PERIODIC SOLUTIONS OF A SECOND ORDER EVOLUTIVE VARIATIONAL INEQUALITY*

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ABSTRACT - Hereafter we shall analyse the existence and regularity properties of the solution of the periodic problem related to a second order evolutive variational inequality.

SUNTO - Si analizzano le questioni di esistenza e regolarità della soluzione per il problema periodico connesso ad una disequazione variazionale di evoluzione del secondo ordine.

INTRODUCTION

Let \( \Omega_1 \) and \( \Omega_2 \) be two open sets of \( \mathbb{R}^N \) with \( \Omega = \Omega_1 \cap \Omega_2 \neq \emptyset \), \( V_i (i = 1, 2) \) a real separable Hilbert space with dense and continuous embedding in \( \mathbb{L}^2 (\Omega_i) \).

Let us denote by:

\[
\begin{align*}
\langle \cdot, \cdot \rangle & \quad \text{the inner product in } \mathbb{L}^2 (\Omega), \\
\langle \cdot, \cdot \rangle_i & \quad \text{the inner product in } \mathbb{L}^2 (\Omega_i), \\
\| \cdot \|_i & \quad \text{the norm in } V_i, \\
\langle \cdot, \cdot \rangle_i & \quad \text{the pairing between } V_i \text{ and its dual } V_i'.
\end{align*}
\]

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Furthermore, let $\mathcal{K}$ be the convex closed part of $V_1 \times V_2$:
\[
\{(z_1, z_2) \in V_1 \times V_2 : z_1 \leq z_2 \text{ a.e. on } \Omega\}.
\]

Given $f_l \in L^2\left(0, T; V_l^*\right)$ $(0 < T < +\infty)$ and operators $A_l, B_l \in L(V_l, V_l^*)$ such that:
\[
\begin{align*}
\langle A_l y, z \rangle_l &= \langle A_l z, y \rangle_l, \quad \forall y, z \in V_l, \\
\langle B_l y, z \rangle_l &= \langle B_l z, y \rangle_l, \quad \forall y, z \in V_l, \\
\langle A_l z, z \rangle_l &\geq a_l \| z \|_{V_l}^2, \quad \langle B_l z, z \rangle_l \geq b_l \| z \|_{V_l}^2, \quad \forall z, \in V_l, \quad (a_l, b_l = \text{const.} > 0),
\end{align*}
\]
we consider the following

**PROBLEM (P)**. Find $(u_1, u_2) \in \prod_{l=1}^{2} H^1(0, T; V_l)$ so that:
\[
\begin{align*}
u''_l &\in L^2\left(0, T; V_l^*\right), \\
u_l(0) &= u_l(0), \quad u'_l(0) = u'_l(T), \\
\left(u'_l(t), u'_2(t)\right) &= \mathcal{K} \text{ a.e. on } [0, T], \\
\frac{1}{2} \sum_{l=1}^{2} \int_0^T \left\langle u''_l(t) + A_l u_l(t) + B_l u'_l(t) - f_l(t), \nu'_l(t) - u'_l(t) \right\rangle_l dt &\geq 0, \\
\forall (\nu_1, \nu_2) \in \prod_{l=1}^{2} H^1(0, T; V_l) \text{ with } \nu_l(0) = \nu_l(T) \text{ and } \left(\nu'_l(t), \nu'_2(t)\right) \in \mathcal{K} \text{ a.e. on } [0, T].
\end{align*}
\]

Of course our problem (P) will not have a unique solution. As a matter of fact:

if $(u_1, u_2)$ is one solution of problem (P), then any other solution will only be of the type $(u_1 + z_1, u_2 + z_2)$, where $z_l$ is an arbitrary element of $V_l$.

Obviously $(u_1 + z_1, u_2 + z_2)$ satisfies our problem (P). On the other hand, assuming that $(\tilde{u}_1, \tilde{u}_2)$ is another solution, we immediately find that:
\[
\|\tilde{u}'_l - u'_l\|_{L^2(0, T; V_l)} = 0,
\]

and therefore $\tilde{u}_l = u_l + z_l$ with $z_l \in V_l$.

We shall now develop two existence theorems for our problem, theorems 4 and 5 (n. 3), with two different hypothesis on $f_l$: in the first we assume $f_l \in L^2\left(0, T; L^2(\Omega_l)\right)$, in the second $f_l \in H^1\left(0, T; V_l^*\right)$, $f_l(0) = f_l(T)$.

For the proof we shall essentially adopt a penalty method [1], [3], [4] based on an existence theorem concerning periodic solutions of an abstract
non linear second order differential equation. Moreover, in proving this theorem \([1, n. 1]\) we follow a technique inspired to the "elliptic regularization" \([3, 3]\). The results given by theorems 2 and 3 \([n. 2]\) for the penalized problem will then be used to prove the above mentioned existence theorems for our problem \((P)\).

