Heinz Quarter Mean Labeling of Graphs

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Abstract

Mean labeling is one of the best-known labeling methods for graphs. Despite the large number of papers published on the subject of graph labeling, there are some particular formulas to be used by researchers to mean-label graphs. In this paper, we introduced the concepts of Heinz Quarter Mean labeling graphs. Heinz Quarter Mean labeling for some graphs like Path, Cycle, Comb, Star graph and Complete Graph is proven as Heinz Quarter Mean Graphs.

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Keywords: Graph, Heinz – Quarter Mean Graph, Path, Cycle, Comb, Star Graph , Complete Graph.‡

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1. Introduction

Graph labeling started in 1960 by Ringel (1964) [1] and Rosa (1967)[2]. Ringel-Kotzig (1964, 1965) [3]conjecture that all trees are graceful. Rosa (1967) introduced $\boldsymbol{\beta}$ and α valuation and other labeling as a tool for the decomposition of complete graphs. Golomb (1972) [11] renamed the $\boldsymbol{\beta}$- valuation as graceful labeling which is widely used. Graham and Sloane (1980)[4] have introduced the concept of harmonious labeling on additive bases problem starting from error-correcting codes. The two basic labeling methods which are widely studied are graceful and harmonious labeling. In the wide field of graph labeling, α -valuation, Elegant labeling and cordial labeling are used entensively.

A. Rosa published a original paper on labeling problems of graphs in 1967. Graph labeling is the way of providing labels, which are mentioned by integers to edges or vertices of a graph. Labeled graphs help as for collection of applications like Radar, X-ray, Crystallography, Communication networks, Circuit design, and models for Constraint programming over finite domain.In the duration of years different labeling of graphs such as graceful, prime, cordial, total cordial, k-graceful and odd graceful labeling etc.

Researchers have introduced and explored many labeling schemes. Graph labelling is widely applied for various problem in mathematics, computer science and communication networks. Yegnanaryanan and Vaidhyanathan [12] have discussed various applications of graph labeling . Though it seems that graph labelling deals with theoretical study, a lot of research works have been carried out in applied fields also.

A detailed literature review of graph labeling, is presented in J.A. Gallian [5]. The standard terminogy and notations are available in Harary [6]. S. Somasundaram and R. Ponraj [7,10] have introduced Mean labeling in 2004. In 2012, the concept of Harmonic mean labeling was developed by S.Somasundaram and S.S. Sandhya introduced [8]. In [9], Bhatia introduced the Heinz mean $f(u,v;\ y)\ =\frac{f(u)^y f(v)^{1-y}+f(u)^{1-y}f(v)^y}{2}$, for $0\leq y\leq\ \frac{1}{2}$.

In graph theory, Path (P_n) is a single edge which is in between any of the two vertices. A closed path in which only the first and last vetices are equal is called as a Cycle (C_n). When the single pendant of each edge is connected to a path, then it is called as a Comb$(P_n \odot K_1)$**.** A Star $(K_{1,n})$ graph is defined as a complete bipartite graph. When each pair of distinct vertices of a graph is adjacent, then it is called as a Complete graph (G), denoted by K_n.

In this paper, we investigate the Heinz Quarter Mean Labeling of Path, Cycle, Comb, Star $K_{1,n}$ and Complete Graph K_n.

2. Main results

2.1 Heinz Mean Graphs

In this section, we first introduced Heinz Mean Graph using Heinz Mean [9]

Definition 2.1.1. A graph $G = (V, E)$ having number of vertices as p and number of edges as q can be called as **Heinz Mean graph,** if each vertex $x \in V$ can be labeled with distinct labels $f(x)$ from 1, 2, 3, $\ldots \ldots$, $q + 1$ and each edge $e = uv$ is labeled with $(e = uv) = \left\lfloor \frac{f(u)^y f(v)^{1-y} + f(u)^{1-y} f(v)^y}{2} \right\rfloor$ or $\left\lceil \frac{f(u)^y f(v)^{1-y} + f(u)^{1-y} f(v)^y}{2} \right\rceil$, with $0 \leq y \leq \frac{1}{2}$, which results in distinct edge labels. Here, f is a called Heinz Quarter mean labeling of G.

