Polynomial collocation methods based on successive integration technique for solving system of neutral delay differential equations

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Abstract

This paper presents a new approach to using polynomials such as Hermite, Bernoulli, Chebyshev, Fibonacci and Bessel to solve a system of linear and nonlinear neutral delay differential equations. The proposed method is based on the truncated polynomial expansion of the function together with collocation points and successive integration techniques. This method reduces the given equation to system of non-linear equations with unknown polynomial coefficients which can be easily calculated. The convergence of the proposed method is discussed with several mild conditions. Numerical examples are considered to demonstrate the efficiency of the method. The numerical results reveal that the proposed new approach gives better results than other conventional methods. It demonstrates the reliability and efficiency of this method for solving a system of linear and nonlinear neutral delay differential equations.

Keywords: Polynomials; Collocation method; successive integration technique; neutral delay differential equations.

2010 AMS subject classification: 65D30, 42C05, 39A13[♥]

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1.Introduction

Delay Differential Equations (DDEs) are a differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. The terms involving previous times are called delay terms. The delay terms may be constant, state-dependent and time-dependent. Neutral delay differential equations (NDDEs) are another type of DDEs in which the highest-order derivative of the unknown function occurs with delay. DDEs arise in the fields of signal processing, digital images, control systems, epidemiology, chemical kinetics, etc. Some notable applications of DDEs and NDDEs are in chemical kinetics [1], climate model [2], SIR epidemic model [3], iterative survival model of red blood cells [4], immunology model [5] and cell growth model [6].

DDEs and NDDEs have been studied by many authors and developed various analytical and numerical methods. Some of them are Adams predictor corrector algorithm [7], Homotopy perturbation method [8], Reproducing kernel Hilbert space method [9], Variational iteration method [10], Elzaki transform method [11], Haar wavelet series method [12], Higher order derivative Runge Kutta method [13], Composite Runge Kutta methods and new one-step techniques [14], Analytical algorithm [15], Hybrid Multistep block method [16] and Generalized Rational multi-step method [17] for solving DDEs and NDDEs.

The Collocation method based on various polynomials is a powerful technique for solving differential equations. Mustafa Gulsu *et al.* [18,19] have proposed collocation method based on Hermite and Chebyshev polynomials for solving DDEs with variable coefficients under mixed conditions. Suvip Yiizbas *et al.* [20] have presented Bessel polynomial operational matrix method for solving NDDEs. Ali H Bhrawy *et al.* [21] have proposed a Legendre-Gauss collocation method for solving NDDEs with proportional delay. Tohidi *et al.* [22] have presented Bernoulli operational matrix for solving DDEs. Ayse Betul Koc *et al.* [23] have presented a matrix method based on Fibonacci polynomial for solving DDEs. Birol Ibis *et al.* [24] have applied Hermite polynomials for solving NDDEs with proportional delays.

The above-mentioned collocation methods using different polynomials are based on operational matrices. In this study, we propose a new approach of using Polynomial Collocation methods based on Successive Integration Technique for solving systems of both linear and non-linear NDDEs.

This paper is organized as follows: In Section 2, the basic definitions of different polynomials are given. In Section 3, the description of the method for solving NDDEs is provided. In Section 4, the convergence analysis of the proposed method is discussed. In Section 5, illustrative examples are provided.

2. Basic definition of polynomials

2.1 Hermite polynomial

The Hermite polynomial $H_n(t)$ of order n is defined on the interval $(-\infty, \infty)$. There are different ways to define for Hermite polynomial, one of them is the so-called Rodrigues' formula

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \tag{1}$$

From Eqn. (1), the recurrence relation for the polynomials can be derived as

$$H_n(t) = 2t\, H_{n-1}(t) - H'_{n-1}(t) \tag{2}$$

$H_0(t)$ can be obtained from Eqn. (1) and the remaining terms are determined by using the recursion relation Eqn. (2).

Thus, we have the following sequence of polynomials:

$H_0(t) = 1$

$H_1(t) = 2t$

$H_2(t) = 4t^2 - 2$

$H_3(t) = 8t^3 - 12t$

$H_4(t) = 16t^4 - 48t^2 + 12$

The n^{th} order Hermite polynomial $H_n(t)$ has a leading coefficient 2^n.

2.2 Bernoulli polynomial

The Bernoulli polynomial is named after Jacob Bernoulli which combines the Bernoulli numbers and binomial coefficients. The generating function for the Bernoulli polynomial of order n is defined by

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{x e^{xt}}{e^x - 1} \tag{3}$$

The recursion formula for Bernoulli polynomial is:

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k(t) = n t^{n-1}, \qquad n = 2, 3, \ldots \tag{4}$$

$B_0(t)$ can be obtained from Eqn. (3) and the remaining terms are determined by using the recursion relation Eqn. (4).

