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Groups having unique faithful irreducible $\mathbb{Q}$-representation

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Abstract

In this paper, we give few sufficient conditions for finite p-group to have unique NEW (i.e faithful irreducible) $\mathbb{Q}$-representation. As a consequence of these conditions we will prove that any finite p-group of nilpotency class 2 has atmost one NEW $\mathbb{Q}$-representation. We also give examples of few classes of finite p-groups which has unique NEW $\mathbb{Q}$-representation.

Keywords: Idempotents, Faithful representation.

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1 Introduction

Let F be a field and G be a finite group of order n such that $F[G]$ is semisimple, or equivalently char$F \nmid |G|$. Primitive central idempotents in $F[G]$ are useful in determining the Wedderburn decomposition of the group algebra $F[G]$. The Wedderburn decomposition helps in finding the group of units of such a group ring and also the group of automorphisms of the group ring. Further, a complete set of orthogonal primitive idempotents gives us enough information to compute all one-sided ideals of the group ring. Finite group algebras and their Wedderburn decomposition also have applications in coding theory. Cyclic codes can be realized as ideals of group algebras over cyclic groups and many other important codes appear as ideals of non-cyclic group algebras. Primitive central idempotents are idempotent generators of minimal abelian codes. Using a complete set of orthogonal primitive idempotents, one would be able to construct all left G-codes, i.e. left ideals of the finite group algebra $F[G]$, which is a much richer class than the (two-sided) G-codes. For finite field $\mathbb{F}_q$ of q elements, primitive central idempotents in $\mathbb{F}_q[G]$ re also useful to determine characters of $\mathbb{F}_q$-representations.

In this paper we study NEW (i.e faithful irreducible) F-representations of finite p-groups G, where F is a field such that characteristic of F does not divide the order of G. Section 2, includes basic definitions, results on representation, characters and idempotents in the semi-simple ring which are useful for our main results. In section 3, we present the concept of lifting idempotent from the group algebra of the quotient group and in the Theorem 3.1, we give method of computing primitive central idempotents in the group algebra $\mathbb{Q}[G]$, where $\mathbb{C}[G]$ contains only one non abelian simple component. In section 4, we discuss the existence of NEW F-representation. We also give method of computing primitive central idempotents in the group algebra $F[G]$, for finite p-group G and F is a field such that characteristic of F does not divide the order of G (Theorem 4.2). In section 5, we prove Theorem 5.1 which gives sufficient conditions for finite p-group to have unique NEW (i.e faithful irreducible) $\mathbb{Q}$-representation. As a consequence of these conditions we also prove that any finite p-group of nilpotency class 2 has at most one NEW $\mathbb{Q}$-representation. We also give examples of few classes of finite p-groups which has unique NEW $\mathbb{Q}$-representation.

2 Preliminaries

In this section, we are presenting some basic concepts on representation, characters and idempotents in the semi-simple ring.

Definition 2.1. *Let G be a group and F be a field. Let V be a vector space over F. An F-representation of G in V is a homomorphism $\rho\colon G \to Aut(V)$. V is* *called as a representation space for ρ, and sometimes V is called a representation of G. The dimension of V is called the degree of the representation (see (7), (8)).*

Definition 2.2. *Let $V \neq \{0\}$ be a representation of group G. Then V is irreducible if it has no subrepresentation W other than $W = \{0\}$ and $W = V$ (see (7), (8)).*

Definition 2.3. *Let $\rho : G \to GL(V)$ be an F-representation of G of finite degree, and let $\mathbf{B}$ be a basis of V. The character of ρ is the function $\chi : G \to F$ defined by*

$$\chi(x) = tr([\rho(x)]_{\mathbf{B}}).$$

$\chi(x)$ is independent of the basis $\mathbf{B}$ (see (7), (8)).

For a field F, let $\overline{F}$ denote the algebraic closure of F. We now state some definitions and results from [6].

