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Recognizability in Stochastic Monoids

A. Kalampakas[a], O. Louscou-Bozapalidou[b], S. Spartalis[c]

[a,c]Department of Production Engineering and Management,
Democritus University of Thrace, 67100, Xanthi, Greece

[b]Section of Mathematics and Informatics,
Technical Institute of West Macedonia, 50100, Kozani, Greece

akalampakas@gmail.duth.gr

sspar@pme.duth.gr

Abstract

Stochastic monoids and stochastic congruences are introduced and the syntactic stochastic monoid M_L associated to a subset L of a stochastic monoid M is constructed. It is shown that M_L is minimal among all stochastic epimorphisms $h : M \to M'$ whose kernel saturates L. The subset L is said to be stochastically recognizable whenever M_L is finite. The so obtained class is closed under boolean operations and inverse morphisms.

Key words: recognizability, stochastic monoids, minimization.

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1 Introduction

A stochastic subset of a set M is a function $F : M \to [0, 1]$ with the additional property $\Sigma_{m\in M} F(m) = 1$, i.e., F is a discrete probability distribution. The corresponding class is denoted by $Stoc(M)$. Our subject of study, in the present paper, are stochastic monoids which were introduced in [4]. A stochastic monoid is a set M equipped with a stochastic multiplication $M \times M \to Stoc(M)$ which is associative and unitary. It can be viewed as a nondeterministic monoid (cf. [1, 2, 3]) with multiplication $M \times M \to \mathcal{P}(M)$ such that for all $m_1, m_2 \in M$ a discrete probability distribution is assigned on the set $m_1 \cdot m_2$.

A congruence on a stochastic monoid M is an equivalence $\sim$ on M such that $m_1 \sim m_1'$ and $m_2 \sim m_2'$ imply

$$\sum_{n\in C}(m_1 \cdot m_2)(n) = \sum_{n\in C}(m_1' \cdot m_2')(n)$$

for all $\sim$-classes C. The quotient $M/\sim$ admits a stochastic monoid structure rendering the canonical function $m \mapsto [m]$ an epimorphism of stochastic monoids. The classical Isomorphism Theorem of Algebra still holds in the stochastic setup, namely

> for any epimorphism of stochastic monoids $h : M \to M'$ and every stochastic congruence $\sim$ on M' its inverse image $h^{-1}(\sim)$ defined by
>
> $$m_1 h^{-1}(\sim) m_2 \quad \text{iff} \quad h(m_1) \sim h(m_2),$$
>
> is again a stochastic congruence and the quotient stochastic monoids $M/h^{-1}(\sim)$ and $M'/\sim$ are isomorphic. In particular if $\sim$ is the equality, then $h^{-1}(=)$ is the kernel congruence of h (denoted by $\sim_h$)
>
> $$m_1 \sim_h m_2 \quad \text{iff} \quad h(m_1) = h(m_2),$$
>
> and the stochastic monoids $M/\sim_h$ and M' are isomorphic.

We show that stochastic congruences are closed under the join operation. This allows us to construct the greatest stochastic congruence included in an equivalence $\sim$. It is the join of all stochastic congruences on M included into $\sim$ and it is denoted by $\sim^{stoc}$. The quotient stochastic monoid $M/\sim^{stoc}$ is denoted by M^{stoc} and has the following universal property:

> given an epimorphism of stochastic monoids $h : M \to M'$ whose kernel $\sim_h$ saturates the equivalence $\sim$ there exists a unique epimorphism of stochastic monoids $h' : M' \to M^{stoc}$ such that $h' \circ h = h^{stoc}$, where $h^{stoc} : M \to M^{stoc}$ is the canonical epimorphism into the quotient.

This result states that h^{stoc} is minimal among all epimorphisms saturating $\sim$.

Let M be a stochastic monoid and $L \subseteq M$. Denote by $\sim_L$ the greatest congruence of M included in the partition (equivalence) $\{L, M - L\}$, i.e., $\sim_L = \{L, M - L\}^{stoc}$. The quotient stochastic monoid $M_L = M/\sim_L$ will be called the syntactic stochastic monoid of L and it is characterized by the following universal property.

> For every stochastic monoid M and every epimorphism $h : M \to M'$ verifying $h^{-1}(h(L)) = L$, there exists a unique epimorphism $h' : M' \to M_L$ such that $h' \circ h = h_L$ where $h_L : M \to M_L$ is the canonical projection into the quotient.

