

Common fixed point theorem for weakly compatible mappings in S_m metric space

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Abstract

In the present paper, at first, we study the structure of the newly S_m -metric space, which is a combination of S-metric space and multiplicative metric space. We have proved a common fixed point theorem for four self-maps in S_m metric space with a new contraction condition by applying the concepts of weakly compatible mappings, semi-compatible mappings, and reciprocally continuous mappings. Further, we also provide some examples to support our results.

Keywords: Multiplicative metric space, S-metric space, S_m -metric space, weakly compatible mappings, reciprocally continuous mappings, and semi-compatible mappings.

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1 Introduction

The notion of multiplicative metric space (MMS) was first developed by Bashirove [1]. Following that, several theorems came to light in this area of MMS [2] ,[3] and [4]. On the other side, Sedghi.S et al.[5] presented a new structure to S-metric space which modified D-metric and G-metric spaces, and then several fixed point theorems [6] and [7] were obtained. Pant et al. [8] generalized the notion of reciprocally continuous mapping which is weaker than continuous and compatible mappings. Recently, Mukesh Kumar Jain [9] introduced a more general form of semi-compatible mappings and proved many fixed point theorems in metric space.

In this article, we use a new generalized metric space referred to as S_m -metric space, which is a combination of both MMS and S -metric space. Using this concept, we establish a common fixed point theorem by applying weakly compatible mappings(WCM), reciprocally continuous mappings, and semi-compatible mappings. Furthermore, some examples are also discussed to support our conclusions.

2 Preliminaries:

Now we give some definitions and examples which are used in this theorem.

Definition 2.1. [1] “Let χ be a non-empty set and $\delta : \chi^2 \rightarrow \mathbb{R}^+$ be a multiplicative metric space (MMS) satisfying the properties :

- (i) $\delta(\psi, \phi) \geq 1$ and $\delta(\psi, \phi) = 1 \iff \psi = \phi$
- (ii) $\delta(\psi, \phi) = \delta(\phi, \psi)$
- (iii) $\delta(\psi, \phi) \leq \delta(\psi, \sigma)\delta(\sigma, \phi), \forall \psi, \phi, \sigma \in \chi.$ ”

Definition 2.2. [5] “ Let χ be a non-empty set defined $S : \chi^3 \rightarrow [0, \infty)$ satisfying:

- (i) $S(\psi, \phi, \sigma) \geq 0$
- (ii) $S(\psi, \phi, \sigma) = 0 \iff \psi = \phi = \sigma$
- (iii) $S(\psi, \phi, \sigma) \leq S(\psi, \psi, \rho) + S(\phi, \phi, \rho) + S(\sigma, \sigma, \rho), \forall \psi, \phi, \sigma, \rho \in \chi.$

A mapping S together with $\chi, (\chi, S)$ is called a S -metric space.”

Definition 2.3. [10] “ Let χ be a non-empty set .A function $S_m : \chi^3 \rightarrow \mathbb{R}^+$ satisfying the conditions :

S_m metric space

(i) $S_m(\psi, \phi, \sigma) \geq 1$

(ii) $S_m(\psi, \phi, \sigma) = 1 \iff \psi = \phi = \sigma$

(iii) $S_m(\psi, \phi, \sigma) \leq S_m(\psi, \psi, \rho)S_m(\phi, \phi, \rho)S_m(\sigma, \sigma, \rho), \forall \psi, \phi, \sigma, \rho \in \chi.$

The pair (χ, S_m) is called as S_m -metric space”.

Definition 2.4. [10] “ Let (χ, S_m) be a S_m -metric space, a sequence $\{\psi_\theta\} \in \chi$ is said to be

(i) *cauchy sequence* $\iff S_m(\psi_\theta, \psi_\theta, \psi_l) \rightarrow 1, \text{ for all } \theta, l \rightarrow \infty;$

(ii) *convergent* $\iff \exists \psi \in \chi$ such that $S_m(\psi_\theta, \psi_\theta, \psi) \rightarrow 1$ as $\theta \rightarrow \infty;$

(iii) *is complete if every cauchy sequence is convergent.”*

Definition 2.5. [11] ” Two self-maps M and K of a S_m metric space are said to be

(i) **Compatible:** if

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) = 1,$$

whenever there exist a sequence $\{\psi_\theta\} \in \chi$ such that

$$\lim_{\theta \rightarrow \infty} S_m(M\psi_\theta, K\psi_\theta, \omega) = 1 \text{ for some } \omega \in \chi.$$

