Some Kinds of Homomorphisms on Hypervector Spaces

Elham Zangiabadi∗, Zohreh Nazari†

Received: 24-05-2018. Accepted: 24-06-2018. Published: 30-06-2018
doi: 10.23755/rm.v34i0.416

Abstract
In this paper, we introduce the concepts of homomorphism of type 1, 2 and 3 and good homomorphism. Then we investigate some properties of them.

Keywords: Hypervector space, Homomorphism, Homomorphism of type 1, 2 and 3, good homomorphism.


1 Introduction and Preliminaries
The concept of hyperstructure was first introduced by Marty [13] in 1934. He defined hypergroups and began to analysis their properties and applied them to groups and rational algebraic functions. Tallini introduced the notion of hypervector spaces [14], [15] and studied basic properties of them. Homomorphisms of hypergroups are studied by several authors ([2] - [12]). Since some kinds of homomorphisms on hypergroup were defined, we encourage to define them on hypervector spaces. In this paper, we introduce the concept of homomorphism of type 1, 2 and 3. And give an example of a homomorphism that is not a homomorphism of type 1, 2 and 3. We show that if $f$ be a homomorphism of type 1, 2 and 3, then $f$ is a homomorphism and every homomorphism of type 2 or 3

∗Department of Mathematics, Vali-e-asr University, Rafsangan, Iran. e.zangiabadi@vru.ac.ir
†Department of Mathematics, Vali-e-asr University, Rafsangan, Iran. z.nazari@vru.ac.ir
is a homomorphism of type 1. Also, we define a good homomorphism and obtain that every homomorphism of type 2 is a good homomorphism and every good homomorphism is a homomorphism. Finally, we prove that every onto strong homomorphism is a good homomorphism.

Let us recall some definitions which are useful in our results.

**Definition 1.1.** A hypervector space over a field $K$ is a quadruplet $(V, +, \circ, K)$ such that $(V, +)$ is an abelian group and

$$
\circ : K \times V \rightarrow P_*(V)
$$
is a mapping of $K \times V$ into the power set of $V$ (deprived of the empty set), such that

1. $(a + b) \circ x \subseteq (a \circ x) + (b \circ x)$, $\forall a, b \in K$, $\forall x \in V$, (1)
2. $a \circ (x + y) \subseteq (a \circ x) + (a \circ y)$, $\forall a \in K$, $\forall x, y \in V$, (2)
3. $a \circ (b \circ x) = (ab) \circ x$, $\forall a, b \in K$, $\forall x \in V$, (3)
4. $x \in 1 \circ x$, $\forall x \in V$, (4)
5. $a \circ (-x) = -a \circ x$, $\forall a \in K$, $\forall x \in V$. (5)

**Definition 1.2.** Let $(V, +, \circ, K)$ be a hypervector space. Then $H \subseteq V$ is a subspace of $V$, if

1) the zero vector, 0, is in $H$,
2) $U, V \in H$, then $U + V \in H$,
3) $U \in H, r \in K$, then $r \circ U \subseteq H$.

**Definition 1.3.** Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces. A mapping

$$f : V \rightarrow W$$
is called

1) a homomorphism, if $\forall r \in K$, $\forall x, y \in V$:

$$f(x + y) = f(x) \oplus f(y),$$ (6)
$$f(r \circ x) \subseteq r * f(x).$$ (7)

2) a strong homomorphism, if $\forall r \in K$, $\forall x, y \in V$:

$$f(x + y) = f(x) \oplus f(y),$$ (8)
$$f(r \circ x) = r * f(x).$$ (9)
2 The main results

In this paper, the ground field of a hypervector space $V$ is presented with $K$. This field is usually considered by $\mathbb{R}$ or $\mathbb{C}$. Let $(V, +, \circ)$ and $(W, \oplus, \ast)$ be two hypervector spaces and $f : V \to W$ be a mapping. We employ for simplicity of notation $x_f = f^{-1}(f(x))$ and for a subset $A$ of $V$, $A_f = f^{-1}(f(A)) = \bigcup\{x_f : x \in A\}$.

**Lemma 2.1.** Let $r \in K$ and $x \in V$. Then the following statements are valid:

i) $r \circ x \subseteq (r \circ x)_f$,

ii) $r \circ x \subseteq r \circ x_f$,

iii) $(r \circ x)_f \subseteq (r \circ x_f)_f$,

iv) $r \circ x_f \subseteq (r \circ x_f)_f$.