Definition 2.4. *Suppose K is a field extension of F and (ρ, U) is an K-representation of G. The representation (ρ, U) is said to be realizable in F if there exists an F-representation (η, T) such that $U \cong T \otimes_F K$. For every $x \in G$, $\rho(x) \in Aut_K(U)$ is extended from $\eta(x) \in Aut_F(T)$.*

Definition 2.5. *Suppose (ρ, U) is an irreducible F-representation of G. We say that (ρ, U) is absolutely irreducible if, for every extension K over F, $U \cong T \otimes_F K$ is irreducible K-representation. A field E is said to be a splitting field for the group G, if every irreducible representation of G over E is absolutely irreducible.*

Definition 2.6. *Let $F \subseteq E$, where E is any splitting field for G. Let (ρ, U) be an irreducible E-representation of G with character χ, and let $F(\chi)$ denote the field obtained by adjoining to F all of the values $\{\chi(g) : g \in G\}$. If K is an extension of F in which (ρ, U) is realizable, then certainly $F(\chi) \subseteq K$. The Schur index of (ρ, U) with respect to F is defined as*

$$m_F(U) := Min\{[K : F(\chi)] \mid U \text{ is realizable in } K\}.$$

Isomorphic representations have the same character and hence the same Schur index. As the character determines the representation up to isomorphism, we shall denote it by $m_F(U)$ also by $m_F(\chi)$.

Theorem 2.1. *Let (ρ, U) be an irreducible $\overline{F}$-representation of G with character χ. There exists a finite field extension K of F in which (ρ, U) is realizable, such that $[K : F(\chi)] = m_F(\chi)$. For any finite extension L of F in which U is realizable, we have $m_F(\chi) | [L : F(\chi)]$. Finally, $m_F(\chi)$ is the smallest integer m such that $m(\rho, U)$ (i.e direct sum of (ρ, U), m times) is realizable in $F(\chi)$.*

Definition 2.7. *Suppose E is a Galois extension of F and a splitting field for G. Let χ be an character of an E-representation of G and $\sigma \in Gal(E/F)$. The function ${}^{\sigma}\chi\colon G \to E$ defined as*

$$ {}^{\sigma}\chi(g) = \sigma(\chi(g)) $$

for all $g \in G$, is a character of an E-representation of G. The character ${}^{\sigma}\chi$ is said to be algebraically conjugate to χ with respect to F.

Theorem 2.2. *Every irreducible F-representation (η, T) of G is completely reducible over $\overline{F}$ into inequivalent $\overline{F}$-representations $(\rho_1, U_1), (\rho_2, U_2), \cdots, (\rho_\delta, U_\delta)$ with the same multiplicity m given by*

$$ m = m_F(\chi_i), \quad 1 \le i \le \delta, $$

where χ_i is the character of (ρ_i, U_i). Infact the number δ of irreducible representation U_i in the decomposition is given by

$$ \delta = [F(\chi_i)\colon F], \quad (1 \le i \le \delta). $$

The representations $(\rho_1, U_1), (\rho_2, U_2), \cdots, (\rho_\delta, U_\delta)$ are algebraically conjugate with respect to F, and they are of the same degree. Also the center of the endomorphism ring of (η, T) over F is isomorphic to the character field $F(\chi_i)$. Conversely each irreducible $\overline{F}$- representation (ρ, U) of G occurs in the decomposition of some irreducible F-representation (η, T) of G over $\overline{F}$.

Corollary 2.1. *If ϕ is the character corresponding to the irreducible F-representation (η, T), and Γ is the Galois group of $F(\chi_i)$ over F, then*

$$ \phi = m(\sum_{\sigma \in \Gamma} {}^{\sigma}\chi_1). $$

Definition 2.8. *Let χ be the character of an F-representation of a group G in V. Then Ker$(\chi) = \{g \in G \,|\, \chi(g) = \chi(1)\}$.*

Remark 2.1.

If χ is the character of the F-representation ρ of G in V, then

$$ \begin{aligned} \text{Ker}\rho &= \{g \in G \mid \rho(g) = I_V\} \\ &= \{g \in G \mid \chi(g) = \chi(1)\} = \text{Ker}(\chi). \end{aligned} $$

As Ker ρ is a normal subgroup of G, so is Ker(χ).