(ii) **Weakly-compatible mappings:** if they commute at their coincidence points,

$$i.e. \omega \in \chi, S_m(M\omega, M\omega, K\omega) = 1, \implies S_m(MK\omega, MK\omega, KM\omega) = 1.”$$

Definition 2.6. [9] “Two self maps M and K of S_m -metric space are said to be **Semi-compatible:** if

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega) = 1$$

whenever there exists a sequence $\{\psi_\theta\} \in X$ such that

$$\lim_{\theta \rightarrow \infty} S_m(M\psi_\theta, K\psi_\theta, \omega) = 1 \text{ for all } \omega \in \chi.”$$

Now we present an example in which semi-compatible is weaker than compatible.

Example 2.6.1

Consider $\chi = [0, \infty)$ with $S_m(\psi, \phi, \sigma) = e^{|\psi-\phi|+|\phi-\sigma|+|\sigma-\psi|}$, for every $\psi, \phi, \sigma \in \chi$. Define two self maps M and K as

$$M(\psi) = \begin{cases} \frac{\cos^2(\pi\psi)+1}{2} & \text{if } 0 < \psi \leq \frac{1}{2}; \\ \sin(\pi\psi) & \text{if } \frac{1}{2} < \psi \leq 3. \end{cases}$$

and

$$K(\psi) = \begin{cases} \frac{2\sin(\pi\psi)-1}{2} & \text{if } 0 < \psi \leq \frac{1}{2}; \\ 1 - \sin(\pi\psi) & \text{if } \frac{1}{2} < \psi \leq 3. \end{cases}$$

Consider a sequence $\{\psi_\theta\}$ as $\psi_\theta = \{\frac{\pi}{2} - \frac{1}{\theta}\}$ for $\theta \geq 0$.

Then

$$\lim_{\theta \rightarrow \infty} M(\psi_\theta) = \lim_{\theta \rightarrow \infty} M\left(\frac{1}{2} - \frac{1}{\theta}\right) = \lim_{\theta \rightarrow \infty} \frac{\cos^2\pi\left(\frac{1}{2} - \frac{1}{\theta}\right) + 1}{2} = \lim_{\theta \rightarrow \infty} \frac{\sin^2\left(\frac{\pi}{\theta}\right) + 1}{2} = \frac{1}{2}$$

and

$$\lim_{\theta \rightarrow \infty} K(\psi_\theta) = \lim_{\theta \rightarrow \infty} K\left(\frac{1}{2} - \frac{1}{\theta}\right) = \lim_{\theta \rightarrow \infty} \frac{2\sin\pi\left(\frac{1}{2} - \frac{1}{\theta}\right) - 1}{2} = \lim_{\theta \rightarrow \infty} \frac{2\cos\left(\frac{\pi}{\theta}\right) - 1}{2} = \frac{1}{2}.$$

$$\text{Therefore } \lim_{\theta \rightarrow \infty} M\psi_\theta = \lim_{\theta \rightarrow \infty} K\psi_\theta = \frac{1}{2} = \omega \text{ (say).}$$

Now

$$\lim_{\theta \rightarrow \infty} MK(\psi_\theta) = \lim_{\theta \rightarrow \infty} M\left(\frac{2\cos\frac{\pi}{\theta} - 1}{2}\right) = \lim_{\theta \rightarrow \infty} \frac{\cos^2\pi\left(\frac{2\cos\frac{\pi}{\theta}-1}{2}\right) + 1}{2} = \frac{\cos^2\frac{\pi}{2} + 1}{2} = \frac{1}{2}$$

and

$$\lim_{\theta \rightarrow \infty} KM(\psi_\theta) = \lim_{\theta \rightarrow \infty} K\left(\frac{\sin^2\frac{\pi}{\theta} + 1}{2}\right) = \lim_{\theta \rightarrow \infty} \left[1 - \sin\pi\left(\frac{\sin^2\frac{\pi}{\theta} + 1}{2}\right)\right] = 0.$$

$$\therefore \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) \neq 0.$$

This implies these two self-maps M and K are not compatible.

But $K(\omega) = K\left(\frac{1}{2}\right) = \frac{1}{2}$.

$$\text{Therefore } \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega) = \lim_{\theta \rightarrow \infty} S_m\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 1.$$

Hence these two self maps M and K are semi-compatible but not compatible.