**Definition 2.1.** Let $(V, +, \circ, K)$ and $(W, \oplus, \ast, K)$ be two hypervector spaces and $f : V \to W$ be a map such that $f(a + b) = f(a) \oplus f(b)$, for all $a, b \in V$. Then, for any $r \in K$ and $x, y \in V$, $f$ is called a homomorphism of

i) type 1, if $f^{-1}(r \ast f(x)) = (r \circ x)_f$,

ii) type 2, if $f^{-1}(r \ast f(x)) = (r \circ x)_f$,

iii) type 3, if $f^{-1}(r \ast f(x)) = (r \circ x_f)$.

**Theorem 2.1.** Let $(V, +, \circ, K)$ and $(W, \oplus, \ast, K)$ be two hypervector spaces, $A$ be a non-empty subset of $V$ and $f : V \to W$ be a map such that $f(a + b) = f(a) \oplus f(b)$, for all $a, b \in V$. Then, $f$ is a homomorphism of

i) type 1 implies $f^{-1}(r \ast f(A)) = (r \circ A)_f$,

ii) type 2 implies $f^{-1}(r \ast f(A)) = (r \circ A)_f$,

iii) type 3 implies $f^{-1}(r \ast f(A)) = (r \circ A_f)$.

**Proof.** Each part is established by a straightforward set theoretic argument.

**Example 2.1.** Let $(W, +, \cdot, K)$ be a classical vector space, $P$ be a proper subspace of $W$, $W_1 = (W, +, \cdot, K)$ and $W_2 = (W, \oplus, \ast, K)$ that $r \circ a = r \ast a + P$ for $r \in K$ and $a \in W$. Then $W_1$ and $W_2$ are hypervector spaces.

Let $f : W_1 \to W_2$ be the function defined by $f(x) = k \cdot x$, where $k \in K$. We show
Elham Zangiabadi and Zohreh Nazari

that $f$ is a homomorphism, but not a homomorphism of type 1, 2 and 3.

For every $r \in K$ and $x \in W_1$ we have

$$f(r \cdot x) = rk \cdot x \subseteq rk \cdot x + P = r \circ f(x).$$

Thus $f$ is a homomorphism. Since $f$ is one to one, we obtain $x_f = x$, for $x \in W$.
It follows that

$$(r \cdot x_f)_f = (r \cdot x)_f = (r \cdot x_f) = (r \cdot x).$$

On the other hand,

$$f^{-1}(r \circ f(x)) = f^{-1}(kr \cdot x + P) = \{ t \in W_1 : f(t) \in kr \cdot x + P \}$$

$$= \{ t \in W_1 : k \cdot t \in kr \cdot x + P \} = \{ t \in W_1 : k \cdot t - kr \cdot x \in P \}.$$

Hence,

$$f^{-1}(r \circ f(x)) \neq r \cdot x.$$

Therefore, $f$ is not a homomorphism of type 1, 2 and 3.

**Theorem 2.2.** Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces and $f : V \to W$ be a homomorphism of type $n$, for $n=1,2,3$. Then $f$ is a homomorphism map.

**Proof.** If $f$ be a homomorphism of type 1. Then by using Lemma 2.1, we have

$$f(r \circ x) \subseteq f(r \circ x_f) \subseteq f((r \circ x_f)_f) = f(f^{-1}(r * f(x))) \subseteq r \circ f(x).$$

Suppose $f$ is a homomorphism of type 2. Then

$$f(r \circ x) \subseteq f((r \circ x)_f) = f(f^{-1}(r * f(x))) \subseteq r \circ f(x).$$

Similarly, if $f$ is a homomorphism of type 3, then

$$f(r \circ x) \subseteq f(r \circ x_f) = f(f^{-1}(r * f(x))) \subseteq r \circ f(x).$$

\[ \square \]

**Lemma 2.2.** Let $f$ be a homomorphism. Then

$$(r \circ x_f)_f \subseteq f^{-1}(r * f(x)).$$
Some Kinds of Homomorphisms on Hypervector Spaces

Proof. Since \( f \) is a homomorphism, for all \( r \in K \) and \( x \in V \), we have
\[
f(r \circ x_f) \subseteq r \ast f(x_f).
\]
Since \( r \ast f(x_f) = r \ast f(f^{-1}(f(x))) \subseteq r \ast f(x) \), hence, \( f(r \circ x_f) \subseteq r \ast f(x) \). Therefore,
\[
(r \circ x_f)_f \subseteq f^{-1}(r \ast f(x)).
\]

Proposition 2.1. Let \((V, +, \circ, K)\) and \((W, \oplus, *, K)\) be two hypervector spaces and \( f : V \to W \) be a homomorphism of type 2 or 3. Then \( f \) is a homomorphism of type 1.