Definition 2.9. *An F-representation (ρ, V) of a group G is said to be faithful if $\rho\colon G \to GL(V)$ injective.*

Definition 2.10. *Let R be a ring. An element e of R is an idempotent if $e^2 = e$. Two idempotents e and f are orthogonal if $ef = fe = 0$. A finite set of idempotents $\{e_i\}$ that are pairwise mutually orthogonal and satisfy $\sum e_i = 1$ are called complementary (See (13)).*

Definition 2.11. *Let R be a ring. An idempotent e of R is central if e is in the center of R. An idempotent e is primitive if $e = f_1 + f_2$ with f_1 and f_2 orthogonal idempotents implies $f_1 = e$ and $f_2 = 0$ or $f_2 = e$ and $f_1 = 0$. A central idempotent e is called central primitive idempotent (or primitive central) if $e = f_1 + f_2$ with f_1 and f_2 orthogonal central idempotents implies $f_1 = e$ and $f_2 = 0$ or $f_2 = e$ and $f_1 = 0$ (See (13)).*

Proposition 2.1. *Let $R = \oplus_{i=1}^{r} R_i$ be a decomposition of a semisimple ring as a direct sum of minimal two-sided ideals. Then, there exists a family $\{e_1, e_2, \cdots, e_r\}$ of elements of R such that:*
(i) $e_i \neq 0$ is a central idempotent, $1 \leq i \leq r$.
(ii) If $i \neq j$ then $e_i e_j = 0$.
(iii) $1 = e_1 + e_2 + \cdots + e_r$.
(iv) e_i cannot be written as $e_i = e_i^{'} + e_i^{''}$ where $e_i^{'}, e_i^{''}$ are central idempotents such that $e_i^{'}, e_i^{''} \neq 0$ and $e_i^{'} e_i^{''} = 0$, $1 \leq i \leq r$. In fact each R_i is the principal ideal of R and we can take $R_i = Re_i$, e_i works as identity of the ring R_i(see (7)).

Definition 2.12. *The elements $e_1, e_2, \cdots, e_r$ in the above proposition are uniquely determined as a set and they are called primitive central idempotents of R (see (7)).*

Let G be a group and $F[G]$ be a group ring of G over field F such that char $F \nmid |G|$. For a finite subgroup N of G, we denote

$$e_N = \widehat{N} = \frac{1}{|N|} \sum_{n \in N} n.$$

Lemma 2.1. *Let G be a group and $F[G]$ be a group ring of G over field F. If N be a finite subgroup of G, then*

1. *$(\widehat{N})^2 = \widehat{N}$ is an idempotent.*
2. *If N is normal in G, then*
   - *(a) $\widehat{N}$ is a central idempotent.*
   - *(b) $(F[G])\widehat{N} \cong F[G/N]$.*
   - *(c) $F[G] = F[G]\widehat{N} \oplus F[G](1 - \widehat{N})$*

3 Lifting central idempotents from group algebra of a quotient group

For a normal subgroup N of a group G, let $\phi : F[G] \to F[G/N]$ be the projection map induced from the natural map $G \to G/N$. For $\bar{e} \in F[G/N]$, an element $e \in F[G]$ is called a **lift** of $\bar{e}$, if $\phi(e) = \bar{e}$ and the element $e_N e \in F[G]$ is called a **pullback** of $\bar{e}$.
Let F be a field and G be a finite group such that char F does not divide order of group G. Let H be a normal subgroup of group G. We have a canonical projection map, $\pi : G \to G/H$.
Correspondingly, we have an injection

$$\pi^* : F[G/H] \to F[G] \text{ given by } \pi^*(\sum a_i(g_iH)) = \sum a_i g_i e_H$$

which maps onto a direct summand of $F[G]$,
$F[G] = \pi^*(F[G/H]) \oplus \cdots$. Let e_H, e_G denote the canonical central idempotents corresponding to H and G. We wish to prove the following useful lemmas.

Lemma 3.1. *Let F be a field and G be a finite group such that char F does not divide order of group G. Let H be a normal subgroup of group G. Then e_H is the sum of the primitive central idempotents in $F[G]$, corresponding to the subring $\pi^*(F[G/H])$.*

Proof. Let $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_r$, be the primitive central idempoents of $F[G/H]$, and $e_1, e_2, \cdots, e_r$ their lifts in $F[G]$. $G/H \cong Ge_H$ as groups, for consider $\phi : G \to Ge_H$ is a group epimorphism and kernel of ϕ is H. As Ge_H is a basis for $F[Ge_H]$ over F, we have $F[Ge_H] \cong F[G/H]$. Hence $e_He_1, e_He_2, \cdots, e_He_r$ are primitive central idempotents in $F[G]$ such that $e_H = e_He_1 + e_He_2 + ... + e_He_r$. □