S_m metric space

Definition 2.7. [8] “Two self-maps M, K of S_m -metric space are said to be **reciprocally continuous** if

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega) = 1 \text{ and } \lim_{\theta \rightarrow \infty} S_m(KM\psi_\theta, KM\psi_\theta, K\omega) = 1,$$

whenever there exist a sequence $\{\psi_\theta\} \in \chi$ such that

$$\lim_{\theta \rightarrow \infty} S_m(M\psi_\theta, K\psi_\theta, \omega) = 1 \text{ some } \omega \in \chi.”$$

Now we present an example in which satisfies reciprocally continuous is weaker but not compatible.

Example 2.7.1 Consider $\chi = (0, \infty)$ with $S_m(\psi, \phi, \sigma) = e^{|\psi-\phi|+|\phi-\sigma|+|\sigma-\psi|}$, for every $\psi, \phi, \sigma \in \chi$. Define two self maps M and K as

$$M(\psi) = \begin{cases} \psi^2 + 2 & \text{if } 0 < \psi \leq 1; \\ 4 - \psi & \text{if } 1 < \psi \leq 3. \end{cases}$$

and

$$K(\psi) = \begin{cases} 1 - 2\psi & \text{if } 0 < \psi \leq 1; \\ \psi - 2 & \text{if } 1 < \psi \leq 3. \end{cases}$$

Consider a sequence $\{\psi_\theta\}$ as $\psi_\theta = \{3 - \frac{1}{\theta}\}$, for $\theta \geq 0$.

Now

$$\lim_{\theta \rightarrow \infty} M(\psi_\theta) = \lim_{\theta \rightarrow \infty} [4 - (3 - \frac{1}{\theta})] = 1 \text{ and } \lim_{\theta \rightarrow \infty} K(\psi_\theta) = \lim_{\theta \rightarrow \infty} [(3 + \frac{1}{\theta}) - 2] = 1$$

$$\therefore \lim_{\theta \rightarrow \infty} M\psi_\theta = \lim_{\theta \rightarrow \infty} K\psi_\theta = 1 = \omega_1 \neq \phi.$$

Also

$$\lim_{\theta \rightarrow \infty} MK(\psi_\theta) = \lim_{\theta \rightarrow \infty} M[(3 - \frac{1}{\theta}) - 2] = \lim_{\theta \rightarrow \infty} M(1 - \frac{1}{\theta}) = 3$$

and

$$\lim_{\theta \rightarrow \infty} KM(\psi_\theta) = \lim_{\theta \rightarrow \infty} K(4 - (3 - \frac{1}{\theta})) = \lim_{\theta \rightarrow \infty} K(1 + \frac{1}{\theta}) = -1.$$

$$\therefore \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) = S_m(3, 3, -1) \neq 1.$$

This gives the self maps M and K are not compatible in S_m - metric space.

Moreover, $M(\omega_1) = 3$ and $K(\omega_1) = -1$.

Which gives

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega_1) = S_m(3, 3, 3) = 1,$$

and

$$\lim_{\theta \rightarrow \infty} S_m(KM\psi_\theta, KM\psi_\theta, K\omega_1) = S_m(-1, -1, -1) = 1.$$

This implies the self-maps M and K are reciprocally continuous but not compatible in S_m metric space.

Now we proceed to the main theorem.

3 Main Theorem

Theorem 3.1. *Let M, H, K, and J be self-mapping of a complete S_m -metric space satisfying the following*

$$(3.1.1) \quad M(\chi) \subseteq J(\chi) \text{ and } H(\chi) \subseteq K(\chi)$$

$$(3.1.2)$$

$$S_m(M\psi, M\psi, H\phi) \leq \left\{ \begin{array}{l} \max[S_m(M\psi, M\psi, K\psi)S_m(H\phi, H\phi, J\phi), \\ S_m(M\psi, M\psi, J\phi)S_m(K\psi, K\psi, H\phi), \\ S_m(M\psi, M\psi, J\phi)S_m(H\phi, H\phi, J\phi), \\ S_m(M\psi, M\psi, K\psi)S_m(H\phi, H\phi, K\psi)] \end{array} \right\}^\lambda$$

where $\lambda \in (0, \frac{1}{2})$

$$(3.1.3) \quad \text{the pair M and K are reciprocally continuous and semi-compatible,}$$

$$(3.1.4) \quad \text{the pair H and J are weakly compatible.}$$

Then the self-maps M, H, K, and J have a unique common fixed point in χ .

