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Common fixed points of self-maps over the generalized cone 𝕓-metric spaces

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Abstract

The main aim of this research paper is to establish the most generalized common fixed-point theorem for two self-maps that are commuting to each other under $\mathfrak{T}$-Kannan type contractive condition over a generalized cone 𝕓-metric spaces. The novelty of this research paper is to find a common fixed point of two weekly compatible self-maps over a generalized cone 𝕓-metric spaces without assuming the normality condition of a cone. We illustrate our main result with a suitable example.

Keywords: weekly compatible self-maps; cone; common fixed points; normality condition of a cone; generalized 𝕓-cone metric space.

2010 AMS subject classification: 54H25, 47H10.[†]

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1. Introduction

M. Frechet [6] introduced the metric spaces. It is well known that the Banach Contraction Principle [2] is the first fixed point theorem in metric spaces. A. Branciari [4] introduced the generalized (rectangular) metric spaces by replacing the triangular inequality in the definition of metric spaces with a similar one that involves four or more points instead of three points. I. A. Bakhtin [1] defined b-metric spaces by replacing triangular inequality in a metric space with a 𝕓-triangular inequality as a generalization of metric spaces. L.G. Huang et al. [7] defined cone metric spaces which are obtained by replacing the set of real numbers in a metric mapping with a Real Banach space. A. Azam et al. [1] defined cone rectangular metric space by replacing triangular inequality in cone metric space with a rectangular inequality. N. Hussain et al. [9] defined cone 𝕓-metric spaces by replacing triangular inequality in cone metric spaces with a 𝕓-triangular inequality as a generalization of cone metric spaces. The authors improved the Banach Contraction Principle in such spaces. Recently, many authors ([5], [8], [14]) established some fixed-point theorems satisfying some generalized contractive conditions in cone 𝕓-metric spaces over Banach algebras.

Reny George et al. [12] defined generalized cone 𝕓-metric spaces by replacing 𝕓-triangular inequality in a cone 𝕓-metric space with a 𝕓-rectangular inequality and proved basic Banach fixed point theorem on these spaces. Thereafter some fixed-point theorems ([10], [11], [13]) proved under various contractive type conditions.

The main objective of this article is to establish a generalized common fixed-point theorem for two self-maps that are commuting to each other under the $\mathfrak{T}$**-**Kannan type contractive condition over a generalized cone 𝕓-metric space. We present an example to illustrate our result. Our main result generalizes many results in generalized cone 𝕓-metric spaces without assuming the normality condition of a cone.

We recall some basic definitions and lemmas required in section 2 and establish main result in section 3 with suitable example.

2. Preliminaries

Definition 2.1 Let $\dot{X}$ be a non-empty set. An element $\dot{x} \in \dot{X}$ is said to be a fixed point of a self-map $\mathfrak{T}: \dot{X} \to \dot{X}$ if $\mathfrak{T}(\dot{x}) = \dot{x}$.
Definition 2.2 A result giving a set of conditions on $\mathfrak{T}$ and $\dot{X}$ under which $\mathfrak{T}$ has a fixed point is known as a fixed-point theorem.
Definition 2.3 (L.G. Huang and X. Zhang. [7]). If $\hat{\mathbf{E}}$ is a real Banach space and $\widehat{P} \subseteq \hat{E}$**,** then $\widehat{P}$ is said to be a cone if it satisfies the following conditions:
(C1) $\widehat{P}$ is a closed and nonempty subset of $\hat{\mathbf{E}}$ with $\widehat{P} \neq \{\hat{\mathbf{0}}\}$;
(C2) If $c_1, c_2 \geq 0$ and $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \in \widehat{P}$**,** then $\boldsymbol{c_1}\hat{\boldsymbol{x}} + \boldsymbol{c_2}\hat{\boldsymbol{y}} \in \widehat{P}$;
(C3) Intersection of $\widehat{P}$ and $-\widehat{P}$ is $\{\hat{\mathbf{0}}\}$.
Definition 2.4 (L.G. Huang and X. Zhang. [7]). A partial-order relation on a cone $\widehat{P} \subseteq \hat{E}$ is denoted by $\lesssim$ and is defined as $\hat{x} \lesssim \hat{y} \Leftrightarrow \hat{y} - \hat{x} \in \widehat{P}$.

Definition 2.5 (L.G. Huang and X. Zhang. [3]). The interior of a cone $\widehat{\mathrm{P}}$ is denoted by $\mathrm{int}(\widehat{\mathrm{P}})$and a partial-order relation $\prec\prec$ on $\mathrm{int}(\widehat{\mathrm{P}}) \subseteq \widehat{\mathrm{P}} \subseteq \hat{\mathrm{E}}$ defined as $\hat{u} \prec\prec \hat{v} \Leftrightarrow \hat{v} - \hat{u} \in \mathrm{int}(\widehat{\mathrm{P}})$.

