Groups of Transformations with a Finite Number of Isometries: the Cases of Tetrahedron and Cube

Ferdinando Casolaro\textsuperscript{1}, Luca Cirillo\textsuperscript{2}, Raffaele Prosperi\textsuperscript{3}

\textsuperscript{1}Department di Architecture University “Federico II” of Naples, Italy  
ferdinando.casolaro@unina.it  
\textsuperscript{2} University of Sannio in Benevento, Italy  
luca.cirillo@unisannio.it  
\textsuperscript{3} DISUFF University of Salerno, Italy  
rprosperi@unisa.it

Received on: 3-11-2016. Accepted on: 11-11-2017. Published on: 28-02-2017  
doi: 10.23755/rm.v31i0.322

© Casolaro et al.

Abstract

This paper deals with groups of transformations with finite number of isometries and extends previous studies (Casolaro, F. L. Cirillo and R. Prosperi 2015) which are related to endless groups of transformations with isometries. In particular, isometries of the tetrahedron and cube, which turn these figures in itself, are presented.

Keywords: Geometric transformations, isometries, symmetry.

2010 AMS subject classification: 97G50; 51N25.
1. Introduction

Compared with the operation of product of isometries, in previous studies, we presented some examples of infinite groups of transformations, whose we highlighted the following properties:

- The isometries of the space form a group.
- The direct isometries of the space form a group, subgroup of the previous group.
- The translations of the space form a group, subgroup of the group of direct isometries.
- Rotations around a straight form a group, subgroup of direct isometries.
- The helical movements all having the same axis form a group, subgroup of the group of direct isometries. In this case, since the helical movements turn out to be products of rotations for translations having the direction of the axis of rotation, also translations (the rotation is reduced to the identity) and rotations (the translation is reduced to the identity) may be considered helical movements.

It is also possible to obtain groups of transformation with a finite number of isometries.

In particular: about the tetrahedron, we show the axial symmetry $\mu$ having as an axis line $r$, rotations $\rho$ of $120^\circ$ and $240^\circ$ around the height of the tetrahedron outgoing from a fixed vertex, planar symmetry $\sigma$ relative to the plan $\pi$ passing through two vertices of the tetrahedron and through the midpoint of the edge that joins the other two vertices; about the cube, rotations $\rho$ around a line $r$ connecting the centers of two opposite faces, rotations $\rho$ around the line $r$ joining the midpoints of two opposite edges, planar symmetry $\sigma$ relative to the plan $\pi$ passing through two vertices of the tetrahedron and through the midpoint of the edge that joins the other two vertices, planar symmetry $\sigma$ relative to the pane $\pi$ parallel to two faces passing through the midpoints of the four edges perpendicular to these two faces, planar symmetries $\sigma$ relative to the pane $\pi$ passing through two opposite edges that do not have face in common and a vertex in common.
Groups of Transformations with a Finite Number of Isometries: the Cases of Tetrahedron and Cube

Consider three straight lines $x$, $y$, $z$, passing through the same point $O$ and perpendicular to each other two by two. The three planes $\alpha$, $\beta$, $\gamma$, respectively determined by the straight lines $x$ and $y$, $x$ and $z$, and $y$ and $z$, are also perpendicular to each other two by two (Figure 1).

Let be:

- $I$ the identity,
- $s_x$ the axial symmetry having as an axis the line $x$,
- $s_y$ the axial symmetry having as an axis the line $y$,
- $s_z$ the axial symmetry having as an axis the line $z$,
- $s_\alpha$ the planar symmetry relative to the plane $\alpha$,
- $s_\beta$ the planar symmetry relative to the plane $\beta$,
- $s_\gamma$ the planar symmetry relative to the plane $\gamma$,
- $s_O$ the symmetry with center $O$,

It occurs that these eight isometries form a group. For this purpose, it is sufficient to prove that the product of any two of them is still one of the eight indicated isometries.

2. Tetrahedron’s Isometries

Other examples of finite groups of isometries can be obtained considering all the isometries which leave fixed a given figure $F$, that is, such that in each of them $F$ is united ($F$ is transformed into itself). $ABCD$ and $A'B'C'D'$ are two congruent tetrahedra. Then there exists one and only one isometry that transforms the vertices $A, B, C, D$ neatly in the vertices $A', B', C', D'$ (Figure 2). This isometry is direct or reverse depending on whether or not the two tetrahedra are equally oriented.
Isometries that turn a tetrahedron $T$ into itself are 24 (twenty-four). They form a group $S_T$, obviously isomorphic to the group $S_4$ of the 24 permutations on four letters A, B, C, D.