A subset L of a stochastic monoid M is stochastically recognizable if there exist a finite stochastic monoid M' and a morphism $h : M \to M'$ such that $h^{-1}(h(L)) = L$. By taking into account the previous result we get that L is recognizable if and only if its syntactic stochastic monoid is finite. Moreover stochastically recognizable subsets are closed under boolean operations and inverse morphisms.

2 Stochastic Subsets

Some useful elementary facts are displayed. Let $(x_i)_{i\in I}$, $(x_{ij})_{i\in I,j\in J}$, $(y_j)_{j\in J}$ be families of nonnegative reals, then

$$\underset{i\in I,j\in J}{sup}\ x_{ij} = \underset{i\in I}{sup}\,\underset{j\in J}{sup}\,x_{ij} = \underset{j\in J}{sup}\,\underset{i\in I}{sup}\,x_{ij}, \qquad \underset{i\in I,j\in J}{sup}\ x_iy_j = \underset{i\in I}{sup}\ x_i \cdot \underset{j\in J}{sup}\ y_j,$$

provided that the above suprema exist. If $\underset{I'\subseteq_{fin} I}{sup}\ \Sigma_{i\in I'} x_i$ exists, then we say that the sum $\Sigma_{i\in I} x_i$ exists and we put

$$\underset{i\in I}{\Sigma}\, x_i = \underset{I'\subseteq_{fin} I}{sup}\ \underset{i\in I'}{\Sigma}\, x_i$$

where the notation $I' \subseteq_{fin} I$ means that I' is a finite subset of I.

It holds

$$\sum_{i\in I,j\in J} x_{ij} = \sum_{i\in I}\sum_{j\in J} x_{ij} = \sum_{j\in J}\sum_{i\in I} x_{ij}, \qquad \sum_{i\in I,j\in J} x_iy_j = \sum_{i\in I} x_i \sum_{j\in J} y_j.$$

Let M be a non empty set and $[0, 1]$ the unit interval, a *stochastic subset* of M is a function $F : M \to [0, 1]$ with the additional property that the sum of its values exists and is equal to 1

$$\sum_{m\in M} F(m) = 1.$$

We denote by $Stoc(M)$ the set of all stochastic subsets of M.

Let $F_i : M \to \mathbb{R}_+$, $i \in I$, be a family of functions such that for every $m \in M$ the sum $\sum_{i\in I} F_i(m)$ exists. Then the assignment

$$m \mapsto \sum_{i\in I} F_i(m)$$

defines a function from M to $\mathbb{R}_+$ denoted by $\sum_{i\in I} F_i$, i.e.,

$$\Big(\sum_{i\in I} F_i\Big)(m) = \sum_{i\in I} F_i(m), \quad m \in M.$$

Now let $(\lambda_i)_{i\in I}$ be a family in $[0,1]$ such that $\sum_{i\in I}\lambda_i = 1$ and $F_i \in Stoc(M)$, $i \in I$. For any finite subset I' of I and any $m \in M$, we have

$$\sum_{i\in I}\lambda_i F_i(m) = \sup_{I'\subseteq_{fin} I}\sum_{i\in I'}\lambda_i F_i(m) \leq 1.$$

Thus $\sum_{i\in I}\lambda_i F_i$ is defined and belongs to $Stoc(M)$ because

$$\begin{aligned}\sum_{m\in M}\Big(\sum_{i\in I}\lambda_i F_i\Big)(m) =& \sum_{m\in M}\sum_{i\in I}\lambda_i F_i(m) = \sum_{i\in I}\sum_{m\in M}\lambda_i F_i(m)\\ =&\Big(\sum_{i\in I}\lambda_i\Big)\Big(\sum_{m\in M}F_i(m)\Big) = 1\cdot 1 = 1.\end{aligned}$$

Thus we can state:

Strong Convexity Lemma (SCL). *The set $Stoc(M)$ is a strongly convex set, i.e., for any stochastic family*

$$\lambda_i \in [0,1], \quad F_i \in Stoc(M), \ i \in I$$

the function $\sum_{i\in I}\lambda_i F_i$ is in $Stoc(M)$.

For arbitrary sets M, M' any function $h : M \to Stoc(M')$ can be extended into a function $\bar{h} : Stoc(M) \to Stoc(M')$ by setting

$$\bar{h}(F) = \sum_{m\in M} F(m)\cdot h(m).$$

In particular, any function $h : M \to M'$ is extended into a function $\bar{h} : Stoc(M) \to Stoc(M')$ by the same as above formula. This formula is legitimate since by the strong convexity lemma

$$\sum_{m\in M} F(m) = 1$$

and $h(m)$ is a stochastic subset of M.