A regularity theorem "with respect to \(x^*\)" is also given in \(n. 3\), theorem 6, when \(V_1 = V_2 = H^1_0(\Omega)\) and operators \(A_j, B_j\), within constant factors, are identical to a uniformly elliptic second order linear differential operator. The uniform ellipticity of this last operator allows us to obtain upper limitations by introducing a "special base" of \(H^1_0(\Omega)\).

Let \(V^*\) and \(H^*\) be real Hilbert spaces: \(V \subset H^*\), with dense and continuous embedding.

1. We identify \(H\) with its dual and denote with

\[
\langle \cdot, \cdot \rangle, | \cdot |, \| \cdot \| \quad \text{the inner product and the norm in } H,
\]

\[\| \cdot \| \quad \text{the norm in } V,\]

\[
\langle \cdot, \cdot \rangle \quad \text{the pairing between } V \text{ and its dual } V^*.
\]

Let also \(f \in L^2 \left(0, T; V^*\right)\) and \(A, B\) the operators from \(V\) into \(V^*\): A linear and continuous, \(B\) strictly monotone and hemicontinuous. Let us suppose that:

\[
\langle Ay, z \rangle = \langle Az, y \rangle \quad \forall y, z \in V,
\]

\[
\langle Az, z \rangle \geq a \| z \|^2 \quad \forall z \in V, \quad (a = \text{const.} > 0)
\]

\[
\langle Bz, z \rangle \geq b \| z \|^2 \quad \forall z \in V, \quad (b = \text{const.} > 0)
\]

for each \(v \in L^2 \left(0, T; V\right)\), \(Bv(\cdot) \in L^2 \left(0, T; V^*\right)\).

Operator \(v \in L^2 \left(0, T; V^*\right) \rightarrow Bv(\cdot)\) is bounded.

**THEOREM 1.** In the above stated assumptions, there exists only one solution \(u \in H^1 \left(0, T; V\right)\) of the following problem:

1. \(u'' \in L^2 \left(0, T; V^*\right),\)
2. \(\langle u''(t), z \rangle + \langle Au(t), z \rangle + \langle Bu'(t), z \rangle = \langle f(t), z \rangle \quad \text{a.e. on } [0, T] \quad \forall z \in V,\)
3. \(u(0) = u(T), u'(0) = u'(T).\)

**PROOF:** Firstly, if \(u_1, u_2 \in H^1 \left(0, T; V\right)\) are solutions of the problem (1), (2), (3), then
\[ \int_0^T \left\{ Bu'_1(t) - Bu'_2(t), \ u'_1(t) - u'_2(t) \right\} dt = 0 \]

must hold and therefore
\[ u'_1(t) = u'_2(t) \quad \text{a.e. on } ]0, T[ \]

owing to the strict monotonicity of \( B \). Thus there exists a \( z_0 \in \mathcal{V} \) such that
\[ u'_1(t) = u'_2(t) + z_0 \quad \forall t \in ]0, T[. \]

This condition in turn is fulfilled iff
\[ \langle Az_0, z \rangle = 0 \quad \forall z \in \mathcal{V}, \]

namely \( z_0 = 0 \). In order to prove the existence of the solution, we introduce the Hilbert space:
\[ \mathcal{W} = \left\{ v \in H^1(0, T; \mathcal{V}) : v'' \in L^2(0, T; H), \ v(0) = v(T), \ v'(0) = v'(T) \right\} \]
equipped with norm
\[ \| v \|_\mathcal{W} = \left( \int_0^T \| v(t) \|^2_2 dt + \int_0^T \| v'(t) \|^2_2 dt + \int_0^T \| v''(t) \|^2_2 dt \right)^{\frac{1}{2}} \quad \forall v \in \mathcal{W} \]
and then denote with \( \langle \cdot, \cdot \rangle \) the pairing between \( \mathcal{W} \) and its dual \( \mathcal{W}' \).

Given \( \varepsilon > 0 \), we set, for any \( u, v \in \mathcal{W} \)

\[ \langle \mathcal{C}_\varepsilon^x u, v \rangle = \epsilon \int_0^T \left( \int_0^T \langle u''(t), v'(t) \rangle dt + \int_0^T \langle Au(t), v(t) \rangle dt \right) dt + \int_0^T \langle u'(t), v'(t) \rangle dt \]

and
\[ + \int_0^T \langle Bu(t), v'(t) \rangle dt - \int_0^T \langle f(t), v'(t) \rangle dt. \]
\( C^*: W \to W^* \) is obviously a bounded, strictly monotone, hemicontinuous and coercive operator. Therefore ([3], theorem 2.1, pg. 171) there exists a unique \( u_0 \in W \) solution of equation

\[
\langle C^* u_0, v \rangle = 0 \quad \forall \, v \in W.
\]