Among the isometries $\phi$ that transform the tetrahedron $T$ into itself, we present the following:

a) The axial symmetry $\mu$ having as an axis the straight line $r$, joining the midpoints of two opposite sides (bimedian), is a rotation of 180° around the straight line $r$.

The symmetries of this type present in the group are 3 (as many as the pairs of opposite sides of the tetrahedron); they have evidently period 2. Therefore there are 3 axial symmetries that leave $T$ globally invariant, as many as the pairs of opposite sides.

A substitution is associated with each of these symmetries (M. Impedovo 1998).

- With symmetry $\mu_1$ about the line $r_1$ joining the midpoints of the sides AB and CD, the following substitution is associated:

$$
\mu_1 : \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}
$$

- With symmetry $\mu_2$ about the line $r_2$ joining the midpoints of the sides AC and BD the following substitution is associated:

$$
\mu_2 : \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}
$$

- With symmetry $\mu_3$ about the line $r_3$ joining the midpoints of the sides AD and BC the following substitution is associated:

$$
\mu_3 : \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}
$$
b) The rotations $\rho$ of 120 ° and 240 ° around the height of the tetrahedron outgoing from a fixed vertex. For each height of the tetrahedron, you have two rotations of period 3 which hold the summit fixed. Since the tetrahedron heights are 4, these rotations are 8; therefore, there are 8 rotations of this type which transform $T$ into itself, two for each height of the tetrahedron.

A substitution is associated with each of these rotations.

- With rotation $\rho_1$ about the height outgoing from $A$ the following substitution is associated:

$$\rho_1 : \begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix}$$

relative to the amplitude of 120°

$$\rho_2 : \begin{pmatrix} A & B & C & D \\ A & D & B & C \end{pmatrix}$$

relative to the amplitude of 240°

- With rotation $\rho_3$ about the height outgoing from $B$ the following substitution is associated:

$$\rho_3 : \begin{pmatrix} A & B & C & D \\ C & B & D & A \end{pmatrix}$$

relative to the amplitude of 120°

$$\rho_4 : \begin{pmatrix} A & B & C & D \\ D & B & A & C \end{pmatrix}$$

relative to the amplitude of 240°

- With rotation $\rho_5$ about the height outgoing from $C$ the following substitution is associated:

$$\rho_5 : \begin{pmatrix} A & B & C & D \\ B & D & C & A \end{pmatrix}$$

relative to the amplitude of 120°

$$\rho_6 : \begin{pmatrix} A & B & C & D \\ D & A & C & B \end{pmatrix}$$

relative to the amplitude of 240°

- With rotation $\rho_7$ about the height outgoing from $D$ the following substitution is associated:
Ferdinando Casolaro, Luca Cirillo, Raffaele Prosperi

\[ \rho_7 : \begin{pmatrix} A & B & C & D \\ B & C & A & D \end{pmatrix} \]
relative to the amplitude of 120°

\[ \rho_8 : \begin{pmatrix} A & B & C & D \\ C & A & B & D \end{pmatrix} \]
relative to the amplitude of 240°

c) The planar symmetry \( \sigma \) relative to the plan \( \pi \) passing through the two vertices of the tetrahedron and the midpoint of the edge that joins the other two vertices. The \( \sigma \) symmetry \( \sigma \) is uniquely determined by the initial vertex. The symmetries of this type are 6 (as many as the pairs of vertices of the tetrahedron), and have period 2.

A substitution is associated with each of these symmetries
- With symmetry about the plane \( ABM_1 \), with \( M_1 \) medium point of \( CD \), the following substitution is associated:

\[ \sigma_1 : \begin{pmatrix} A & B & C & D \\ A & B & D & C \end{pmatrix} \]

- With symmetry about the plane \( ACM_2 \), with \( M_2 \) medium point of \( BD \), the following substitution is associated:

\[ \sigma_2 : \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix} \]

- With symmetry about the plane \( ADM_3 \), with \( M_3 \) medium point of \( BC \), the following substitution is associated:

\[ \sigma_3 : \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix} \]

- With symmetry about the plane \( BCM_4 \), with \( M_4 \) medium point of \( AD \), the following substitution is associated:

\[ \sigma_4 : \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix} \]
Groups of Transformations with a Finite Number of Isometries: the Cases of Tetrahedron and Cube

- With symmetry about the plane $BDM_5$, with $M_5$ medium point of $AC$, the following substitution is associated:

$$\sigma_5: \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix}$$

- With symmetry about the plane $CDM_6$, with $M_6$ medium point of $AB$, the following substitution is associated:

$$\sigma_6: \begin{pmatrix} A & B & C & D \\ B & A & C & D \end{pmatrix}$$

It is observed that the two sets of isometries described in points a) and b) each supplemented with the identity

$$I: \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}$$

are closed about to the product.

