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Non-linear difference polynomials sharing a polynomial with finite weight

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Abstract

The uniqueness theory of meromorphic function mainly studies the conditions under which there exists only one function satisfying these conditions. The uniqueness theory of entire and meromorphic functions has grown up as an extensive sub-field of value distribution theory and the Nevanlinna's Five value and Four value theorems serves as the starting point of this uniqueness theory. In this paper, we consider a linear difference polynomial $\mathcal{L}_\eta(\mathfrak{f}) = \mathfrak{f}(z+\eta) + \eta_0\mathfrak{f}(z)$, of the finite ordered non-constant meromorphic function $\mathfrak{f}$, with η and η_0 being finite non-zero complex constants, and with the help of Nevanlinna theory, we analyse the uniqueness results between two finite ordered non-constant meromorphic functions $\mathfrak{f}$ and $\mathfrak{g}$, when their non-linear difference polynomials $\mathfrak{f}^n(z)\mathcal{L}_\eta(\mathfrak{f})$ and $\mathfrak{g}^n(z)\mathcal{L}_\eta(\mathfrak{g})$, with $n \geq 2$ being a positive integer shares a non-zero polynomial $p(z)$ with finite weights 0,1 and 2. Our results extend and improve some of the earlier results of Majumder (*Applied Mathematics E-Notes, (17): 114-123, 2017)*.

Keywords: Meromorphic functions; difference polynomials; weighted sharing; uniqueness.
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1 Introduction

The Nevanlinna theory, is one of the several branches of complex analysis which has seen many fine works. It plays a key role in studying oscillation of complex differential equations. This theory mainly deals with studying the distribution of the zeros of the equation $\mathfrak{f}(z) = \beta$ in a disc $|z| \leq r$, where $\mathfrak{f}$ is an entire or meromorphic function in the complex plane $\mathbb{C}$, $z \in \mathbb{C}$ and $\beta \in \mathbb{C} \cup \{\infty\}$. One can look into (Hayman [8], Yi and Yang [21], Yang [23]) for the standard definitions and notations of the theory.

Let $\mathcal{T} = \{\mathfrak{f}(z)/\, \mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ is a non-constant meromorphic function$\}$. For $\mathfrak{f}, \mathfrak{g} \in \mathcal{T}$, $\beta \in \mathbb{C}\cup\{\infty\}$ and $s \in \mathbb{Z}^+\cup\{\infty\}$, the set $E(\beta, \mathfrak{f}) = \{z : \mathfrak{f}(z)-\beta = 0\}$ denotes all those β-points of $\mathfrak{f}$, where each β-point of $\mathfrak{f}$ with multiplicity s is counted s times in the set and the set $\overline{E}(\beta, \mathfrak{f}) = \{z : \mathfrak{f}(z) - \beta = 0\}$, denotes all those β-points of $\mathfrak{f}$, where the multiplicities are ignored. If $\mathfrak{f}(z) - \beta$ and $\mathfrak{g}(z) - \beta$ assumes the same zeros with the same multiplicities, then we say that $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ share the value β CM (counting multiplicity) and we have $E(\beta, \mathfrak{f}) = E(\beta, \mathfrak{g})$; Suppose, if $\mathfrak{f}(z) - \beta$ and $\mathfrak{g}(z) - \beta$ assumes the same zeros ignoring the multiplicities, then we say that $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ share the value β IM (ignoring multiplicity) and we will have $\overline{E}(\beta, \mathfrak{f}) = \overline{E}(\beta, \mathfrak{g})$. If $\frac{1}{\mathfrak{f}}$ and $\frac{1}{\mathfrak{g}}$ share 0 CM, then $\mathfrak{f}$ and $\mathfrak{g}$ share ∞ CM.

In general, for a meromorphic function $\mathfrak{f}(z)$, the quantity $m(r, \mathfrak{f})$ denotes the proximity function of $\mathfrak{f}(z)$, while $N(r, \mathfrak{f})$ denotes the counting function of poles of $\mathfrak{f}(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r, \mathfrak{f})$ denotes the reduced counting function when multiplicities are ignored). The quantity $N(r, \beta; \mathfrak{f})$ denotes the counting function of β points of $\mathfrak{f}(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r, \beta; \mathfrak{f})$ denotes the reduced counting function when multiplicities are ignored). The notation $N(r, \beta; \mathfrak{f} \mid= 1)$ denotes the counting function of simple β-points of $\mathfrak{f}$ and the notation $N(r, \beta; \mathfrak{f} \mid\geq 2)$ denotes the counting function of those β-points of $\mathfrak{f}$ whose multiplicities are atleast 2 (respectively, $\overline{N}(r, \beta; \mathfrak{f} \mid= 1)$ and $\overline{N}(r, \beta; \mathfrak{f} \mid\geq 2)$ denotes the reduced counting functions).

Suppose $\mathfrak{f}$ and $\mathfrak{g}$ share 1 IM and z_0 is a zero of $\mathfrak{f}(z) - 1$ of order s and also a zero of $\mathfrak{g}(z) - 1$ of order t, then $\overline{N}_L(r, 1; \mathfrak{f})$ counts those 1-points of $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ where $s > t$, $\overline{N}_E^{1)}(r, 1; \mathfrak{f})$ counts those 1-points of $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ where $s = t = 1$, $\overline{N}_E^{(2}(r, 1; \mathfrak{f})$ counts those 1-points of $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ where $s = t \geq 2$, $\overline{N}_{\mathfrak{f}>2}(r, 1; \mathfrak{g})$ counts those 1-points of $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ where $s > t = 2$. It is to be noted that each point in these counting functions are counted only once. Similarly $\overline{N}_L(r, 1; \mathfrak{g})$, $\overline{N}_E^{(2}(r, 1; \mathfrak{g})$, $\overline{N}_{\mathfrak{g}>2}(r, 1; \mathfrak{f})$ are defined.

The Nevanlinna characteristic function of a meromorphic function $\mathfrak{f}$ plays a very important role in the value distribution theory and it is denoted by $T(r, \mathfrak{f})$. We have $T(r, \mathfrak{f}) = m(r, \mathfrak{f}) + N(r, \mathfrak{f})$, which clearly shows that $T(r, \mathfrak{f})$ is non-negative. A meromorphic function $\alpha(z)$ is called a small function with respect to $\mathfrak{f}(z)$, if $T(r,\alpha) = S(r,\mathfrak{f})$, where $S(r,\mathfrak{f})$ denotes any quantity, which satisfies $S(r,\mathfrak{f}) = o(T(r,\mathfrak{f}))$ as $r \to +\infty$ possibly outside a set I with finite linear measure $\lim\limits_{r\to\infty} \int\limits_{(1,r]\cap I} \frac{dt}{t} < \infty$.

2 Definitions and Theorems

Let us recall the following standard definitions of Nevanlinna theory.

Definition 2.1. *[8] The order $\rho(\mathfrak{f})$ of a meromorphic function $\mathfrak{f}(z)$ is defined as,*

$$\rho(\mathfrak{f}) = \overline{\lim_{r\to\infty}} \frac{\log T(r,\mathfrak{f})}{\log r}.$$

Definition 2.2. *[11] Let k be a non-negative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$, we denote by $E_k(a,\mathfrak{f})$ the set of all a-points of $\mathfrak{f}$, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,\mathfrak{f}) = E_k(a,\mathfrak{g})$, then we say that $\mathfrak{f}$ and $\mathfrak{g}$ share the value a with the weight k.*

The definition implies that if $\mathfrak{f}$ and $\mathfrak{g}$ share a value a with the weight k, then z_0 is a zero of $\mathfrak{f} - a$ with multiplicity $m(\leq k)$ if and only if z_0 is a zero of $\mathfrak{g} - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $\mathfrak{f} - a$ with multiplicity $m(> k)$ if and only if z_0 is a zero of $\mathfrak{g} - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write $\mathfrak{f}$, $\mathfrak{g}$ share (a,k) to mean that, $\mathfrak{f}$, $\mathfrak{g}$ share the value a with the weight k. Clearly, if $\mathfrak{f}$, $\mathfrak{g}$ share (a,k), then $\mathfrak{f}$, $\mathfrak{g}$ share (a,p) for any integer p, such that, $0 \leq p < k$. Also we note that, $\mathfrak{f}$, $\mathfrak{g}$ share a value a IM or CM if and only if $\mathfrak{f}$, $\mathfrak{g}$ share $(a,0)$ or (a,∞) respectively.