Eq. (4), written with \( v = u_0 \), gives the upper limitations

\[
\varepsilon \left[ \int_0^T \left| u^*_0(t) \right|^2 dt + \int_0^T \left\| u_0(t) \right\|^2 dt \right] \leq c,
\]

\[
\int_0^T \left\| u^*_0(t) \right\|^2 dt \leq c,
\]

\( (c = \text{const.} > 0 \text{ indep. from } \varepsilon) \)

as well as

\[
\int_0^T \left\| \bar{u}_0(t) \right\|^2 dt \leq c
\]

where \( \bar{u}_0(t) = u_0(t) - u_0(0) \). Inequalities (5), (6), (7) imply the existence of \( \bar{u} \in \mathcal{H}^1(0,T;V) \), with \( \bar{u}(0) = \bar{u}(T) = 0 \), of \( g \in \mathcal{L}^2(0,T;V') \) and of a positive numerical infinitesimal sequence \( \{ \varepsilon_n \} \) such that for \( n \to +\infty \)

\[
\bar{u}_{\varepsilon_n} \rightharpoonup \bar{u} \quad \text{weakly in } \mathcal{L}^2(0,T;V).
\]

\[
u_{\varepsilon_n} \rightharpoonup \bar{u}' \quad \text{weakly in } \mathcal{L}^2(0,T;V'),
\]

\( B\nu'_{\varepsilon_n} (\cdot) \to g \quad \text{weakly in } \mathcal{L}^2(0,T;V'), \)

\[
\varepsilon_n \left[ \left\| u_{\varepsilon_n} \right\|_{\mathcal{L}^2(0,T;H)}^2 + \left\| u_{\varepsilon_n} \right\|_{\mathcal{L}^2(0,T;V')}^2 \right] \to 0.
\]

Starting from (4) and using (8), we come to relation

\[
\int_0^T \langle \bar{u}'(t), v''(t) \rangle dt = \int_0^T \left\langle A\bar{u}(t) + g(t) - f(t), v'(t) \right\rangle dt \quad \forall \, v \in W.
\]
Let now \( \varphi_0 \in C^\infty_0([0,T]) \) with \( \int_0^T \varphi_0(t)dt = 1, \; \varphi \in C^\infty_0([0,T]) \) and \( z \in V \). From (9), setting

\[
\nu(t) = \left( \int_0^t \varphi(s) - \varphi_0(s) \int_0^T \varphi(t)dt \right) \hat{z} \quad \forall t \in [0,T]
\]

we get:

\[
\left\{ \int_0^T \bar{u}'(t)\varphi'(t)dt, z \right\} = 
\left\{ \int_0^T \left[ A\bar{u}(t) + g(t) - f(t) \right] \varphi(t)dt, z \right\} + 
\left\{ \int_0^T \bar{u}'(t)\varphi_0'(t)dt \right\} \left\{ \int_0^T \varphi(t)dt, z \right\} + 
\left\{ \int_0^T \left( A\bar{u}(t) + g(t) - f(t) \right)\varphi_0(t)dt \right\} \left\{ \int_0^T \varphi(t)dt, z \right\}
\]

or

\[
\bar{u}''(t) = -\left[ A\bar{u}(t) + g(t) - f(t) \right] - \theta \quad \text{a.e. on } [0,T] \text{ in the } V' \text{ sense}
\]

being \( \theta \) the element of \( V' \)

\[
\int_0^T \bar{u}'(t)\varphi_0'(t) dt - \int_0^T \left( A\bar{u}(t) + g(t) - f(t) \right)\varphi_0(t) dt.
\]

Let \( u_0 \) be the element of \( V \) for which \( Au_0 = \theta \), and given \( u = \bar{u} + u_0 \), it is obvious that \( u \) satisfies (1), the first of (3) and that:

(10)

\[
\langle u''(t), z \rangle + \langle Au(t), z \rangle + \langle g(t), z \rangle = \langle f(t), z \rangle \quad \text{a.e. on } [0,T] \forall z \in V.
\]
We may also note that the second of (3) holds too. Indeed, chosen \( \psi \in C^2([0, T]) \) with \( \psi(0) = \psi(T) \) and \( \psi'(0) = \psi'(T) = 1 \), recalling (9) and (10), we get \( \forall z \in V : \)

\[
(u'(T) - u'(0), z) = (u'(T)\psi'(T) - u'(0)\psi'(0), z) = \left\langle \int_0^T [u'(t)\psi'(t)]' \, dt, z \right\rangle = \\
\int_0^T \left\langle u''(t), \psi'(t)z \right\rangle \, dt + \int_0^T [u'(t), \psi''(t)z] \, dt = \\
\int_0^T \left\langle u''(t), \psi'(t)z \right\rangle \, dt + \int_0^T \{Au(t) + g(t) - f(t), \psi'(t)z \} \, dt = \\
-0.
\]