Remark 2.1.2. When we assume $y = 0$, Heinz Mean is called Heinz-0-Mean and it can be viewed as Arithmetic mean. Heinz-0-Mean : $H\left(f(u), f(v)\right) = \frac{f(u)+f(v)}{2}$ = A.M

When we assume $y = \frac{1}{2}$, Heinz Mean is called Heinz -$\frac{1}{2}$- Mean and it can be viewed as Geometric mean. Heinz -$\frac{1}{2}$- Mean : $H\left(f(u), f(v)\right) = \frac{2\sqrt{f(u)f(v)}}{2}$ = G.M

2.2 Heinz Quarter Mean Graphs

In this section, using Heinz Mean Graph we assume $y = \frac{1}{4}$, newly formed Heinz Quarter Mean Graph.

Definition 2.2.1. A graph $G = (V, E)$ having number of vertices as p and number of edges as q can be called as **Heinz Quarter Mean graph,** if the vertices $x \in V$ can be labeled with distinct labels $f(x)$ from 1, 2, 3, $\ldots \ldots$, $q + 1$ and each edge $e = uv$ is labeled with $(e{=}uv){=}\left\lfloor \frac{\sqrt[4]{f(u)f(v)}\left(\sqrt{f(u)}+\sqrt{f(v)}\right)}{2} \right\rfloor$ or $\left\lceil \frac{\sqrt[4]{f(u)f(v)}\left(\sqrt{f(u)}+\sqrt{f(v)}\right)}{2} \right\rceil$, then the resulting edge labels are distinct. Here, f is called Heinz Quarter mean labeling of G. This graph is said to Heinz Quarter Mean Graph.

Observation 2.2.2. If G is a Heinz Quarter mean graph, since one of the edge is getting label "1", then, one of the vertices should get the label "1".

Observation 2.2.3. If vertex u gets label 1, then any edges which are incident with u should get label 1 or 2 or 3.

The above statement we have provided seems to describe a labeling rule for the edges incident with a vertex u in a graph . Specifically it states that if vertex u is assigned the label 1 then any edges connected to u should be labeled with either 1, 2, or 3. Obviously the degree of this vertex $d(u)$ should be less than or equal to 3.

Theorem: 2.2.4. For a Heinz – Quarter mean graph G, the vertex are labeled from $1,2,\dots,q+1$, and the edges are labeled from $1,2,\dots,q$.

Proof . Assign the labels from the set $1,2,\dots.,q+1$ to the vertices of Heinz Quarter mean graph. Each vertex should be labeled with one of these numbers.

In a Heinz Quarter mean graph edges are labeled based on the difference between the labels of the vertices they connect. Specifically we assign the label from the set $(1,2,\dots,q)$ to each edge based on the absolute difference between the labels of its two end points.

For each edge connecting vertices labeled i and j, where $i < j$, assign the label $[i-j]$ to that edge. We do this for all edges in the graph.

Theorem 2.2.5. Every Path P_n is Heinz – Quarter mean graph.

Proof. Consider a path graph G, with number of vertices as n and number of edges as *n-1*, with vertices $u_1, u_2, u_3, \dots\dots, u_n$.

A function defined by $f: V(P_n) \to \{1,2,3,\dots\dots,q+1\}$ by $f(u_i) = i$, $1 \le i \le n$.

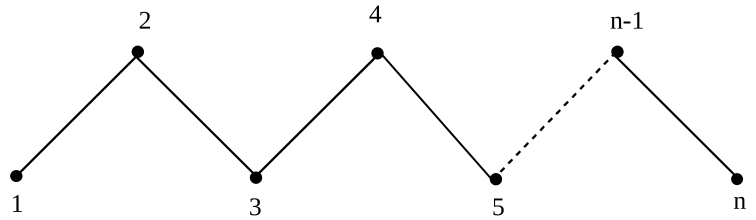

Figure 2.2.1 Path P_n

$f(u_1) = 1, f(u_2) = 2, f(u_3) = 3, \dots.., f(u_{n-1}) = n-1, f(u_n) = n$)

In view of the above defined labeling pattern f is the Heinz Quarter Mean labeling for the Path graph

Therefore, path P_n is Heinz – Quarter mean graph.