Hence, for any stochastic subset $F : M \to [0,1]$ we have the expansion formula

$$F = \sum_{m\in M} F(m)\hat{m}$$

where $\hat{m} : M \to [0,1]$ stands for the singleton function

$$\hat{m}(n) = \begin{cases} 1, & \text{if } n = m;\\ 0, & \text{if } n \neq m.\end{cases}$$

Often $\hat{m}$ is identified with m itself.

3 Stochastic Congruences

Our main interest is focused on equivalences in the stochastic setup. Any equivalence relation $\sim$ on the set M, can be extended into an equivalence relation $\approx$ on the set $Stoc(M)$ as follows: for $F, F' \in Stoc(M)$ we set $F \approx F'$ if and only if for each $\sim$-class C it holds

$$\sum_{m \in C} F(m) = \sum_{m \in C} F'(m),$$

that is both F, F' behave stochastically on C in similar way. The above sums exist because F, F' are stochastic subsets of M:

$$\sum_{m \in C} F(m) \leq \sum_{m \in M} F(m) = 1.$$

The equivalence $\approx$ has a fundamental property, it is compatible with strong convex combinations.

Proposition 3.1. *Assume that $(\lambda_i)_{i \in I}$ is a stochastic family of numbers in $[0,1]$ and $F_i, F_i' \in Stoc(M)$, for all $i \in I$. Then*

$$F_i \approx F_i', \text{ for all } i \in I, \quad \text{implies} \quad \sum_{i \in I} \lambda_i F_i \approx \sum_{i \in I} \lambda_i F_i'.$$

Proof. By hypothesis we have

$$\sum_{m \in C} F_i(m) = \sum_{m \in C} F_i'(m)$$

for any $\sim$-class C in M, and thus

$$\begin{aligned}
\sum_{m \in C} \left(\sum_{i \in I} \lambda_i F_i \right)(m) &= \sum_{m \in C} \sum_{i \in I} \lambda_i F_i(m) = \sum_{i \in I} \lambda_i \sum_{m \in C} F_i(m) \\
&= \sum_{i \in I} \lambda_i \sum_{m \in C} F_i'(m) = \sum_{m \in C} \sum_{i \in I} \lambda_i F_i'(m) \\
&= \sum_{m \in C} \left(\sum_{i \in I} \lambda_i F_i' \right)(m)
\end{aligned}$$

that is

$$\sum_{i \in I} \lambda_i F_i \approx \sum_{i \in I} \lambda_i F_i'$$

as wanted. □

4 Stochastic Monoids

A stochastic monoid is a set M equipped with a stochastic multiplication, i.e. a function

$$M \times M \to Stoc(M), \quad (m_1, m_2) \mapsto m_1 m_2$$

which is associative

$$\sum_{n \in M} (m_1 m_2)(n)(n m_3) = \sum_{n \in M} (m_2 m_3)(n)(m_1 n)$$

and unitary i.e. there is an element $e \in M$ such that

$$me = m = em, \text{ for all } m \in M.$$

For instance any ordinary monoid can be viewed as a stochastic monoid. In the present study it is important to have a congruence notion. More precisely, let M be a stochastic monoid and $\sim$ an equivalence relation on the set M, such that: $m_1 \sim m_1'$ and $m_2 \sim m_2'$ implies

$$\sum_{m \in C} (m_1 m_2)(m) = \sum_{m \in C} (m_1' m_2')(m)$$

for all $\sim$-classes C, then $\sim$ is called a *stochastic congruence* on M. This condition can be reformulated as follows: $m_1 \sim m_1'$ and $m_2 \sim m_2'$ implies

$$m_1 m_2 \approx m_1' m_2'.$$

Proposition 4.1. *The quotient set $M/\sim$ is structured into a stochastic monoid by defining the stochastic multiplication via the formula*

$$([m_1][m_2])([n]) = \sum_{m \in [n]} (m_1 m_2)(m).$$

Proof. First observe that the above multiplication is well defined. Next for every $\sim$-class $[b]$ we have

$$\begin{aligned}(([m_1][m_2])\,[m_3])\,([b]) &= \sum_{[n] \in M/\sim} ([m_1][m_2])([n])([n][m_3])([b]) \\ &= \sum_{[n] \in M/\sim} \sum_{n_1 \in [n]} (m_1 m_2)(n_1) \sum_{b' \in [b]} (n m_3)(b')\end{aligned}$$