Because of (10), (2) is acquired as soon as we are able to prove that

\[
(11) \quad Bu'(\cdot) = g.
\]

From (4), with \( v = u_n' \), we obtain:

\[
\int_0^T \left\langle Bu_n'(t), u_n'(t) \right\rangle \, dt \leq \int_0^T \left\langle f(t), u_n'(t) \right\rangle \, dt
\]

from which, because of the second of (8),

\[
(12) \quad \lim_{n \to \infty} \int_0^T \left\langle Bu_n'(t), u_n'(t) \right\rangle \, dt \leq \int_0^T \left\langle f(t), u'(t) \right\rangle \, dt.
\]

Assuming \( z = u'(t) \), (10) produces the following equality:

\[
(13) \quad \int_0^T \left\langle g(t), u'(t) \right\rangle \, dt = \int_0^T \left\langle f(t), u'(t) \right\rangle \, dt.
\]

The second and third of (8), together with (12) and (13), imply (11), since operator

\[
\nu \in L^2(0, T; \nu') \rightarrow B\nu(\cdot)
\]

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is bounded, monotone and hemicontinuous ([3], proposition 2.5, pg. 179).

**REMARK.** Proof of the existence is essentially the same when assuming "B monotone" instead of "B strictly monotone".

2. Let us suppose

\[ V = V_1 \times V_2, \quad H = L^2(\Omega_1) \times L^2(\Omega_2) \]

and write for each \( z = (z_1, z_2), \ y = (y_1, y_2) \in V \)

\[
\langle Az, y \rangle = \langle A_1 z_1, y_1 \rangle_1 + \langle A_2 z_2, y_2 \rangle_2,
\]

\[
\langle Lz, y \rangle = \frac{1}{\varepsilon} \left( \left[ z_1 - z_2 \right]^*, y_1 - y_2 \right) \text{ with } \varepsilon > 0,
\]

\[
\langle Bz, y \rangle = \langle B_1 z_1, y_1 \rangle_1 + \langle B_2 z_2, y_2 \rangle_2 + \langle Lz, y \rangle.
\]

Of course Hilbertian spaces \( V, H \) and operators \( A, B \), from \( V \) into \( V' \), meet the assumptions stated at the beginning of n. 1. Therefore, from theorem 1, there exists a unique \( (u_{le}, u_{2e}) \in \prod_{i=1}^{2} H^1(0,T; V_i) \) solution of the problem

\[
(14) \quad \begin{align*}
\ddot{u}_{le} & \in L^2(0,T; V_i), \\
\sum \int_0^T \langle u_{le}^\prime(t) + A_{1le}(t) + B_{1le}(t) - f_1(t), z_1 \rangle_1 + \\
\int_0^T \int_{\Omega_i} \left[ u_{le}(t) - u_{2e}(t) \right]^* z_i - z_2 = 0 \\
\text{a.c. on } & \{ 0, T \}, \forall (z_1, z_2) \in V_1 \times V_2,
\end{align*}
\]

\[
(15) \quad u_{le}(0) = u_{le}(T), \quad u_{le}^\prime(0) = u_{le}^\prime(T).
\]

**THEOREM 2.** If for \( i = 1, 2 \) \( f_i \in L^2(0, T; L^2(\Omega_i)) \) we then have:

\[
(16) \quad u_{le} \in L^2(0, T; L^2(\Omega_i)),
\]

\[
(17) \quad \| u_{le} \|_{L^2(0, T; L^2(\Omega_i))} \leq c. \quad (c = \text{const.} > 0 \text{ indep. from } \varepsilon)
\]
PROOF. Let \( \{z_j\} \) be a base of \( V_j \) and, for each \( n \in \mathbb{N} \), \( V_{ln} \) be the space spanned by \( \{z_1, \ldots, z_{ln}\} \). Theorem 1 and the finite dimensions of \( V_{ln} \) assure the existence of a unique \( (w_{ln}, w'_{2n}) \in \prod_{l=1}^{2} H^2(0, T; V_{ln}) \) such that
\[
\begin{align*}
\sum_{l=1}^{2} I \left( \left( w''_{ln}(t), z_l \right)_l + \left( A_l w_{ln}(t) + B_l w'_{ln}(t), z_l \right)_l - (f_l(t), z_l)_l \right) + \\
\frac{1}{\varepsilon} \left[ (w'_{ln}(t) - w'_{2n}(t))^+, z_1 - z_2 \right] = 0 \\
\quad \text{a.e. on } [0, T] \quad \forall (z_1, z_2) \in V_{ln} \times V_{2n},
\end{align*}
\]
(18)

\( w_{ln}(0) = w_{ln}(T), \quad w'_{ln}(0) = w'_{ln}(T) \).