Proof:

Let there is a point $\psi_0 \in \chi$, and the sequence $\{\psi_\theta\}$ be defined as $M\psi_0 = J\psi_1 = \phi_0$. For this point ψ_1 then there exists $\psi_2 \in \chi$ such that $H\psi_1 = K\psi_2 = \phi_1$. In general, by induction choose $\psi_{\theta+1}$, construct a sequence $\{\phi_\theta\} \in \chi$ such that

$$\phi_{2\theta} = M\psi_{2\theta} = J\psi_{2\theta+1} \text{ and } \phi_{2\theta+1} = H\psi_{2\theta+1} = K\psi_{2\theta+2}, \text{ for } \theta \geq 0.$$

S_m metric space

On putting $\psi = \psi_{2\theta}$ and $\phi = \phi_{2\theta+1}$ in (3.1.2) we get.

$$\begin{aligned} S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) &= S_m(M\psi_{2\theta}, M\psi_{2\theta}, H\psi_{2\theta+1}) \\ &\leq \max \left\{ S_m(M\psi_{2\theta}, M\psi_{2\theta}, \theta\psi_{2\theta}) S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, J\psi_{2\theta+1}), \right. \\ &\quad S_m(M\psi_{2\theta}, M\psi_{2\theta}, J\psi_{2\theta+1}) S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, \theta\psi_{2\theta}), \\ &\quad S_m(M\psi_{2\theta}, M\psi_{2\theta}, J\psi_{2\theta+1}) S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, J\psi_{2\theta+1}), \\ &\quad \left. S_m(M\psi_{2\theta}, M\psi_{2\theta}, K\psi_{2\theta}) S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, K\psi_{2\theta}) \right\}^\lambda \end{aligned}$$

$$\begin{aligned} S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) &\leq \max \left\{ S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta-1}) S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta}), \right. \\ &\quad S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta}) S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta-1}), \\ &\quad S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta}) S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta}), \\ &\quad \left. S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta-1}) S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta-1}) \right\}^\lambda \end{aligned}$$

this implies that

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta+1})^\lambda.$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq \{S_m(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}) S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1})\}^\lambda.$$

$$S_m^{1-\lambda}(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m^\lambda(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}).$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m^{\frac{\lambda}{1-\lambda}}(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}).$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m^p(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}). \text{ where } p = \frac{\lambda}{1-\lambda}.$$

Now this gives

$$S_m(\phi_\theta, \phi_\theta, \phi_{\theta+1}) \leq S_m^p(\phi_{\theta-1}, \phi_{\theta-1}, \phi_\theta) \leq S_m^{p^2}(\phi_{\theta-2}, \phi_{\theta-2}, \phi_{\theta-1}) \leq \cdots S_m^{p^n}(\phi_0, \phi_0, \phi_n).$$

By using triangular inequality

$$S_m(\phi_\theta, \phi_\theta, \phi_n) \leq S_m^{p^\theta}(\phi_0, \phi_0, \phi_l) \leq S_m^{p^{\theta+1}}(\phi_0, \phi_0, \phi_n) \leq \cdots S_m^{p^{n-1}}(\phi_0, \phi_0, \phi_n)$$

$$S_m(\phi_\theta, \phi_\theta, \phi_n) \leq S_m^{\frac{p^\theta}{1-p}}(\phi_0, \phi_0, \phi_l) \text{ for all } \theta \geq 1.$$

Hence $\{\phi_\theta\}$ is a cauchy sequence in S_m -metric space.

Since the self-maps, M and K are weakly reciprocally continuous.

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega) = 1 \text{ or } \lim_{\theta \rightarrow \infty} S_m(KM\psi_\theta, KM\psi_\theta, \theta\omega) = 1. \quad (1)$$

Also, the pair (M, K) is semi compatible, we have

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega) = 1. \quad (2)$$

From (1) and (2) we get

$$S_m(M\omega, M\omega, K\omega) = 1. \quad (3)$$

Since $M(\chi) \subseteq J(\chi)$ which gives then there exists $\nu \in \chi$ such that $J\nu = M\psi_\theta$, since $M\psi_\theta \rightarrow \omega$ as $\theta \rightarrow \infty$. Which implies

$$S_m(J\nu, J\nu, \omega) = 1. \quad (4)$$

Now, we have to prove $S_m(J\nu, H\nu, \omega) = 1$.