Lemma 3.2. *Let F be a field and G be a finite group such that char F does not divide order of group G. Let H be a normal subgroup of group G. Then $1 - e_H$ is the sum of the primitive central idempotents in $F[G]$, which corresponds to the endomorphism rings of the irreducible representations of G, which, in turn, are induced from the irreducible representations of some subgroup between G and H (see (7)).*

Lemma 3.3. *Let $F = \mathbb{Q}$, Let G be a finite group, and H be the commutator subgroup of G. For a canonical projection map, $\pi : G \to G/H$. correspondingly, we have an injection $\pi^* : \mathbb{Q}[G/H] \to \mathbb{Q}[G]$, which maps onto a direct summand of $\mathbb{Q}[G]$.*
$\mathbb{Q}[G] = \mathbb{Q}[G^{ab}] \oplus$ non-abelian part (see (7)).

Theorem 3.1. *Let G be a finite group and G' be the commutator subgroup of G. If $\mathbb{C}[G]$ contains only non abelian simple component, $e_{G'}$ is the canonical idempotent corresponding to G' and $\{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_r\}$ be the primitive central idempotents in $\mathbb{Q}[G/G']$, then $e_{G'}e_1, e_{G'}e_2, \cdots, e_{G'}e_r, 1 - e_{G'}$ are the primitive central idempotents in $\mathbb{Q}[G]$.*

Proof. Let G be a finite group and G' be the commutator subgroup of G. We have a canonical projection map, $\pi : G \to G/G'$. Correspondingly, we have an injection $\pi^* : \mathbb{Q}[G/G'] \to \mathbb{Q}[G]$, which maps onto a direct summand of $\mathbb{Q}[G]$, so $\mathbb{Q}[G] = \mathbb{Q}[G^{ab}]\oplus$ non-abelian part. By hypothesis, we have that $\mathbb{C}[G]$ contains only non abelian simple component, so $\mathbb{Q}[G]$ contains only one non abelian simple component. Thus

$$\mathbb{Q}[G] \;=\; \mathbb{Q}[G^{ab}] \oplus M = \pi^*(\mathbb{Q}[G/G']) \oplus M$$

Let $e_{G'}$ be the canonical idempotent corresponding to G' and $\{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_r\}$ be the primitive central idempotents in $\mathbb{Q}[G/G']$, then from above we know that $e_{G'}e_1, e_{G'}e_2, \cdots, e_{G'}e_r$ are primitive central idempotents in $\mathbb{Q}[G]$ such that $e_{G'} = e_{G'}e_1 + e_{G'}e_2 + ... + e_{G'}e_r$. Next $1 - e_{G'}$ is the sum of primitive central idempotents corresponding to the non-abelian simple components of $\mathbb{Q}[G]$, but $\mathbb{Q}[G]$ has only one simple component. Thus $1 - e_{G'}$ is the primitive central idempotent corresponding to non-abelian simple components M of $\mathbb{Q}[G]$. Hence
$e_{G'}e_1, e_{G'}e_2, \cdots, e_{G'}e_r, 1 - e_{G'}$ are all the primitive central idempotents in $\mathbb{Q}[G]$. □

Remark 3.1. (i) *In* 1968*, Seitz in (11) characterized groups with only one non-linear K-representation, where K is an algebraically closed field of characteristic zero.*
(ii) *Let F be a field and G be a finite group such that char$F \nmid |G|$. If $F[G]$ contains only one non-abelian simple component, then we can construct primitive central idempotents in $F[G]$ in the same way as in the above theorem.*

Theorem 3.2. *Let K be an algebraically closed field of characteristic zero. A group G has exactly one irreducible K-representation of degree greater than one if and only if*
i) $|G| = 2^k, k$ is odd, $G' = Z(G)$, and $|G'| = 2$, or (ii) G is isomorphic to the group of all transformations $x \to ax + b, a \neq 0$, on a field of order $p^n \neq 2$.