Proof. Suppose that \( r \in K \), \( x \in V \) and \( f : V \to W \) be a homomorphism of type 2, then by Lemma 2.2 we have
\[
(r \circ x_f)_f \subseteq (r \circ x_f)_f \subseteq f^{-1}(r \ast f(x)) = (r \circ x)_f.
\]
Similarly, if \( f \) is a homomorphism of type 3, then
\[
r \circ x_f \subseteq (r \circ x_f)_f \subseteq f^{-1}(r \ast f(x)) = r \circ x_f.
\]

Proposition 2.2. Let \((V, +, \circ, K)\) and \((W, \oplus, *, K)\) be two hypervector spaces and \( f : V \to W \) be an onto mapping. Then, given \( r \in K \) and \( x \in V \), \( f \) is a homomorphism of

\begin{itemize}
  \item[i)] type 1 if and only if \( f(r \circ x_f) = r \ast f(x) \),
  \item[ii)] type 2 if and only if \( f(r \circ x) = r \ast f(x) \).
\end{itemize}

Proof. Since \( f \) is onto, we obtain
\[
f f^{-1}(r \ast f(x)) = r \ast f(x).
\]
Thus, (i) and (ii) are trivial.

Corollary 2.1. Let \((V, +, \circ, K)\) and \((W, \oplus, *, K)\) be two hypervector spaces, \( A \) and \( B \) be non-empty subsets of \( V \) and \( f : V \to W \) be an onto mapping. Then, \( f \) is homomorphism of

\begin{itemize}
  \item[i)] type 1 implies \( f(r \circ A_f) = r \ast f(A) \),
  \item[ii)] type 2 implies \( f(r \circ A) = r \ast f(A) \).
\end{itemize}
Remark 2.1. On onto homomorphisms between hypervector spaces, a homomorphism of type 2 is equivalent with a strong homomorphism.

Theorem 2.3. Let $(V_1, +_1, \circ_1, K)$, $(V_2, +_2, \circ_2, K)$ and $(V_3, +_3, \circ_3, K)$ be hypervector spaces. For $n = 1, 2, 3$, let $f$ be a homomorphism of type $n$ of $V_1$ onto $V_2$ and $g$ be a homomorphism of type $n$ of $V_2$ onto $V_3$. Then, $gf$ is a homomorphism of type $n$ of $V_1$ onto $V_3$.

Proof. Let $x, y \in V$. We have $gf(x +_1 y) = g(f(x) +_2 f(y)) = g f(x) +_3 g f(y))$. One can easily seen that $x_{gf} = f^{-1}(f(x)g)$. Let $n = 1$. By above relation, we obtain

$$g f(r \circ x_{gf}) = g f(r \circ f^{-1}(f(x)g)).$$

Since $f$ is onto, there exists a subset $A$ of $V$ such that $f(A) = f^{-1}(f(x)g)$. By Corollary 2.1, we obtain

$$g f(r \circ f^{-1}(f(x)g)) = g(r \circ_2 f(x)g).$$

Then, by Proposition 2.2, we have

$$g(r \circ_2 f(x)g) = r \circ_3 g f(x).$$

Let $n = 2$. Similar to the previous case, but simpler. Let $n = 3$. Since $g$ is of type 3,

$$(gf)^{-1}(r \circ_3 (gf)(x)) = f^{-1}g^{-1}r \circ_3 (gf)(x)) = f^{-1}(r * f(x)g).$$

Since $f$ is onto, the item (iii) of Theorem 2.1 implies

$$f^{-1}(r \circ_2 f(x)g) = r \circ_1 f^{-1}(f(x)g) = r \circ_1 x_{gf}.$$ 

Definition 2.2. Let $a \in V$ and $r \in K$. We define

$$a/r = \{x \in V : a \in r \circ x\}.$$

Proposition 2.3. Let $(V_1, +, \circ, K)$ and $(V_2, \oplus, *, K)$ be two hypervector spaces. If $f : V_1 \to V_2$ be an onto mapping. Then we have

1) $f(a/r) = f(a)/r$, if $f$ is a homomorphism of type 2.