Definition 2.6 (L.G. Huang and X. Zhang. [7]). If $\dot{\mathrm{X}}$ is a non-empty set and a mapping $\mathrm{d}_{\hat{\mathrm{E}}}$: $\dot{\mathrm{X}} \times \dot{\mathrm{X}} \rightarrow \hat{\mathbf{E}}$ satisfies:
(CMS 1) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) > \widehat{\mathbf{0}}$, $\forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$ and $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \widehat{\mathbf{0}} \Leftrightarrow \dot{x} = \dot{y}$;
(CMS 2) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, \dot{x})$, $\forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$;
(CMS 3) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) \lesssim \mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{z}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{z}, \dot{y})$, $\forall\, \dot{x}, \dot{y}, \dot{z} \in \dot{\mathrm{X}}$,
then $\mathrm{d}_{\hat{\mathrm{E}}}$ is called a cone metric on $\dot{\mathrm{X}}$ and $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is called a cone metric space.
Definition 2.7 (A. Azam et al. [1]). If $\dot{\mathrm{X}}$ is a non-empty set and a mapping $\mathrm{d}_{\hat{\mathrm{E}}}$: $\dot{\mathrm{X}} \times \dot{\mathrm{X}} \rightarrow \hat{\mathbf{E}}$ satisfies:
(CRMS 1) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) > \widehat{\mathbf{0}}$, $\forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$ and $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \widehat{\mathbf{0}} \Leftrightarrow \dot{x} = \dot{y}$;
(CRMS 2) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, \dot{x})$, $\forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$;
(CRMS 3) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) \lesssim \mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{u}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{u}, \dot{v}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{v}, \dot{y})$, $\forall\, \dot{x}, \dot{y}, \dot{u}, \dot{v} \in \dot{\mathrm{X}}$ with $\dot{u} \neq \dot{v}$,
then $\mathrm{d}_{\hat{\mathrm{E}}}$ is called a cone rectangular metric on $\dot{\mathrm{X}}$ and $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is called a cone rectangular metric space.
Definition 2.8 (N. Hussain et al. [9]). If $\dot{\mathrm{X}}$ is a non-empty set and a mapping $\mathrm{d}_{\hat{\mathrm{E}}}$: $\dot{\mathrm{X}} \times \dot{\mathrm{X}} \rightarrow \hat{\mathbf{E}}$ satisfies:
(CbMS 1) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) > \widehat{\mathbf{0}}$, $\forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$ and $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \widehat{\mathbf{0}} \Leftrightarrow \dot{x} = \dot{y}$;
(CbMS 2) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, \dot{x})$, $\forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$;
(CbMS 3) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) \lesssim b[\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{z}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{z}, \dot{y})]$, $\forall\, \dot{x}, \dot{y}, \dot{z} \in \dot{\mathrm{X}}$,
then $\mathrm{d}_{\hat{\mathrm{E}}}$ is called a cone 𝕓-metric on $\dot{\mathrm{X}}$ and $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is called a cone 𝕓-metric space.
Definition 2.9 (R. George et al. [12]). If $\dot{\mathrm{X}}$ is a non-empty set and a mapping $\mathrm{d}_{\hat{\mathrm{E}}}$: $\dot{\mathrm{X}} \times \dot{\mathrm{X}} \rightarrow \hat{\mathbf{E}}$ satisfies:
(GCbMS 1) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) > \widehat{\mathbf{0}}$, $\forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$ and $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \widehat{\mathbf{0}} \Leftrightarrow \dot{x} = \dot{y}$;
(GCbMS 2) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, \dot{x})$, $\forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$;
(GCbMS 3) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) \lesssim b[\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{u}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{u}, \dot{v}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{v}, \dot{y})]$, $\forall\, \dot{x}, \dot{y}, \dot{u}, \dot{v} \in \dot{\mathrm{X}}$ with $\dot{u} \neq \dot{v}$,
then $\mathrm{d}_{\hat{\mathrm{E}}}$ is called a generalized cone 𝕓-metric on $\dot{\mathrm{X}}$ and $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is called a generalized cone 𝕓-metric space.
Definition 2.10 (R. George et al. [12]). Let $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ be a generalized cone 𝕓-metric space. Then

(1) If for every $\hat{t} \in \hat{\mathbf{E}}$ with $\hat{0} \prec\prec \hat{t}$, there exists a natural number N such that $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}_n, \dot{x}) \prec\prec \hat{t}$, for all $n > \mathrm{N}$, then $\{\dot{x}_n\}$ converges to $\dot{x}$
(2) If for every $\hat{t} \in \hat{\mathbf{E}}$ with $\hat{0} \prec\prec \hat{t}$, there exists a natural number N such that $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}_m, \dot{x}_n) \prec\prec \hat{t}$, for all $m, n > \mathrm{N}$, then $\{\dot{x}_n\}$ converges to $\dot{x}$
(3) If every Cauchy sequence is convergence in $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$, then $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is a complete generalized cone 𝕓-metric space.

Lemma 2.11 (R. George et al. [12]). Let $\widehat{\mathrm{P}}$ be a cone and $\{\dot{x}_n\}$ be a sequence in $\hat{\mathbf{E}}$. If $\hat{t} \in int\,(\widehat{\mathrm{P}})$ and $\hat{0} \leq \dot{x}_n \rightarrow \hat{0}$ as there exists a natural number N such that for all $n > \mathrm{N}$, we have $\dot{x}_n \prec\prec \hat{t}$.

