OVER THE CONSTRUCTION OF AN
HYPERSTRUCTURE OF QUOTIENTS
FOR A MULTIPLICATIVE HYPERRING

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Summary - In this paper we construct a weak hyperfield of quotients for a class of multiplicative hyperrings.

First of all we want to recall some algebraic definitions that will be used through the paper.

An hyperring \((A, \oplus, \cdot)\) is a set \(A\) with an hyperoperation \(\oplus\) and a product \(\cdot\) such that the following properties hold:

\[
\begin{align*}
\text{i) } & \forall a,b,c \in A : a \oplus (b \oplus c) = (a \oplus b) \oplus c, \\
\text{ii) } & \forall a,b \in A : a \oplus b = b \oplus a, \\
\text{iii) } & \exists 0 \in A / \forall a \in A : 0 \oplus a = a \oplus 0 = a, \\
\text{iv) } & \forall a \in A \exists! a' \in A : a \oplus a' = 0, (a' = -a), \\
\text{v) } & \forall a,b,c \in A \forall x \in b \oplus c \Rightarrow c = a - b (c = a \oplus b), \\
\text{vi) } & \forall a,b,c \in A : (a \cdot b) \cdot c = a \cdot (b \cdot c), \\
\text{vii) } & \forall a,b,c \in A : (a \oplus b) \cdot c = a \cdot c \oplus b \cdot c, \\
\text{viii) } & \forall a,b,c \in A : a \cdot (b \oplus c) = a \cdot b \oplus a \cdot c. \\
\end{align*}
\]

We recall that \((A, \oplus)\) satisfying i), ii), iii), iv) and v) is called canonical hypergroup.

Let us observe ([3]) that axiom v) is equivalent to v)' and also to v)'':

\[
\begin{align*}
v)' & \forall a,b \in A : -(a \oplus b) = -a - b,
\end{align*}
\]

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v) \( \forall a, b, c, d \in A : (a \oplus b) \cap (c \oplus d) \neq \emptyset \Rightarrow (c - a) \cap (b - d) \neq \emptyset. \)

An hyperring A is called hyperfield if \((A^*, \cdot)\) is a group, where \(A^* = A \setminus \{0\}\).

A multiplicative hyperring \((A, +, \cdot)\) is an abelian group \((A, +)\) together with an hyperproduct satisfying the following properties:

i) \( \forall a, b, c \in A : a \cdot (b \cdot c) = (a \cdot b) \cdot c; \)

ii) \( \forall a, b, c \in A : (a + b) \cdot c = a \cdot c + b \cdot c; \)

iii) \( \forall a, b, c \in A : a \cdot (b + c) = a \cdot b + a \cdot c; \)

iv) \( \forall a, b \in A : (-a) \cdot b = a \cdot (-b) = -(a \cdot b). \)

If a multiplicative hyperring satisfies, instead of properties ii) and iii), the following ones:

ii) \( \forall a, b, c \in A : (a + b) \cdot c = a \cdot c + b \cdot c \)

iii) \( \forall a, b, c \in A : a \cdot (b + c) = a \cdot b + a \cdot c \)

then \((A, +, \cdot)\) is called strongly distributive. Moreover \((A, +, \cdot)\) is strongly left (right) distributive if ii)' (iii)') holds.

In [2] J. Mittas studied the possibility of immersion for an hyperring in a hyperfield and constructed the hyperfield of quotients for a particular hyperring. In this paper we want to solve the analogous problem for a multiplicative hyperring.

Let \((A, +, \cdot)\) be a multiplicative, strongly distributive, commutative hyperring, from now on we will write ab instead of \(a \cdot b\) and, for any two sets \(X\) and \(Y\), \(X \approx Y\) if and only if \(X \setminus Y \neq \emptyset\). We suppose that the hyperring \(A\) satisfies the following properties: i) if \(ab = 0, a \neq 0 \Rightarrow b = 0\); ii) \(\forall X, Y, Z, W \in P^*(A) = \{P(A)\setminus \emptyset\} / XY = \emptyset \Rightarrow (\forall x \in X \text{ and } \forall w \in W \exists y \in Y \text{ and } z \in Z / xy = zw).\)

For this structure we can prove the following lemmas:

I.- For any \(a, b \in A\) if \(0 \in ab\) and \(a \neq 0\) then \(b = 0.\)

Proof. - Since \(A\) is strongly distributive \(0 \in ac \forall c \in A\), thus from i) \(b = 0\) follows.