2) $f(a)/r \subseteq f(a_f)/r$, if $f$ is a homomorphism of type 3.
Proof. 1) We know that an onto homomorphism of type 2 is a strong homomorphism. Suppose that $y \in f(a/r)$. Then, there exists $t \in a/r$ such that $f(t) = y$, so $a \in r \circ t$ and $f(a) \in r \ast f(t)$. It implies that $y = f(t) \in f(a)/r$. Therefore, $f(a/r) \subset f(a)/r$. Note that the inverse inclusion is always true. 2) If $y \in f(a)/r$, there is $t \in V_1$ such that $f(t) = y$. Since $f$ is homomorphism of type 3, we have $a_f \in r \circ t_f$, which means that $t_f \in a_f/r$, therefore $y \in f(a_f)/r$. \hfill \qedsymbol

Definition 2.3. Let $(V, +, \circ, K)$ and $(W, *, \oplus, K)$ be two hypervector spaces and $f : V \to W$ be a map such that $f(a + b) = f(a) \oplus f(b)$. Then, $f$ is called a good homomorphism if

$$f(a/r) = f(a)/r,$$

for any $a, b \in V$ and $r \in K$.

Remark 2.2. According to Proposition 2.3, if $f$ is a homomorphism of type 2, then $f$ is a good homomorphism.

Theorem 2.4. Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces. If $f : V \to W$ be a good homomorphism then, $f$ is a homomorphism.

Proof. Let $r \in K$ and $a \in V_1$. If $y \in f(r \circ a)$, then, there exists $t \in r \circ a$ such that $y = f(t)$. Hence, $f(a) \in f(t/r) = f(t)/r$. Abviously, $y \in f(t) \in r \ast f(a)$. \hfill \qedsymbol

Theorem 2.5. Let $(V_1, +_1, \circ_1, K)$, $(V_2, +_2, \circ_2, K)$, and $(V_3, +_3, \circ_3, K)$ be hypervector spaces. Let $f$ be a good homomorphism of $V_1$ to $V_2$ and $g$ be a good homomorphism of $V_2$ to $V_3$. Then, $gf$ is a good homomorphism of $V_1$ to $V_3$.

Proof. For every $r \in K$ and $a \in V_1$, we have

$$gf(a/r) = g(f(a)/r) = gf(a)/r.$$

\hfill \qedsymbol

Proposition 2.4. Let $V$ and $W$ be two hypervector spaces over $K$ and $f : V \to W$ be a good homomorphism. Then

$$f(A/K) = f(A)/K,$$

where $A \subseteq V$ and $A/K = \bigcup\{a/r : a \in A, r \in K\}$.

Proof. Let $y \in f(A/K)$. There exist $r \in K$ and $a \in A$ such that $y \in f(a/r) = f(a)/r \subseteq f(A)/K$. Conversely, let $y \in f(A)/k$. Then, there exist $r \in k$ and $a \in V$ such that $y \in f(a)/r = f(a)/r$ and so $y \in f(A/K)$. \hfill \qedsymbol

Theorem 2.6. Let $(V, +, \circ, K)$ and $(W, \oplus, *, K)$ be two hypervector spaces, $f$ be onto strong homomorphism from $V$ to $W$. Then $f$ is a good homomorphism.
Elham Zangiabadi and Zohreh Nazari

Proof. Let \( f(t) \in f(x/r) \). So \( x \in r \circ t \). It followes that \( f(t) \in f(x)/r \). Therefore \( f(x/r) \subseteq f(x)/r \).

On the other hand, let \( y \in f(x)/r \). Since \( f \) is an onto mapping, there exists a \( t \in V \) such that \( y = f(t) \). Hence, \( f(x) \in r \ast f(t) = f(r \circ t) \). Thus \( x \in r \circ t \) and then we have \( t \in x/r \) and \( y = f(t) \in f(x/r) \). Therefore \( f(x)/r \subseteq f(x/r) \). This implies that \( f(x/r) = f(x)/r \).

\[ \square \]

References


92
Some Kinds of Homomorphisms on Hypervector Spaces