3. Main Result

Theorem 3.1 Let $(\dot{X}, d_{\hat{E}})$ be a generalized cone $\mathbb{b}$-metric space. If the two self-maps $\mathfrak{f}$ and $\mathfrak{T}$ defined on $\dot{X}$ satisfy the following conditions:

(3.1) $d_{\hat{E}}\left(\mathfrak{T}\mathfrak{f}\dot{x},\mathfrak{T}\mathfrak{f}\dot{y}\right)\lesssim\gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{x},\mathfrak{T}\mathfrak{f}\dot{x}\right)+d_{\hat{E}}\left(\mathfrak{T}\dot{y},\mathfrak{T}\mathfrak{f}\dot{y}\right)\right]$, $\forall\,\dot{x},\dot{y}\in\dot{X}$, where, $0\le\gamma<\frac{1}{b+1}$,

(3.2) $\mathfrak{T}$ is one-to-one mapping and

(3.3) $\mathfrak{T}(\dot{X})$ is a complete subspace of $\dot{X}$, then the mapping $\mathfrak{f}$ has a unique fixed point in $\dot{X}$. Further,

(3.4) If $\mathfrak{f}$ and $\mathfrak{T}$ are commutes at the fixed point of $\mathfrak{f}$, then there exists a unique common fixed point of $\mathfrak{f}$ and $\mathfrak{T}$ in $\dot{X}$.

Pf. Let $\{\dot{y}_p\}$ be a sequence in $\dot{X}$ such that $\dot{y}_{p+1} = \mathfrak{f}\dot{y}_p$, $\forall\, p = 0,1,2,\ldots$. If $\dot{y}_m = \dot{y}_{m+1}$, for some natural number m, then $\dot{y}_m = \mathfrak{f}\dot{y}_m$, i.e., $\dot{y}_m$ is a fixed point of $\mathfrak{f}$ in $\dot{X}$. Suppose, $\dot{y}_p \neq \dot{y}_{p+1}$, $\forall\, p = 0,1,2,\ldots$, Then from (3.1) we obtain,

$$
\begin{aligned}
d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right) &= d_{\hat{E}}\left(\mathfrak{T}\mathfrak{f}\dot{y}_{p-1},\mathfrak{T}\mathfrak{f}\dot{y}_p\right)\\
&\lesssim\gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\mathfrak{T}\mathfrak{f}\dot{y}_{p-1}\right)+d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\mathfrak{f}\dot{y}_p\right)\right]\\
&\lesssim\gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\mathfrak{T}\dot{y}_p\right)+d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)\right]\\
&\lesssim\frac{\gamma}{1-\gamma}d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\mathfrak{T}\dot{y}_p\right),
\end{aligned}
$$

which implies that, $d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)\lesssim\eta d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\mathfrak{T}\dot{y}_p\right)$, $\forall\, p = 1,2,\ldots$, where $0\le\eta=\frac{\gamma}{1-\gamma}<\frac{1}{b}$. Therefore, by induction, we have,

$$
\begin{aligned}
d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right) &\lesssim\eta^2 d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-2},\mathfrak{T}\dot{y}_{p-1}\right)\\
&\;\vdots\\
&\lesssim\eta^p d_{\hat{E}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right),\ \forall\, p=1,2,\ldots,\ \text{where } 0\le\eta<\frac{1}{b} \qquad (3.5)
\end{aligned}
$$

Since, $\gamma\le\eta$ and therefore by using Kannan inequality (3.1) and $\mathbb{b}$-rectangular inequality, we get,

$$
\begin{aligned}
d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+2}\right) &= d_{\hat{E}}\left(\mathfrak{T}\mathfrak{f}\dot{y}_{p-1},\mathfrak{T}\mathfrak{f}\dot{y}_{p+1}\right)\\
&\lesssim\gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\mathfrak{T}\mathfrak{f}\dot{y}_{p-1}\right)+d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\mathfrak{f}\dot{y}_{p+1}\right)\right]\\
&\lesssim\gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\mathfrak{T}\dot{y}_p\right)+d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]
\end{aligned}
$$

$$\precsim b\gamma\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\ \mathfrak{T}\dot{y}_{p}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\ \mathfrak{T}\dot{y}_{p+2}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+1}\right)\right]$$
$$+\gamma\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)$$
$$=b\gamma\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\ \mathfrak{T}\dot{y}_{p}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\ \mathfrak{T}\dot{y}_{p+2}\right)\right]+(1+b)\gamma\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)$$
$$\precsim b\gamma\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\ \mathfrak{T}\dot{y}_{p}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\ \mathfrak{T}\dot{y}_{p+2}\right)\right]+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)$$