Substitute $\psi = \psi_\theta$ and $\phi = \nu$ in (3.1.2) we have

$$S_m(M\psi_\theta, M\psi_\theta, H\nu) \leq \left\{ \begin{array}{l} \max[S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\nu, H\nu, J\nu), \\ S_m(M\psi_\theta, M\psi_\theta, J\nu)S_m(K\psi_1, K\psi_1, H\nu), \\ S_m(M\psi_\theta, M\psi_\theta, J\nu)S_m(H\nu, H\nu, J\nu), \\ S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\nu, H\nu, K\psi_\theta)] \end{array} \right\}^\lambda$$

$$S_m(\omega, \omega, H\nu) \leq \left\{ \begin{array}{l} \max[S_m(\omega, \omega, \omega)S_m(H\nu, H\nu, \omega), S_m(\omega, \omega, \omega)S_m(\omega, \omega, H\nu), \\ S_m(\omega, \omega, \omega)S_m(H\nu, H\nu, \omega), S_m(\omega, \omega, \omega)S_m(H\nu, H\nu, \omega)] \end{array} \right\}^\lambda$$

$$S_m(\omega, \omega, H\nu) \leq \{S_m(\omega, \omega, H\nu)\}^\lambda$$

$$S_m^{(1-\lambda)}(\omega, \omega, H\nu) \leq 1 \implies S_m(H\nu, H\nu, \omega) = 1.$$

$$\therefore S_m(J\nu, H\nu, \omega) = 1.$$

S_m metric space

Since the pair $(H.J)$ is WCM and ν is a coincidence point then $HJ\nu = JH\nu$

$$S_m(H\omega, H\omega, J\omega) = 1. \quad (5)$$

Substitute $\psi = \psi_\theta$ and $\phi = \omega$ in (3.1.2) we have

$$S_m(M\psi_\theta, M\psi_\theta, H\omega) \leq \left\{ \begin{array}{l} \max[S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\omega, H\omega, J\omega), \\ S_m(M\psi_\theta, M\psi_\theta, J\omega)S_m(K\psi_\theta, K\psi_\theta, H\omega), \\ S_m(M\psi_\theta, M\psi_\theta, J\omega)S_m(H\omega, H\omega, J\omega), \\ S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\omega, H\omega, K\psi_\theta)] \end{array} \right\}^\lambda$$

also

$$S_m(H\omega, \omega, \omega) \leq \left\{ \begin{array}{l} \max[S_m(\omega, \omega, \omega)S_m(H\omega, H\omega, \omega), S_m(\omega, \omega, \omega)S_m(\omega, \omega, H\omega), \\ S_m(\omega, \omega, \omega)S_m(H\omega, H\omega, \omega), S_m(\omega, \omega, \omega)S_m(H\omega, H\omega, \omega)] \end{array} \right\}^\lambda$$

and this gives

$$S_m(H\omega, \omega, \omega) \leq S_m(H\omega, \omega, \omega)^\lambda$$

$$S_m^{(1-\lambda)}(H\omega, \omega, \omega) \leq 1 \implies H\omega = \omega$$

$$\therefore S_m(H\omega, J\omega, \omega) = 1. \quad (6)$$

Replace $\psi = \omega$ and $\phi = \nu$ in (3.1.2) then we have

$$S_m(M\omega, M\omega, H\nu) \leq \left\{ \begin{array}{l} \max[S_m(M\omega, M\omega, K\omega)S_m(J\nu, H\nu, H\nu), \\ S_m(M\omega, M\omega, J\nu)S_m(K\omega, K\omega, H\nu), \\ S_m(M\omega, M\omega, J\nu)S_m(J\nu, J\nu, H\nu), \\ S_m(M\omega, M\omega, K\omega)S_m(H\nu, H\nu, K\omega)] \end{array} \right\}^\lambda$$

$$S_m(M\omega, M\omega, \omega) \leq \left\{ \begin{array}{l} \max[S_m(M\omega, M\omega, M\omega)S_m(\omega, \omega, \omega), \\ S_m(M\omega, M\omega, \omega)S_m(M\omega, M\omega, \omega), \\ S_m(M\omega, M\omega, \omega)S_m(\omega, \omega, \omega), \\ S_m(M\omega, M\omega, M\omega)S_m(\omega, \omega, M\omega)] \end{array} \right\}^\lambda$$

$$S_m(M\omega, M\omega, \omega) \leq \{S_m(M\omega, M\omega, \omega)\}^\lambda$$

$$S_m^{(1-\lambda)}(M\omega, M\omega, \omega) \leq 1 \implies M\omega = \omega$$

$$\therefore S_m(M\omega, J\omega, \omega) = 1. \quad (7)$$

From (6) and (7) we get

$$M\omega = J\omega = H\omega = K\omega = \omega. \quad (8)$$

Therefore “ ω ” is a common fixed point of M , H , K , and J .