Since $n \sim n_1$ we get

$$
\begin{aligned}
&= \sum_{[n]\in M/\sim} \sum_{n_1\in[n]} (m_1m_2)(n_1) \sum_{b'\in[b]} (n_1m_3)(b') \\
&= \sum_{[n]\in M/\sim} \sum_{b'\in[b]} \sum_{n_1\in[n]} (m_1m_2)(n_1)(n_1m_3)(b') \\
&= \sum_{b'\in[b]} \sum_{n_1\in M} (m_1m_2)(n_1)(n_1m_3)(b').
\end{aligned}
$$

By taking into account the associativity of M we obtain:

$$
\begin{aligned}
&= \sum_{b'\in[b]} \sum_{n_1\in M} (m_2m_3)(n_1)(m_1n_1)(b') \\
&= ([m_1]([m_2][m_3]))([b]). \qquad \square
\end{aligned}
$$

Congruences on an ordinary monoid M coincide with stochastic congruences when M is viewed as a stochastic monoid. The first question arising is whether stochastic congruence is a good algebraic notion. This is checked by the validity of the known isomorphism theorems in their stochastic variant.

Given stochastic monoids M and M', a strict morphism from M to M' is a function $h : M \to M'$ preserving stochastic multiplication and units, i.e.,

$$\bar{h}(m_1m_2) = h(m_1)h(m_2), \quad h(e) = e',$$

for all $m_1, m_2 \in M$, where e, e' are the units of M, M' respectively, and $\bar{h} : Stoc(M) \to Stoc(M')$ the canonical extension of h defined in Section 2.

Theorem 4.1. *Given an epimorphism of stochastic monoids $h : M \to M'$ and a stochastic congruence $\sim$ on M', its inverse image $h^{-1}(\sim)$ defined by*

$$m_1 h^{-1}(\sim) m_2 \quad \textit{if} \quad h(m_1) \sim h(m_2)$$

is also a stochastic congruence and the stochastic quotient monoids $M/h^{-1}(\sim)$ and $M'/\sim$ are isomorphic.

Proof. Assume that

$$m_1 h^{-1}(\sim) m_1' \text{ and } m_2 h^{-1}(\sim) m_2'$$

that is

$$h(m_1) \sim h(m_1') \text{ and } h(m_2) \sim h(m_2').$$

Then

$$\bar{h}(m_1m_2) = h(m_1)h(m_2) \approx h(m_1')h(m_2') = \bar{h}(m_1'm_2'),$$

that is for all $C \in M'/\sim$, we have

$$\sum_{c\in C}\bar{h}(m_1m_2)(c) = \sum_{c\in C}\bar{h}(m_1'm_2')(c),$$

but

$$\sum_{c\in C}\bar{h}(m_1m_2)(c) = \sum_{c\in C}\sum_{m\in M}(m_1m_2)(m)h(m)(c) = \sum_{m\in M}(m_1m_2)(m)\sum_{c\in C}h(m)(c)$$
$$= \sum_{m\in h^{-1}(C)}(m_1m_2)(m).$$

Recall that all $h^{-1}(\sim)$-classes are of the form $h^{-1}(C)$, $C \in M'/\sim$. Consequently,

$$= \sum_{m\in h^{-1}(C)}(m_1m_2)(m) = \sum_{m\in h^{-1}(C)}(m_1'm_2')(m)$$

which shows that $h^{-1}(\sim)$ is indeed a congruence of the stochastic monoid M. The desired isomorphism $\hat{h} : M/h^{-1}(\sim) \to M'/\sim$ is given by

$$\hat{h}([m]_{h^{-1}(\sim)}) = [h(m)]_\sim. \qquad \square$$

Corolary 4.1. *Let $h : M \to M'$ be an epimorphism of stochastic monoids. Then the kernel equivalence*

$$m_1 \sim_h m_2 \text{ if } h(m_1) = h(m_2)$$

is a congruence on M and the stochastic quotient monoid $M/\sim_h$ is isomorphic to M'.