The first set is a $G_4$ group of order 4 of involutoric transformations. The second set is a $G_5$ group of order 9 of periodic transformations of order 3.

The union of the two groups is a $G_6$ group of order 12, which is the group of direct isometries of $T$.

We will now examine the product of three symmetries, or we will fix an isometry $\sigma_t$ of type c) (planar symmetry), and we will consider an isometry $\alpha_t$ ($t = 1, 2, \ldots, 12$) variable in the $G_3$ group. The product $\sigma_t \cdot \alpha_t$ is still an isometry that changes the tetrahedron $T$ into itself.

They are in number of 12; in fact, if we fix, for example, the isometry

$$\sigma_1: \begin{pmatrix} A & B & C & D \\ A & B & D & C \end{pmatrix}$$

multiplying each isometry of the $G_3$ Group for $\sigma_1$, we will get 12 reverse isometries reverse, which can be summarized as:
It is easily seen that it results:

$$\phi_{12} = \sigma_1, \quad \phi_5 = \sigma_2, \quad \phi_4 = \sigma_3, \quad \phi_7 = \sigma_4, \quad \phi_6 = \sigma_5, \quad \phi_1 = \sigma_6$$

That is the 12 isometries $\sigma_k \circ \alpha_i$ are given by the 6 planar symmetries $\sigma_3$ of the type c) and by the 6 antirotations $\phi_k$, with period 4. The isometries $\phi_k$ do not take firm no vertex and no edge of the tetrahedron.

In summary, we can say that the three axial symmetries of the $G_1$ group, the 8 rotations of the $G_2$ group, the 6 planar symmetries, the 6 latest found isometries, along with the identity, are the 24 isometries that leave the tetrahedron $T$ globally invariant; their set is the $S_T$ group of isometries of $T$.

$S_T$ is the group of isometries that change the tetrahedron $T$ in itself.
3. Isometries of Cube

Some examples of finite groups of isometries can be had considering all isometries leaving globally invariant a cube (A. Morelli, 1989).

$ABCDEFGH$ and $A'B'C'D'E'F'G'H'$ are two equal cubes. Then there exists one and only one isometry that transforms the vertices $A, B, C, D, E, F, G, H$, neatly in the vertices $A', B', C', D', E', F', G', H'$ (Figure 3). This isometry is direct or reverse depending on whether or not the two cubes are equally oriented.

Among the isometries that transform the C Cube itself there are obviously the following:

a) The rotations $\rho$ around a straight line $r$ which joins the centers of two opposite faces.
   Since the faces of the cube are six, these lines are three; for each of these straight lines the cube is transformed into itself by the amplitude rotations, respectively, $90^\circ$, $180^\circ$, $270^\circ$.
   Therefore you have nine rotations of this type which transform C itself. For each of these rotations it is associated a substitution.
   - To $\rho_1$ rotation around the straight through $M_1M_2$, with $M_1$ the center of the $ABCD$ face and $M_2$ the center of the $EFGH$ face, is associated the substitution:

$$\begin{bmatrix}
A & B & C & D & E & F & G & H \\
D & A & B & C & F & G & H & E
\end{bmatrix}$$ relative to the amplitude of $90^\circ$
Ferdinando Casolaro, Luca Cirillo, Raffaele Prosperi

\[ \rho_2 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ C & D & A & B & G & H & E & F \end{pmatrix} \] relative to the amplitude of 180°

\[ \rho_3 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ B & C & D & A & H & E & F & G \end{pmatrix} \] relative to the amplitude of 270°

- To \( \rho_4 \) rotation around the straight through \( M_3 M_4 \), with \( M_3 \) the center of the \( ABFE \) face and \( M_4 \) the center of the \( DCGH \) face, is associated the substitution:

\[ \rho_4 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ B & G & F & C & D & E & H & A \end{pmatrix} \] relative to the amplitude of 90°

\[ \rho_5 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ G & H & E & F & C & D & A & B \end{pmatrix} \] relative to the amplitude of 180°

\[ \rho_6 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & A & D & E & F & C & B & G \end{pmatrix} \] relative to the amplitude of 270°

