by the contrary. Since by Lemma 3.5(iv) and (i), $0 \oplus a = \{0, a\}$ and $1 \in 1 \oplus a$, then

$$(0 \oplus 1) \oplus a = \{0, 1\} \oplus a = (0 \oplus a) \cup (1 \oplus a) = M.$$

Thus by (HMV_1),

$$M = (0 \oplus 1) \oplus a = (0 \oplus a) \oplus 1 = \{0, 1\} \cup (1 \oplus a).$$

and so $a \in 1 \oplus a$. Hence $a \oplus 1 \neq \{0, 1\}$ and so by lemma 3.5(iv), $a \oplus 1 = \{a, 1\}$ or M.

(iii). By Lemma 3.5(i), $0 \oplus a = \{0, a\}$. Now, since $1 \in a \oplus a$, then

$$(0 \oplus a) \oplus a = \{0, a\} \oplus a = (0 \oplus a) \cup (a \oplus a) = M.$$

Hence, by (HMV_1), $0 \oplus (a \oplus a) = (0 \oplus a) \oplus a = M$. Since $a \notin 0 \oplus 0$ and $a \notin 0 \oplus 1$, then $a \in a \oplus a$. Hence $a \oplus a = \{a, 1\}$ or M.

(iv). Let $a \in 1 \oplus 1$. By (HMV_5),

$$a \oplus 1 = a^* \oplus 1 \subseteq (1 \oplus 1)^* \oplus 1 = (0 \oplus 0)^* \oplus 0 = \{0, 1\}.$$

which is a contradiction by (i). Hence $a \notin 1 \oplus 1$ and so by Lemma 3.5(v), $1 \oplus 1 = \{0, 1\}$ or $\{1\}$.

Theorem 3.9. *There are 6 non-isomorphic proper hyper MV-algebras of order 3 such that $0 \oplus 1 = \{0, 1\}$.*

Proof. By Lemma 3.8 (iii), $a \oplus a = \{a, 1\}$ or M. If $a \oplus a = \{a, 1\}$, then by Lemma 3.8 (ii), $a \oplus 1 = \{a, 1\}$ or M. By Lemma 3.8 (i), if $a \oplus a = \{a, 1\}$, then $a\oplus 1\neq\{a,1\}$. Hence we must investigate 2 following tables which both of them are hyper MV-algebras.

$\oplus_4$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,1\}$
a	$\{0,a\}$	$\{a,1\}$	$\{0,a,1\}$
1	$\{0,1\}$	$\{0,a,1\}$	$\{1\}$

$\oplus_5$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,1\}$
a	$\{0,a\}$	$\{a,1\}$	$\{0,a,1\}$
1	$\{1\}$	$\{0,a,1\}$	$\{0,1\}$

Now, if $a\oplus a=M$, then by Lemma 3.8 (ii) and (iv), $a\oplus 1=\{a,1\}$ or M and $1\oplus 1=\{0,1\}$ or $\{1\}$. Thus we must investigate 4 following tables, which all of them are hyper MV-algebras.

$\oplus_6$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{a,1\}$
1	$\{0,1\}$	$\{a,1\}$	$\{1\}$

$\oplus_7$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{a,1\}$
1	$\{1\}$	$\{a,1\}$	$\{0,1\}$

$\oplus_8$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,1\}$	$\{0,a,1\}$	$\{1\}$

$\oplus_9$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,1\}$	$\{0,a,1\}$	$\{0,1\}$

Case 3: $0\oplus 1=M$

Lemma 3.10. *Let $M=\{0,a,1\}$ be a proper hyper MV-algebra of order 3 such that $0\oplus 1=M$. Then*

(i) $(a\oplus a)\cup(1\oplus a)=M$,

(ii) If $a\oplus a=\{1\}$, then $a\oplus 1=1\oplus 1=M$,

(iii) If $a\oplus a=\{0,1\}$, then $a\oplus 1=\{a,1\}$ or M and if $a\oplus 1=\{a,1\}$, then $1\oplus 1=\{1\},\{0,1\}$ or M,

(iv) If $a\oplus a=\{a,1\}$, then $a\oplus 1=\{0,1\}$ or M and if $a\oplus 1=\{0,1\}$, then $1\oplus 1=\{a,1\}$ or M,

(v) If $a\oplus a=M$ and $a\oplus 1=\{1,a\}$, then $1\oplus 1=\{1\},\{0,1\}$ or M,

(vi) If $a\oplus a=M$ and $a\oplus 1=\{0,1\}$, then $1\oplus 1=\{0,1\},\{a,1\}$ or M.

Proof.