Thus, we have few terms of the Bernoulli polynomials as:

$B_0(t) = 1$

$B_1(t) = t - \frac{1}{2}$

$B_2(t) = t^2 - t + \frac{1}{6}$

$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$

$B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}$

2.3 Chebyshev polynomial

The Chebyshev polynomial related to cosine functions on the interval $[-1, 1]$ of order n is defined as

$$T_n(\cos t) = \cos(nt) \tag{5}$$

The recursion relation of Chebyshev polynomial is:

$$T_{n+1}(t) = 2t\, T_n(t) - T_{n-1}(t) \tag{6}$$

$T_0(t)$ and $T_1(t)$ can be obtained from Eqn. (5). Then the remaining terms are determined from Eqn. (6). Thus, we have the following sequence of polynomials:

$T_0(t) = 1$

$T_1(t) = t$

$T_2(t) = 2t^2 - 1$

$T_3(t) = 4t^3 - 3t$

$T_4(t) = 8t^4 - 8t^2 + 1$

2.4 Fibonacci polynomial

The Fibonacci polynomials are a polynomial sequence which can be considered of Fibonacci numbers. The Fibonacci polynomials are defined by a recurrence relation

$$F_n(t) = \begin{cases} 0, & if\ n = 0 \\ 1, & if\ n = 1 \\ tF_{n-1}(t) + F_{n-2}(t), & if\ n \geq 2. \end{cases}$$

The first few Fibonacci polynomials are:

$F_0(t) = 0$

$F_1(t) = 1$

$F_2(t) = t$

$F_3(t) = t^2 + 1$

$F_4(t) = t^3 + 2t$

2.5 Bessel polynomial

The Bessel polynomial is defined by

$$y_n(t) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\,k!} \left(\frac{t}{2}\right)^k$$

The recursion equation for the Bessel polynomial is: $y_n(t) = (2n-1)ty_{n-1}(t) + y_{n-2}(t)$

The first few Bessel polynomials are:

$y_0(t) = 1$

$y_1(t) = t + 1$

$y_2(t) = 3t^2 + 3t + 1$

$y_3(t) = 15t^3 + 15t^2 + 6t + 1$

3. Description of the proposed method

Consider the n[th] order NDDE of the form

$$y^{(n)}(t) = f\left(t, y(t), y'(t), \ldots, y^{(n-1)}(t), y(t-\tau_0), y'(t-\tau_1), \ldots, y^{(n)}(t-\tau_n)\right), \quad t > t_0 \tag{7}$$

with initial conditions

$$y^{(i)}(t_0) = \emptyset(t), \; i = 1, 2, 3, \ldots, (n-1) \text{ for } \quad t \le t_0 \tag{8}$$

Here $\emptyset(t)$ is the initial function and τ is the delay term.

Let P(t) represent any orthogonal polynomials. For the proposed method, we assume that

$$y^{(n)}(t) \approx B^T P(t) = \sum_{j=0}^{N} c_j P_j(t) \tag{9}$$

where N being any positive integer,

$B^T = (c_0, c_1, \dots c_N)$

$P(t) = \left(P_0(t), P_1(t) \dots P_N(t)\right)^T$

Here T stands for transpose of the matrix.

Our aim is to determine the polynomial coefficients $c_j's$. For this, we integrate Eqn. (9) with respect to t from t_0 to t,

$$\left.\begin{array}{c} y^{(n-1)}(t) = y(t_0) + \int_{t_0}^{t} B^T\, P(t)\, dt \\ y^{(n-2)}(t) = y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T\, P(t)\, dt \\ \vdots \\ y'(t) = \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T\, P(t)\, dt \\ y(t) = \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T\, P(t)\, dt \end{array}\right\} \quad (10)$$

Now, for delay terms

$$\left.\begin{array}{c} y^{(n)}(t-\tau_n) = B^T P(t-\tau_n) \\ y^{(n-1)}(t-\tau_{n-1}) = y(t_0) + \int_{t_0}^{t} B^T\, P(t-\tau_{n-1})\, dt \\ y^{(n-2)}(t-\tau_{n-2}) = y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T\, P(t-\tau_{n-2})\, dt \\ \vdots \\ y'(t-\tau_1) = \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T\, P(t-\tau_1)\, dt \\ y(t-\tau_0) = \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T\, P(t-\tau_0)\, dt \end{array}\right\} \quad (11)$$

Then we substitute (10) and (11) in (7) and use the collocating points $t_i = \frac{i}{N}$, where i = 0, 1…N. This yields a system of linear or nonlinear equations subject to the linear and nonlinear terms in Eqn. (7). On solving this system of equations, we get the respective polynomial coefficients c_j's from which the solution of the NDDE (7) can be obtained.