Immediate consequence of (18) are the upper limitations:
\[
\left\| w_{ln} \right\|_{L^2(0, T; V_j)} \leq c.
\]
(19) \( c = \text{const.} > 0 \) indep. from \( \varepsilon \) and \( n \)
\[
\left\| w''_{ln} \right\|_{L^2(0, T; L^2(\Omega_j))} \leq c.
\]
and also
\[
\left\| \overline{w}_{ln} \right\|_{L^2(0, T; V_j)} \leq c,
\]
where \( \overline{w}_{ln}(t) = w_{ln}(t) - w_{ln}(0) \) Therefore \( \overline{w}_j \in H'(0, T; V_j) \), with
\[
\overline{w}_j \in L^2\left(0, T; L^2(\Omega_j)\right), \quad \overline{w}_j(0) = \overline{w}_j(T) = 0, \quad \overline{w}'_j(0) = \overline{w}'_j(T).
\]
(20)

and \( h \in L^2\left(0, T; L^2(\Omega)\right) \) exist so that, to within a subsequence, for \( n \to +\infty \):
\[
\begin{align*}
\overline{w}_{ln} & \to \overline{w}_j \quad \text{weakly in } L^2(0, T; V_j), \\
w'_{ln} & \to w'_j \quad \text{weakly in } L^2(0, T; V_j), \\
w''_{ln} & \to w'' \quad \text{weakly in } L^2\left(0, T; L^2(\Omega_j)\right), \\
\left[w'_{ln} - w'_{2n}\right]^+ & \to h \quad \text{weakly in } L^2\left(0, T; L^2(\Omega)\right)
\end{align*}
\]
(21)
Using (21) and equality

\[ \bigcup_{n \in \mathbb{N}} V_{f_n} = V_f, \]

we easily derive from the first of (18) this relation:

\[
\frac{2}{\varepsilon} \sum_{i=1}^{\tau} \int_{0}^{T} \left\{ \left( \bar{w}_i''(t), \varphi'(t)z_i \right)_I + \left( A_i \bar{w}_i(t) + B_i \bar{w}'_i(t), \varphi'(t)z_i \right)_I - \left( f_i(t), \varphi(t)z_i \right)_I \right\} \, dt + \\
+ \frac{1}{\varepsilon} \left[ \int_{0}^{T} \left( h(t), \varphi'(t)(z_1 - z_2) \right) \, dt = 0 \quad \forall \varphi \in C^0_0([0,T]) \text{ and } \forall (z_1, z_2) \in V_1 \times V_2, \right. 
\]

the latter being equivalent, a.e. on \([0,T]\), to:

\[
\frac{d}{dt} \left[ \bar{w}_1''(t) + A_1 \bar{w}_1(t) + B_1 \bar{w}'_1(t) + \frac{1}{\varepsilon} h(t) - f_1(t) \right] = 0 \quad \text{in the sense of } V_1', \\
\frac{d}{dt} \left[ \bar{w}_2''(t) + A_2 \bar{w}_2(t) + B_2 \bar{w}'_2(t) - \frac{1}{\varepsilon} h(t) - f_2(t) \right] = 0 \quad \text{in the sense of } V_2'.
\]

which in turn lead to existence of \( F_i \in \mathcal{V}_i' \) such that, a.e. on \([0,T]\):

\[
\bar{w}_1''(t) + A_1 \bar{w}_1(t) + B_1 \bar{w}'_1(t) + \frac{1}{\varepsilon} h(t) - f_1(t) = F_1, \\
\bar{w}_2''(t) + A_2 \bar{w}_2(t) + B_2 \bar{w}'_2(t) - \frac{1}{\varepsilon} h(t) - f_2(t) = F_2. \tag{22}
\]

Given \( w_{i0} = A_i^{-1} F_i \) and \( w_i = \bar{w}_i - w_{i0} \in H^1(0,T; \mathcal{V}_i) \), from (20), (22) we get respectively:

\[
(23) \quad w_i'' \in L^2(0,T; L^2(\Omega_i)), \quad w_i(0) = w_i(T), \quad w_i'(0) = w_i'(T).
\]
\[ \sum_{i=1}^{2} \left\{ \left( w^*_i(t) + \frac{1}{A_i} B_i w^*_i(t) - f^*_i(t) \right) \cdot x_i \right\} + \frac{1}{e} (h(t), z_1 - z_2) = 0 \quad \text{a.e. on] } 0, T [ \quad \forall (z_1, z_2) \in V_1 \times V_2. \]