Uniqueness

Let ρ be one more fixed point, we assume that $\rho \neq \omega$ then we have

$$M\rho = K\rho = H\rho = J\rho = \rho.$$

In the condition (3.1.2) put $\psi = \omega$ and $\phi = \rho$ we get

$$S_m(M\omega, M\omega, H\rho) \leq \left\{ \begin{array}{l} \max[S_m(M\omega, M\omega, K\omega)S_m(H\rho, H\rho, J\rho), \\ S_m(M\omega, M\omega, J\rho)S_m(K\omega, K\omega, H\rho), \\ S_m(M\omega, M\omega, J\rho)S_m(H\rho, H\rho, J\rho), \\ S_m(M\omega, M\omega, K\omega)S_m(H\rho, H\rho, K\omega)] \end{array} \right\}^\lambda$$

$$S_m(\omega, \omega, \rho) \leq$$

$$\left\{ \begin{array}{l} \max[S_m(\omega, \omega, \omega)S_m(\rho, \rho, \rho), S_m(\omega, \omega, \rho)S_m(\omega, \omega, \rho), \\ S_m(\omega, \omega, \rho)S_m(\rho, \rho, \rho), S_m(\omega, \omega, K\omega)S_m(\rho, \rho, \omega)] \end{array} \right\}^\lambda$$

$$S_m(\omega, \omega, \rho) \leq \left\{ S_m(\omega, \omega, \rho) \right\}^\lambda$$

this implies that $S_m(\omega, \omega, \rho) = 1 \implies \omega = \rho$.

This shows that “ ω ” is the unique common fixed point of $M.H.J$ and K .

S_m metric space

Now, the following example substantiates our theorem.

Example 3.2

Suppose $\chi = (0, 1)$, S_m - metric space by $S_m(\psi, \phi, \sigma) = e^{|\psi-\phi|+|\phi-\sigma|+|\sigma-\psi|}$,

when $\psi, \phi, \sigma \in \chi$. Define $M, K, H, J: \chi \times \chi \times \chi \rightarrow \chi$ as follows

$$M(\psi) = \begin{cases} \frac{2-\psi}{5} & \text{if } 0 < \psi \leq \frac{1}{3}; \\ \psi & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

$$K(\psi) = \begin{cases} 1 - 2\psi & \text{if } 0 < \psi \leq \frac{1}{3}; \\ \frac{1+\psi}{2} & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

$$H(\psi) = \begin{cases} 3\psi^2 - 3\psi + 1 & \text{if } 0 < \psi \leq \frac{1}{3}; \\ \frac{2+\psi}{7} & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

$$J(\psi) = \begin{cases} 1 - 6\psi^2 & \text{if } 0 < \psi \leq \frac{1}{3}; \\ 1 - \psi & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

Then $M(\chi) = (\frac{1}{3}, 1] \subseteq J(\chi) = (0, 1]$ and $H(\chi) = (\frac{1}{3}, 1] \subseteq K(\chi) = (\frac{1}{3}, 1]$.

Therefore the condition (3.1.1) holds.

Consider a sequence $\{\psi_\theta\}$ as $\psi_\theta = \{\frac{1}{3} - \frac{1}{\theta}\}$ as $\theta \geq 0$.

$$\text{Then } \lim_{\theta \rightarrow \infty} M(\psi_\theta) = \lim_{\theta \rightarrow \infty} M\left(\frac{1}{3} - \frac{1}{\theta}\right) = \lim_{\theta \rightarrow \infty} \frac{2 - (\frac{1}{3} - \frac{1}{\theta})}{5} = \frac{1}{3}$$

and

$$\lim_{\theta \rightarrow \infty} K(\psi_\theta) = \lim_{\theta \rightarrow \infty} K\left(\frac{1}{3} - \frac{1}{\theta}\right) = \lim_{\theta \rightarrow \infty} [1 - 2(\frac{1}{3} - \frac{1}{\theta})] = \frac{1}{3}.$$

$$\text{Therefore } \lim_{\theta \rightarrow \infty} M(\psi_\theta) = \lim_{\theta \rightarrow \infty} K(\psi_\theta) = \frac{1}{3} = \omega_1.$$