In similar manner, the above proposed method can be applied to solve system of linear and nonlinear NDDEs.

4. Convergency analysis

Consider the first order NDDE of the form

$$\left(y(t) + a(t)y(q_0 t)\right)' = b(t)y(t) + c(t)y(q_1 t) + f(t), t \in [0, T] \quad (12)$$

with initial value condition

$$y(0) = y_0. \quad (13)$$

Here $a(t), b(t)$ and $c(t)$ are given analytical functions.

The convergence of the proposed method will be provided under several mild conditions such as the solution boundedness of the Eqn. (12). Some definitions and lemmas are provided in order to clarify the main convergence theorem of this section.

Definition 4.1 [21]

A function $\xi: [-1, 1] \to \Re$ belongs to the Sobolev space $\mathcal{W}^{m,p}$, if its j^{th} weak derivative $\xi^{(j)}$ lies in $L^p[-1, 1]$ for all $0 \leq j \leq m$ with the norm

$$\|\xi\|_{\mathcal{W}^{m,p}} = \sum_{j=0}^{m} \left\|\xi^{(j)}\right\|_{L^p}$$

where $\|\xi\|_{L^p}$ denotes the usual Lebesgue norm

$$\|\xi\|_{L^p} = \left(\int_{-1}^{1} \|\xi\|^p dx\right)^{\frac{1}{p}}$$

and $\|\xi\|$ stands for any finite dimensional norm in $\Re^n$.

Lemma 4.1 [21]

For a given function $\xi \in \mathcal{W}^{m,\infty}$, there exists a polynomial $y_N(t)$ of degree $\leq$ N such that

$$\|\xi(t) - y_N(t)\|_{L^\infty} \leq C C_0 N^{-m}, \;\; t \in [-1, 1]$$

where $C_0 = \|\xi\|_{\mathcal{W}^{m,p}}$, C is a constant independent of N and m is the order of smoothness of ξ. Here $y_N(t)$ with the smallest norm $\|\xi(t) - y_N(t)\|_{L^\infty}$ is known as the N^{th} order best polynomial approximation of $\xi(t)$ in the norm of L^∞.

Note that if $\xi \in C^\infty$, then $m = \infty$. This implies that $y_N(t)$ converges to ξ at a spectral rate, that is it would be faster than any given polynomial rate. Moreover, let us denote the set of continuous functions in a linear space on $[0, T]$ by $C[0, T]$ and the uniform norm in $C[0, T]$ by

$$\|f\|_\infty = \max_{0\le t\le x}|f(t)|, \qquad f\in C[0,T].$$

Now integrating the Eqn. (12) in the interval $[0,T]$ and using the initial condition (13), we get

$$\big(y(t)+a(t)y(q_0t)\big)-(y_0+a(0)y_0)=\int_0^t b(x)y(x)+c(x)y(q_1x)dx$$

Taking $\theta = q_0x$ we rewrite the above equation in the following form

$$y(t)=g(t)-a(t)y(q_0t)+\int_0^t b(x)y(x)\,dx+\int_0^{q_1t}\hat{c}(\theta)y(\theta)\,d\theta \quad (14)$$

where

$$g(t)=(y_0+a(0)y_0)+\int_0^t f(x)\,dx$$

and

$$\hat{c}(\theta)=\frac{1}{q_1}c\left(\frac{\theta}{q_1}\right).$$

In the following Theorem 4.1, we show that the approximate solution expressed in terms of the orthogonal polynomials converges to the exact solution under several mild conditions.

Theorem 4.1

Let $y(t)$ and $y_N(t)$ be the exact and numerical solutions of Eqn. (12). Also, assume the approximations of $a(t), b(t), g(t)$ and $\hat{c}(t)$ be $a_N(t), b_N(t), g_N(t)$ and $\hat{c}_N(t)$ respectively. Moreover, suppose that $\|a(t)\|_\infty \le A$, $\|b(t)\|_\infty \le B$, $\|\hat{c}(t)\|_\infty \le C$ and $\|y_N(t)\|_\infty \le Y_N$ where $t\in[0,T]$.

Then, $\lim_{N\to\infty} y_N(t)=y(t)$ subject to the condition $A+TB+TC \ll 1$.

Proof:

Suppose that the unknown functions $a(t), b(t), \hat{c}(t)$ and $g(t)$ are approximated in terms of any orthogonal polynomials. Then the numerical solution is an approximated polynomial in the form of $y_N(t)$. We need to find an upper bound for the error between $y(t)$ and $y_N(t)$ for Eqn. (14).