Since from (18) and the second of (21)

\[ \lim_{n \to \infty} \frac{1}{n} \int_{0}^{T} \left( \left\{ w_{2n}^*(t) - w_{\infty}^*(t) \right\} \cdot \left( w_{\infty}^*(t) - w_{n}^*(t) \right) \right) dt \leq 2 \sum_{i=1}^{2} \int_{0}^{T} \left( f_i(t), w_{n}^*(t) \right) dt - \frac{2}{T} \sum_{i=1}^{2} \int_{0}^{T} \left( B_i w_{n}^*(t), w_{n}^*(t) \right) dt \]

and from (23), (24)

\[ \frac{1}{e} \int_{0}^{T} \left( h(t), w_{n}^*(t) - w_2^*(t) \right) dt = \sum_{i=1}^{2} \int_{0}^{T} \left( f_i(t), w_{n}^*(t) \right) dt + \frac{2}{T} \sum_{i=1}^{2} \int_{0}^{T} \left( B_i w_{n}^*(t), w_{n}^*(t) \right) dt \]

we may write

\[ h(t) = \left[ w_1^*(t) - w_2^*(t) \right] \quad \forall t \in [0, T] \]

since

\[ v \in L^2(0, T; \nu) \to L^2(\cdot) \]

is a bounded, monotone and hemi-continuous operator. From (23), (24) and (25) we see that \( w_{\infty} = u_{\infty} \). Consequently (17) hold: the first of (17) is true because of the first of (20), the second because of (19) and the third of (21).
THEOREM 3. If for \( l = 1, 2 \) \( f_l \in H^1(0, T; V_l) \) and \( f_l(0) = f_l(T) \), then we have:

\[
\varepsilon \to L^2(0, T; V_l), \quad (c = \text{const. } > 0 \text{ indep. from } \varepsilon)
\]

\[
\left\| \varepsilon \right\|_{L^2(0, T; V_l)} \leq c.
\]

PROOF. Proceeding as with the beginning of the proof of theorem 2 and considering the present hypothesis on \( f_l \), we see that there exists a unique

\[
(w_{1n}, w_{2n}) \in \prod_{l=1}^2 H^1(0, T; V_l)
\]

which fulfills the following conditions:

(26)

\[
\sum_{l=1}^2 \left\{ \left( w_{ln}(t), z_l \right) + \left( A_l w_{ln}(t) + B_l \dot{w}_{ln}(t) - f_l(t), z_l \right) \right\} + \\
+ \frac{1}{\varepsilon} \left( \left[ \dot{w}_{ln}(t) - \dot{w}_{2n}(t) \right]^+, z_l - z_2 \right) = 0 \quad \forall t \in [0, T], \text{ and } \forall (z_1, z_2) \in V_1 \times V_2.
\]

(27)

\[
w_{ln}(0) = w_{ln}(T), \quad \dot{w}_{ln}(0) = w_{ln}'(T), \quad \ddot{w}_{ln}(0) = w_{ln}''(T).
\]

By differentiating the left member of (26), accounting for both the second and third of (27) and inequality:

\[
\left( \frac{d}{dt} \left[ \dot{w}_{ln}(t) - \dot{w}_{2n}(t) \right]^+, w_{ln}'(t) - w_{2n}'(t) \right) \geq 0 \quad \forall t \in [0, T],
\]

we obtain

\[
\left\| \ddot{w}_{ln} \right\|_{L^2(0, T; V_l)} \leq c. \quad (c = \text{const. } > 0 \text{ indep. from } \varepsilon \text{ and } n)
\]

This proof is completed by proceeding similarly to what reasoned with theorem 2.

3. Results obtained in n. 2 will now allow us to produce some existence theorems for problem \((P)\).
THEOREM 4. If for $l = 1, 2$, $f_l \in L^2\left(0, T; L^2(\Omega_l)\right)$ then there exists a 
$(u_1, u_2) \in \prod_{l=1}^2 H^1(0, T; V_l)$, with $u_l^\prime \in L^2\left(0, T; L^2(\Omega_l)\right)$, which is solution of 
problem (P).

PROOF. For any $\varepsilon > 0$ let $(u_{1\varepsilon}, u_{2\varepsilon}) \in \prod_{l=1}^2 H^1(0, T; V_l)$ be a solution of 
problem (14), (15), (16). Because of theorem 2 $u_{l\varepsilon}^\prime \in L^2\left(0, T; L^2(\Omega_l)\right)$ and 
we have:

\[ \left\| u_{l\varepsilon}^\prime \right\|_{L^2(0, T; L^2(\Omega_l))} \leq c. \] 
($c = \text{const.} > 0$ indep. from $\varepsilon$)

(28) Consequence of (15), (16) are the upper limitations:

\[ \left\| u_{l\varepsilon} \right\|_{L^2(0, T; L^2(\Omega_l))} \leq c. \] 
($c = \text{const.} > 0$ indep. from $\varepsilon$)

(29) \[ \left\| u_{l\varepsilon} - u_{2\varepsilon} \right\|_{L^2(0, T; L^2(\Omega_l))} \leq c\varepsilon. \] 

(30)