Theorem 2.2.6. For $n \ge 3$, all cycles C_n are Heinz-Quarter mean graphs.

Proof. Let G be a cycle graph of length *n*, with vertices $u_{1,}u_{2,}u_3,\dots\dots,u_n$

Consider $n = \begin{cases} 2m & if\ n = even \\ 2m+1 & if\ n = odd \end{cases}$

A function defined by $f: V(C_n) \to \{1,2,3,\dots\dots,q+1\}$

Where, $f(u_i) = \begin{cases} 2i-1 & 1 \le j \le m+1 & if\ n\ is\ even \\ 2i-1 & 1 \le j \le m & if\ n\ is\ odd \end{cases}$

$f(u_{m+j}) = \begin{cases} n-2j+2 & 2 \le j \le m & if\ n\ is\ even \\ n-2j+3 & 1 \le j \le m+1 & if\ n\ is\ odd \end{cases}$

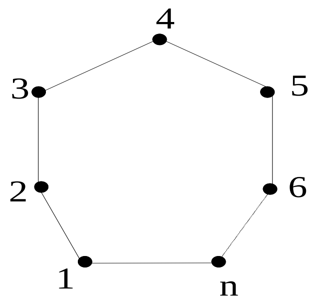

Figure 2.2.2 Cycle C_n

In view of the above defined labeling pattern, the edge labels are distinct.Hence f is the Heinz Quarter Mean labeling for the Cycle graph.

Hence Cycle C_n is a Heinz-Quarter mean graph.

Theorem 2.2.7. For all values of n, Comb $P_n \odot K_1$ is a Heinz - Quarter mean graph.

Proof. Let $P_n = u_1 u_2 u_3 \ldots \ldots u_n$ be a path and v_i be the vertices corresponds to every vertex u_i , where $(1 \leq i \leq n)$. Let $P_n \odot K_1$ be a graph obtained by the number of vertices be $2n$ and the number of edges be $2n-1$ in the graph G.

A function defined as $f: V(P_n \odot K_1) \rightarrow \{1, 2, 3, \ldots, q+1\}$

Where,

$$f(u_i) = 2i - 1, \ 1 \leq i \leq n$$

and $f(v_i) = 2i$, $1 \leq i \leq n$.

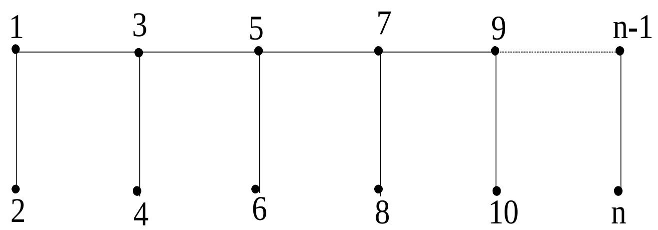

Figure 2.2.3 Comb ($P_6 \odot K_1$)

Here, $f(u_i u_{i+1}) = 2i$, $\quad 1 \leq i \leq n-1$

$f(u_i v_i) = 2i - 1$, $\quad 1 \leq i \leq n$, are the labels of edges.

In the above labeling pattern, we get the distinct edge labels .Hence $P_n \odot K_1$ is a Heinz-Quarter mean labeling. Therefore $P_n \odot K_1$ is a Heinz-Quarter mean graph.

Theorem: 2.2.8. Star $K_{1,n}$ is Heinz - Quarter mean graph if and only if $n \leq 5$.

Proof. Star $K_{1,1}$ and Star $K_{1,2}$ are same as P_2 and P_3 respectively. Clearly $K_{1,1}$ and $K_{1,2}$ are Heinz - Quarter Mean graphs. The central vertex of $K_{1,n}$ is considered as u and the remaining vertices are $v_1, v_2, v_3, \dots \dots, v_n$ respectively.

Case (i). The vertex u is assigned with label 3 and the vertex v_i is assigned with i $(1 \leq i \leq 2)$.