According to the assumptions, Eqn. (14) can be written as

$$y_N(t)=g_N(t)-a_N(t)y(q_0t)+\int_0^t b_N(x)y(x)\,dx+\int_0^{q_1t}\hat{c}_N(\theta)y(\theta)\,d\theta \quad (15)$$

Now, Eqn. (14) – Eqn. (15) yields

$$\|y(t)-y_N(t)\|_\infty = \Bigg\| g(t)-g_N(t)+\big(-a(t)y(q_0t)+a_N(t)y(q_0t)\big) + \int_0^t b(x)y(x)\,dx - \int_0^t b_N(x)y(x)\,dx + \int_0^{q_1t} \hat{c}(\theta)y(\theta)\,d\theta - \int_0^{q_1t} \hat{c}_N(\theta)y(\theta)\,d\theta \Bigg\|_\infty$$

By using the triangle inequality, we get

$$\|y(t)-y_N(t)\|_\infty \le \|g(t)-g_N(t)\|_\infty + \|-a(t)y(q_0t)+a_N(t)y(q_0t)\|_\infty + \left\|\int_0^t b(x)y(x)\,dx - \int_0^t b_N(x)y(x)\,dx\right\|_\infty + \left\|\int_0^{q_1t} \hat{c}(\theta)y(\theta)\,d\theta - \int_0^{q_1t} \hat{c}_N(\theta)y(\theta)\,d\theta\right\|_\infty$$

Since $0 \le q_1 \le 1$ and $t \in [0,T]$, the above inequality reduces to

$$\|y(t)-y_N(t)\|_\infty \le \|g(t)-g_N(t)\|_\infty + \|-a(t)y(q_0t)+a_N(t)y(q_0t)\|_\infty T\|b(t)y(t)-b_N(t)y(t)\|_\infty + T\|\hat{c}(t)y(t)-\hat{c}_N(t)y(t)\|_\infty$$

This can be rewritten as

$$\|y(t)-y_N(t)\|_\infty \le \|g(t)-g_N(t)\|_\infty + \big\|-a(t)\big(y(t)-y_N(t)\big)\big\|_\infty + \big\|-\big(a(t)-a_N(t)\big)y_N(t)\big\|_\infty + T\big\|b(t)\big(y(t)-y_N(t)\big)\big\|_\infty + T\big\|\big(b(t)-b_N(t)\big)y_N(t)\big\|_\infty + T\big\|\hat{c}(t)\big(y(t)-y_N(t)\big)\big\|_\infty + T\big\|(\widehat{c}(t)-\hat{c}_N(t))y_N(t)\big\|_\infty$$

Using the assumptions

$$\|-a(t)\|_\infty \le A, \quad \|b(t)\|_\infty \le B,$$

$$\|\hat{c}(t)\|_\infty \le C\,, \qquad \|y_N(t)\|_\infty \le Y_N\,.$$

in the above inequality, we get

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty &\\ &\leq \|g(t)-g_N(t)\|_\infty + A\left\|\left(y(t)-y_N(t)\right)\right\|_\infty + Y_N\|(a(t)-a_N(t))\|_\infty \\ &+ TB\left\|\left(y(t)-y_N(t)\right)\right\|_\infty + TY_N\|(b(t)-b_N(t))\|_\infty \\ &+ TC\|(y(t)-y_N(t))\|_\infty + TY_N\|\hat{c}(t)-\hat{c}_N(t)\|_\infty\end{aligned}$$

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty - A\left\|\left(y(t)-y_N(t)\right)\right\|_\infty - TB\left\|\left(y(t)-y_N(t)\right)\right\|_\infty &\\ - TC\left\|\left(y(t)-y_N(t)\right)\right\|_\infty &\\ \leq \|g(t)-g_N(t)\|_\infty + Y_N\left\|\left(a(t)-a_N(t)\right)\right\|_\infty + TY_N\|(b(t)-b_N(t))\|_\infty &\\ + TY_N\|\hat{c}(t)-\hat{c}_N(t)\|_\infty &\end{aligned}$$

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty(1-A-TB-TC) &\\ \leq \|g(t)-g_N(t)\|_\infty + Y_N\left\|\left(a(t)-a_N(t)\right)\right\|_\infty + TY_N\|(b(t)-b_N(t))\|_\infty &\\ + TY_N\|\hat{c}(t)-\hat{c}_N(t)\|_\infty &\end{aligned}$$

Let us introduce the following notations:

$$G_1(t) = g(t) - g_N(t), \qquad G_2(t) = a(t) - a_N(t),$$

$$G_3(t) = b(t) - b_N(t), \qquad G_4(t) = \hat{c}(t) - \hat{c}_N(t).$$

Then the above equation becomes

$$\|y(t)-y_N(t)\|_\infty \leq \frac{\|G_1(t)\|_\infty + Y_N\|G_2(t)\|_\infty + TY_N\|G_3(t)\|_\infty + TY_N\|G_4(t)\|_\infty}{(1-A-TB-TC)}$$