Definition 2.3. *[11] Let $\mathfrak{f}$ and $\mathfrak{g}$ share the value a IM.We denote by $\overline{N}_*(r,a;\mathfrak{f},\mathfrak{g})$ the reduced counting function of those a-points of $\mathfrak{f}$ whose multiplicities differ from the multiplicities of the corresponding a-points of $\mathfrak{g}$.*
Clearly, $\overline{N}_*(r,a;\mathfrak{f},\mathfrak{g}) \equiv \overline{N}_*(r,a;\mathfrak{g},\mathfrak{f})$ and $\overline{N}_*(r,a;\mathfrak{f},\mathfrak{g}) = \overline{N}_L(r,a;\mathfrak{f}) + \overline{N}_L(r,a;\mathfrak{g})$.

Around 2006, with the development of the difference analogue of Logarithmic Derivative Lemma [6], Halburd and Korhonen laid the foundation for the difference analogue of Nevanlinna theory and developed the Nevanlinna theory for the difference operator [5] wherein they extended the classical Nevanlinna theory based on certain estimates involving the derivative $\mathfrak{f} \mapsto \mathfrak{f}'$ of a meromorphic function to a theory for the exact difference $\mathfrak{f} \mapsto \Delta\mathfrak{f} = \mathfrak{f}(z+c) - \mathfrak{f}(z)$. Their results gave the difference analogue of the second main theorem of Nevanlinna theory, as well as the difference analogues of the Nevanlinna defect relation, Picards theorem and

Nevanlinna's five value theorem. In 2008, Chiang and Feng [4] investigated the growth of Nevanlinna characteristic of $\mathfrak{f}(z+c)$ for a fixed $c \in \mathbb{C}$ and obtained a precise asymptotic relation between $T(r, \mathfrak{f}(z+c))$ and $T(r, \mathfrak{f})$, which is only true for finite ordered meromorphic functions. They also obtained the proximity function and pointwise estimates of $\mathfrak{f}(z+c)/\mathfrak{f}(z)$ which were discrete versions of the classical logarithmic derivative estimates of $\mathfrak{f}(z)$. Then they applied those results to give new growth estimates of meromorphic solutions to higher order linear difference equations. Since then many mathematicians showed keen interest on value distribution and uniqueness problems of meromorphic functions, their shifts, difference equations and difference operators. Some fine works can be seen in ([3], [7], [9], [10], [13], [14], [15], [16], [18], [19], [20], [24]). One such studies can be seen in Liu et al., [15], wherein they have studied the uniqueness of the difference monomials and obtained the following results.

Theorem A. *[15] Let* $\mathcal{T} = \{\mathfrak{f}(z)\,/\mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ *is a finite ordered transcendental meromorphic function*$\}$. *Let* $\mathfrak{f}, \mathfrak{g} \in \mathcal{T}$. *Suppose* $\eta \in \mathbb{C} - \{0\}$ *and* n *be any natural number, such that* $n \geq 14$, *with* $\mathfrak{f}^n(z)\mathfrak{f}(z+\eta)$ *and* $\mathfrak{g}^n(z)\mathfrak{g}(z+\eta)$ *share 1 CM, then either* $\mathfrak{f}(z) \equiv t\mathfrak{g}(z)$ *or* $\mathfrak{f}(z)\mathfrak{g}(z) \equiv t$, *where* $t^{n+1} = 1$.

Theorem B. *[15] Let* $\mathcal{T} = \{\mathfrak{f}(z)\,/\mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ *is a finite ordered transcendental meromorphic function*$\}$. *Let* $\mathfrak{f}, \mathfrak{g} \in \mathcal{T}$. *Suppose* $\eta \in \mathbb{C} - \{0\}$, n *be any natural number, such that* $n \geq 26$, *with* $\mathfrak{f}^n(z)\mathfrak{f}(z+\eta)$ *and* $\mathfrak{g}^n(z)\mathfrak{g}(z+\eta)$ *share 1 IM, then either* $\mathfrak{f}(z) \equiv t\mathfrak{g}(z)$ *or* $\mathfrak{f}(z)\mathfrak{g}(z) \equiv t$, *where* $t^{n+1} = 1$.

Later, using the concept of weighted sharing introduced by Lahiri [11] (given in Definition 2.2), in 2015 Liu et al., [16] improved Theorems A, B and obtained the following results.

Theorem C. *[16] Let* $\eta \in \mathbb{C} - \{0\}$ *and* $\mathcal{T} = \{\mathfrak{f}(z)\,/\mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ *is a finite ordered transcendental meromorphic function*$\}$. *Suppose* $n, k \in \mathbb{N}$, *such that* $n \geq 14$, $k \geq 3$ *and if* $E_k(1, \mathfrak{f}^n(z)\mathfrak{f}(z+\eta))$=$E_k(1, \mathfrak{g}^n(z)\mathfrak{g}(z+\eta))$, *then either* $\mathfrak{f}(z) \equiv t_1\mathfrak{g}(z)$ *or* $\mathfrak{f}(z)\mathfrak{g}(z) \equiv t_2$, *for some constants* t_1 *and* t_2 *satisfying* $t_1^{n+1} = 1$ *and* $t_2^{n+1} = 1$.

Theorem D. *[16] Let* $\eta \in \mathbb{C} - \{0\}$ *and* $\mathcal{T} = \{\mathfrak{f}(z)\,/\mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ *is a finite ordered transcendental meromorphic function*$\}$. *Suppose* $n \in \mathbb{N}$, *such that* $n \geq 16$, *and if* $E_2(1, \mathfrak{f}^n(z)\mathfrak{f}(z+\eta))$=$E_2(1, \mathfrak{g}^n(z)\mathfrak{g}(z+\eta))$, *then either* $\mathfrak{f}(z) \equiv t_1\mathfrak{g}(z)$ *or* $\mathfrak{f}(z)\mathfrak{g}(z) \equiv t_2$, *for some constants* t_1 *and* t_2 *satisfying* $t_1^{n+1} = 1$ *and* $t_2^{n+1} = 1$.

Theorem E. *[16] Let* $\eta \in \mathbb{C} - \{0\}$ *and* $\mathcal{T} = \{\mathfrak{f}(z)\,/\mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ *is a finite ordered transcendental meromorphic function*$\}$. *Suppose* $n \in \mathbb{N}$, *such that* $n \geq 22$, *and if* $E_1(1, \mathfrak{f}^n(z)\mathfrak{f}(z+\eta))$=$E_1(1, \mathfrak{g}^n(z)\mathfrak{g}(z+\eta))$, *then either* $\mathfrak{f}(z) \equiv t_1\mathfrak{g}(z)$ *or* $\mathfrak{f}(z)\mathfrak{g}(z) \equiv t_2$, *for some constants* t_1 *and* t_2 *satisfying* $t_1^{n+1} = 1$ *and* $t_2^{n+1} = 1$.

Later, in 2017 Majumder [17] studied Theorems C, D and E for polynomial sharing instead of sharing value 1 and obtained the following generalized results.

Theorem F. *[17] Let $\eta \in \mathbb{C} - \{0\}$ and $\mathcal{T} = \{\mathfrak{f}(z)\,/\mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ is a finite ordered transcendental meromorphic function$\}$. Suppose $n \in \mathbb{N}$, such that $n \geq 14$, $p(z)(\not\equiv 0)$ be a polynomial such that $deg(p) < (n-1)/2$ and if $\mathfrak{f}^n(z)\mathfrak{f}(z+\eta) - p(z)$ and $\mathfrak{g}^n(z)\mathfrak{g}(z+\eta) - p(z)$, share $(0,2)$, then one of the following two cases holds:*

(1) $\mathfrak{f}(z) \equiv t\mathfrak{g}(z)$ for some constant t such that $t^{n+1} = 1$,

(2) $\mathfrak{f}(z)\mathfrak{g}(z) \equiv t$, where $p(z)$ reduces to a non-zero constant η and t is a constant such that $t^{n+1} = \eta^2$.