4 New representations of p-groups

Definition 4.1. *Let F be a field, char F does not divide $|G|$. An irreducible faithful(i.e. injective) representation is called a "NEW F-representation" in the* *sense that it is not lifted from any proper normal quotient. An representation is called a "OLD F-representation" if it is not a "NEW F-representation".*

Remark 4.1. *The NEW F-representations may not exist. We now prove that the existence of NEW F-representations does not depend on whether or not F is algebraically closed. Here we have developed the following results.*

Theorem 4.1. *Let G be a finite group and F be a field such that char F does not divide order of group G. Then G has NEW F-representation if and only if G has NEW $\overline{F}$-representation.*

Proof. Let ρ be any F-representation of G and χ be the character of ρ, then

$$\chi = m(\sum_{\sigma \in \Gamma} {}^{\sigma}\chi_1)$$

where χ_1 is a $\overline{F}$-representation of G and Γ is the Gal$(F(\chi_1)/F)$. It is easy to see that

$$\begin{aligned} \ker(\chi) &= \bigcap_{\sigma \in \Gamma} \ker({}^{\sigma}\chi_1) \quad \text{and} \\ \ker(\chi_1) &= \ker({}^{\sigma}\chi_1), \quad \forall\ 1 \neq \sigma \in \Gamma. \end{aligned}$$

Thus for any $\sigma \in \Gamma$, we have $\ker(\chi) = \ker({}^{\sigma}\chi_1)$. Hence we see that F-representation corresponding to character χ is NEW if and only if $\overline{F}$-representation corresponding to character χ_1 is NEW. □

Lemma 4.1. *If G admits a NEW $\overline{F}$-representation, then $Z(G)$ is cyclic.*

Proof. Since in any irreducible representation ρ, by Schur's lemma, for $a \in Z(G)$, $\rho(a)$ must act as a homothety. So in a faithful irreducible (NEW) $\overline{F}$-representation, $Z(G)$ is isomorphic to a finite commutative subgroup of F^*. q.e.d. □

Lemma 4.2. *If G is a finite group with a unique minimal normal subgroup, and a field F with characteristic not dividing the order of G, then there exists a NEW F-representation.*

Proof. By Maschke's theorem, regular representation is a direct sum of irreducible representations, since characteristic of F does not divide order of G.
Suppose there does not exists a faithful irreducible F-representation. Then all the irreducible representations contain the unique minimal normal subgroup in their kernel and hence regular representation is not faithful. This is a contradiction. □

Corollary 4.1. *Let G be a p-group, and a field F with characteristic not dividing the order of G. If $A := Z(G)$ is cyclic subgroup of G. Then G admits a NEW F-representation.*

Thus for a p-group G, if char$F \nmid |G|$, then G admits a NEW F-representation if and only if $Z(G)$ is cyclic. W. Gaschütz ((5), §42) has given neccessary and sufficient conditions for a group G to have a NEW representation. It requires that the product of all minimal abelian normal subgroups of G is generated by one conjugacy class of G.

Lemma 4.3. *Let G be a p-group, and ρ a non-trivial, irreducible F-representation. If ρ is OLD, then G factors through a proper central subgroup.*

Proof. Let $H = \ker \rho$. Since ρ is non-trivial and not NEW, H is a proper normal subgroup. Thus $H \cap Z(G) \neq \{e\}$. Hence G factors through a proper central subgroup. □

Example 4.1.

By Corollary 4.1 a cyclic p-group G of order p^k admits a NEW rep μ_0 over a field F such that char$F \nmid |G|$. If F is algebraically closed then deg μ_0 is 1, and there are precisely $\phi(p^k) = p^{k-1}(p-1)$ NEW F-representations, where ϕ is the Euler's totient function. The other extreme is $F = \mathbb{Q}$, when deg μ_0 is $\phi(p^k)$, and there is a unique NEW F-representation.

More generally suppose F is not necessarily algebraically closed. Let $\overline{F}$ be its algebraic closure. Let ζ be a primitive p^k-th root of unity in $\overline{F}$. Let $q(X)$ be the minimal (monic) polynomial of ζ, then the degree of $q(X)$ is the degree of μ_0. In this case deg μ_0 divides $\phi(p^k)$, and there are $\phi(p^k)/\text{deg}(\mu_0)$ NEW F-representations.