Further

$$\lim_{\theta \rightarrow \infty} H(\psi_\theta) = \lim_{\theta \rightarrow \infty} H\left(\frac{1}{3} - \frac{1}{\theta}\right) = \lim_{\theta \rightarrow \infty} [3(\frac{1}{3} - \frac{1}{\theta})^2 - 3(\frac{1}{3} - \frac{1}{\theta}) + 1] = \frac{1}{3}$$

and

$$\lim_{\theta \rightarrow \infty} J(\psi_\theta) = \lim_{\theta \rightarrow \infty} J\left(\frac{1}{3} - \frac{1}{\theta}\right) = \lim_{\theta \rightarrow \infty} [1 - 6(\frac{1}{3} - \frac{1}{\theta})^2] = \frac{1}{3}.$$

$$\text{Therefore } \lim_{\theta \rightarrow \infty} H(\psi_\theta) = \lim_{\theta \rightarrow \infty} J(\psi_\theta) = \frac{1}{3} = \omega_1.$$

Moreover

$$\lim_{\theta \rightarrow \infty} MK(\psi_\theta) = \lim_{\theta \rightarrow \infty} M[1 - (\frac{2}{3} - \frac{2}{\theta})] = \lim_{\theta \rightarrow \infty} M(\frac{1}{3} + \frac{2}{\theta}) = \frac{1}{3}$$

and

$$\lim_{\theta \rightarrow \infty} KM(\psi_\theta) = \lim_{\theta \rightarrow \infty} K(\frac{1}{3} + \frac{1}{5\theta}) = \lim_{\theta \rightarrow \infty} \frac{1 + 2(\frac{1}{3} + \frac{1}{5\theta})}{2} = \frac{2}{3}.$$

$$\therefore \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}) \neq 1$$

which implies that the pair (M, K) is not compatible.

Furthermore

$$\lim_{\theta \rightarrow \infty} HJ(\psi_\theta) = \lim_{\theta \rightarrow \infty} H(\frac{1}{3} + \frac{4}{\theta} - \frac{1}{\theta^2}) = \lim_{\theta \rightarrow \infty} (\frac{2 + (\frac{1}{3} + \frac{4}{\theta} - \frac{1}{\theta^2})}{7}) = \frac{1}{3}$$

and

$$\lim_{\theta \rightarrow \infty} JH(\psi_\theta) = \lim_{\theta \rightarrow \infty} J(\frac{1}{3} + \frac{4}{\theta} - \frac{1}{\theta^2}) = \lim_{\theta \rightarrow \infty} [1 - (\frac{1}{3} + \frac{4}{\theta} - \frac{1}{\theta^2})] = \frac{2}{3}$$

$$\text{Therefore } \lim_{\theta \rightarrow \infty} S_m(HJ\psi_\theta, HJ\psi_\theta, JH\psi_\theta) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}) \neq 1.$$

Which shows that the pair(H,J) is not compatible .

$$\text{Also } M(\frac{1}{3}) = \frac{1}{3}, K(\frac{1}{3}) = \frac{1}{3}.$$

$$\text{This implies } \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega_1) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$$

$$\text{and } \lim_{\theta \rightarrow \infty} S_m(KM\psi_\theta, KM\psi_\theta, K\omega_1) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1.$$

This shows that the pair (M, K) is reciprocally continuous in S_m metric space.

$$\text{Also } \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega_1) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1.$$

This shows that the pair (M, K) is semi-compatible in S_m metric space.

Hence the inequality (3.1.3) holds.

Further

$$S_m(H(\frac{1}{3}), J(\frac{1}{3}), \frac{1}{3}) = 1 \text{ and } S_m(HJ(\frac{1}{3}), JH(\frac{1}{3}), \frac{1}{3}) = 1.$$

S_m metric space

This implies that $S_m(HJ(\frac{1}{3}), HJ(\frac{1}{3}), JH(\frac{1}{3})) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$.
Which indicates that the pair (H, J) is weakly compatible.

Now, we prove the condition (3.1.2) in various cases

CASE-I

Let $\psi, \phi \in [0, \frac{1}{2}]$, while we have $S_m(\psi, \phi, \sigma) = e^{|\psi-\sigma|+|\phi-\sigma|}$.