Theorem G. *[17] Let $\eta \in \mathbb{C} - \{0\}$ and $\mathcal{T} = \{\mathfrak{f}(z)\,/\mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ is a finite ordered transcendental meromorphic function$\}$. Suppose $n \in \mathbb{N}$, such that $n \geq 16$, $p(z)(\not\equiv 0)$ be a polynomial such that $deg(p) < (n-1)/2$ and if $\mathfrak{f}^n(z)\mathfrak{f}(z+\eta) - p(z)$ and $\mathfrak{g}^n(z)\mathfrak{g}(z+\eta) - p(z)$, share $(0,1)$, then conclusion of Theorem F holds.*

Theorem H. *[17] Let $\eta \in \mathbb{C} - \{0\}$ and $\mathcal{T} = \{\mathfrak{f}(z)\,/\mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ is a finite order transcendental meromorphic function $\}$. Suppose $n \in \mathbb{N}$, such that $n \geq 26$, $p(z)(\not\equiv 0)$ be a polynomial such that $deg(p) < (n-1)/2$ and if $\mathfrak{f}^n(z)\mathfrak{f}(z+\eta) - p(z)$ and $\mathfrak{g}^n(z)\mathfrak{g}(z+\eta) - p(z)$, share $(0,0)$, then conclusion of Theorem F holds.*

In the light of above mentioned results, it becomes sense to pose the following questions.

Question 1. Suppose $\mathcal{L}_\eta(\mathfrak{f}) = \mathfrak{f}(z+\eta)+\eta_0\mathfrak{f}(z)$, is a linear difference polynomial of the finite ordered non-constant meromorphic functions $\mathfrak{f}$, with η and η_0 being finite non-zero complex constants, then what can we say about the relation between two finite ordered non-constant meromorphic functions $\mathfrak{f}$ and $\mathfrak{g}$, if their non-linear difference polynomials $\mathfrak{f}^n(z)\mathcal{L}_\eta(\mathfrak{f})$ and $\mathfrak{g}^n(z)\mathcal{L}_\eta(\mathfrak{g})$ share a non-zero polynomial $p(z)$, with $n \geq 2$ being a positive integer?

Question 2. Is it possible to reduce the lower bound of n any further in Theorems F, G and H?

In this paper, we have attemped to answer the above posed questions successfully.

3 Main Results

Following are the main results of our paper.

Theorem 3.1. *Let* $\mathcal{T} = \{\mathfrak{f}(z) / \mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ *is a finite ordered transcendental meromorphic function*$\}$. *Let* $\eta \in \mathbb{C} - \{0\}$ *and* $n \in \mathbb{N}$, *such that* $n > 11$. *Suppose* $\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f}) - p(z)$ *and* $\mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g}) - p(z)$ *share* $(0, 2)$, *where* $p(z)$ *be a non-zero polynomial such that* $2deg(p) < n - 2$ *then one of the following two cases holds:*

(1) $\mathfrak{f} \equiv t\mathfrak{g}$, *for some constant* t, *such that* $t^{n+1} = 1$,

(2) $\mathfrak{f}\mathfrak{g} \equiv d^2$, *where* $p(z)$ *reduces to a non-zero constant* d, *such that*

(i) when $\eta_0 = 0$, *then* $\mathfrak{f}$ *and* $\mathfrak{g}$ *takes the form* $\mathfrak{f} = e^U$ *and* $\mathfrak{g} = te^{-U}$, *where* U *is a non-constant polynomial and* t *is a constant such that* $t^{n+1} = d^2$.

(ii) when $\eta_0 \neq 0$, *then* $\mathfrak{f}$ *and* $\mathfrak{g}$ *takes the form* $\mathfrak{f}(z) = \eta_1 e^{az}$ *and* $\mathfrak{g}(z) = \eta_2 e^{-az}$, *where* a, η_1, η_2 *are non-zero constants satisfying* $(\eta_1\eta_2)^{n+1}(e^{a\eta} + \eta_0)(e^{-a\eta} + \eta_0) = d^2$.

Theorem 3.2. *Let* $\mathcal{T} = \{\mathfrak{f}(z) / \mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ *is a finite ordered transcendental meromorphic function*$\}$. *Let* $\eta \in \mathbb{C} - \{0\}$ *and* $n \in \mathbb{N}$, *such that* $n > 25/2$. *Suppose* $\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f}) - p(z)$ *and* $\mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g}) - p(z)$ *share* $(0, 1)$, *where* $p(z)$ *be a non-zero polynomial such that* $2deg(p) < n - 2$, *then one of the conclusion of Theorem 3.1 holds.*

Theorem 3.3. *Let* $\mathcal{T} = \{\mathfrak{f}(z) / \mathfrak{f}(z) : \mathbb{C} \to \mathbb{C}$ *is a finite ordered transcendental meromorphic function*$\}$. *Let* $\eta \in \mathbb{C} - \{0\}$ *and* $n \in \mathbb{N}$, *such that* $n > 14$. *Suppose* $\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f}) - p(z)$ *and* $\mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g}) - p(z)$ *share* $(0, 0)$, *where* $p(z)$ *be a non-zero polynomial such that* $2deg(p) < n - 2$, *then one of the conclusion of Theorem 3.1 holds.*

4 Remarks

If we take $\eta_0 = 0$ then,
1. Theorem 3.1, reduces to Theorem F with $n > 11$,
2. Theorem 3.2, reduces to Theorem G with $n > 25/2$,
3. Theorem 3.3, reduces to Theorem H with $n > 14$.
Thus, Theorems 3.1, 3.2, 3.3 are improvements of Theorems F, G, H respectively.

Clearly, for a specific choice of the constant $\eta_0 = -1$, we have $\mathcal{L}_\eta(\mathfrak{f}) = \Delta_\eta \mathfrak{f}$, and we can obtain similar uniqueness results for the relationship between $\mathfrak{f}^n \Delta_\eta \mathfrak{f}$ and $\mathfrak{g}^n \Delta_\eta \mathfrak{g}$, where $\Delta_\eta \mathfrak{f} = \mathfrak{f}(z + \eta) - \mathfrak{f}(z)$.

5 Lemmas

In this section we provide all the necessary lemmas required to prove our theorems.
Let us define,

$$H=\left(\frac{\mathfrak{F}''}{\mathfrak{F}'}-\frac{2\mathfrak{F}'}{\mathfrak{F}-1}\right)-\left(\frac{\mathfrak{G}''}{\mathfrak{G}'}-\frac{2\mathfrak{G}'}{\mathfrak{G}-1}\right). \tag{5.1}$$

Lemma 5.1. *[4] Let $\eta\in\mathbb{C}-\{0\}$ be fixed and $\mathfrak{f}(z)$ be a non-constant finite ordered meromorphic function. Then for each $\epsilon>0$, we have*

$$m\left(r,\frac{\mathfrak{f}(z+\eta)}{\mathfrak{f}(z)}\right)+m\left(r,\frac{\mathfrak{f}(z)}{\mathfrak{f}(z+\eta)}\right)=O(r^{\rho-1+\epsilon})=S(r,f).$$

Lemma 5.2. *[22] Let $\mathfrak{f}(z)$ be a non-constant meromorphic function and for $i=0,1,2,...,n$, let $\alpha_n(z)$ $(\not\equiv 0)$, $\alpha_{n-1}(z)$,..., $\alpha_0(z)$ be meromorphic functions satisfying $T(r,\alpha_i(z))=S(r,\mathfrak{f})$. Then*

$$T(r,\alpha_n\mathfrak{f}^n+\alpha_{n-1}\mathfrak{f}^{n-1}+\cdots+\alpha_1\mathfrak{f}+\alpha_0)=nT(r,\mathfrak{f})+s(r,\mathfrak{f}).$$

Lemma 5.3. *[9] Let $\eta\in\mathbb{C}$ be fixed and $\mathfrak{f}(z)$ be a non-constant finite ordered meromorphic function. Then*

$$N(r,0;\mathfrak{f}(z+\eta))\leq N(r,0;\mathfrak{f}(z))+S(r,\mathfrak{f}),$$
$$N(r,\infty;\mathfrak{f}(z+\eta))\leq N(r,\infty;\mathfrak{f})+S(r,\mathfrak{f}),$$
$$\overline{N}(r,0;\mathfrak{f}(z+\eta))\leq N(r,0;\mathfrak{f}(z))+S(r,\mathfrak{f}),$$
$$\overline{N}(r,\infty;\mathfrak{f}(z+\eta))\leq \overline{N}(r,\infty;\mathfrak{f})+S(r,\mathfrak{f}).$$