The existence of $(u_1, u_2) \in \prod_{l=1}^2 H^1(0, T; V_l)$ with

\[ u_l^\prime \in L^2\left(0, T; L^2(\Omega_l)\right), \quad u_l(0) = u_l(T) = u_l^0(T), \]

and of a positive numerical infinitesimal sequence $\{e_n\}$ such that for $n \to +\infty$:

\[ u_{1e_n} - u_{2e_n} (0) \to u_l \quad \text{weakly in} \quad L^2\left(0, T; V_l\right), \]

(31) \[ u_{1e_n}^\prime \to u_l^\prime \quad \text{weakly in} \quad L^2\left(0, T; V_l\right), \]

\[ u_{1e_n}^\prime \to u_l^\prime \quad \text{weakly in} \quad L^2\left(0, T; L^2(\Omega_l)\right). \]
is guaranteed by (28) and (29).

Thus, solution of problem \((P)\) is \((u_1, u_2)\). Indeed, on one hand

\[
\left( u_1(t), u_2(t) \right) \in K \quad \text{a.e. on } \left] 0, T \right[
\]

since, holding (30) and the second of (31), we have:

\[
\begin{align*}
\int_0^T \left\| u_1'(t) - u_2'(t) \right\|^2 dt & \leq \lim_{\varepsilon \to 0} \int_0^T \left\| u_{1\varepsilon}(t) - u_{2\varepsilon}(t) \right\|^2 dt = 0.
\end{align*}
\]

On the other hand, by virtue of inequality

\[
\int_0^T \left( \left[ u_{1\varepsilon}(t) - u_{2\varepsilon}(t) \right] - \left[ \nu_1(t) - \nu_{1\varepsilon}(t) \right] - \left[ \nu_2(t) - \nu_{2\varepsilon}(t) \right] \right) dt \leq 0,
\]

where \((\nu_1, \nu_2)\) is an arbitrary element of \(\prod_{i=1}^2 H^1(0, T; \mathbb{R}_e)\), satisfying conditions:

\[
\left( \nu_1(t), \nu_2(t) \right) \in K \quad \text{a.e. on } \left] 0, T \right[, \quad \nu_i(0) = \nu_i(T),
\]

from (15), (16) we get:

\[
\begin{align*}
\sum_{k=1}^2 \int_0^T & \left\{ \left\langle u_{1\varepsilon}(t), \nu_1(t) \right\rangle_1 + \left\langle A_t \left[ u_{1\varepsilon}(t) - u_{1\varepsilon}(0) \right], \nu_1(t) \right\rangle_1 + \left\langle B_t \left[ u_{1\varepsilon}(t), \nu_1(t) \right], \nu_1(t) \right\rangle_1 + \\
& \left\langle f_1(t), \nu_1(t) - u_{1\varepsilon}(t) \right\rangle_1 \right\} dt \geq \sum_{k=1}^2 \int_0^T \left\langle B_t \left[ u_{1\varepsilon}(t), u_{1\varepsilon}(t) \right], \nu_1(t) \right\rangle_1 dt
\end{align*}
\]

and from here, taking the limit as \(n \to +\infty\), we obtain because of (31):
\[
\sum_{l=1}^{2} \int_{0}^{T} \left\{ \left( u''_{l}(t), v_{l}(t) \right) + \left( \mathbf{a}_{l} u_{l}(t), v_{l}(t) \right) + \left( B_{l} u'_{l}(t), v_{l}(t) \right) \right\} dt - \int_{0}^{T} \left\{ f_{l}(t) v_{l}(t) - u'_{l}(t) \right\} dt \geq \sum_{l=1}^{2} \int_{0}^{T} \left\{ \mathbf{b}_{l} u'_{l}(t), u_{l}(t) \right\} dt.
\]

Using theorem 3, with the above procedure we prove the following:

**THEOREM 5.** If for \( l = 1,2 \) \( f_{l} \in H^{1}_{0}(0,T;\mathcal{V}_{l}) \) and \( f_{l}(0) = f_{l}(T) \) then there exists a \((u_{1}, u_{2}) \in H^{2}(0,T;\mathcal{V}_{l})\) solution of problem \((P)\).

Let us complete the study of problem \((P)\) by analysing a particular case. Let: \( \Omega_{1} = \Omega_{2} = \Omega \) be a \( C^{1,1} \) open, bounded, connected set of \( \mathbb{R}^{n} \) and \( \mathcal{V}_{1} = \mathcal{V}_{2} = H^{1}_{0}(\Omega) \).

We now consider the uniformly elliptic second order linear differential operator

\[
A = -\sum_{i,j} a_{ij} \frac{\partial}{\partial x_{j}} \left( \frac{\partial}{\partial x_{i}} \right)
\]

with \( a_{ij} = a_{ji} \in C^{0,1}(\Omega) \), and let

\[
A_{l} = \alpha_{l} A, \quad B_{l} = \beta_{l} A,
\]

\( \alpha_{1} \) and \( \beta_{1} \) being positive constants.