- To \( \rho_7 \) rotation around the straight through \( M_5 M_6 \), with \( M_5 \) the center of the \( AEHD \) face and \( M_6 \) the center of the \( BFGC \) face, is associated the substitution:

\[ \rho_7 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & B & A & D & C & F & E \end{pmatrix} \] relative to the amplitude of 90°

\[ \rho_8 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & F & G & H & A & B & C & D \end{pmatrix} \] relative to the amplitude of 180°

\[ \rho_9 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & F & E & H & G & B & A \end{pmatrix} \] relative to the amplitude of 270°

b) The rotations \( \rho \) around the straight line \( r \) that connects the midpoints of two opposite edges. Since the edges of the cube are twelve, these lines are six; for
Groups of Transformations with a Finite Number of Isometries: the Cases of Tetrahedron and Cube

each of these straight lines the cube is transformed into itself by rotations of 180 ° amplitude.
For each of these rotations it is associated a substitution.
- To rotation $\rho_{10}$ around the straight line joining the midpoints of AB and EF edges, is associated the substitution:

$$
\rho_{10} : \begin{pmatrix} A & B & C & D & E & F & G & H \\
A & B & G & H & E & F & C & D &
\end{pmatrix}
$$

- To rotation $\rho_{11}$ around the straight line joining the midpoints of CD and HG edges, is associated the substitution:

$$
\rho_{11} : \begin{pmatrix} A & B & C & D & E & F & G & H \\
E & F & C & D & A & B & G & H &
\end{pmatrix}
$$

- To rotation $\rho_{12}$ around the straight line joining the midpoints of BC and HE edges, is associated the substitution:

$$
\rho_{12} : \begin{pmatrix} A & B & C & D & E & F & G & H \\
G & B & C & F & E & D & A & H &
\end{pmatrix}
$$

- To rotation $\rho_{13}$ around the straight line joining the midpoints of AD and FG edges, is associated the substitution:

$$
\rho_{13} : \begin{pmatrix} A & B & C & D & E & F & G & H \\
A & H & E & D & C & F & G & B &
\end{pmatrix}
$$

- To rotation $\rho_{14}$ around the straight line joining the midpoints of BC and HE edges, is associated the substitution:

$$
\rho_{14} : \begin{pmatrix} A & B & C & D & E & F & G & H \\
G & B & C & F & E & D & A & D &
\end{pmatrix}
$$

- To rotation $\rho_{15}$ around the straight line joining the midpoints of AD and FG edges, is associated the substitution:
c) The rotations $\rho$ around the straight line $r$ that contains a diagonal. The number of these lines is four; for each of these straight lines the cube is transformed into itself by the amplitude rotations respectively $120^\circ$ and $240^\circ$. Therefore there are eight rotations of this type which transform $C$ to itself. For each of these rotations it is associated a substitution.

- To rotation $\rho_{16}$ around the diagonal $AF$, it is associated the substitution:

$$\rho_{16} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & A & H & E & D & C & F \end{pmatrix}$$

relative to the amplitude of $120^\circ$

$$\rho_{17} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & G & F & E & H & A & D \end{pmatrix}$$

relative to the amplitude of $240^\circ$

- To rotation $\rho_{18}$ around the diagonal $BE$, it is associated the substitution:

$$\rho_{18} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & A & H & E & D & C & F \end{pmatrix}$$

relative to the amplitude of $120^\circ$

$$\rho_{19} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & G & F & E & H & A & D \end{pmatrix}$$

relative to the amplitude of $240^\circ$

- To rotation $\rho_{20}$ around the diagonal $CH$, it is associated the substitution:

$$\rho_{20} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ G & F & C & B & A & D & E & H \end{pmatrix}$$

relative to the amplitude of $120^\circ$

$$\rho_{21} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & D & C & F & G & B & A & H \end{pmatrix}$$

relative to the amplitude of $240^\circ$

- To rotation $\rho_{21}$ around the diagonal $DG$, it is associated the substitution:
Groups of Transformations with a Finite Number of Isometries: the Cases of Tetrahedron and Cube

\[
\rho_{22} := \begin{pmatrix} A & B & C & D & E & F & G & H \\ C & F & E & D & A & H & G & B \end{pmatrix}
\]\[\text{relative to the amplitude of } 120^\circ\]

\[
\rho_{23} := \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & H & A & D & C & B & G & F \end{pmatrix}
\]\[\text{relative to the amplitude of } 240^\circ\]

d) The planar symmetry \(\sigma\) with respect to \(\pi\) plane parallel to two faces through the midpoints of the four edges perpendicular to these two faces. The symmetries of the type indicated are three.