Lemma 5.4. *[4] Let $\eta\in\mathbb{C}$ be fixed and $\mathfrak{f}$ be a finite ordered transcendental meromorphic function. Then*

$$T(r,\mathfrak{f}(z+\eta))=T(r,\mathfrak{f})+S(r,\mathfrak{f}).$$

Lemma 5.5. *[23] Let $\mathfrak{f}$ and $\mathfrak{g}$ be any two non-constant meromorphic functions. Then*

$$N\left(r,\infty;\frac{\mathfrak{f}}{\mathfrak{g}}\right)-N\left(r,\infty;\frac{\mathfrak{g}}{\mathfrak{f}}\right)=N(r,\infty;\mathfrak{f})+N(r,0;\mathfrak{g})-N(r,\infty;\mathfrak{g})-N(r,0;\mathfrak{f}).$$

Lemma 5.6. *[1] Let $\mathfrak{f}$ and $\mathfrak{g}$ be any two non-constant meromorphic functions. Suppose $\mathfrak{f}$, $\mathfrak{g}$ share $(1,1)$, then*

$$2\overline{N}_L(r,1;\mathfrak{f})+2\overline{N}_L(r,1;\mathfrak{g})+\overline{N}_E^{(2}(r,1;\mathfrak{f})-\overline{N}_{\mathfrak{f}>2}(r,1;\mathfrak{g})\leq N(r,1;\mathfrak{g})-\overline{N}(r,1;\mathfrak{g}).$$

Lemma 5.7. *[2] Let* $\mathfrak{f}$ *and* $\mathfrak{g}$ *be any two non-constant meromorphic functions. Suppose* $\mathfrak{f}$, $\mathfrak{g}$ *share* $(1,1)$, *then*

$$\overline{N}_{\mathfrak{f}>2}(r,1;\mathfrak{g}) \leq \frac{1}{2}\overline{N}(r,0;\mathfrak{f}) + \frac{1}{2}\overline{N}(r,\infty;\mathfrak{f}) - \frac{1}{2}N_0(r,0;\mathfrak{f}') + S(r,\mathfrak{f}),$$

where $N_0(r,0;\mathfrak{f}')$ *represents counting function of those zeros of* $\mathfrak{f}'$ *which are not the zeros of* $\mathfrak{f}(\mathfrak{f}-1)$.

Lemma 5.8. *[2] Let* $\mathfrak{f}$ *and* $\mathfrak{g}$ *be any two non-constant meromorphic functions. Suppose* $\mathfrak{f}$, $\mathfrak{g}$ *share* $(1,0)$, *then*

(i) $\overline{N}_{\mathfrak{f}>1}(r,1;\mathfrak{g}) \leq \overline{N}(r,0;\mathfrak{f}) + \overline{N}(r,\infty;\mathfrak{f}) - N_0(r,0;\mathfrak{f}') + S(r,\mathfrak{f}),$

(ii) $\overline{N}_{\mathfrak{g}>1}(r,1;\mathfrak{f}) \leq \overline{N}(r,0;\mathfrak{g}) + \overline{N}(r,\infty;\mathfrak{g}) - N_0(r,0;\mathfrak{g}') + S(r,\mathfrak{g}).$

Lemma 5.9. *[2] Let* $\mathfrak{f}$ *and* $\mathfrak{g}$ *be any two non-constant meromorphic functions. Suppose* $\mathfrak{f}$, $\mathfrak{g}$ *share* $(1,0)$, *then*

$$\overline{N}_L(r,1;\mathfrak{f}) \leq \overline{N}(r,0;\mathfrak{f}) + \overline{N}(r,\infty;\mathfrak{f}) + S(r,\mathfrak{f}).$$

Lemma 5.10. *[2] Let* $\mathfrak{f}$ *and* $\mathfrak{g}$ *be any two non-constant meromorphic functions. Suppose* $\mathfrak{f}$, $\mathfrak{g}$ *share* $(1,0)$, *then*

$$\begin{aligned}\overline{N}_L(r,1;\mathfrak{f}) + 2\overline{N}_L(r,1;\mathfrak{g}) + \overline{N}_E^{(2}(r,1;\mathfrak{f}) - \overline{N}_{\mathfrak{f}>1}(r,1;\mathfrak{g}) - \overline{N}_{\mathfrak{g}>1}(r,1;\mathfrak{g})&\\ \leq N(r,1;\mathfrak{g}) - \overline{N}(r,1;\mathfrak{g}).&\end{aligned}$$

Lemma 5.11. *[12] If* $N(r,0;\mathfrak{f}^{(k)}|\mathfrak{f}\neq 0)$ *denotes the counting function of those zeros of* $\mathfrak{f}^{(k)}(z)$ *which are not the zeros of* $\mathfrak{f}(z)$, *where a zero of* $\mathfrak{f}^{(k)}(z)$ *is counted according to its multiplicity, then*

$$N(r,0;\mathfrak{f}^{(k)}|\mathfrak{f}\neq 0) \leq k\overline{N}(r,\infty;\mathfrak{f}) + N(r,0;\mathfrak{f}|<k) + k\overline{N}(r,0;\mathfrak{f}|\geq k) + S(r,\mathfrak{f}).$$

Lemma 5.12. *Let* $\mathfrak{f}$ *be a finite ordered transcendental meromorphic function and let* $\mathfrak{F} = \mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f})$, *where* $n\in\mathbb{N}$. *Then*

$$(n-1)T(r,\mathfrak{f}) \leq T(r,\mathfrak{F}) + S(r,\mathfrak{f}).$$

Proof. From Lemmas 5.1, 5.2 and the First fundamental theorem, we obtain

$$\begin{aligned}(n+1)T(r,\mathfrak{f}) &= T(r,\mathfrak{f}^{n+1}) + S(r,\mathfrak{f})\\ &\leq T\left(r,\frac{\mathfrak{F}\mathfrak{f}(z)}{\mathcal{L}_\eta(\mathfrak{f})}\right) + S(r,\mathfrak{f})\\ &\leq T(r,\mathfrak{F}) + T\left(r,\frac{\mathcal{L}_\eta(\mathfrak{f})}{\mathfrak{f}(z)}\right) + S(r,\mathfrak{f})\\ &\leq T(r,\mathfrak{F}) + N\left(r,\frac{\mathcal{L}_\eta(\mathfrak{f})}{\mathfrak{f}(z)}\right) + S(r,\mathfrak{f})\\ &\leq T(r,\mathfrak{F}) + 2T(r,\mathfrak{f}) + S(r,\mathfrak{f}).\end{aligned}$$

Therefore, we have $(n-1)T(r,\mathfrak{f}) \leq T(r,\mathfrak{F}) + S(r,\mathfrak{f})$. This completes the proof of Lemma 5.12. $\square$

Lemma 5.13. *Let $\eta \in \mathbb{C} - \{0\}$, $n \in \mathbb{N}$, such that $n > 2$. Let $\mathfrak{f}$, $\mathfrak{g}$ be two finite ordered transcendental meromorphic functions. Let $p(z)$ be a non-zero polynomial such that $2deg(p) < n-2$. Then*

(1) if $deg(p(z)) \geq 1$, then $\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f})\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g}) \not\equiv p^2(z)$,

(2) if $p(z)$ reduces to a non-zero constant d, then

- *(i) when $\eta_0 = 0$, $\mathfrak{f}$, $\mathfrak{g}$ takes the form, $\mathfrak{f} = e^U$ and $\mathfrak{g} = te^{-U}$, where U is a non-constant polynomial and t is a constant, such that $t^{n+1} = d^2$, or*
- *(ii) when $\eta_0 \neq 0$, $\mathfrak{f}$, $\mathfrak{g}$ takes the form, $\mathfrak{f} = \eta_1 e^{az}$ and $\mathfrak{g} = \eta_2 e^{-az}$, where a, η_1, η_2 and d are non-zero constants satisfying $(\eta_1\eta_2)^{n+1}(e^{a\eta}+\eta_0)(e^{-a\eta}+\eta_0) = d^2$.*

Proof. Suppose
$$\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f})\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g}) \equiv p^2(z). \tag{5.2}$$
Let $h_1 = \mathfrak{f}\mathfrak{g}$, then by (5.2), we have
$$h_1^n(z) = \frac{p^2(z)}{\mathcal{L}_\eta(\mathfrak{f})\mathcal{L}_\eta(\mathfrak{g})}. \tag{5.3}$$