For $3 \leq i \leq n$, the vertex labels are $i + 1$. For the above, it is a Heinz - Quarter Mean Labeling, as shown below.

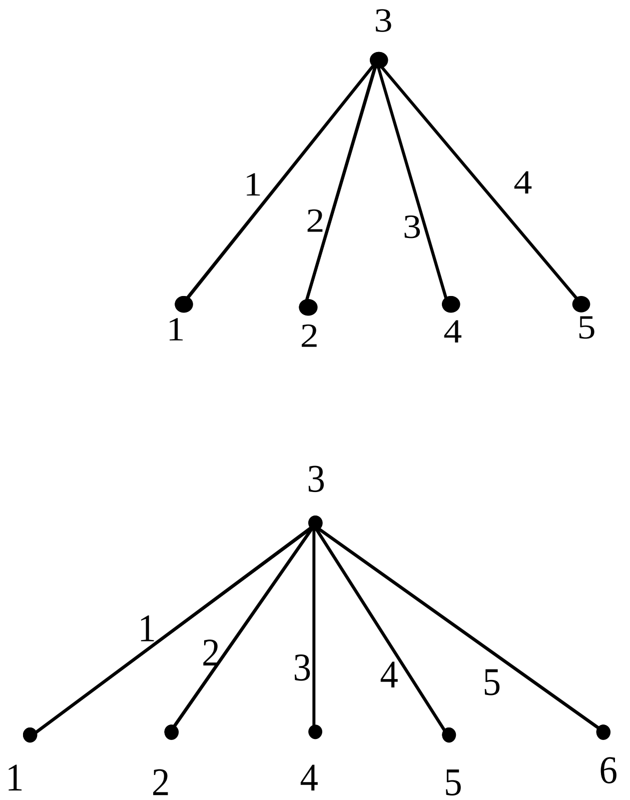

Figure 2.2.4 Star Graph $K_{1,4}$ and $K_{1,5}$

Case (ii) . Assume $n > 5$, if $K_{1,n}$ is Heinz - Quarter Mean labeling.

Here the following subcases are considered.

Subcase (ii) (a). Assuming that the central vertex u is labeled as 2.

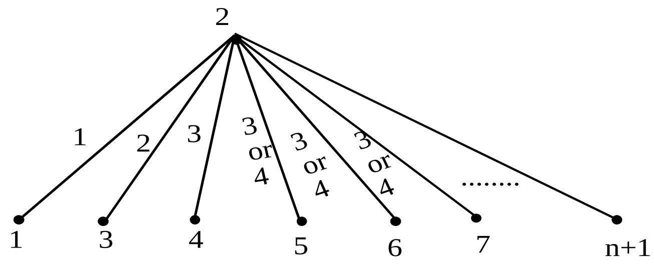

Figure 2.2.5 Star Graph $K_{1,n}$

The labeling of other vertices v_1, v_3, v_4, v_5, v_6 ..... are taken as 1, 3, 4, 5, 6, … respectively.

The edge labels of uv_4 is 3 and for uv_5 , uv_6 and uv_7 are 3 or 4 (repeated for all 3 edges), which is not possible. Because f is a injective function.

Subcase (ii) (b). Assuming that the central vertex is labeled as 4.

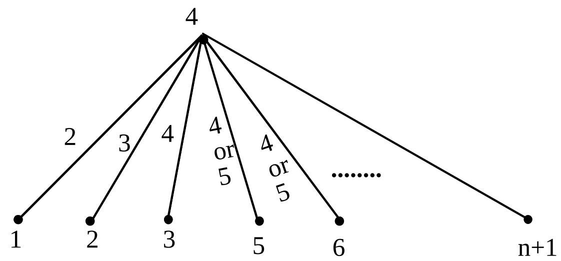

Figure 2.2.6 Star Graph $K_{1,n}$

The labeling of other vertices $v_1, v_2, v_3, v_5, v_6, v_7$ .... are taken as 1,2,3,5,6,7…respectively. Here , no edges are given label 1.

The edge labels of uv_3 , uv_5 and uv_6 are 4 , 4 and 5, respectively, which is not possible. Because the function f is injective.