(i). Since by Lemma 3.5(iv), $1\in 1\oplus a$, then $M=0\oplus 1=1^*\oplus 1\subseteq (a\oplus 1)^*\oplus 1$ and so $(a\oplus 1)^*\oplus 1=M$. Hence by (HMV_5), $(0\oplus a)^*\oplus a=(a\oplus 1)^*\oplus 1=M$ and so by Lemma 3.5(i),

$$M=(0\oplus a)^*\oplus a=\{0,a\}^*\oplus a=\{1,a\}\oplus a=(1\oplus a)\cup a\oplus a.$$

(ii). Let $a \oplus a = \{1\}$. Since $1 \in 1 \oplus a$, then by (HMV_5) and Lemma 3.5(i),

$$\begin{aligned} 1 \oplus a &= (1 \oplus a) \cup (a \oplus a) = \{1, a\} \oplus a = \{0, a\}^* \oplus a = (0 \oplus a)^* \oplus a \\ &= (a \oplus 0)^* \oplus 0 = \{1, a\} \oplus 0 = (1 \oplus 0) \cup (a \oplus 0) \\ &= M \end{aligned}$$

Now, since $a \oplus a = \{1\}$ and $1 \oplus a = M$, then by (HMV_1),

$$\begin{aligned} 1 \oplus 1 &= (a \oplus a) \oplus (a \oplus a) = a \oplus (a \oplus (a \oplus a)) \\ &= a \oplus (a \oplus 1) = a \oplus M = (a \oplus 1) \cup (a \oplus a) \cup (a \oplus 0) = M. \end{aligned}$$

(iii). If $a \oplus a = \{0, 1\}$, then by (i) and Lemma 3.5(iv), $a \oplus 1 = \{a, 1\}$ or M. Let $a \oplus 1 = \{a, 1\}$. If $1 \oplus 1 = \{a, 1\}$, then by (HMV_1) and (i),

$$\begin{aligned} M &= (a \oplus a) \cup (1 \oplus a) = \{a, 1\} \oplus a = (1 \oplus 1) \oplus a \\ &= 1 \oplus (1 \oplus a) = 1 \oplus \{1, a\} = (1 \oplus 1) \cup (1 \oplus a) \\ &= (1 \oplus 1) \ \cup \ \{1, a\} \end{aligned}$$

Hence $0 \in 1 \oplus 1 = \{a, 1\}$, which is a contradiction. Thus $1 \oplus 1 \neq \{a, 1\}$ and so by Lemma 3.5(v), $1 \oplus 1 = \{1\}, \{0, 1\}$ or M.

(iv). By (i), if $a \oplus a = \{a, 1\}$, then $a \oplus 1 = \{0, 1\}$ or M.

If $a \oplus 1 = \{0, 1\}$, then by (HMV_1),

$$\begin{aligned} M &= \{0, a\} \cup (1 \oplus a) = \{0, 1\} \oplus a = (1 \oplus a) \oplus a \\ &= 1 \oplus (a \oplus a) = 1 \oplus \{a, 1\} = (1 \oplus a) \cup (1 \oplus 1) \\ &= \{0, 1\} \cup (1 \oplus 1) \end{aligned}$$

Hence $a \in 1 \oplus 1$. By Lemma 3.5(v), $1 \oplus 1 = \{1, a\}$ or M.

(v). Let $a \oplus a = M$ and $1 \oplus a = \{1, a\}$. If $1 \oplus 1 = \{a, 1\}$, then by (HMV_1),

$$\begin{aligned} M &= (a \oplus a) \cup (1 \oplus a) = \{1, a\} \oplus a = (1 \oplus 1) \oplus a = 1 \oplus (1 \oplus a) \\ &= 1 \oplus \{1, a\} = (1 \oplus 1) \cup (1 \oplus a) \\ &= (1 \oplus 1) \cup \{1, a\} \end{aligned}$$

Hence $0 \in 1 \oplus 1 = \{a, 1\}$, which is impossible. Thus $1 \oplus 1 \neq \{1, a\}$ and so by Lemma 3.5(v), $1 \oplus 1 = \{1\}, \ \{0, 1\}$ or M.

(vi). Let $a \oplus a = M$ and $1 \oplus a = \{0, 1\}$. Then by (HMV_1),

$$(1 \oplus 1) \oplus a = 1 \oplus (1 \oplus a) = 1 \oplus \{0, 1\} = (0 \oplus 1) \cup (1 \oplus 1) = M.$$

Now, if $1 \oplus 1 = \{1\}$, then $1 \oplus a = (1 \oplus 1) \oplus a = M$, which is a contradiction. Hence $1 \oplus 1 \neq \{1\}$ and so by Theorem 3.5(v), $1 \oplus 1 = \{0, 1\}, \{a, 1\}$ or M

Theorem 3.11. *There are 24 non-isomorphic proper hyper MV-algebras of order 3 such that* $0 \oplus 1 = M$.

Proof. By Lemma 3.5 (*iii*), $a \oplus a = \{1\}$, $\{0, 1\}$, $\{1, a\}$ or M. If $a \oplus a = \{1\}$, then by Lemma 3.10 (*ii*), $a \oplus 1 = 1 \oplus 1 = M$ and so we must investigate the following table, which is a hyper MV-algebra.