We now consider the following three cases.
Case 1. Suppose h_1 is a transcendental meromorphic function, then by Lemmas 5.1, 5.2 and 5.3, we get

$$\begin{aligned} nT(r,h_1) &= T(r,h_1^n) + S(r,h_1) \\ &= T\left(r, \frac{p^2(z)}{\mathcal{L}_\eta(\mathfrak{f})\mathcal{L}_\eta(\mathfrak{g})}\right) + S(r,h_1) \\ &\leq N(r,0;\mathcal{L}_\eta(\mathfrak{f})) + N(r,0;\mathcal{L}_\eta(\mathfrak{g})) + S(r,h_1) \\ &\leq 2[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] + S(r,h_1), \end{aligned}$$

from $h_1 = \mathfrak{f}\mathfrak{g}$, which implies $T(r,h_1) = T(r,\mathfrak{f}) + T(r,\mathfrak{g})$, we get

$$n\left[T(r,\mathfrak{f}) + T(r,\mathfrak{g})\right] \leq 2\left[T(r,\mathfrak{f}) + T(r,\mathfrak{g})\right] + S(r,h_1),$$

which is a contradiction.
Case 2. Suppose h_1 is a rational function. Let
$$h_1 = \frac{h_2}{h_3}, \tag{5.4}$$

where h_2, h_3 are two non-zero relatively prime polynomials. By (5.4), we have

$$T(r,h_1)=\max\left\{deg(h_2),deg(h_3)\right\}\log r+O(1). \tag{5.5}$$

Now by (5.3)-(5.5), we have

$$\begin{aligned} n\max\left\{deg(h_2),deg(h_3)\right\}\log r &= T(r,h_1^n)+O(1)\\ &\le 2\left[T(r,\mathfrak{f})+T(r,\mathfrak{g})\right]+2T(r,p)+O(1)\\ &\le 2\max\left\{deg(h_2),deg(h_3)\right\}\log r+2T(r,p)+O(1). \end{aligned} \tag{5.6}$$

We see that $\max\left\{deg(h_2),deg(h_3)\right\}\ge 1$.
Now by (5.6), we deduce that $(n-2)\le 2deg(p(z))$, which contradicts our assumption that $2deg(p(z))<n-2$. Hence h_1 must be a non-zero constant. Let

$$h_1=t\in\mathbb{C}-\{0\}. \tag{5.7}$$

Now, when $deg(p)\ge 1$, by (5.3) and (5.7), we arrive at a contradiction. In this case we have

$$\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f})\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g})\not\equiv p^2(z).$$

Case 3. suppose $p(z)$ is not a constant then, from (5.2), we have

$$\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f})\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g})\equiv p^2(z). \tag{5.8}$$

From (5.8) and the fact that $\mathfrak{f}$ and $\mathfrak{g}$ are meromorphic functions, one can immediately say that both $\mathfrak{f}$ and $\mathfrak{g}$ have either no or atmost finitely many zeros. So, we may write

$$\mathfrak{f}(z)=p_1(z)e^{Q_1(z)}\qquad\text{and}\qquad\mathfrak{g}(z)=p_2(z)e^{Q_2(z)}, \tag{5.9}$$

where p_1,p_2,Q_1,Q_2 are polynomials and Q_1, Q_2 are non-constants. Substituting (5.9) in (5.8), we obtain

$$\begin{aligned} (p_1p_2)^n e^{n(Q_1(z)+Q_2(z))}\Big[&p_1(z+\eta)p_2(z+\eta)e^{Q_1(z+\eta)+Q_2(z+\eta)}\\ &+\eta_0^2p_1(z)p_2(z)e^{Q_1(z)+Q_2(z)}+\eta_0p_1(z)p_2(z+\eta)e^{Q_1(z)+Q_2(z+\eta)}\\ &+\eta_0p_1(z+\eta)p_2(z)e^{Q_1(z+\eta)+Q_2(z)}\Big]=p^2(z). \end{aligned} \tag{5.10}$$

Keeping in view of (5.9), we must have

$$n(Q_1(z)+Q_2(z))+Q_1(z+\eta)+Q_2(z+\eta)=A_1, \tag{5.11}$$

$$n(Q_1(z)+Q_2(z))+Q_1(z)+Q_2(z+\eta)=A_2, \tag{5.12}$$

$$n(Q_1(z)+Q_2(z))+Q_1(z+\eta)+Q_2(z)=A_3, \tag{5.13}$$

$$(n+1)(Q_1(z)+Q_2(z))=A_4, \tag{5.14}$$

where A_1, A_2, A_3, A_4 are constants. Let $Q_1(z)+Q_2(z)=W(z)$. Then (5.11) can be written as

$$nW(z)+W(z+\eta)=A_1, \tag{5.15}$$

for all $z \in \mathbb{C}$. Therefore, from (5.15), we must have $W=B$, where B is a constant, and therefore,we have

$$Q_2 = B - Q_1, \tag{5.16}$$

keeping in view of (5.16), we can write (5.9) as

$$\mathfrak{f}(z)=p_1(z)e^{Q_1(z)} \qquad \text{and} \qquad \mathfrak{g}(z)=p_2(z)e^{B}e^{-Q_1(z)}. \tag{5.17}$$

Now, (5.10) can be written as

$$(p_1p_2)^n[p_1(z+\eta)p_2(z+\eta)e^{A_1}+\eta_0 p_1(z+\eta)p_2(z)e^{A_3} + \eta_0 p_1(z)p_2(z+\eta)e^{A_2}+\eta_0^2 p_1(z)p_2(z)e^{A_4}] = p^2(z). \tag{5.18}$$

If p_1p_2 is not a constant, then the degree of the left side of (5.18) is atleast $n+1$. But the condition $2deg(p) < n-2$ implies that the degree of the right side of (5.18) is less than $n-2$, which is a contradiction. Thus p_1p_2 and p reduce to non-zero constants.
Since p_1, p_2 are both polynomials and their product is constant, each of them must be constant. Therefore, (5.17) can be written as

$$\mathfrak{f}(z)=e^{U} \qquad \text{and} \qquad \mathfrak{g}(z)=e^{B}e^{-U}, \tag{5.19}$$

where U is a non-constant polynomial. Using the above forms of $\mathfrak{f}$, $\mathfrak{g}$ and keeping in mind that p is a constant, say d, (5.8) reduces to

$$e^{(n+1)B}\left(e^{[U(z+\eta)-U(z)]}+\eta_0\right)\left(e^{-[U(z+\eta)-U(z)]}+\eta_0\right)=d^2 \tag{5.20}$$

Subcase 3.1. If $\eta_0 = 0$, (5.20) reduces to $e^{(n+1)B} = d^2$. Set $e^B = t$. Then (5.19) can be written as $\mathfrak{f}(z)=e^U$, $\mathfrak{g}(z)=te^{-U}$, where $p(z)$ reduces to a non-zero constant d and t is a constant, such that $t^{n+1}=d^2$, and U is a non-constant polynomial.
Subcase 3.2. If $\eta_0 \neq 0$, then from (5.20), one can say that $e^{[U(z+\eta)-U(z)]}+\eta_0$ has no zeros. Then $\phi(z)=e^{[U(z+\eta)-U(z)]} \neq 0, -\infty, \infty$. By Picard's theorem, ϕ is constant and so $deg(U(z))=1$. Therefore, from (5.19), one may obtain $\mathfrak{f}(z)=\eta_1 e^{az}$ and $\mathfrak{g}(z)=\eta_2 e^{-az}$, where a, η_1, η_2 are non-zero constants. Using these in (5.8), we obtain $(\eta_1\eta_2)^{n+1}(e^{a\eta}+\eta_0)(e^{-a\eta}+\eta_0)=d^2$.
This completes the proof of Lemma 5.13. □

Lemma 5.14. *Let* $\mathfrak{f}$, $\mathfrak{g}$ *be two finite ordered transcendental meromorphic functions. Let* $n \in \mathbb{N}$, *such that* $n > 9$. *Let* $\mathfrak{F} = \frac{\mathfrak{f}^n(z)\mathcal{L}_\eta(\mathfrak{f})}{p(z)}$ *and* $\mathfrak{G} = \frac{\mathfrak{g}^n(z)\mathcal{L}_\eta(\mathfrak{g})}{p(z)}$, *where* $p(z)$ *is a non-zero polynomial. If* $H \equiv 0$ *(*H *as defined in (5.1)), then either*

(i) $\mathfrak{f}^n(z)\mathcal{L}_\eta(\mathfrak{f})\mathfrak{g}^n(z)\mathcal{L}_\eta(\mathfrak{g}) \equiv p^2(z)$, *where* $\mathfrak{f}^n(z)\mathcal{L}_\eta(\mathfrak{f}) - p(z)$ *and* $\mathfrak{g}^n(z)\mathcal{L}_\eta(\mathfrak{g}) - p(z)$ *share 0 CM; or*

(ii) $\mathfrak{f} \equiv t\mathfrak{g}$, *for a constant* t *with* $t^{n+1} = 1$.