Take $\psi = \frac{1}{4}$ and $\phi = \frac{1}{5}$ then $M(\frac{1}{4}) = \frac{7}{20}$, $K(\frac{1}{4}) = \frac{1}{2}$, $H(\frac{1}{5}) = \frac{13}{25}$ and $J(\frac{1}{5}) = \frac{19}{25}$
substitute the above values in (3.1.2)

$$S_m(\frac{7}{20}, \frac{7}{20}, \frac{13}{25}) \leq \left\{ \max[S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{13}{25}, \frac{13}{25}, \frac{19}{25}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{19}{25})S_m(\frac{13}{25}, \frac{13}{25}, \frac{1}{2}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{19}{25})S_m(\frac{13}{25}, \frac{13}{25}, \frac{19}{25}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{13}{25}, \frac{13}{25}, \frac{1}{2})] \right\}^\lambda$$

$$we\ have\ e^{0.34} \leq \left\{ \max[e^{0.3}e^{0.48}, e^{0.82}e^{0.34}, e^{0.3}e^{0.04}, e^{0.82}e^{0.48}] \right\}^\lambda$$

$$e^{0.34} \leq \{ \max[e^{0.78}, e^{1.16}, e^{0.034}, e^{1.3}] \}^\lambda \implies e^{0.34} \leq e^{1.16\lambda}$$

which gives $\lambda = 0.2$ where $\lambda \in (0, \frac{1}{3})$.

CASE-II

Let $\psi, \phi \in (\frac{1}{2}, 1]$, then $S_m(\psi, \phi, \sigma) = e^{|\psi-\sigma|+|\phi-\sigma|}$.

Take $\psi = \frac{1}{2}$ and $\phi = \frac{1}{2}$ then $M(\frac{1}{2}) = \frac{1}{2}$, $K(\frac{1}{2}) = \frac{3}{4}$, $H(\frac{1}{2}) = \frac{5}{14}$ and $J(\frac{1}{2}) = \frac{3}{4}$
substitute the above values in (3.1.2)

$$S_m(\frac{1}{2}, \frac{1}{2}, \frac{5}{14}) \leq \left\{ \max[S_m(\frac{1}{2}, \frac{1}{2}, \frac{3}{4})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{3}{4}), S_m(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{1}{2}, \frac{1}{2}, \frac{3}{4})S_m(\frac{5}{14}, \frac{5}{14}, \frac{3}{4})] \right\}^\lambda$$

which implies that

$$e^{0.285} \leq \left\{ \max[e^{0.5}e^{0.285}, e^{0.0}e^{0.786}, e^{0.0}e^{0.28}, e^{0.5}e^{0.786}] \right\}^\lambda$$

$$e^{0.285} \leq \{ \max[e^{0.785}, e^{0.786}, e^{0.28}, e^{1.286}] \}^\lambda \implies e^{0.285} \leq e^{1.286\lambda}$$

which gives $\lambda = 0.22$ where $\lambda \in (0, \frac{1}{2})$.

CASE-III

Let $\psi, \phi \in (\frac{1}{2}, 1]$, then $S_m(\psi, \phi, \sigma) = e^{|\psi-\sigma|+|\phi-\sigma|}$

Take $\psi = \frac{1}{4}$ and $\phi = \frac{1}{2}$ then $M(\frac{1}{4}) = \frac{7}{20}$, $K(\frac{1}{4}) = \frac{1}{2}$, $H(\frac{1}{5}) = \frac{5}{14}$ and $J(\frac{1}{5}) = \frac{1}{2}$ substitute the above values in **(3.1.2)**

$$S_m\left(\frac{7}{20}, 0, \frac{7}{20}, \frac{5}{14}\right) \leq \left\{ \max\left[S_m\left(\frac{7}{20}, \frac{7}{20}, \frac{1}{2}\right)S_m\left(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}\right), S_m\left(\frac{7}{20}, \frac{7}{20}, \frac{1}{2}\right)S_m\left(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}\right), S_m\left(\frac{7}{20}, \frac{7}{20}, \frac{1}{2}\right)S_m\left(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}\right)\right] \right\}^\lambda$$

which implies that

$$e^{0.014} \leq \left\{ \max[e^{0.3}e^{0.28}, e^{0.3}e^{0.28}, e^{0.3}e^{0.28}, e^{0.3}e^{0.28}] \right\}^\lambda$$

$$e^{0.014} \leq \{ \max[e^{0.58}, e^{0.58}, e^{0.58}, e^{0.58}] \}^\lambda \implies e^{0.014} \leq e^{0.5.8\lambda}$$

this gives that $\lambda = 0.14$ where $\lambda \in (0, \frac{1}{2})$.

Hence the inequality **(3.3.2)** holds.

It can be seen that “ $\frac{1}{2}$ ” is a unique common fixed point for four self mappings M, K H, and J.

4 Conclusions

In this article, we established a common fixed point theorem in S_m -metric space by using weakly-compatible mappings, semi-compatible mappings, and reciprocally continuous mappings for four self-maps. Furthermore, our results are also justified with suitable examples.

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