**THEOREM 6.** If for \( l = 1,2 \) \( f_{l} \in L^{2}(0,T;L^{2}(\Omega)) \), then there exists a \((u_{1}, u_{2}) \in \left[ H^{1}(0,T;H^{1}_{0}(\Omega) \cap H^{2}(\Omega)) \right]^{2} \) with \( u''_{l} \in L^{2}(0,T;L^{2}(\Omega)) \), solution to problem \((P)\).

**PROOF.** In the light of the proof given for theorem 4 and of statements made in the introduction concerning solutions of problem \((P)\), it is evidently enough to prove that for the solution \((u_{1e}, u_{2e})\) of the problem \((14), (15), (16)\) we have:
\[ u^{\prime \prime}_{n} \in L^{2}(0, T; H^{1}_{0}(\Omega)) \cap H^{2}(\Omega), \quad u^{\prime 
olimits}_{n} \in L^{2}(0, T; L^{2}(\Omega)). \]

\[(c = \text{const.} > 0 \text{ indep. from } n) \]

\[ \left\| u^{\prime 
olimits}_{n} \right\|_{L^{2}(0, T; H^{1}_{0}(\Omega) \cap H^{2}(\Omega))} + \left\| u^{\prime 
olimits}_{n} \right\|_{L^{2}(0, T; L^{2}(\Omega))} \leq c. \]

Assumptions made for $\Omega$ and operator $A$ assure ([2], Remark 31, pg. 308; [7], theorem 2.1, pg. 201) the existence of a base $\{z_{j}\}$ of $H^{1}_{0}(\Omega)$ of functions of $H^{2}(\Omega)$ such that

\[(a_{j} = \lambda_{j}z_{j})\]

where $\{\lambda_{j}\}$ is a positively diverging sequence of positive numbers. Let $V_{n}$ be the space spanned by $\{z_{1}, \ldots, z_{n}\}$, from theorem 1 there is a unique $(w_{n}, w_{2n}) \in \left[ H^{2}(0, T; V_{n}) \right]^{2}$ solution of the problem:

\[ \sum_{i=1}^{2} \left( w_{i}^{\prime 
olimits}_{n}(t) + \alpha_{i} A w_{n}(t) + \beta_{i} A w_{n}^{\prime}(t) - f_{i}(t), y_{j} \right) + \]

\[ + \frac{1}{\varepsilon} \left( w_{n}^{\prime}(t) - w_{2n}(t) \right)^{\top}, y_{1} - y_{2} = 0 \quad \text{a.e. on } 0, T \quad \forall (y_{1}, y_{2}) \in V_{n}^{2}. \]

\[(w_{n}(0) = w_{n}(T), \quad w_{n}^{\prime}(0) = w_{n}^{\prime}(T)).\]

Recalling (33), (34), may also be written as:

\[ \sum_{i=1}^{2} \left( w^{\prime 
olimits}_{i}(t) + \alpha_{i} A w_{n}(t) + \beta_{i} A w_{n}^{\prime}(t) - f_{i}(t), A y_{j} \right) + \]

\[ + \frac{1}{\varepsilon} \left( w_{i}^{\prime}(t) - w_{2n}(t) \right)^{\top}, A[y_{1} - y_{2}] = 0 \quad \text{a.e. on } 0, T \quad \forall (y_{1}, y_{2}) \in V_{n}^{2} \]
and this, together with (35) and the following inequality
\[
\left( \left[ w'_{1n}(t) - w'_{2n}(t) \right]^+ , A \left[ w'_{1n}(t) - w'_{2n}(t) \right] \right) \geq 0 \quad \forall t \in [0,T],
\]
allow us to acquire the upper limitation:
\[
\left\| A w'_{ln} \right\|_{L^2(0,T;L^2(\Omega))] \leq c \quad (c = \text{const.} > 0 \text{ indep. from } \varepsilon \text{ and } n)
\]
from which ([7], theor. 2.1, pg. 201):
\[
(36) \quad \left\| w'_{ln} \right\|_{L^2(0,T;H^1_0(\Omega) \cap H^2(\Omega))] \leq c.
\]

The further upper limitation
\[
(37) \quad \left\| w''_{ln} \right\|_{L^2(0,T;L^2(\Omega))] \leq c \quad (c = \text{const.} > 0 \text{ indep. from } \varepsilon \text{ and } n)
\]
follows because of (34), (35), (36).

Inequalities (36), (37) bring us to (32) with the same technique used for theorem 2 considering that now $F_j \in L^2(\Omega)$ and that $w_{10} = A_j^{-1} F_j \in H^1_0(\Omega) \cap H^2(\Omega)$ ([7], theor. 2.1, pg. 201).
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