Proof. Let H be as defined in (5.1). Since $H \equiv 0$, by integration, we get

$$\frac{1}{\mathfrak{F} - 1} = \frac{B\mathfrak{G} + A - B}{\mathfrak{G}}, \tag{5.21}$$

where $A \neq 0$, B are constants. From (5.21) it is clear that $\mathfrak{F}$ and $\mathfrak{G}$ share $(1, \infty)$. We now consider the following cases.

Case 4. Let $B \neq 0$ and $A \neq B$.

If $B = -1$, then from (5.21), we have

$$\mathfrak{F} = \frac{-A}{\mathfrak{G} - A - 1}.$$

Therefore, $\overline{N}(r, A+1; \mathfrak{G}) = \overline{N}(r, \mathfrak{F}) \leq N\left(r, \frac{1}{p}\right) \leq T(r, p) = S(r, \mathfrak{g})$.

So in view of Lemma 5.12 and the Second fundamental theorem, we get

$$\begin{aligned}(n-1)T(r, \mathfrak{g}) &\leq T(r, \mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g})) + S(r, \mathfrak{g})\\ &\leq T(r, \mathfrak{G}) + S(r, \mathfrak{g})\\ &\leq \overline{N}(r, \infty; \mathfrak{G}) + \overline{N}(r, 0; \mathfrak{G}) + \overline{N}(r, (A+1); \mathfrak{G}) + S(r, \mathfrak{g})\\ &\leq 2T(r, \mathfrak{g}) + 3T(r, \mathfrak{g}) + S(r, \mathfrak{g}),\end{aligned}$$

which is a contradiction, since $n > 6$.

If $B \neq 1$, from (5.21), we obtain that

$$\mathfrak{F} - \left(1 + \frac{1}{B}\right) = \frac{-A}{B^2\left(\mathfrak{G} + \frac{A-B}{B}\right)}.$$

This implies that

$$\overline{N}\left(r, \frac{B-A}{B}; \mathfrak{G}\right) = S(r, \mathfrak{g}).$$

Using Lemma 5.12 and the same argument as used in the case when $B = -1$, we can get a contradiction.

Case 5. Let $B \neq 0$ and $A = B$.
If $B = -1$, then from (5.21), we have

$$\mathfrak{F}(z)\mathfrak{G}(z) \equiv 1,$$

i.e.,

$$\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f})\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g}) \equiv p^2(z),$$

where $\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f}) - p(z)$ and $\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g}) - p(z)$ share 0 CM.
If $B \neq -1$, then from (5.21), we have

$$\frac{1}{\mathfrak{F}} = \frac{B\mathfrak{G}}{(1+B)\mathfrak{G} - 1}.$$

Therefore,

$$\overline{N}\left(r, \frac{1}{1+B}; \mathfrak{G}\right) = \overline{N}(r, 0; \mathfrak{F}) + S(r, \mathfrak{f}).$$

So, in view of Lemmas 5.3, 5.12 and the Second fundamental theorem, we get

$$\begin{aligned}
(n-1)T(r,\mathfrak{g}) &\leq T(r, \mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g})) + S(r,\mathfrak{g}) \\
&\leq T(r,\mathfrak{G}) + S(r,\mathfrak{g}) \\
&\leq \overline{N}(r,\infty;\mathfrak{G}) + \overline{N}(r,0;\mathfrak{G}) + \overline{N}\left(r, \frac{1}{1+B}; \mathfrak{G}\right) + S(r,\mathfrak{g}) \\
&\leq \overline{N}(r,\infty;\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g})) + \overline{N}(r,0;\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g})) + \overline{N}(r,0;\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f})) \\
&\quad + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\
&\leq 2T(r,\mathfrak{g}) + 3T(r,\mathfrak{g}) + 3T(r,\mathfrak{f}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}),
\end{aligned}$$

for $r \in I$, we have

$$(n-9)T(r,\mathfrak{g}) \leq s(r,\mathfrak{g}),$$

which is a contradiction, since $n > 9$.
Case 6. Let $B = 0$. From (5.21), we obtain

$$\mathfrak{F} = \frac{\mathfrak{G} + A - 1}{A}. \tag{5.22}$$

If $A \neq 1$,then from (5.22), we obtain

$$\overline{N}(r, 1-A; \mathfrak{G}) = \overline{N}(r, 0; \mathfrak{F}).$$

We can similarly deduce a contradiction as in Case 5. Therefore $A = 1$ and from (5.22), we obtain

$$\mathfrak{F}(z) \equiv \mathfrak{G}(z),$$

i.e.,

$$\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f}) \equiv \mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g}). \tag{5.23}$$

Let $h = \mathfrak{f}/\mathfrak{g}$, and then substituting $\mathfrak{f} = \mathfrak{g}h$ in (5.23), we deduce

$$h^{n+1} = \frac{\mathfrak{f}\mathcal{L}_\eta(\mathfrak{g})}{\mathcal{L}_\eta(\mathfrak{f})\mathfrak{g}}.$$

If h is not a constant, then we have

$$\begin{aligned}(n+1)T(r,h) &\le T\left(r, \frac{\mathfrak{f}}{\mathcal{L}_\eta(\mathfrak{f})}\right) + T\left(r, \frac{\mathcal{L}_\eta(\mathfrak{g})}{\mathfrak{g}}\right) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\ &\le N\left(r, \frac{\mathcal{L}_\eta(\mathfrak{f})}{\mathfrak{f}}\right) + N\left(r, \frac{\mathcal{L}_\eta(\mathfrak{g})}{\mathfrak{g}}\right) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\ &\le 2\left[T(r,\mathfrak{f}) + T(r,\mathfrak{g})\right] + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).\end{aligned}$$

Combining the above inequality with $T(r,h) = T\left(r, \frac{\mathfrak{f}}{\mathfrak{g}}\right) = T(r,\mathfrak{f}) + T(r,\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g})$, we obtain $(n-1)[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] \le S(r,\mathfrak{f}) + S(r,\mathfrak{g})$ which is impossible. Therefore h is a constant, then substituting $\mathfrak{f} = \mathfrak{g}h$ in to (5.23), we have $h^{n+1} \equiv 1$. Therefore $f \equiv t\mathfrak{g}$, where t is a constant with $t^{n+1} = 1$.
This completes the proof of Lemma 5.14. □

6 Proof of Theorems

6.1 Proof of Theorem 2.1.

Proof. Let $\mathfrak{F} = \frac{\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f})}{p(z)}$ and $\mathfrak{G} = \frac{\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g})}{p(z)}$. It follows that $\mathfrak{F}$ and $\mathfrak{G}$ share (1,2) except the zeros of $p(z)$.
Case 7. Let $H \not\equiv 0$, from (5.1), we obtain

$$\begin{aligned}N(r,\infty;H) \le& \overline{N}_*(r,1;\mathfrak{F},\mathfrak{G}) + \overline{N}(r,0;\mathfrak{F}| \ge 2) + \overline{N}(r,0;\mathfrak{G}| \ge 2) \\ &+ \overline{N}_0(r,0;\mathfrak{F}') + \overline{N}_0(r,0;\mathfrak{G}'),\end{aligned} \tag{6.1}$$

where $\overline{N}_0(r,0;\mathfrak{F}')$ is the reduced ounting function of those zeros of $\mathfrak{F}'$ which are not the zeros of $\mathfrak{F}(\mathfrak{F}-1)$. $\overline{N}_0(r,0;\mathfrak{G}')$ is similarly defined.
Let z_0 be a simple zero of $\mathfrak{F}-1$ such that $p(z_0) \ne 0$. Then z_0 is a simple zero of $\mathfrak{G}-1$ and a zero of H. So

$$N(r,1;\mathfrak{F}| = 1) \le N(r,0;H) \le N(r,\infty;H) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \tag{6.2}$$