which implies that,

$$\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+2}\right)\precsim\frac{1}{1-b\gamma}\left[b\gamma\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1},\ \mathfrak{T}\dot{y}_{p}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]$$
$$\precsim\frac{1}{1-b\gamma}\left[b\eta\eta^{p-1}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)+\eta^{p+1}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\right]$$
$$\precsim\left[\frac{b+\eta}{1-b\gamma}\right]\eta^{p}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\ \mathfrak{T}\dot{y}_{1}\right)$$
$$=\alpha\eta^{p}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\ \mathfrak{T}\dot{y}_{1}\right),\tag{3.6}$$

where $\alpha=\dfrac{b+\eta}{1-b\gamma}\geq 0,\ \forall\, p\geq 1$. For the sequence $\{\mathfrak{T}\dot{y}_p\}$, we consider $\mathrm{d}_{\hat{E}}(\mathfrak{T}\dot{y}_p, \mathfrak{T}_{p+n})$ in two different cases. If $n = 2r + 1$, where $r \geq 1$, then by applying 𝕓-rectangular inequality we get,

$$\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+2r+1}\right)\precsim b\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+2r+1}\right)\right]$$
$$\precsim b\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]+b^{2}\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+3}\right)\right.$$
$$\left.+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+3},\mathfrak{T}\dot{y}_{p+4}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+4},\mathfrak{T}\dot{y}_{p+2r-1}\right)\right]$$
$$\precsim b\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]+b^{2}\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+3}\right)\right.$$
$$\left.+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+3},\mathfrak{T}\dot{y}_{p+4}\right)\right]+\cdots+b^{r}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2r},\mathfrak{T}\dot{y}_{p+2r+1}\right)$$
$$\precsim b\left[\eta^{p}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)+\eta^{p+1}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\right]+b^{2}\left[\eta^{p+2}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\right.$$
$$\left.+\eta^{p+3}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\right]+\cdots+b^{r}\eta^{p+2r}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)$$

$$\lesssim \eta^{p}\left[1+b\eta^{2}+\cdots\right]\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)$$

$$+b\eta^{p+1}\left[1+b\eta^{2}+\cdots\right]\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)$$

$$=(1+\eta)b\eta^{p}\left[1+b\eta^{2}+\cdots\right]\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)$$

Hence, $\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+2r+1}\right)\lesssim\left(\frac{1+\eta}{1-b\eta^{2}}\right)b\mu^{p}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)$, for all natural numbers p and r.

Suppose, $\hat{0}\prec\prec\hat{k}$. Since, $b\eta^2<1$, we notice that $\left(\frac{1+\eta}{1-b\eta^{2}}\right)b\eta^{p}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\to\hat{0}$, as $n\to\infty$. Therefore for some $\hat{k}\in$ int $(\widehat{\mathrm{P}})$, we obtain a natural number $N_1\in\mathbb{N}\ni$ $\left(\frac{1+\eta}{1-b\eta^{2}}\right)b\eta^{p}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\prec\prec\hat{k},\ \forall p>N_{1}$.

Thus, $\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+2r+1}\right)\lesssim\left(\frac{1+\eta}{1-b\eta^{2}}\right)b\eta^{p}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\prec\prec\hat{k},\forall p>N_1$ and $r\geq 1$.

Suppose, $p=2r$, for $r\geq 1$. Since, $b\eta^2<1$ and by using inequalities (3.5) and (3.6) we get,

$$\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+2r}\right)\lesssim b\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+2r}\right)\right]$$

$$\lesssim b\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]+b^{2}\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+3}\right)\right.$$

$$\left.+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+3},\mathfrak{T}\dot{y}_{p+4}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+4},\mathfrak{T}\dot{y}_{p+2r}\right)\right]$$

$$\lesssim b\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]+b^{2}\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+3}\right)\right.$$

$$\left.+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+3},\mathfrak{T}\dot{y}_{p+4}\right)\right]+\cdots+b^{r-1}\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2r-4},\mathfrak{T}\dot{y}_{p+2r-3}\right)\right.$$

$$\left.+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2r-3},T\dot{y}_{p+2r-2}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2r-2},\mathfrak{T}\dot{y}_{p+2r}\right)\right]$$

$$\lesssim b\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]+b^{2}\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+3}\right)\right.$$

$$\left.+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+3},\mathfrak{T}\dot{y}_{p+4}\right)\right]+\cdots+b^{r-1}\left[\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2r-4},\mathfrak{T}\dot{y}_{p+2r-3}\right)\right.$$

$$\left.+\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2r-3},\mathfrak{T}\dot{y}_{p+2r-2}\right)\right]+b^{r-1}\mathrm{d}_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+2r-2},\mathfrak{T}\dot{y}_{p+2r}\right)$$

$$\lesssim b\left[\eta^{p}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)+\eta^{p+1}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right]+b^{2}\left[\eta^{p+2}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right.$$

$$\left.+\eta^{p+3}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right]+\cdots+\eta^{r-1}\left[\eta^{p+2r-4}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right.$$

$$\left.+\eta^{p+2r-3}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right]+b^{m-1}\alpha\eta^{p+2r-2}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$$

$$\lesssim b\eta^{p}\left[1+b\eta^{2}+\cdots\right]\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$$

$$+b\eta^{p+1}\left[1+b\eta^{2}+\cdots\right]\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)+b^{r-1}\alpha\eta^{p+2r-2}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$$

$$=(1+\eta)b\eta^{p}\left[1+b\eta^{2}+\cdots\right]\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)+b^{r-1}\alpha\eta^{p+2r-2}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$$

That is, $\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+2r}\right)\lesssim\left(\frac{1+\eta}{1-b\eta^{2}}\right)b\eta^{p}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)+b^{r-1}\alpha\eta^{p+2r-2}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$

i.e., $\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+2r}\right)\lesssim\left(\frac{1+\eta}{1-b\eta^{2}}+b^{r-2}\alpha\eta^{2r-2}\right)b\eta^{p}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$, for all natural numbers p and r, where $\alpha\geq 0$.