$\oplus_{10}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{0, a, 1\}$

If $a \oplus a = \{0, 1\}$, then by Lemma 3.10 (*iii*), $a \oplus 1 = \{a, 1\}$ or M and if $a \oplus 1 = \{a, 1\}$, then $1 \oplus 1 = \{1\}$, $\{0, 1\}$ or M. Thus we must investigate the following 3 cases which all of them are hyper MV-algebras.

$\oplus_{11}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, 1\}$	$\{a, 1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{1\}$

$\oplus_{12}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, 1\}$	$\{a, 1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{0, 1\}$

$\oplus_{13}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, 1\}$	$\{a, 1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{0, a, 1\}$

If $a \oplus 1 = M$, then by Lemma 3.5 (*v*), $1 \oplus 1 = \{1\}$, $\{0, 1\}$, $\{1, a\}$ or M. Hence we must investigate the following 4 cases which all of them are hyper MV-algebras.

$\oplus_{14}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{1\}$

$\oplus_{15}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{0, 1\}$

$\oplus_{16}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{a, 1\}$

$\oplus_{17}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{0, a, 1\}$

Now, if $a \oplus a = \{a, 1\}$, then by Lemma 3.10 (iv), $a \oplus 1 = \{0, 1\}$ or M and if $a \oplus 1 = \{0, 1\}$, then $1 \oplus 1 = \{a, 1\}$ or M. Hence we must investigate the following 2 cases which both of them are hyper MV-algebras.

$\oplus_{18}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, 1\}$
1	$\{0, a, 1\}$	$\{0, 1\}$	$\{a, 1\}$

$\oplus_{19}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, 1\}$
1	$\{0, a, 1\}$	$\{0, 1\}$	$\{0, a, 1\}$

If $a \oplus 1 = M$, then by Lemma 3.5 (v), $1 \oplus 1 = \{1\}$, $\{0, 1\}$, $\{a, 1\}$ or M and so we must investigate the following 4 cases which all of them are hyper MV-algebras.

$\oplus_{20}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{1\}$

$\oplus_{21}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{0, 1\}$

$\oplus_{22}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{a, 1\}$

$\oplus_{23}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{0, a, 1\}$
1	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{0, a, 1\}$

Now, let $a \oplus a = M$. Then by Lemma 3.10 (v), $a \oplus 1 = \{1, a\}$, $\{0, 1\}$ or M. If $a \oplus 1 = \{1, a\}$, then $1 \oplus 1 = \{1\}, \{0, 1\}$ or M. Thus we must investigate the following 3 cases which all of them are hyper MV-algebras.

$\oplus_{24}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{a, 1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{1\}$

$\oplus_{25}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{a, 1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{0, 1\}$

$\oplus_{26}$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{a, 1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{0, a, 1\}$

Also by Lemma 3.10 (v), if $a \oplus 1 = \{0, 1\}$, then $1 \oplus 1 = \{0, 1\}, \{a, 1\}$ or M. Hence we must investigate the following 3 cases which all of them are hyper

MV-algebras.

$\oplus_{27}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,1\}$
1	$\{0,a,1\}$	$\{0,1\}$	$\{0,1\}$

$\oplus_{28}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,1\}$
1	$\{0,a,1\}$	$\{0,1\}$	$\{a,1\}$

$\oplus_{29}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,1\}$
1	$\{0,a,1\}$	$\{0,1\}$	$\{0,a,1\}$

Finally, if $a\oplus 1 = M$, then by Lemma 3.5 (v), $1\oplus 1 = \{1\}, \{0,1\}, \{a,1\}$ or M. Hence we must investigate the following 4 cases which all of them are hyper MV-algebras.

$\oplus_{30}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{1\}$

$\oplus_{31}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{0,1\}$

$\oplus_{32}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{a,1\}$

$\oplus_{33}$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,a,1\}$	$\{0,a,1\}$	$\{0,a,1\}$

Corolary 3.12. *There are* 33 *non-isomorphic hyper MV-algebras of order* 3.

Proof. By Theorems 3.3, 3.7, 3.9 and 3.11, we have 33 non-isomorphic hyper MV-algebras of order 3. □

References

[1] C. C. Chang, *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc, 88 (1958), 467–490.

[2] G. Georgescu, A. Iorgulescu, *Pseudo-MV algebras*, Multi Valued Logic, **6**, (2001), 95-135.

[3] S. Ghorbani, E. Eslami and A. Hasankhani, *Quotient hyper MV-algebras*, Scientiae Mathematicae Japonicae, 3 (2007) 371–386.

[4] Sh. Ghorbani, A. Hasankhani, and E. Eslami, *Hyper MV-algebras*, Set-Valued Math. Appl, 1 (2008), 205–222.

[5] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandin aves, Stockholm (1934), 45–49.

[6] D. Mundici, *Interpretation of AFC^*-algebras in Lukasiewicz sentential calculus*, J. Funct. Anal, **65**, (1986), 15–63.