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Note on Heisenberg Characters of Heisenberg Groups

Alieh Zolfi [*]and Ali Reza Ashrafi [†]

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Abstract

An irreducible character χ of a group G is called a Heisenberg character, if $Ker\chi \supseteq [G, [G, G]]$. In this paper, the Heisenberg characters of the quaternion Heisenberg, generalized Heisenberg, polarised Heisenberg and three other types of infinite Heisenberg groups are computed.

Keywords: Heisenberg character, Heisenberg group.

1 Introduction

Suppose G is a finite group and V is a vector space over the complex field $\mathbb{C}$. A representation of G is a homomorphism $\varphi : G \longrightarrow GL(V)$, where $GL(V)$ denotes the group of all invertible linear transformations $V \longrightarrow V$ equipped with

[*]Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317-53153, I. R. Iran

[†]Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317-53153, I. R. Iran; ashrafi@kashanu.ac.ir

composition of functions. The commutator subgroup $[G, G]$ is the subgroup generated by all the commutators $[x, y] = xyx^{-1}y^{-1}$ of the group G.

An irreducible character χ of a group G is called a **Heisenberg character**, if $ker\chi \supseteq [G, [G, G]]$ [1]. Suppose $\varphi : G \longrightarrow GL(V)$ is an irreducible representation with irreducible character χ. Since $[G, G'] \leq Ker\chi$, $\overline{\varphi} : \frac{G}{[G,G']} \longrightarrow GL(V)$ is an irreducible representation of $\frac{G}{[G,G']}$. Conversely, we assume that $\chi \in Irr\left(\frac{G}{[G,G']}\right)$ and $\delta : \frac{G}{[G,G']} \longrightarrow GL(V)$ affords the irreducible character χ. If $\gamma : G \longrightarrow \frac{G}{[G,G']}$ denotes the canonical homomorphism then $\delta o\gamma : G \longrightarrow \frac{G}{[G,G']}$ is an irreducible representation for G and $Ker\ \delta o\gamma \supseteq [G, G']$. This proves that there is a one to one correspondence between Heisenberg characters of G and irreducible characters of $\frac{G}{[G,G']}$, see [2, 8] for details.

Marberg [8] in his interesting paper proved that the number of Heisenberg characters of the group $U_n(q)$ is a polynomial in $q - 1$ with nonnegative integer coefficients, with degree $n - 1$, and whose leading coefficient is the $(n - 1)-$th Fibonacci number. The present authors [1], characterized groups with at most five Heisenberg characters. The aim of this paper is to compute all Heisenberg characters of five classes of infinite Heisenberg groups. These are as follows:

1. Suppose $\mathbf{T}$ denotes the set of all complex numbers of unit modulus and $H = \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{T}$. Define $(y_1, x_1, z_1)(y_2, x_2, z_2) = (y_1 + y_2, x_1 + x_2, e^{-2\pi i y_2.x_1} z_1 z_2)$. It is easy to see that H is a group under this operation. This group is called the Heisenberg group of second type [5].

2. The polarised Heisenberg group H_n^3 is defined as the set of all triples in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ under the multiplication
$$(x, y, z)(a, b, c) = (x + a, y + b, z + c + \frac{1}{2}(x.b - y.a)),$$
see [3] for details.

3. Suppose $a = (a_1, a_2, \ldots, a_n)$ is an $n-$tuple in $\mathbb{R}^n$, where a_i's are positive real constants, $1 \leq i \leq n$. Following Tianwu and Jianxun [10], we define a group operation on $H_n^a = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ given by
$$(x, y, z)(r, s, t) = (x + r, y + s, z + t + \frac{1}{2}\sum_{j=1}^{n} a_j(r_j y_j - s_j x_j)).$$

This group is called the generalized Heisenberg group. In the mentioned paper, the authors proved that the group operation of the generalized Heisenberg group can be simplified in the following way:

Suppose $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Define $x * y = (x_1y_1, x_2y_2, \ldots, x_ny_n)$ and $ab = \sum_{j=1}^{n} a_jb_j$. If $c = (c_1, c_2, \ldots, c_n)$ and $\lambda \in \mathbb{R}$ then we can see that (i) $x * (y + c) = x * y + x * c$; (ii) $(x * y)c = x(y * c)$; and (iii) $(\lambda x) * y = \lambda(x * y)$. Therefore, the group operation of the generalized Heisenberg group can be written as $(x, y, z)(r, s, t) = (x + r, y + s, z + t + \frac{1}{2}((a * r)y - (a * s)x))$.