From (6.1) and (6.2), we have

$$\begin{aligned}\overline{N}(r,1;\mathfrak{F}) \leq \overline{N}(r,0;\mathfrak{F}|\geq 2) + \overline{N}(r,0;\mathfrak{G}|\geq 2) + \overline{N}_*(r,1;\mathfrak{F},\mathfrak{G}) + \overline{N}_0(r,0;\mathfrak{F}') \\ + \overline{N}(r,1;\mathfrak{F}|\geq 2) + \overline{N}_0(r,0;\mathfrak{G}') + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \end{aligned} \tag{6.3}$$

Using Lemma 5.11, we get

$$\begin{aligned}\overline{N}_0(r,0;\mathfrak{G}') + \overline{N}(r,1;\mathfrak{F}|\geq 2) + \overline{N}_*(r,1;\mathfrak{F},\mathfrak{G}) &\leq \overline{N}_0(r,0;\mathfrak{G}') + \overline{N}(r,1;\mathfrak{F}|\geq 2) \\ &\quad + \overline{N}(r,1;\mathfrak{F}|\geq 3) \\ &\leq N(r,0;\mathfrak{G}'|\mathfrak{G}\neq 0) \\ &\leq \overline{N}(r,0;\mathfrak{G}) + S(r,\mathfrak{g}). \end{aligned} \tag{6.4}$$

Using (6.3), (6.4), Lemmas 5.1, 5.12, and the Second fundamental theorem, we get

$$\begin{aligned}(n-1)T(r,\mathfrak{f}) &\leq T(r,\mathfrak{F}) + S(r,\mathfrak{f}), \\ &\leq \overline{N}(r,\infty;\mathfrak{F}) + \overline{N}(r,0;\mathfrak{F}) + \overline{N}(r,1;\mathfrak{F}) - \overline{N}_0(r,0;\mathfrak{F}') + s(r,\mathfrak{f}) \\ &\leq \overline{N}(r,\infty;\mathfrak{F}) + N_2(r,0;\mathfrak{F}) + N_2(r,0;\mathfrak{G}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\ &\leq 2\overline{N}(r,\mathfrak{f}) + 2\overline{N}(r,0;\mathfrak{f}) + N(r,0;\mathcal{L}_\eta(\mathfrak{f})) + N(r,0;\mathcal{L}_\eta(\mathfrak{g})) \\ &\quad + 2\overline{N}(r,0;\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\ &\leq 6T(r,\mathfrak{f}) + 4T(r,\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \end{aligned} \tag{6.5}$$

Similarly, we obtain

$$(n-1)T(r,\mathfrak{g}) \leq 6T(r,\mathfrak{g}) + 4T(r,\mathfrak{f}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \tag{6.6}$$

Combining (6.5) and (6.6), we have

$$(n-11)\left[T(r,\mathfrak{f}) + T(r,\mathfrak{g})\right] \leq S(r,\mathfrak{f}) + S(r,\mathfrak{g}), \tag{6.7}$$

which is a contradiction, since $n > 11$.
Case 8. Let $H \equiv 0$. Then the proof follows from Lemmas 5.13 and 5.14.
This completes the proof of Theorem 3.1. □

6.2 Proof of Theorem 2.2.

Proof. Let $\mathfrak{F}(z) = \frac{\mathfrak{f}^n\mathcal{L}_\eta(\mathfrak{f})}{p(z)}$ and $\mathfrak{G}(z) = \frac{\mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g})}{p(z)}$. Then $\mathfrak{F}$ and $\mathfrak{G}$ share $(1,1)$ except for the zeros of $p(z)$. We now consider the following two cases.

Case 9. Suppose $H \not\equiv 0$.
Using Lemmas 5.12, 5.14, 5.6 and equations (6.1), (6.2), we get

$$\begin{aligned}
\overline{N}(r,1;\mathfrak{F}) &\leq N(r,1;\mathfrak{F}|=1) + \overline{N}_L(r,1;\mathfrak{F}) + \overline{N}_L(r,1;\mathfrak{G}) + \overline{N}_E^{(2}(r,1;\mathfrak{F}) \\
&\leq \overline{N}(r,0;\mathfrak{F}|\geq 2) + \overline{N}(r,0;\mathfrak{G}|\geq 2) + \overline{N}_*(r,1;\mathfrak{F},\mathfrak{G}) + \overline{N}_L(r,1;\mathfrak{F}) \\
&\quad + \overline{N}_L(r,1;\mathfrak{G}) + \overline{N}_E^{(2}(r,1;\mathfrak{F}) + \overline{N}_0(r,0;\mathfrak{F}') + \overline{N}_0(r,0;\mathfrak{G}') \\
&\quad + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\
&\leq \overline{N}(r,0;\mathfrak{F}|\geq 2) + \overline{N}(r,0;\mathfrak{G}|\geq 2) + 2\overline{N}_L(r,1;\mathfrak{F}) + 2\overline{N}_L(r,1;\mathfrak{G}) \\
&\quad + \overline{N}_E^{(2}(r,1;\mathfrak{F}) + \overline{N}_0(r,0;\mathfrak{F}') + \overline{N}_0(r,0;\mathfrak{G}') + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\
&\leq \overline{N}(r,0;\mathfrak{F}|\geq 2) + \overline{N}(r,0;\mathfrak{G}|\geq 2) + \overline{N}_{\mathfrak{F}>2}(r,1;\mathfrak{G}) + N(r,1;\mathfrak{G}) \\
&\quad - \overline{N}(r,1;\mathfrak{G}) + \overline{N}_0(r,0;\mathfrak{F}') + \overline{N}_0(r,0;\mathfrak{G}') + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\
&\leq \overline{N}(r,0;\mathfrak{F}|\geq 2) + \frac{1}{2}\overline{N}(r,0;\mathfrak{F}) + \overline{N}(r,0;\mathfrak{G}|\geq 2) + N(r,0;\mathfrak{G}'|\mathfrak{G}\neq 0) \\
&\quad + \overline{N}_0(r,0;\mathfrak{F}') + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\
\overline{N}(r,1;\mathfrak{F}) &\leq \overline{N}(r,0;\mathfrak{F})|\geq 2) + \frac{1}{2}\overline{N}(r,0;\mathfrak{F}) + N_2(r,0;\mathfrak{G}) + \overline{N}_0(r,0;\mathfrak{F}') \\
&\quad + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \qquad (6.8)
\end{aligned}$$

Hence by using Second fundamental theorem, (6.8), Lemmas 5.1 and 5.12, we get

$$\begin{aligned}
(n-1)T(r,\mathfrak{f}) &\leq T(r,\mathfrak{F}) + S(r,\mathfrak{f}) \\
&\leq \overline{N}(r,\infty;\mathfrak{F}) + \overline{N}(r,0;\mathfrak{F}) + \overline{N}(r,1;\mathfrak{F}) - N_0(r,0;\mathfrak{F}') + S(r,\mathfrak{f}) \\
&\leq \overline{N}(r,\infty;\mathfrak{F}) + \frac{1}{2}\overline{N}(r,0;\mathfrak{F}) + N_2(r,0;\mathfrak{F}) + N_2(r,0;\mathfrak{G}) \\
&\quad + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\
&\leq 2\overline{N}(r,\mathfrak{f}) + \frac{3}{2}T(r,\mathfrak{f}) + 2\overline{N}(r,0;\mathfrak{f}) + N(r,0;\mathcal{L}_\eta(\mathfrak{f})) \\
&\quad + 2\overline{N}(r,0;\mathfrak{g}) + N(r,0;\mathcal{L}_\eta(\mathfrak{g})) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\
&\leq 2\overline{N}(r,\mathfrak{f}) + \frac{3}{2}T(r,\mathfrak{f}) + 2T(r,\mathfrak{f}) + 2T(r,\mathfrak{f}) + 2T(r,\mathfrak{g}) \\
&\quad + 2T(r,\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \\
(n-1)T(r,\mathfrak{f}) &\leq \frac{15}{2}T(r,\mathfrak{f}) + 4T(r,\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \qquad (6.9)
\end{aligned}$$