Given stochastic monoids $M_1, \ldots, M_k$ the stochastic multiplication

$$[(m_1, \ldots, m_k) \cdot (m_1', \ldots, m_k')](n_1, \cdots, n_k) = (m_1m_1')(n_1)\cdots(m_km_k')(n_k)$$

structures the set $M_1\times\cdots\times M_k$ into a stochastic monoid so that the canonical projection

$$\pi_i : M_1 \times \cdots \times M_k \to M_i, \quad \pi_i(m_1, \ldots, m_k) = m_i$$

becomes a morphism of stochastic monoids. Notice that the above multiplication is stochastic because

$$\sum_{\substack{n_i\in M_i\\1\le i\le k}}(m_1m_1')(n_1)\cdots(m_km_k')(n_k) = \sum_{n_1\in M_1}(m_1m_1')(n_1)\cdots\sum_{n_k\in M_k}(m_km_k')(n_k)$$
$$= 1\cdots 1 = 1.$$

Theorem 4.2. *Let* $\sim_i$ *be a stochastic congruence on the stochastic monoid* M_i $(1 \leq i \leq k)$. *Then* $\sim_1 \times \cdots \times \sim_k$ *is a stochastic congruence on the stochastic monoid* $M_1 \times \cdots \times M_k$ *and the stochastic monoids* $M_1 \times \cdots \times M_k / \sim_1 \times \cdots \times \sim_k$ *and* $M_1/\sim_1 \times \cdots \times M_k/\sim_k$ *are isomorphic.*

5 Greatest Stochastic Congruence Saturating an Equivalence

First observe that, due to the symmetric property which an equivalence relation satisfies, the sumability condition in the definition of a congruence can be replaced by the weaker condition: $m_1 \sim m_1'$ and $m_2 \sim m_2'$ implies

$$\sum_{m \in C} (m_1 m_2)(m) \leq \sum_{m \in C} (m_1' m_2')(m)$$

for all $\sim$-classes C.

Lemma 5.1. *The equivalence* $\sim$ *on the stochastic monoid* M *is a congruence if and only if the following condition is fulfilled:* $m \sim m'$, *implies*

$$\sum_{b \in C} (m \cdot n)(b) \leq \sum_{b \in C} (m' \cdot n)(b) \quad \text{and} \quad \sum_{b \in C} (n \cdot m)(b) \leq \sum_{b \in C} (n \cdot m')(b).$$

Proof. One direction is immediate whereas for the opposite direction we have: $m_1 \sim m_1'$ and $m_2 \sim m_2'$ imply

$$\sum_{b \in C} (m_1 \cdot m_2)(b) \leq \sum_{b \in C} (m_1' \cdot m_2)(b) \leq \sum_{b \in C} (m_1' \cdot m_2')(b). \qquad \square$$

Next we demonstrate that stochastic congruences are closed under the join operation. We recall that the join $\bigvee_{i \in I} \sim_i$ of a family of equivalences $(\sim_i)_{i \in I}$ on a set A is the reflexive and transitive closure of their union:

$$\bigvee_{i \in I} \sim_i = \left(\bigcup_{i \in I} \sim_i \right)^* .$$

Theorem 5.1. *If* $(\sim_i)_{i \in I}$ *is a family of stochastic congruences on* M, *then their join* $\bigvee_{i \in I} \sim_i$ *is also a stochastic congruence.*

Proof. Let $\sim_1, \sim_2$ be two congruences on M and $\sim = \sim_1 \vee \sim_2$. First we show that $m \sim_1 m'$ implies

$$\sum_{b \in C} (m \cdot n)(b) \leq \sum_{b \in C} (m' \cdot n)(b),$$

for all $\sim$-classes C. From the inclusion $\sim_1 \subseteq \sim$ we get that C is the disjoint union

$$C = \bigcup_{j=1}^{m} C_j^1$$

where C_j^1 denote $\sim_1$-classes. Then

$$\sum_{b\in C}(m\cdot n)(b) = \sum_{j=1}^{m}\sum_{b\in C_j^1}(m\cdot n)(b) \leq \sum_{j=1}^{m}\sum_{b\in C_j^1}(m'\cdot n)(b) = \sum_{b\in C}(m'\cdot n)(b).$$

By a similar argument we show that $m \sim_2 m'$ implies

$$\sum_{b\in C}(m\cdot n)(b) \leq \sum_{b\in C}(m'\cdot n)(b),$$

for all $\sim$-classes C. Now, if $m \sim m'$, without any loss we may assume that

$$m \sim_1 m_1 \sim_2 m_2 \sim_1 \cdots \sim_1 m_{2\lambda-1} \sim_2 m'$$

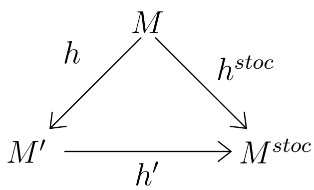
for some elements $m_1, \ldots, m_{2\lambda-1} \in M$. Applying successively the previous facts, we obtain

$$\sum_{b\in C}(m\cdot n)(b) \leq \sum_{b\in C}(m_1\cdot n)(b) \leq \cdots \leq \sum_{b\in C}(m_{2\lambda-1}\cdot n)(b) \leq \sum_{b\in C}(m'\cdot n)(b).$$