4. Suppose $\mathbb{H}$ denotes the set of all of quaternion numbers with three imaginary units i, j and k such that $i^2 = j^2 = k^2 = ijk = -1$. Following Liu and Wang [7], we define the quaternion Heisenberg group $\mathcal{N}$ as a nilpotent Lie group with underlying manifold $\mathbb{R}^4 \times \mathbb{R}^3$. The group structure is given by
$$(q, t)(p, s) = (q + p, t + s + \frac{1}{2}Im(\overline{p}q)),$$
where $p, q \in \mathbb{R}^4$ and $t, s \in \mathbb{R}^3$.

5. Following Qingyan and Zunwei [9], the Heisenberg group $\mathbb{H}^n$ of third type is a non-commutative nilpotent Lie group, with the underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$. The group operation can be given as:
$$(x_1, x_2, \ldots, x_{2n}, x_{2n+1})(x_1^{'}, x_2^{'}, \ldots, x_{2n}^{'}, x_{2n+1}^{'}) =$$
$$(x_1 + x_1^{'}, x_2 + x_2^{'}, \ldots, x_{2n} + x_{2n}^{'}, x_{2n+1} + x_{2n+1}^{'} + 2\sum_{j=1}^{n}(x_j^{'}x_{n+j} - x_jx_{n+j}^{'}).$$

6. Suppose $\mathcal{H}^n = \mathbb{C}^n \times \mathbb{R}$ with group law defined by $(z, t) \cdot (w, s) = (z + w, t + s + 2Im(z.\overline{w}))$. This is our sixth class of Heisenberg groups. Following Chang et al. [4], this group can be realized as the boundary of the Siegel upper half-space $\mathcal{U}_{n+1}$ in $\mathbb{C}^{n+1}$, where the group operation gives a group action on the hypersurface.

Throughout this paper our notation is standard and can be taken from the famous book of Isaacs [6]. Suppose G is a group and $\{\{e\} = A_0, A_1, \ldots, A_n = G\}$ is a set of normal subgroups of G such that
$$A_0 \lhd A_1 \lhd \ldots \lhd A_n = G. \tag{1}$$

The sequence (1) is called a **central series** for G, if $[G, A_{i+1}] \leq A_i$ in which $[G, H]$ denotes the subgroup of G generated by all commutators $ghg^{-1}h^{-1}$, where $g \in G, h \in H$. The group G is called **nilpotent**, if it has a central series. The **nilpotency class** of G, $nc(G)$, is the length of its central series. The set of all irreducible characters of G is denoted by $Irr(G)$ and the trivial character of G is denoted by 1_G.

2 Main Results

The aim of this section is to compute the Heisenberg characters of five different types of Heisenberg groups. To do this, we first note that every linear character of a group G is Heisenberg. This proves that all irreducible characters of abelian groups are Heisenberg.

Lemma 2.1. *All irreducible characters of a group G are Heisenberg if and only if G is nilpotent of class two.*

Proof. Suppose $nc(G) = 2$. Then $[G, G'] = 1$ and so all irreducible characters are Heisenberg. If all irreducible characters are Heisenberg then $[G, G'] \leq \cap_{\chi \in Irr(G)}$ = $\{e\}$, as desired. $\square$

Theorem 2.2. *All irreducible characters of the Heisenberg groups H, H_n^3, H_n^a, $\mathcal{N}$, $\mathbb{H}^n$ and $\mathcal{H}^n$ are Heisenberg.*

Proof. Apply Lemma 2.1. Our main proof will consider five separate cases as follows:

1. *The Heisenberg Group H*. We first compute the derived subgroup H'. We have,

$$\begin{aligned}
H' &= \left\langle [(x, y, z), (x^{'}, y^{'}, z^{'})] \mid (x, y, z), (x^{'}, y^{'}, z^{'}) \in H, z = e^{i\Theta_1}, z^{'} = e^{i\Theta_2} \right\rangle \\
&= \langle (x + x^{'}, y + y^{'}, e^{i(\Theta_1+\Theta_2-2\pi x^{'}y)})(-x - x^{'}, -y - y^{'}, \\
&\quad e^{i(\Theta_1+\Theta_2+2\pi xy+2\pi x^{'}y^{'}+2\pi x^{'}y)}) \mid (x, y, z), (x^{'}, y^{'}, z^{'}) \in H, z = e^{i\Theta_1}, z^{'} = e^{i\Theta_2} \rangle \\
&= \left\langle (0, 0, e^{2\pi i(x.y^{'}-x^{'}.y)}) \mid x, y, x', y' \in \mathbb{R}^n \right\rangle .
\end{aligned}$$

Therefore,

$$[H, H'] = \big\langle (x, y, e^{i\Theta_1})(0, 0, e^{i\Theta_2})(-x, -y, e^{-i\Theta_1 - 2\pi ixy})(0, 0, e^{-i\Theta_2}) \mid (x, y, e^{i\Theta_1}) \in H, (0, 0, e^{i\Theta_2}) \in H' \big\rangle$$
$$= \langle (x, y, e^{i\Theta_1 + i\Theta_2})(-x, -y, e^{-i(\Theta_1 + \Theta_2 + 2\pi ixy)}) \mid (x, y, e^{i\Theta_1}) \in H, (0, 0, e^{i\Theta_2}) \in H' \rangle = \{(0, 0, 1)\}.$$