Similarly, we obtain

$$(n-1)T(r,\mathfrak{g}) \leq \frac{15}{2}T(r,\mathfrak{g}) + 4T(r,\mathfrak{f}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \qquad (6.10)$$

Now, combining (6.9) and (6.10), we get

$$(n-1)\left[T(r,\mathfrak{f})+T(r,\mathfrak{g})\right] \leq \frac{23}{2}\left[T(r,\mathfrak{f})+T(r,\mathfrak{g})\right]+S(r,\mathfrak{f})+S(r,\mathfrak{g})$$
$$\left(n-\frac{25}{2}\right)\left[T(r,\mathfrak{f})+T(r,\mathfrak{g})\right] \leq S(r,\mathfrak{f})+S(r,\mathfrak{g}),$$

which is a contradiction, since $n > \frac{25}{2}$.
Case 10. Suppose $H \equiv 0$.
Then the proof follows from Lemmas 5.13 and 5.14.
This completes the proof of Theorem 3.2. □

6.3 Proof of Theorem 2.3.

Proof. Let $\mathfrak{F}(z) = \frac{\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f})}{p(z)}$ and $\mathfrak{G}(z) = \frac{\mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g})}{p(z)}$. Then $\mathfrak{F}$ and $\mathfrak{G}$ share $(1,0)$ except for the zeros of $p(z)$. We now consider the following two cases.
Case 11. Suppose $H \not\equiv 0$.
In this case (6.2) changes to

$$N_E^{1)}(r,1;\mathfrak{F}) \leq N(r,0;H) \leq N(r,\infty;H)+S(r,\mathfrak{F})+S(r,\mathfrak{G}) \tag{6.11}$$

Using (6.2), (6.11) and Lemmas 5.8, 5.9, 5.10, 5.11, we get

$$\begin{aligned}
\overline{N}(r,1;\mathfrak{F}) &\leq N_E^{1)}(r,1;\mathfrak{F})+\overline{N}_L(r,1;\mathfrak{F})+\overline{N}_L(r,1;\mathfrak{G})+\overline{N}_E^{(2}(r,1;\mathfrak{F})\\
&\leq \overline{N}_*(r,1;\mathfrak{F},\mathfrak{G})+\overline{N}(r,0;\mathfrak{F}|\geq 2)+\overline{N}(r,0;\mathfrak{G}|\geq 2)+\overline{N}_L(r,1;\mathfrak{F})\\
&\quad+\overline{N}_L(r,1;\mathfrak{G})+\overline{N}_E^{(2}(r,1;\mathfrak{F})+\overline{N}_0(r,0;\mathfrak{F}')+\overline{N}_0(r,0;\mathfrak{G}')\\
&\quad+S(r,\mathfrak{f})+S(r,\mathfrak{g})\\
&\leq \overline{N}(r,0;\mathfrak{F}|\geq 2)+\overline{N}(r,0;\mathfrak{G}|\geq 2)+2\overline{N}_L(r,1;\mathfrak{F})+2\overline{N}_L(r,1;\mathfrak{G})\\
&\quad+\overline{N}_E^{(2}(r,1;\mathfrak{F})+\overline{N}_0(r,0;\mathfrak{F}')+\overline{N}_0(r,0;\mathfrak{G}')+S(r,\mathfrak{f})+S(r,\mathfrak{g})\\
&\leq \overline{N}(r,0;\mathfrak{F}|\geq 2)+\overline{N}(r,0;\mathfrak{G}|\geq 2)+\overline{N}_{\mathfrak{F}>1}(r,1;\mathfrak{G})+\overline{N}_{\mathfrak{G}>1}(r,1;\mathfrak{F})\\
&\quad+\overline{N}_L(r,1;\mathfrak{F})+N(r,1;\mathfrak{G})-\overline{N}(r,1;\mathfrak{G})+\overline{N}_0(r,0;\mathfrak{F}')\\
&\quad+\overline{N}_0(r,0;\mathfrak{G}')+S(r,\mathfrak{f})+S(r,\mathfrak{g})\\
&\leq N_2(r,0;\mathfrak{F})+\overline{N}(r,0;\mathfrak{F})+N_2(r,0;\mathfrak{G})+N(r,0;\mathfrak{G}'|\mathfrak{G}|\neq 0)\\
&\quad+\overline{N}_0(r,0;\mathfrak{F}')+S(r,\mathfrak{f})+S(r,\mathfrak{g})\\
\overline{N}(r,1;\mathfrak{F}) &\leq N_2(r,0;\mathfrak{F})+\overline{N}(r,0;\mathfrak{F})+N_2(r,0;\mathfrak{G})+\overline{N}(r,0;\mathfrak{G})\\
&\quad+\overline{N}_0(r,0;\mathfrak{F}')+S(r,\mathfrak{f})+S(r,\mathfrak{g}).
\end{aligned} \tag{6.12}$$

Hence by using the Second fundamental theorem and Lemmas 5.1, 5.12, we obtain

$$\begin{aligned}(n-1)T(r,\mathfrak{f}) &\leq T(r,\mathfrak{F}) + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,\infty;\mathfrak{F}) + \overline{N}(r,0;\mathfrak{F}) + \overline{N}(r,1;\mathfrak{F}) - N_0(r,0;\mathfrak{F}') + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,\infty;\mathfrak{F}) + 2N_2(r,0;\mathfrak{F}) + N_2(r,0;\mathfrak{G}) + \overline{N}(r,0;\mathfrak{G}) + S(r,\mathfrak{f}) \\ &\quad + S(r,\mathfrak{g}) \\ &\leq 2\overline{N}(r,\mathfrak{f}) + 4\overline{N}(r,0;\mathfrak{f}) + 2N(r,0;\mathcal{L}_\eta(\mathfrak{f})) + 2\overline{N}(r,0;\mathfrak{g}) \\ &\quad + \overline{N}(r,0;\mathfrak{g}) + N(r,0;\mathcal{L}_\eta(\mathfrak{g})) + \overline{N}(r,0;\mathcal{L}_\eta(\mathfrak{g})) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}) \\ &\leq 2\overline{N}(r,\mathfrak{f}) + 4T(r,\mathfrak{f}) + 4T(r,\mathfrak{f}) + 2T(r,\mathfrak{g}) + 2T(r,\mathfrak{g}) \\ &\quad + T(r,\mathfrak{g}) + 2T(r,\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \\ (n-1)T(r,\mathfrak{f}) &\leq 6T(r,\mathfrak{f}) + 7T(r,\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \end{aligned} \tag{6.13}$$

Similarly, we obtain

$$(n-1)T(r,\mathfrak{g}) \leq 6T(r,\mathfrak{g}) + 7T(r,\mathfrak{f}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \tag{6.14}$$

Now, combining (6.13) and (6.14), we get

$$(n-14)\left[T(r,\mathfrak{f}) + T(r,\mathfrak{g})\right] \leq +S(r,\mathfrak{f}) + S(r,\mathfrak{g}), \tag{6.15}$$

which is a contradiction, since $n > 14$.
Case 12. Suppose $H \equiv 0$.
Then the proof follows from Lemmas 5.13 and 5.14.
This completes the proof of Theorem 3.3. □

7 Conclusions

Nevanlinna theory is an effective analytic technique with several applications in a variety of mathematical fields because of its insights into the behaviour of meromorphic functions, the distribution of values, and the characteristics of differential equation solutions. Mathematicians can develop mathematical theory in areas like communication networks, signal processing, the design of filters and controllers for systems, as well as certain practical applications, by using Nevanlinna's theory to better comprehend complex functions and their behaviour.

In this article, with the help of the Nevanlinna theory, we have studied the uniqueness results of two non-linear difference polynomials generated by two finite ordered non-constant meromorphic functions when they share a non-zero polynomial with finite weight and obtain some results which extend and improve some of the earlier results of Majumder [17].

Further, we would like to pose the following open questions.
Question 1. Can the linear difference polynomial $\mathcal{L}_\eta(\mathfrak{f})$ in Theorems 2.1-2.3 be further generalized to a larger class of function $\Delta^n_\eta\mathfrak{f}$, where $\Delta^n_\eta\mathfrak{f} = \sum_{r=0}^{n-1}(-1)^r\binom{n}{r}\mathfrak{f}(z+(n-r)\eta)$?
Question 2. Is the condition for 'n', in Theorem 2.1, Theorem 2.2 and Theorem 2.3 sharp?

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