Remark 4.2. *Let G be a p-group, and $A := Z(G)$ cyclic. Let A_0 be the subgroup of A of order p. An F-representation of G is faithful if and only if it is injective on A_0. Any OLD representation of G will factor through A_0. So the sum of the contributions to dimension of $F[G]$ by OLD F-representations is p^{n-1}. Hence the sum of the contributions to dimension of $F[G]$ by NEW F-representations is $p^n - p^{n-1} = \phi(p^n)$.*

Theorem 4.2. *Let G be a p-group.*
a) If G has no NEW F-representation, then the primitive central idempotents of $F[G]$ may be computed as the pull-backs of the primitive central idempotents of G/A, where A is a central subgroup of order p. In other words, all primitive central idempotents of G are contained among the pullbacks of the primitive central idempotents of G/A, where A is a subgroup of order p, of the elementary abelian

subgroup of $Z(G)$.
b) If G admits a NEW F-representation , let A_o be the subgroup of order p of $A = Z(G)$. The OLD primitive central idempotents add up to e_{A_o}, and the NEW primitive central idempotents add up to $1 - e_{A_o}$.

Corollary 4.2. *A finite p-group G has unique NEW F-representation if and only if $Z(G)$ is cyclic and $1 - e_{A_o}$ is a primitive central idempotent in $F[G]$, where A_o is the central subgroup of order p.*

5 Groups admitting unique NEW representation

By Irr(G), we denote the set of irreducible complex characters of group G.

Definition 5.1. *Let χ be a complex character of group G. Then*
$Z(\chi) = \{g \in G \,|\, |\chi(g)| = \chi(1)\}$.

Lemma 5.1. *Let χ be a complex character of group G and ρ be e representation of G which affords χ. Then*
(i) $Z(\chi) = \{g \in G \,|\, \rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}\}$;
(ii) $Z(\chi)/ker(\chi)$ is cyclic
Furthermore, if $\chi \in Irr(G)$, then
(iii) $Z(\chi)/ker(\chi) = Z(G/ker(\chi))$.

Let K be a subgroup of a finite group G and χ_K a linear character on K. Let χ_K^G be the induced character on G: such a character of G is called a monomial character. For normal subgroup K of G and $\lambda \in$ Irr(K)

$$I_G(\lambda) = \{x \in G | \lambda(xkx^{-1}) = \lambda(x),\ \forall\ k \in K\} \tag{1}$$

is the inertia subgroup of G. For subgroup K of G, $Z_G(K)$ denotes the centralizer of K in G.

Definition 5.2. *A pair (H, K) of subgroups of G is called a Shoda pair if it satisfies the following conditions:*
(i) $H \trianglelefteq K$;
(ii) K/H is cyclic, and
(iii) If $g \in G$ and $[K, g] \cap K \trianglelefteq H$, then $g \in K$.

If χ is a linear character of a subgroup K of G with kernel H, then the induced character G is irreducible if and only if (H, K) is a Shoda pair(see Olivieri et al. [6]).

Proposition 5.1. *A finite noncyclic p-group G has a maximal normal Abelian subgroup which is cyclic of order p^n, if and only if G is of one of the following types:*

1. $Q_{2^{n+1}} =< a, b | a^{2^n} = 1, b^2 = a^{2^{n-1}}, a^b = a^{-1} >, n \geq 2$;
2. $D_{2^{n+1}} =< a, b | a^{2^n} = b^2 = 1, a^b = a^{-1} >, n \geq 2$;
3. $M_p(n, m) =< a, b | a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-m}} >, n > m,$ *and* $n \geq m + 2$ *if* $p = 2$;
4. $S(n, m) =< a, b | a^{2^n} = b^{2^m} = 1, a^b = a^{-1-2^{n-m}} >$;
5. $M^*(n, m) =< a, b, c | a^{2^n} = b^{2^m} = c^2 = 1, a^b = a^{1+2^{n-m}}, a^c = a^{1-2^{n-1}}, b^c = b >$;
6. $M_*(n, m) =< a, b, c | a^{2^n} = b^{2^m} = c^2 = 1, a^b = a^{1+2^{n-m}}, a^c = a^{1-2^{n-1}}, b^c = aba^{-1} >$.

Proof. See Hao[Theorem 1, 6]. □

For $\chi \in \text{Irr}(G)$, it is known that $\chi(1) \leq [G : Z(\chi)]$.