So, all irreducible characters of H are Heisenberg.
2. *The Heisenberg group* H_n^3. The commutator subgroup of H_n^3 can be computed as follows:

$$(H_n^3)' = \langle [(x, y, z), (x', y', z')] \mid (x, y, z), (x', y', z') \in H_n^3 \rangle$$
$$= \langle (x, y, z)(x', y', z')(x, y, z)^{-1}(x', y', z')^{-1} \mid (x, y, z), (x', y', z') \in H_n^3 \rangle$$
$$= \langle (0, 0, (x.y' - y.x') \mid x, y, x', y' \in \mathbb{R}^n \rangle.$$

On the other hand, $[H_n^3, (H_n^3)'] = \{(0, 0, 0)\}$ and so all irreducible characters of this group are Heisenberg.
3. *The Heisenberg group* H_n^a. Again, we first compute the commutator subgroup $(H_n^a)'$. We have,

$$(H_n^a)' = \langle [(x, y, z), (x', y', z')] \mid (x, y, z), (x', y', z') \in H_n^a, a \in \mathbb{R}_+^n \rangle$$
$$= \Big\langle (x + x', y + y', \frac{1}{2}((a * x')y - (a * y')x))(-x - x', -y - y',$$
$$+ \frac{1}{2}\big((a * -x')(-y) - (a * -y')(-x)\big) \mid (x, y, z), (x', y', z') \in H_n^a, a \in \mathbb{R}_+^n \rangle$$
$$= \Big\langle (0, 0, (a * x')y - (a * y')x) \mid x, y, x', y' \in \mathbb{R}^n, a \in \mathbb{R}_+^n \Big\rangle.$$

Therefore, $[H_n^a, (H_n^a)'] = \{(0, 0, 0)\}$. This shows that all irreducible characters are Heisenberg.
4. *The Heisenberg group* $\mathcal{N}$. By definition of this group, we have

$$\mathcal{N}' = \langle [(p, t), (q, s)] \mid (p, t), (q, s) \in \mathcal{N} \rangle$$
$$= \langle (p + q, t + s + \frac{1}{2} Im(\overline{q}p))(-p - q, -t - s + \frac{1}{2} Im(\overline{q}p)) \mid p, q \in \mathbb{R}^4, t, s \in \mathbb{R}^3 \rangle$$
$$= \langle (0, Im(\overline{q}p)) \mid p, q \in \mathbb{R}^4 \rangle.$$

Therefore, we have again $[\mathcal{N}, \mathcal{N}'] = \{(0, 0)\}$. Now apply Lemma 2.1 to deduce that all irreducible characters of this group are Heisenberg.

5. *The Heisenberg group* $\mathbb{H}^n$. By definition of this group,

$$\begin{aligned}
(\mathbb{H}^n)' &= \big\langle [(x_1,\ldots,x_{2n},x_{2n+1}),(x_1^{'},\ldots,x_{2n}^{'},x_{2n+1}^{'})] \mid \\
&\qquad (x_1,\ldots,x_{2n},x_{2n+1}),(x_1^{'},\ldots,x_{2n}^{'},x_{2n+1}^{'}) \in \mathbb{H}^n \big\rangle \\
&= \langle (x_1+x_1^{'},\ldots,x_{2n}+x_{2n}^{'},x_{2n+1}+x_{2n+1}^{'}+2\big(\sum_{j=1}^{n}(x_j^{'}x_{n+j}-x_jx_{n+j}^{'})\big) \\
&\quad (-x_1-x_1^{'},\ldots,-x_{2n}-x_{2n}^{'},-x_{2n+1}-x_{2n+1}^{'}+2\big(\sum_{j=1}^{n}(x_j^{'}x_{n+j}-x_jx_{n+j}^{'})\big) \\
&\qquad \mid (x_1,\ldots,x_{2n},x_{2n+1}),(x_1^{'},\ldots,x_{2n}^{'},x_{2n+1}^{'}) \in \mathbb{H}^n \rangle \\
&= \big\langle (0,\ldots,4\big(\sum_{j=1}^{n}(x_j^{'}x_{n+j}-x_jx_{n+j}^{'})\big) \\
&\qquad \mid x_1,\ldots,x_{2n},x_{2n+1},x_1^{'},\ldots,x_{2n}^{'},x_{2n+1}^{'} \in \mathbb{R}\big\rangle.
\end{aligned}$$

Therefore, $[\mathbb{H}^n,(\mathbb{H}^n)'] = \{(0,\ldots,0,0)\}$ and by Lemma 2.1 all irreducible characters of this group are Heisenberg.