If $A + TB + TC \ll 1$, then

$$\operatorname*{Lim}_{N\to\infty} \|y(t)-y_N(t)\|_\infty = 0$$

i.e., $\lim\limits_{N\to\infty} \|y_N(t)\|_\infty = \lim\limits_{N\to\infty} \|y(t)\|_\infty$

This is possible because of the smoothness of $a(t), b(t), \hat{c}(t)$ and $g(t)$. Now Lemma 4.1 implies

$$\operatorname*{Lim}_{N\to\infty} \|a(t)-a_N(t)\|_\infty = 0,$$

$$\operatorname*{Lim}_{N\to\infty} \|b(t)-b_N(t)\|_\infty = 0,$$

$$\lim_{N\to\infty} \|\hat{c}(t)-\hat{c}_N(t)\|_\infty = 0 \text{ and}$$

$$\lim_{N\to\infty} \|g(t)-g_N(t)\|_\infty = 0$$

This completes the proof.

5. Numerical examples

In this section, numerical examples of linear and non-linear system of NDDEs and an example of NDDE which arises in the modelling of E.coli growth are given to demonstrate the accuracy and effectiveness of the proposed collocation method based on successive integration technique.

Example 1 [15]

Consider the 2 – dimensional linear system of NDDEs with constant delay

$$y_1'(t) = y_1'(t-1) + 4y_2(t), \qquad 0 \le t \le 2$$

$$y_2'(t) = y_1(t) - 4y_1(t-1), \qquad 0 \le t \le 2$$

with history function

$$y_1(t) = e^{-2t}, \quad y_2(t) = \frac{1}{2}\left(e^{-2(t-1)} - e^{-2t}\right). \qquad t \in [-1,0].$$

The exact solution is $y_1(t) = e^{-2t}, \quad y_2(t) = \frac{1}{2}\left(e^{-2(t-1)} - e^{-2t}\right).$

The numerical results of the proposed method using by Hermite polynomial for different values of N are given in Table 1. The numerical results of the proposed method using Bessel polynomial, Bernoulli polynomial, Chebyshev polynomial, Hermite polynomial and Fibonacci polynomials for different values of N at time t = 1 are given in Table 2. The solution graph obtained by using the proposed method with N = 10 is presented in Figure 1.

Time t	**Hermite polynomial collocation method**					
	N = 5		**N = 7**		**N = 10**	
	y_1	y_2	y_1	y_2	y_1	y_2
0.2	1.11 e-07	1.13 e-08	2.63 e-11	2.08 e-11	6.06 e-17	9.91 e-17
0.4	6.24 e-08	3.99 e-08	5.27 e-13	2.83 e-11	1.03 e-16	5.85 e-17
0.6	1.77 e-07	3.15 e-07	5.31 e-11	5.69 e-11	1.95 e-17	8.14 e-17
0.8	3.95 e-07	3.32 e-07	2.25 e-11	5.97 e-11	5.97 e-17	9.52 e-17
1.0	1.43 e-08	0.00 e+00	1.81 e-10	0.00 e+00	3.61 e-16	0.00 e+00
1.2	2.91 e-08	4.88 e-08	6.77 e-12	1.02 e-11	6.45 e-18	2.16 e-17
1.4	5.12 e-08	3.15 e-08	8.31 e-12	6.81 e-13	3.55 e-17	3.71 e-17
1.6	1.02 e-08	6.77 e-08	1.03 e-11	1.99 e-11	2.12 e-17	6.73 e-18
1.8	1.26 e-07	1.64 e-07	2.17 e-11	9.06 e-12	2.91 e-17	2.17 e-17
2.0	1.45 e-07	2.22 e-08	2.19 e-11	6.84 e-11	3.34 e-17	1.28 e-16

Table 1: Absolute errors for Example 1

Polynomials	**N = 5**		**N = 7**		**N = 10**	
	y_1	y_2	y_1	y_2	y_1	y_2
Bessel	9.77 e-07	1.55 e-06	2.23 e-10	4.39 e-10	2.68 e-16	7.25 e-16
Bernoulli	8.30 e-08	2.43 e-07	7.02 e-12	2.75 e-11	1.06 e-18	8.13 e-18
Chebyshev	3.66 e-07	7.09 e-07	6.84 e-11	1.68 e-10	6.18 e-17	2.19 e-16
Hermite	1.45 e-07	2.22 e-08	2.19 e-11	6.84 e-11	3.34 e-17	1.28 e-16
Fibonacci	1.58 e-06	2.28 e-06	5.80 e-10	9.64 e-10	1.37 e-15	2.84 e-15