Suppose, $\hat{0}\prec\prec\hat{t}$. Since, $b\eta^2<1$, we obtain that $\left(\frac{1+\eta}{1-b\eta^{2}}+b^{r-2}\alpha\eta^{2r-2}\right)b\eta^{p}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\to\hat{0}$, as $p\to\infty$. Therefore, for any $\hat{t}\in$ int $(\widehat{\mathrm{P}})$, we obtain $N_2\in\mathbb{N}\ni\left(\frac{1+\eta}{1-b\eta^{2}}+b^{r-2}\alpha\eta^{2r-3}\right)b\eta^{p}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\prec\prec\hat{t}$, $\forall p>N_2$, $r\geq 1$ and $\alpha=\frac{b+\eta}{1-b\eta}\geq 0$. Thus, $\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+2r}\right)\lesssim\left(\frac{1+\eta}{1-b\eta^{2}}+b^{r-2}\alpha\eta^{2r-2}\right)b\eta^{p}\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\prec\prec\hat{z}$, $\forall p>N_2$ and $r\geq 1$. Let $N_0=\max\{N_1,N_2\}$. Thus for each $\hat{t}\in$ Int $(\widehat{\mathrm{P}})$, we get, $\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+n}\right)\prec\prec\hat{k}$, $\forall p>N_0$ and $n\geq 1$.

Hence in both cases, $\{\mathfrak{T}\dot{y}_p\}$ is a Cauchy sequence in $\dot{\mathrm{X}}$. By the hypothesis we have, $\mathfrak{T}(\dot{\mathrm{X}})$ is a complete subspace of $\dot{\mathrm{X}}$, therefore $\exists\ \dot{z}\in\mathfrak{T}(\dot{\mathrm{X}})\ni\lim_{p\to\infty}\mathfrak{T}\dot{y}_{p+1}=\lim_{p\to\infty}\mathfrak{T}\mathfrak{f}\dot{y}_p=z$. Here, we find $\dot{y}\in\dot{\mathrm{X}}\ni\dot{z}=\mathfrak{T}\dot{y}$. Suppose, $\hat{0}\prec\prec\hat{c}$. we find $N_3, N_4\in\mathbb{N}\ni\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,z\right)\prec\prec\frac{\hat{c}(1-b\gamma)}{2b}$, $\forall p>N_3$ and $\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)\prec\prec\frac{\hat{c}(1-b\gamma)}{2b(1+\gamma)}$, $\forall p>N_4$. Let $N=\max\{N_3,N_4\}$. Using 𝕓-rectangular inequality and (3.5) we obtain,

$$\mathrm{d}_{\hat{\mathrm{E}}}\left(\dot{z},\mathfrak{T}\mathfrak{f}\dot{y}\right)\lesssim b\left[\mathrm{d}_{\hat{\mathrm{E}}}\left(\dot{z},\mathfrak{T}\dot{y}_p\right)+\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\mathfrak{f}\dot{y}_p\right)+\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\mathfrak{f}\dot{y}_p,\mathfrak{T}\mathfrak{f}\dot{y}\right)\right]$$

$$\lesssim b\mathrm{d}_{\hat{\mathrm{E}}}\left(\dot{z},\mathfrak{T}\dot{y}_p\right)+b\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)+b\gamma\left[\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\mathfrak{f}\dot{y}_p\right)+\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y},\mathfrak{T}\mathfrak{f}\dot{y}\right)\right]$$

$$=b\mathrm{d}_{\hat{\mathrm{E}}}\left(\dot{z},\mathfrak{T}\dot{y}_p\right)+b\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)+b\gamma\left[\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{\mathrm{E}}}\left(\dot{z},\mathfrak{T}\mathfrak{f}\dot{y}\right)\right]$$

Hence, $\mathrm{d}_{\hat{\mathrm{E}}}\left(\dot{z},\mathfrak{T}\mathfrak{f}\dot{y}\right)\lesssim\frac{1}{1-b\gamma}\left[b\mathrm{d}_{\hat{\mathrm{E}}}\left(\dot{z},\mathfrak{T}\dot{y}_n\right)+b(1+\gamma)\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)\right]\prec\prec\frac{\hat{c}}{2}+\frac{\hat{c}}{2}=\hat{c},\forall p\geq 1.$