For each of these symmetries it is associated a substitution.

- At the planar symmetry \(\sigma_1\) with respect to the plane \(\pi_1\) parallel to \(ABGH\) and \(EFCD\) faces, is associated the substitution:

\[
\sigma_1 := \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & B & A & H & G & F & E \end{pmatrix}
\]

- At the planar symmetry \(\sigma_2\) with respect to the plane \(\pi_2\) parallel to \(ABDC\) and \(HGEF\) faces, is associated the substitution:

\[
\sigma_2 := \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & F & E & D & C & B & A \end{pmatrix}
\]

- At the planar symmetry \(\sigma_3\) with respect to the plane \(\pi_3\) parallel to \(BCGH\) and \(ADHE\) faces, is associated the substitution:

\[
\sigma_3 := \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & F & E & D & C & B & A \end{pmatrix}
\]

e) The symmetries \(\sigma\) with respect to the \(\pi\) plan through two opposite edges that do not have common face and vertex. The symmetries of the type indicated are six.

For each of these symmetries it is associated a substitution.

- At the planar symmetry \(\sigma_4\) respect to the \(\pi_4\) plan through the edges \(AD\) and \(GF\) is associated with the substitution:
At the planar symmetry $\sigma_5$ respect to the $\pi_5$ plan through the edges $BC$ and $HE$ is associated with the substitution:

$$\sigma_5 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & C & F & E & D & A & H \end{pmatrix}$$

At the planar symmetry $\sigma_6$ respect to the $\pi_6$ plan through the edges $AB$ and $EF$ is associated with the substitution:

$$\sigma_6 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & G & H & E & F & C & D \end{pmatrix}$$

At the planar symmetry $\sigma_7$ with respect to the $\pi_7$ plan through the edges $CD$ and $HG$ is associated with the substitution:

$$\sigma_7 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & F & C & D & A & B & G & H \end{pmatrix}$$

At the planar symmetry $\sigma_8$ with respect to the $\pi_8$ plan through the edges $AH$ and $CF$ is associated with the substitution:

$$\sigma_8 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & D & C & B & G & F & E & H \end{pmatrix}$$

At the planar symmetry $\sigma_9$ with respect to the $\pi_9$ plan through the edges $BG$ and $DE$ is associated with the substitution:

$$\sigma_9 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & A & D & E & H & G & F \end{pmatrix}$$

Note that the two sets of isometry described in points a), b) and c), each supplemented with the identity:
Groups of Transformations with a Finite Number of Isometries: the Cases of Tetrahedron and Cube

$I: \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & C & D & E & F & G & H \end{pmatrix}$,

are closed with respect to the product.

The first set $G_1$ is a group of order ten, the second set is a group $G_2$ of order seven, the third set is a group $G_3$ of order nine. The union of these three groups is a $G_4$ group of order twenty four which constitutes the group of direct isometries of C.

Let us now examine the product of three symmetries, that is fix an type d) isometry $\sigma_k$ (planar symmetry), and consider an isometry $\alpha_t$ ($t = 1, 2, ..., 24$) variable in the $G_4$ group. The product $\sigma_k \circ \alpha_t$ is still an isometry which changes the C Cube to itself.

The number of these product is twenty four; in fact, it fixed eg. the isometry $\sigma_1: \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & B & A & H & G & F & E \end{pmatrix}$, multiplying each isometry of the $G_4$ group $\rho_1$, you get twenty four reverse isometries, which can be summarized as:

$\sigma_1 \circ \rho_1 = \varphi_1: \begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & A & D & E & H & G & F \end{pmatrix}$,

$\sigma_1 \circ \rho_2 = \varphi_2: \begin{pmatrix} A & B & C & D & E & F & G & H \\ B & A & D & C & F & E & H & G \end{pmatrix}$,

$\sigma_1 \circ \rho_3 = \varphi_3: \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & D & C & B & G & F & E & H \end{pmatrix}$,

$\sigma_1 \circ \rho_4 = \varphi_4: \begin{pmatrix} A & B & C & D & E & F & G & H \\ C & F & G & B & A & H & E & D \end{pmatrix}$,