For an arbitrary set of congruences we proceed in a similar way. □

The previous result leads us to introduce the greatest stochastic congruence included into an equivalence $\sim$ of M. It is the join of all stochastic congruences on M included into $\sim$ and it is denoted by $\sim^{stoc}$. The quotient stochastic monoid $M/\sim^{stoc}$ is denoted by M^{stoc} and has the following universal property

Theorem 5.2. *Given an epimorphism of stochastic monoids $h : M \to M'$ whose kernel $\sim_h$ saturates the equivalence $\sim$ there exists a unique epimorphism of stochastic monoids $h' : M' \to M^{stoc}$ rendering commutative the triangle*



where $h^{stoc} : M \to M^{stoc}$ *is the canonical projection* $m \mapsto [m]_{stoc}$ *sending every element* $m \in M$ *on its* $\sim^{stoc}$*-class.*

Proof. By virtue of the Isomorphism Theorem the stochastic monoid M' is isomorphic to the quotient $M/\sim_h$. Since by assumption $\sim_h \subseteq \sim^{stoc}$, h' is the following composition

$$M' \xrightarrow{\sim} M/\sim_h \xrightarrow{f} M/\sim^{stoc} = M^{stoc},$$

with $f([m]_h) = [m]_{stoc}$, $[m]_h$ being the $\sim_h$-class of m. □

The previous result states that h^{stoc} is minimal among all epimorphisms saturating $\sim$.

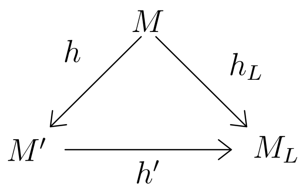
6 Syntactic Stochastic Monoids

Let M be a stochastic monoid and $L \subseteq M$. Denote by $\sim_L$ the greatest congruence of M included in the partition (equivalence) $\{L, M - L\}$, i.e.,

$$\sim_L = \{L, M - L\}^{stoc}.$$

The quotient stochastic monoid $M_L = M/\sim_L$ will be called the *syntactic stochastic monoid* of L and it is characterized by the following universal property.

Theorem 6.1. *For every stochastic monoid* M *and every epimorphism* $h : M \to M'$ *verifying* $h^{-1}(h(L)) = L$*, there exists a unique epimorphism* $h' : M' \to M_L$ *rendering commutative the triangle*



where h_L *is the canonical morphism sending every element* $m \in M$ *to its* $\sim_L$*-class.*

Proof. The hypothesis $h^{-1}(h(L)) = L$ means that $\sim_h$ saturates L and so the statement follows immediately by Theorem 5.2. □

Given stochastic monoids M, M' we write $M' < M$ if there is a stochastic monoid $\bar{M}$ and a situation

$$M' \xleftarrow{h} \bar{M} \xrightarrow{i} M$$

where i (resp. h) is a monomorphism (resp. epimorphism).

Theorem 6.2. *Given subsets L_1, L_2, L of a stochastic monoid M it holds*

i) $M_{L_1 \cap L_2} < M_{L_1} \times M_{L_2}$,

ii) $M_L = M_{\bar{L}}$, *where $\bar{L}$ designates the set theoretic complement of L,*

iii) $M_{L_1 \cup L_2} < M_{L_1} \times M_{L_2}$,

iv) If $h : M \to N$ is an epimorphism of ND-monoids and $L \subseteq N$, then $M_{h^{-1}(L)} = M_L$.

Proof. The proof follows by applying Theorem 6.1. □

A subset L of a stochastic monoid M is *stochastically recognizable* if there exist a finite stochastic monoid M' and a morphism $h : M \to M'$ such that $h^{-1}(h(L)) = L$. The class of stochastically recognizable subsets of M is denoted by *StocRec*(M). By taking into account Theorem 6.1 we get

Proposition 6.1. *$L \subseteq M$ is recognizable if and only if its syntactic stochastic monoid is finite, $card(M_L) < \infty$.*

Putting this result together with Theorem 6.2 we yield

Proposition 6.2. *The class StocRec(M) is closed under boolean operations and inverse morphisms.*

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