Thus $\mathrm{d}_{\hat{\mathrm{E}}}\left(\dot{z},\mathfrak{T}\mathfrak{f}\dot{y}\right)\prec\prec\hat{k}$, $\forall\,\hat{c}\in$ Int ($\widehat{\mathrm{P}}$), since $\hat{c}$ is any arbitrary in Int ($\widehat{\mathrm{P}}$), we get

$\mathrm{d}_{\hat{\mathrm{E}}}(\dot{z},\mathfrak{T}\mathfrak{f}\dot{y})=\hat{0}$. That is, $\mathfrak{T}\mathfrak{f}\,\dot{y}=\mathfrak{T}\dot{y}=\dot{z}$. Since $\mathfrak{T}$ is one-to-one it implies that $\dot{y}=\mathfrak{f}\dot{y}$, i.e., there exists a fixed point $\dot{y}\in\dot{\mathrm{X}}$. Suppose $\dot{y}'\in\dot{\mathrm{X}}$ is also a fixed point of $\mathfrak{f}$, i.e., $\dot{y}'=\mathfrak{f}\dot{y}'$. Therefore, by using Kannan Contraction (3.1), we get,

$\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y},\mathfrak{T}\dot{y}'\right)=\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\mathfrak{f}\dot{y},\mathfrak{T}\mathfrak{f}\dot{y}'\right)\lesssim\gamma\left[\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y},\mathfrak{T}\dot{y}\right)+\mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}',\mathfrak{T}\dot{y}'\right)\right]=\hat{0}$, and hence we get, $\mathfrak{T}\dot{y}=\mathfrak{T}\dot{y}'$. Since $\mathfrak{T}$ is one-to-one mapping, we get that $\dot{y}=\dot{y}'$. Since, the two self-maps $\mathfrak{f}$ and $\mathfrak{T}$ are commute at the fixed point $\dot{y}$ of $\mathfrak{f}$, we get that, $\mathfrak{f}\mathfrak{T}\dot{y}=\mathfrak{T}\mathfrak{f}\dot{y}$, i.e., $\mathfrak{f}\mathfrak{T}\dot{y}=\mathfrak{T}\dot{y}$, i.e., $\mathfrak{T}\dot{y}$ is also a fixed point of $\mathfrak{f}$. Since, we know that $\mathfrak{f}$ has a unique fixed point, therefore $\mathfrak{T}\dot{y}=\dot{y}$ and hence $\mathfrak{T}\dot{y}=\mathfrak{f}\dot{y}=\dot{y}$. Therefore, there exists a unique common fixed point $\dot{y}\in\dot{\mathrm{X}}$ of $\mathfrak{f}$ and $\mathfrak{T}$. □

Put $\mathfrak{T}=\mathfrak{I}$, the identity mapping of $\dot{\mathrm{X}}$ in Theorem 3.1, we get the following corollary.

Corollary 3.2 If a self-map $\mathfrak{f}\colon\dot{\mathrm{X}}\to\dot{\mathrm{X}}$ defined on a complete generalized cone 𝕓-metric space $(\dot{\mathrm{X}},\mathrm{d}_{\hat{\mathrm{E}}})$ satisfying an inequality:

$$\mathrm{d}_{\hat{\mathrm{E}}}\,(\mathfrak{f}\,\dot{x},\mathfrak{f}\,\dot{y})\lesssim\gamma\,[\mathrm{d}_{\hat{\mathrm{E}}}\,(\dot{x},\mathfrak{f}\dot{x})\;+\;\mathrm{d}_{\hat{\mathrm{E}}}\,(\dot{y},\mathfrak{f}\dot{y})],$$

$\forall\,\dot{x},\dot{y}\in\dot{\mathrm{X}}$, $0\leq\gamma<\frac{1}{b+1}$ and $b>1$, then there exists a unique fixed point of $\mathfrak{f}$ in $\dot{\mathrm{X}}$.

We illustrate our main theorem 3,1 with the following example.

Example 3.3 Suppose $\dot{\mathrm{X}}=\dot{\mathcal{A}}\cup\dot{\mathcal{B}}$, if $\dot{\mathcal{A}}=\{0\}\cup\left\{\frac{1}{n}:n\in\{2,3,4\}\right\}$ and $\dot{\mathcal{B}}=[1,2]$, $\hat{\mathbf{E}}=C_{\mathbb{R}}(\dot{\mathrm{X}})$, $\widehat{\mathrm{P}}=\{\hat{f}:f(\dot{x})\geq 0,\ \dot{x}\in\dot{\mathrm{X}}\}\subset\hat{\mathbf{E}}$, It is known that $\widehat{\mathrm{P}}$ is a cone in $\hat{\mathbf{E}}$. Define $\mathrm{d}_{\hat{\mathrm{E}}}\colon\dot{\mathrm{X}}\times\dot{\mathrm{X}}\to\hat{\mathbf{E}}$ as $\mathrm{d}_{\hat{\mathrm{E}}}\,(\dot{t},\dot{s})=\mathrm{d}_{\hat{\mathrm{E}}}\,(\dot{s},\dot{t})$, $\forall\dot{t},\dot{s}\in\dot{\mathrm{X}}$ and