$\sigma_1 \circ \rho_5 = \varphi_5: \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & E & H & G & B & A & D & C \end{pmatrix}$,

$\sigma_1 \circ \rho_6 = \varphi_6: \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & D & A & H & G & B & C & F \end{pmatrix}$,
Ferdinando Casolaro, Luca Cirillo, Raffaele Prosperi

\[
\sigma_1 \circ \rho_7 = \varphi_7 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & G & H & E & F & C & D \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_8 = \varphi_8 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & F & E & D & C & B & A \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_9 = \varphi_9 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & F & C & D & A & B & G & H \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_{10} = \varphi_{10} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & B & A & D & C & F & E \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_{11} = \varphi_{11} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & F & E & H & G & B & E \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_{12} = \varphi_{12} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & C & B & G & H & A & D & E \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_{13} = \varphi_{13} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & E & H & A & B & G & F & C \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_{14} = \varphi_{14} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & C & B & G & H & A & D & E \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_{15} = \varphi_{15} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & E & H & A & B & G & F & C \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_{16} = \varphi_{16} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & E & D & A & B & C & F & G \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_{17} = \varphi_{17} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ B & G & H & A & D & E & F & C \end{pmatrix},
\]

\[
\sigma_1 \circ \rho_{18} = \varphi_{18} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & A & B & G & F & C & D & E \end{pmatrix},
\]

108
Groups of Transformations with a Finite Number of Isometries: the Cases of Tetrahedron and Cube

\[
\sigma_1 \circ \rho_{19} = \varphi_{19} : \begin{bmatrix} A & B & C & D & E & F & G & H \\ F & G & B & C & D & A & H & E \end{bmatrix},
\]

\[
\sigma_1 \circ \rho_{20} = \varphi_{20} : \begin{bmatrix} A & B & C & D & E & F & G & H \\ B & C & F & G & H & E & D & A \end{bmatrix},
\]

\[
\sigma_1 \circ \rho_{21} = \varphi_{21} : \begin{bmatrix} A & B & C & D & E & F & G & H \\ F & C & D & E & H & A & B & G \end{bmatrix},
\]

\[
\sigma_1 \circ \rho_{22} = \varphi_{22} : \begin{bmatrix} A & B & C & D & E & F & G & H \\ D & E & F & C & B & G & H & A \end{bmatrix},
\]

\[
\sigma_1 \circ \rho_{23} = \varphi_{23} : \begin{bmatrix} A & B & C & D & E & F & G & H \\ D & A & H & E & F & G & B & C \end{bmatrix},
\]

\[
\sigma_1 \circ I = \varphi_{24} : \begin{bmatrix} A & B & C & D & E & F & G & H \\ D & C & B & A & H & G & F & E \end{bmatrix}.
\]

It is easily seen that results:

\[
\varphi_{24} = \sigma_1 , \; \varphi_8 = \sigma_2 , \; \varphi_2 = \sigma_3 , \; \varphi_7 = \sigma_6 , \; \varphi_9 = \sigma_1 , \; \varphi_3 = \sigma_8 , \; \varphi_1 = \sigma_9
\]

that is, the twentyfour isometries \( \sigma_k \circ \alpha_t \) are given from nine symmetries \( \sigma_k \) planar type d), e), and fifteen anti rotations \( \varphi_k \).

In summary therefore it can be said that the twenty three rotations of the \( G_4 \) group, the nine planar symmetries and the latest isometries found, along with the identity, are the forty eight isometries which leave the cube \( C \) globally invariant; their set is the \( S_C \) group of isometries of the cube \( C \).

\( S_C \) is the group of the isometries that change \( C \) cube to itself.

**Conclusions**

As already shown in a previous work (Casolaro, F., Cirillo, L. and Prosperi, R. 2015), the geometric Universe is three-dimensional, so the transformations taking place in it are generated in space. Then, we believe, for a correct analysis of the physical phenomena that occur in the universe, that it is essential to the knowledge of the real transformations that take place in it. Recent results of other branches of mathematics, in particular the modern algebra, have
highlighted the interrelationships between movements in the plane and in space with some properties of the Theory of Groups (Casolaro, F. 1992), for which we consider essential to the deepening of these issues both in education and in the field of pure research (Casolaro, F. and Eugeni, F. 1996). Unfortunately, teaching (Casolaro F. 2014) in both the Secondary School that the University has been anchored to old programs that do not take into account the development of mathematics in the last 150 years, so we hope that this work will stimulate teachers and researchers to expand their views.

References