Table 2: Absolute errors for Example 1

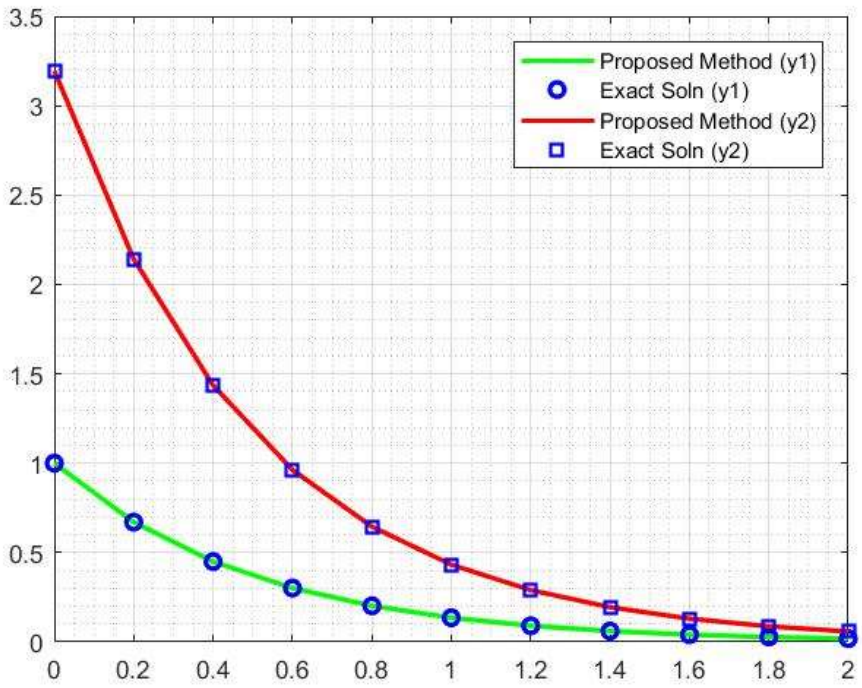

Figure 1: Solution graph for Example 1

Example 2 [15]

Consider the third-order non-linear system of NDDE with both constant and proportional delay.

$$y_1'''(t) = y_1'''(t-2)y_1\left(\frac{t}{3}\right) + \left(y_1(t)\right)^{\frac{2}{3}} + 2t + e^{-t}$$

$$y_2'''(t) = \frac{1}{2}y_2'''\left(\frac{t}{2}\right) + y_2'(t-1)y_1\left(\frac{t}{3}\right), \qquad t \geq 1$$

with history function $y_1(t) = e^t$ and $y_2(t) = t^2, \quad t \in [-2,0]$.

The given initial conditions are

$$y_1(0) = 1, \quad y_1'(0) = 1, \quad y_1''(0) = 1$$

$$y_2(0) = 0, \quad y_2'(0) = 0, \quad y_2''(0) = 2$$

For this example, the numerical results are obtained by using the proposed method on the above mentioned five polynomials. The solution graph by the proposed method is given in Figure 2.

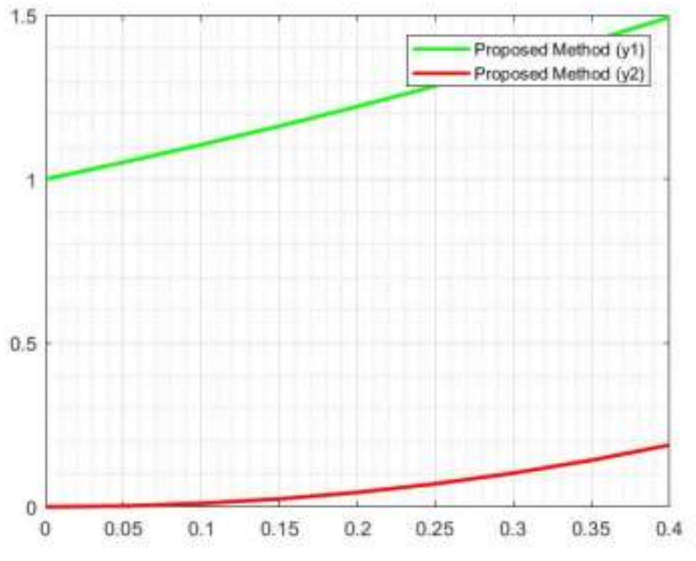

Figure 2: Solution graph for Example 2

Example 3 [6]

Consider the following NDDEs arise in modelling the growth patterns of Escheria coli (E. coli) cells, which clearly exhibit the presence of a time delay in their cell division stage.

$$y'(t) = \rho_0\, y(t) + \rho_1\, y(t-\tau) + \rho_2\, y'(t-\tau) \quad t \geq 0$$

For this example, the numerical results are obtained by using the proposed method based on the above-mentioned five polynomials. The numerical simulation by the proposed method at N = 5 is compared with the graphical solution given in [6]. These have been shown in Figure 3 and 4.