Theorem 5.1. *Let G be a finite non-abelian p-group.*

1. *If G has a faithful irreducible character χ with $\chi(1)^2 = [G : Z(\chi)]$, then G has unique NEW $\mathbb{Q}$-representation.*
2. *Let K be a cyclic normal subgroup of G and χ_K be the irreducible complex character on K. If the induced character χ_K^G is irreducible, then G has unique NEW $\mathbb{Q}$-representation.*

Proof.

1. Let χ be a faithful irreducible character of G such that $\chi(1)^2 = [G : Z(\chi)]$. Since χ is faithful, we have $Z(\chi) = Z(G)$ is cyclic and by [6, Corollary 1], $1 - e_L$ is a primitive central idempotent in $\mathbb{Q}[G]$, where L is the unique subgroup of $Z(G)$ of order p. Hence by corollary 4.2, G has unique NEW $\mathbb{Q}$-representation.
2. Let K be a cyclic normal subgroup of G, and χ_K be the irreducible complex character on K, such that induced character χ_K^G is irreducible. By Problem 6.1 of [6], we must have $I_G(\chi_K) = K$. Clearly $Z_G(K) \subseteq I_G(\chi_K)$, hence $Z_G(K) = K$ i.e. K is a maximal normal abelian subgroup of G. As $Z(G)$ always lies in a maximal normal abelian subgroup, we have $Z(G) \subseteq K$, and hence $Z(G)$ is cyclic. Thus NEW $\mathbb{Q}$-representation exists. Also as K is cyclic, there exists faithful linear character λ of K. It follows that

$I_G(\lambda) = Z_G(K) = K$. By Problem 6.1 of [6] again, we see that λ^G is irreducible. Since both Ker(λ) and K are normal subgroups of G, by [6, Theorem 5], $1 - e_L$ is a primitive central idempotent in $\mathbb{Q}[G]$, where L is the unique subgroup of $Z(G)$ of order p. Hence by corollary 4.2, G has unique NEW $\mathbb{Q}$-representation.

□

Remark 5.1. *A finite nilpotent group admits a NEW F-representation if and only if each of its p-Sylow subgroups admits a NEW F-representation. It is also easy to see that finite abelian group(i.e Nilpotent group of class 1) has NEW F-representation if and only if it is cyclic. Moreover, let G is any cyclic group of order n and F have characteristic not dividing n. Let $E = F(\zeta)$, where ζ is a primitive nth root of unity. Let $m = [E : F]$. Then every faithful irreducible F-representation of G has degree m and that there are exactly $\phi(n)/m$ similarity classes of such representation. In particular we see that G has unique NEW $\mathbb{Q}$-representation. As a consequence of above theorem we have the following result.*

Corollary 5.1. *If G is a finite p-group of class 2, then G has atmost one NEW $\mathbb{Q}$-representation.*

Proof. Let G is a finite p-group of class 2. If $Z(G)$ is not cyclic, then G has no NEW $\mathbb{Q}$-representation. If $Z(G)$ is cyclic, then G admits NEW $\mathbb{C}$-representation say ρ. Let χ be the character of representation ρ. Since χ is faithful, we have $Z(\chi) = Z(G)$. Also G is of class 2, hence $G/Z(\chi) = G/Z(G)$ is abelian. By Issacs[6, Corollary 2.30], $\chi(1)^2 = [G : Z(\chi)]$. Hence by above theorem G has unique NEW $\mathbb{Q}$-representation. □

Corollary 5.2. *Let G be a finite non-abelian p-group. If G has a cyclic normal subgroup K and the pair $(1, K)$ is a Shoda pair, then G has unique NEW $\mathbb{Q}$-representation.*

Example 5.1.

Let G be a extra-special p-group. We know that G is of class 2, hence by the above theorem 5.1, G has unique NEW $\mathbb{Q}$-representation.

Example 5.2.

If G is isomorphic to one of the groups listed in proposition 5.1, then G has unique NEW $\mathbb{Q}$-representation.

6 Conclusion

In this paper, we have obtained few sufficient conditions for finite p-group to have unique NEW (i.e faithful irreducible) $\mathbb{Q}$-representation. In future researchers can take up the problem of finding necessary and sufficient conditions for finite groups to have unique NEW (i.e faithful irreducible) $\mathbb{F}$-representation, char$F \nmid |G|$ and computing complete set of primitive central idempotents in the group algebra $F[G]$.

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