$$\begin{cases} d_{\hat{E}}\left(0,\frac{1}{2}\right)=d_{\hat{E}}\left(\frac{1}{3},\frac{1}{4}\right)=0.03e^{\dot{x}}; \\ d_{\hat{E}}\left(0,\frac{1}{3}\right)=d_{\hat{E}}\left(\frac{1}{4},\frac{1}{2}\right)=0.06e^{\dot{x}}. \\ d_{\hat{E}}\left(0,\frac{1}{4}\right)=d_{\hat{E}}\left(\frac{1}{2},\frac{1}{3}\right)=0.02e^{\dot{x}}; \\ d_{\hat{E}}\left(\dot{t},\dot{s}\right)=|\dot{t}-\dot{s}|^{2}\ e^{\dot{x}},\ \text{otherwise}, \end{cases}$$

Here $e^{\dot{x}} \in \hat{\mathbf{E}}$. Then $(\dot{X}, d_{\hat{E}})$ is neither a cone metric space, since, $d_{\hat{E}}\left(0, \frac{1}{3}\right) = 0.06\ e^{\dot{x}} > d_{\hat{E}}\left(0, \frac{1}{2}\right) + d_{\hat{E}}\left(\frac{1}{2}, \frac{1}{3}\right) = 0.03\ e^{\dot{x}} + 0.02\ e^{\dot{x}} = 0.05\ e^{\dot{x}}$ nor a cone rectangular metric space, since, $d_{\hat{E}}(0, 1) = e^{\dot{x}} > d_{\hat{E}}\left(1, \frac{1}{2}\right) + d_{\hat{E}}\left(\frac{1}{2}, \frac{1}{3}\right) + d_{\hat{E}}\left(\frac{1}{3}, 0\right) = 0.25\ e^{\dot{x}} + 0.02\ e^{\dot{x}} + 0.06e^{\dot{x}} = 0.33\ e^{\dot{x}}$. However, it is easy to see that $(\dot{X}, d_{\hat{E}})$ is a complete generalized cone 𝕓-metric space with coefficient $b = 4 > 1$.

Further, let $\mathfrak{f}$ and $\mathfrak{T}$: $\dot{X} \to \dot{X}$ be the mappings defined by:

$$\mathfrak{f}\left(\dot{x}\right)=\begin{cases}\frac{1}{4}, & \dot{x}\in\dot{\mathcal{A}}\\ \frac{1}{2}, & \dot{x}\in\dot{\mathcal{B}}\end{cases}$$

$$\text{and } \mathfrak{T}(\dot{x})=\begin{cases}\frac{1}{2}-\dot{x}, & \dot{x}\in\{0\}\cup\left\{\frac{1}{n}:n\in\{2,4\}\right\}\\ \dot{x}, & \dot{x}\in\dot{\mathcal{B}}\cup\left\{\frac{1}{3}\right\}\end{cases}$$

We observe that $\mathfrak{T}$ is one-to-one and self-maps $\mathfrak{f}$ and $\mathfrak{T}$ are satisfying Kannan type contraction (3.1) with $\gamma = \frac{1}{5}$. Suppose $\dot{x} \in \dot{\mathcal{A}}$ and $y \in \dot{\mathcal{B}}$. Therefore, $d_{\hat{E}}\ (\mathfrak{T}\mathfrak{f}\dot{x}, \mathfrak{T}\mathfrak{f}y) = d_{\hat{E}}\left(\mathfrak{T}\left(\frac{1}{4}\right), \mathfrak{T}\left(\frac{1}{2}\right)\right) = d_{\hat{E}}\left(\frac{1}{4}, 0\right) = 0.02\ e^{\dot{x}}$. Suppose $\dot{x} \in \{0\} \cup \left\{\frac{1}{n} : n \in \{2,4\}\right\}$ and $\dot{y} \in \dot{\mathcal{B}}$. Therefore, $d_{\hat{E}}(\mathfrak{T}\dot{x}, \mathfrak{T}\mathfrak{f}\dot{x}) + d_{\hat{E}}(\mathfrak{T}\dot{y}, \mathfrak{T}\mathfrak{f}\dot{y}) = d_{\hat{E}}\left(\mathfrak{T}\dot{x}, \mathfrak{T}\left(\frac{1}{4}\right)\right) + d_{\hat{E}}\left(\mathfrak{T}\dot{y}, \mathfrak{T}\left(\frac{1}{2}\right)\right) = d_{\hat{E}}\left(\frac{1}{2} - \dot{x}, \frac{1}{4}\right) + d_{\hat{E}}(\dot{y}, 0)$. Then clearly, we can find $\gamma = \frac{1}{5} \in \left(0, \frac{1}{4}\right)$ satisfying $\mathfrak{T}$-Kannan type contractive condition (3.1).