Parameters	**Biological Interpretation**	**Values**
$\tau > 0$	The average cell-division time	20.2229
$\rho_0 \geq 0$	The rate of cell-death	-0.0057
$\rho_1 \geq 0$	The rate of commitment to the cell-division process	0.0131
$0 \leq \rho_2 \geq 2$	The gradual dispersal of synchronization of cell-division	1.8407

Table 3: The biological interpretation of the parameters and their values of Example 3

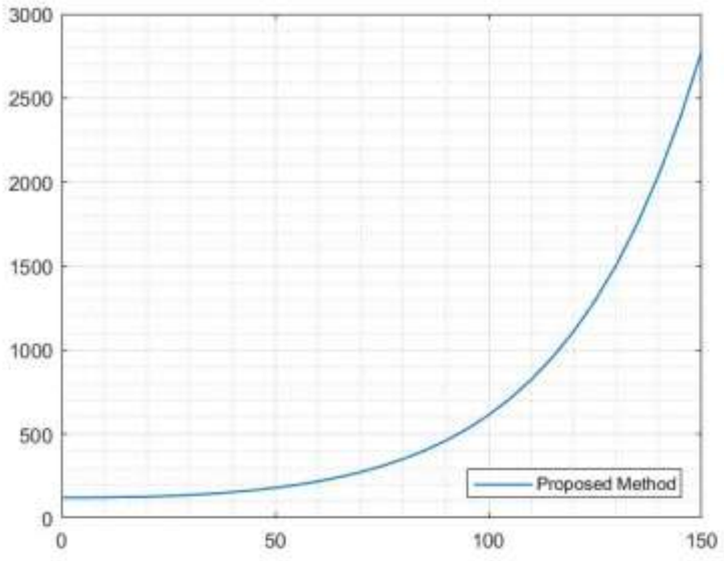

Figure 3: Solution graph for Example 3

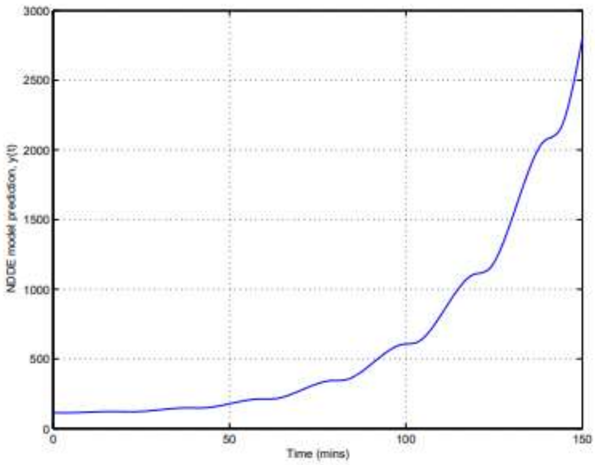

Figure 4: Solution graph of Example 3 in [6]

6. Conclusion

In this paper, a new application of the Polynomial collocation method based on successive integration techniques is presented for solving a system of neutral delay differential equations. The convergence analysis of the proposed method has been discussed. Numerical examples of linear and non-linear systems of neutral delay differential equations are considered to demonstrate the efficiency of the proposed method.

The Tables show that the proposed polynomial collocation method based on successive integration techniques gives results with reasonable accuracy. Also, it is observed that accuracy increases as N increases. The proposed method is computationally simple. Hence it is concluded that the proposed method is suitable for solving linear and non-linear systems of neutral delay differential equations in real-world problems that exist in different fields of science and engineering.

References

[1] R. Irving Epstein and Yin Luo, Differential delay equations in chemical kinetics. Nonlinear models: The cross-shaped phase diagram and the Oregonator. J.Chem. Phys., 95(1) (1991) 244 – 254.

[2] Andrew Keane, Bernd Krauskopf and M. Claire Postlewaite, Climate models with delay differential equations. An Interdisciplinary Journal of Nonlinear Science, 27(11) (2017) 114309.

[3] Kapula Rajendra Prasad, Mahammad Khuddush and Kuparala Venkata Vidyasagar, Almost periodic positive solutions for a time-delayed SIR epidemic model with saturated treatment on time scales. Journal of Mathematical Modeling, 9(1) (2021) 45-60.