6. *The Heisenberg group* $\mathcal{H}^n$. The derived subgroup of this group can be computed as follows:

$$\begin{aligned}
(\mathcal{H}^n)' &= \langle [(z,t),(w,s)] \mid (z,t),(w,s) \in \mathcal{H}^n \rangle \\
&= \langle (z,t)(w,s)(-z,-t)(-w,-s) \mid (z,t),(w,s) \in \mathcal{H}^n \rangle \\
&= \langle (z+w,t+s+2Im(z\overline{w}))(-z-w,-t-s+2Im(z\overline{w})) \mid z,w \in \mathbb{C}^n, \\
&\qquad\quad t,s \in \mathbb{R} \rangle \\
&= \langle (0,2Im(z\overline{w}-w\overline{z})) \mid z,w \in \mathbb{C}^n \rangle.
\end{aligned}$$

Therefore, $[\mathcal{H}^n,(\mathcal{H}^n)'] = \{(0,0,0)\}$ and by Lemma 2.1, all irreducible characters of this group are again Heisenberg.

This completes our argument. □

In the end of this paper we compute the factor groups of six types of Heisenberg groups modulus their centers.

Theorem 2.3. *The factor groups of all Heisenberg groups modulus their centers* *can be computed as:*

$$\begin{aligned}\frac{H}{Z(H)} &\cong \mathbb{R}^n \times \mathbb{R}^n, \frac{H_n^3}{Z(H_n^3)} \cong \mathbb{R}^n \times \mathbb{R}^n, \frac{H_n^a}{Z(H_n^a)} \cong \mathbb{R}^n \times \mathbb{R}^n \\ \frac{\mathcal{N}}{Z(\mathcal{N})} &\cong \mathbb{R}^4, \frac{\mathbb{H}^n}{Z(\mathbb{H}^n)} \cong \mathbb{R}^n \times \mathbb{R}^n, \frac{\mathcal{H}}{Z(\mathcal{H})} \cong \mathbb{C}^n.\end{aligned}$$

Proof. An easy calculations show that $Z(H) = \{(0,0,z)|z \in \mathbf{T}\}$, $Z(H_n^3) = \{(0,0,s)|s \in \mathbb{R}\}$, $Z(H_n^a) = \{(0,0,s)|s \in \mathbb{R}\}$, $Z(\mathbb{H}^n) = \{(0,\dots,0,x_{2n+1})| x_{2n+1} \in \mathbb{R}\}$ and $Z(\mathcal{H}^n) \cong \mathbb{R}$. Therefore, $\frac{H}{Z(H)} \cong \mathbb{R}^n \times \mathbb{R}^n$, $\frac{H_n^3}{Z(H_n^3)} \cong \mathbb{R}^n \times \mathbb{R}^n$, $\frac{H_n^a}{Z(H_n^a)} \cong \mathbb{R}^n \times \mathbb{R}^n$, $\frac{\mathbb{H}^n}{Z(\mathbb{H}^n)} \cong \mathbb{R}^n \times \mathbb{R}^n$ and $\frac{\mathcal{H}}{Z(\mathcal{H})} \cong \mathbb{C}^n$. So, it is enough to compute $\frac{\mathcal{N}}{Z(\mathcal{N})} \cong \mathbb{R}^4$. To do this, we assume that $(p,t) \in Z(\mathcal{N})$ is arbitrary. Hence for each pair (q,s), $(p,t)(q,s) = (q,s)(p,t)$. This proves that $(p+q, s+t+\frac{1}{2}Im(\overline{q}p)) = (q+p, s+t+\frac{1}{2}Im(\overline{p}q))$ and so $Im(\overline{p}q) = 0$. Suppose $p = p_0 + ip_1 + jp_2 + kp_3$. Then by considering three different values $q = (1,0,0,0), (0,1,0,0), (0,0,1,0)$, we will have the following system of equations:

$$\begin{cases} p_1 + p_2 + p_3 = 0, \\ p_0 + p_2 - p_3 = 0, \\ p_0 - p_1 + p_3 = 0. \end{cases}$$

Hence $Z(\mathcal{N}) \cong \mathbb{R}^3$ and $\frac{\mathcal{N}}{Z(\mathcal{N})} \cong \mathbb{R}^4$ that completes the proof. □

3 Concluding Remarks

In this paper the Heisenberg characters of six classes of Heisenberg groups were computed. It is proved that all irreducible characters of these Heisenberg groups are Heisenberg. We also compute all factor groups of these Heisenberg groups which show these factor groups are abelian and so all irreducible characters of these factor groups are again Heisenberg.

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