From the above, it is concluded that $K_{1,n}$, $n > 5$ is not a Heinz - Quarter Mean Graph.

Theorem 2.2.9. The complete graph K_n is Heinz-Quarter mean graph if and only if $n \leq 4$.

Proof . The following cases are studied

Case (i). $n = 2$ *or* 3 *or* 4

By theorem 2.2.5 and 2.2.6 K_2 and K_3 are Heinz – Quarter mean graph.

Also K_4 is Heinz-Quarter mean graph.

The pattern of labeling K_2, K_3 and K_4 are shown in Figure 7.

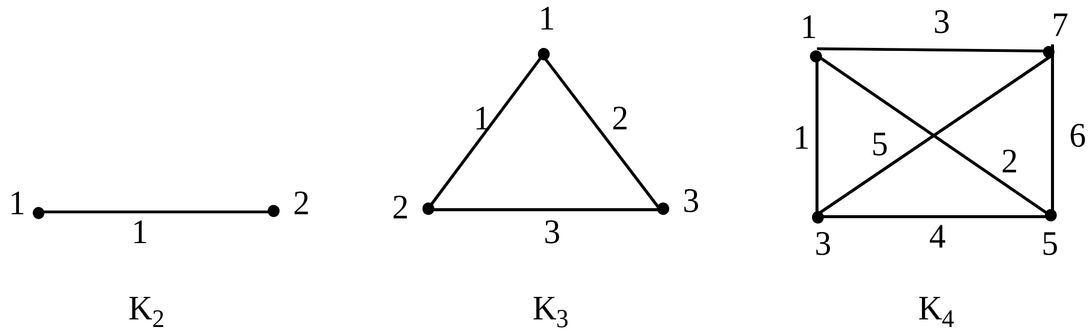

Figure 2.2.7 Complete Graph K_2, K_3 and K_4

Case (ii). $n > 4$

In order to obtain an edge label 1, a vertex u having label 1 is essential . There are 4 more vertices namely u_1 to u_4 are incident with u. By considering observation 2.2.2, this is not allowed.

Therefore, the complete graph K_n is not Heinz – Quarter mean graph if $n > 4$.

The following example gives the labeling pattern of K_5

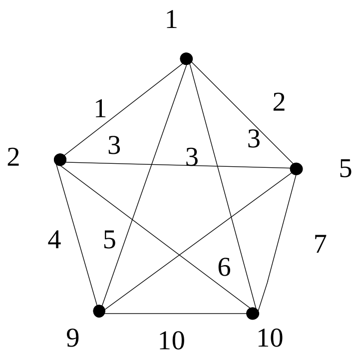

Figure 2.2.8 Complete Graph K_5

Hence K_n is Heinz mean graph if and only if $n \leq 4$

3. Conclusions

The Heinz Quarter mean labeling of various graphs, including Path, Cycle, Comb, Complete, and Star graphs, provides a unique and intriguing perspective on graph labeling schemes. The application of Heinz Quarter mean labeling involves assigning weights to the vertices of a graph based on the arithmetic mean of their distances from a designated set of vertices. This labeling scheme has been studied extensively for different graph classes.

In the case of Path graphs, Heinz Quarter mean labeling has been shown to yield interesting results, providing a systematic way to assign distinct labels to vertices. Similarly, for Cycle graphs, Comb graphs, Complete graphs, and Star graphs, the Heinz Quarter mean labeling presents a methodical approach to graph labeling, showcasing the versatility of this technique across various graph structures.

The exploration of Heinz Quarter mean labeling in different graph types contributes to the broader understanding of graph theory and labeling strategies. As researchers delve deeper into the properties and applications of this labeling scheme, it opens up avenues for further investigations into the mathematical characteristics and practical implications of Heinz Quarter mean labeled graphs.

Finally the study of Heinz Quarter mean labeling on Path, Cycle, Comb, Complete, and Star graphs enriches our understanding of graph labeling techniques, offering a systematic and structured approach for assigning labels to vertices in diverse graph structures. The insights gained from such investigations contribute to the ongoing development of graph theory and its applications in various fields.

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