[4] Marwa Khemis, Ahleme Bouakkaz and Rabah Khemis, Existence, uniqueness and stability results of an iterative survival model of red blook cells with a delayed nonlinear harvesting term. Journal of Mathematical Modeling, 10(3) (2022) 515-528.

[5] M. Sarah Andrew, T. H. Christopher Baker and A. Gennady Bocharov, Rival approaches to mathematical modelling in immunology. Journal of Computational and Applied Mathematics, 205 (2007) 669 – 686.

[6] Christopher T. H. Baker, Gennady A. Bocharov and Fathalla A. Rihan, Neutral Delay Differential Equations in the Modelling of Cell Growth. Applied Mathematics Group Research Report, University of Chester, UK, (2008) 1 – 29.

[7] Z. Jackiewixa and E. Lo, The numerical solution of neutral functional differential equations by Adams predictor-corrector methods. Applied Numerical Mathematics, 8 (1991) 477 – 491.

[8] Jafar Biazar and Behzad Ghanbari, The homotopy perturbation method for solving neutral functional-differential equations with proportional delays. Journal of King Saud University-Science, 24 (2012) 33 – 37.

[9] Xueqin Lv and Yue Gao, The RKHSM for solving neutral functional-differential equations with proportional delays. Mathematical Methods in the Applied Sciences, 36(6) (2012) 642 – 6.

[10] M.M. Khader, Numerical and theoretical treatment for solving linear and nonlinear delay differential equations using variational iteration method. Arab Journal of Mathematical Sciences,19(2) 2013 243-256.

[11] J. Ebimene Mamadu and N. Ignatius Njoseh, Solving Delay Differential Equations by Elzaki Transform Method. Boson Journal of Modern Physics, 3(1) (2017) 214 – 219.

[12] Akmal Raza and Arshad Khand, Haar wavelet series solution for solving neutral delay differential equations, Journal of King Saud University - Science, (2018)1-7.

[13] C. Dhinesh Kumar and A. Emimal Kanaga Pushpam, Higher Order Derivative Runge Kutta Method for Solving Delay Differential Equations. Advances in Mathematics:Scientific Journal, 8(3) (2019) 26 – 34.

[14] J. Vinci Shaalini and A. Emimal Kanaga Pushpam, Analysis of Composite Runge Kutta Methods and New One-Step Technique for Stiff Delay Differential Equations, IAENG International Journal of Applied Mathematics, 49(3) (2019) 1 – 10.

[15] Aminu Barde and Normah Maan, Analytical Algorithm for Systems of Neutral Delay Differential Equations, *Applied Mathematics,* 10(9) (2019) 753-768.

[16] Nur Inshirah Naziah Ismail, Zanariah Abdul Majid and Norzak Senu, Hybrid Multistep Block Method for Solving Neutral Delay Differential Equations. Sains Malaysiana, 49 (2020) 929 – 940.
[17] J. Vinci Shaalini and A. Emimal Kanaga Pushpam, Generalized Rational Multi-step Method for Delay Differential Equations, IAENG International Journal of Applied Mathematics, 50(1) (2020) 1 – 10.
[18] Mustafa Gulsu, Hatice Yalman, Yalcin Oxturk and Mehmet Sezer. A New Hermite Collocation Method for Solving Differential Difference Equations. Application and Applied Mathematics, 6(11) (2011) 116-129.
[19] Mustafa Gulsu, Yalcin Ozturk and Mehmet Sezer, A new Chebyshev polynomial approximation for solving delay differential equations. Journal of Difference Equations and Applications, 18(6) (2012) 1043-1065.
[20] Suvip Yiizbas, Niyazi Sahin and Mehmet Sezer, A Bessel polynomial approach for solving linear neutral delay differential equations with variable coefficients. Journal of Advanced Research in Differential Equations, 3(1) (2011) 81-101.
[21] Ali H Bhrawy, Laila M Assas, Emran Tohidi and Mohammed A Alghamdi, A Legendre-Gauss collocation method for neutral functional-differential equations with proportional delays, Advances in Difference Equations, 63 (2013) 1 – 16.
[22] E. Tohidi, A.H. Bhrawy and K. Erfani, A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation. Applied Mathematical Modelling, 37 (2013) 4238-4294.
[23] Ayse Betul Koc, Musa Cakmak and Aydin Kurnaz, A Matrix Method Based on the Fibonacci Polynomials to the Generalized Pantograph Equations with Functional Arguments, Advances in Mathematical Physics, 2014 (2014) 1 – 5.
[24] Birol Ibis and Mustafa Bayram, Numerical solution of the neutral functional-differential equations with proportional delays via collocation method based on Hermite polynomials. Communication in Mathematical Modeling and Applications, 3 (2016) 22 – 30.