Suppose $\dot{x} \in \{0\} \cup \left\{\frac{1}{n} : n \in \{2,4\}\right\}$ and $\dot{y} \in \left\{\frac{1}{3}\right\}$. Therefore we can find $\gamma = \frac{1}{5} \in \left(0, \frac{1}{4}\right)$ satisfying $\mathfrak{T}$-Kannan type contractive condition (3.1). Similarly in all other cases, $\mathfrak{f}$ satisfies $\mathfrak{T}$-Kannan type contractive condition (3.1). Thus $\mathfrak{f}$ satisfies all the conditions of Theorem 3.1 and there exists a unique fixed point $\frac{1}{4} \in \dot{X}$ of $\mathfrak{f}$. Finally, $\mathfrak{f}$ and $\mathfrak{T}$ are commuting at $\frac{1}{4} \in \dot{X}$, and hence $\dot{x} = \frac{1}{4} \in \dot{X}$ is a unique common fixed point of the self-maps $\mathfrak{f}$ and $\mathfrak{T}$. However, $\mathfrak{f}$ does not satisfy $\mathfrak{T}$-contractive condition of Corollary 3.2 at $\dot{x} =$

0 and $\dot{y}=1$ as $d_{\hat{E}}(\mathfrak{f}\dot{x},\mathfrak{f}\dot{y}) = d_{\hat{E}}(\mathfrak{f}(0),\mathfrak{f}(1)) = d_{\hat{E}}\left(\frac{1}{4},\frac{1}{2}\right) = 0.06\, e^{\dot{x}} > \frac{1}{5}[d_{\hat{E}}(\dot{x},\mathfrak{f}\dot{x}) + d_{\hat{E}}(\dot{y},\mathfrak{f}\dot{y})] = \frac{1}{5}[d_{\hat{E}}(0,\mathfrak{f}(0)) + d_{\hat{E}}(1,\mathfrak{f}(1))] = \frac{1}{5}\left[d_{\hat{E}}\left(0,\frac{1}{4}\right) + d_{\hat{E}}\left(1,\frac{1}{2}\right)\right] = 0.054e^{\dot{x}}$

4 Discussion and Conclusions

In Theorem 3.1 we have constructed a common fixed-point theorem for two self-maps that are commuting to each other under the $\mathfrak{T}$-Kannan-type contractive condition over a generalized cone $\mathbb{b}$-metric space without assuming the normality condition of a cone. If we put $\mathfrak{T} = \mathfrak{I}$, the identity mapping of $\dot{X}$ in Theorem 3.1, we get the fixed-point theorem as corollary 3.2. We have also given an example 3.3 which satisfies all the conditions of our main result. Our result is generalized and extended to many recent results. Our result may be the vision for other authors to extend and improve many results in generalized cone $\mathbb{b}$-metric spaces.

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References

[1] A. Azam, M. Arshad and I. Beg, Banach contraction principle on cone rectangular metric spaces, Applicable Analysis and Discrete Mathematics, 3**,** 236–241.2009.

[2] S. Banach, Sur les op´erations dans les ensembles abstraits et leur application aux´ equations int´egrales. Fundamenta Mathematicae, 3(1), 133–181, 1922.

[3] IA. Bakhtin, The contraction mapping principle in almost metric spaces. Functional Analysis, Gos. Ped. Inst. Unianowsk, 30, 26-37, 1989.

[4] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces**,** Publicationes Mathematicae Debrecen, 57(1-2), 31-37 2000.

[5] Chuanzhi Bai. Common fixed point theorems for generalized ordered contractive mappings on cone b-metric spaces over Banach algebras. J. Nonlinear Sci. Appl., 9, 5766–5771, 2016.

[6] M. Frechet, Sur quelques points du calcul fonctionnel, Rendiconti Circolo Mat. Palermo, 22(1), 1–72, 1906.

[7] L.G. Huang and X. Zhang. Cone Metric Spaces and Fixed Point Theorems of Contractive mappings. J.Math. Anal. Appl., 332, 1468-1476, 2007.

[8] HP. Huang, S. Radenovic and G. Deng. A sharp generalization on cone b-metric space over Banach algebra. J. Nonlinear Sci. and Appl., 10, 429-435, 2017.

[9] N. Hussain, M.H. Shah. KKM mappings in cone b-metric spaces. Comput. Math. Appl., 62, 1677–1684, 2011.

[10] M. Rangamma, P. Mallikarjun Reddy, A fixed point theorem for Reich contractions in generalized cone b-metric spaces. International Journal of Analysis, 10 (6), 271-278, 2016.

[11] M. Rangamma, P. Mallikarjun Reddy, A common fixed point theorem for T-contractions on generalized cone b-metric spaces. Communications in Korean Mathematical Society, 32(1), 65-74, 2017.

[12] R. George, A. Hossam Nabwey, K. P. Reshma, and R. Rajagopalan. Generalized cone b-metric spaces and contraction principles. Mathematiqki Vesnik, 67 (4), pp. 246–257, 2015.

[13] Virath Singh1 and Pravin Singh. Fixed point theorems in a generalized cone b-metric space. Advances in Mathematics: Scientific Journal, 10 (4), 2083–2094, 2021.

[14] Yan Han and Shaoyuan Xu. Generalized Reich–C´iric´–Rus-type and Kannan -type Contractions in cone b-metric spaces over Banach algebras. Journal of Mathematics, Article ID 7549981, 11 pages, 2021.