II.- For any \(X, Y \in P^*(A)\) and for any \(d \in A \setminus \{0\}\), \(dX \approx dY \Leftrightarrow X \approx Y.\)

Proof. - Since \(dX = \bigcup dx, x \in X\) and \(dY = \bigcup dy, y \in Y\), thus, from the hypothesis, there exists \(z \in dX \setminus dY\), that is \(z \in dx\) and \(z \in dy\), for some \(x \in X\)
and \( y \in Y \); this implies, from the strong distributivity, \( 0 \in d(x-y)=dx-dy \) and, since \( d \neq 0 \), \( x = y \), thus \( X = Y \). The inverse implication is obvious.

\[ \text{III. For any } X, Y \in P^{\ast}(A) \text{ and for any } d_i \in A \setminus \{0\}, (X+Y)d_i \bullet \cdots \bullet d_k = (Xd_1 \bullet \cdots \bullet d_k)+(Yd_1 \bullet \cdots \bullet d_k). \]

Proof. We can prove the lemma for \( k = 1 \); the general case will follow as a consequence of the associative property. Now \((X+Y)d=(z \in z \in (x+y)d, x \in X, y \in Y) = z / z \in xd + yd, x \in X, y \in Y) = Xd + Yd.\)

Denoting by \( A^* \) the set \( A \setminus \{0\} \) and by \( H \) the set \( A \times A^* = \{ (a,b) / a \in A, b \in A^* \} \), we define in \( H \) the following relation \((a,b) \rho (c,d) \iff ad = bc; \) for this relation the following properties hold:

\[ \text{IV. } (a,b) \rho (0,d) \iff a = 0. \]

Proof. - Since \((a,b) \rho (0,d) \iff ad = 0b \) then, from i) and \( d \neq 0 \), \( a = 0 \) follows. Vice-versa, since \( 0 \in 0d = 0b \), then \((0,b) \rho (0,d), \forall b, d \in A^* \).

\[ \text{V. For any } a, b, c, d, f \in A, \text{ if } ad = bc, \text{ then } adf = bcf. \]

Proof. - If \( z \in ad \sim bc \) then \( zf \subseteq adf \) and \( zf \subseteq bcf \); thus \( adf = bcf \).

\[ \text{VI. } \rho \text{ is an equivalence relation.} \]

Proof. - Reflexivity and symmetry are for \( \rho \) immediate consequence of commutativity in \( A \). As for transitivity let \((a,b) \rho (c,d) \) and \((c,d) \rho (e,f) \); then \( ad = bc \) and \( cf = de \) that is there exist \( z, w \in A \) such that \( z \in ad \sim bc, \) \( w \in cf \sim de, \) thus \( zw = adef = bced. \) So \( adef = bced \) and, since \( d \neq 0 \), from proposition II we obtain \( adef = bced \); now, if \( c \neq 0 \), again from proposition II \( af = bc \), while if \( c = 0 \) then, from proposition IV, \( a = e = 0. \) In both cases it results \((a,b) \rho (c,f) \) as requested.

Let now \( K = H/\rho \), we want to define in \( K \) two commutative hyperoperations \( \oplus \) and \( \otimes \) in such a way that the following conditions hold:

1) \((K, \oplus) \) is a canonical hyergroup;
2) \( \forall x, y, z \in K : x \oplus (y \otimes z) = (x \otimes y) \otimes z; \)
3) \( \forall x, y, z \in K : x \oplus (y \otimes z) = (x \otimes y) \oplus (x \otimes z); \)
4) \( \forall x, y \in K : -(x \otimes y) = (-x) \otimes y = x \otimes (-y); \)
5) \( \exists 1 \in K / \forall x \in K : x \otimes 1 = x; \)
6) \( \forall x \in K \setminus \{0\} \exists y \in K \setminus \{0\} / 1 \in x \otimes y. \)

For any \([a,b], [c,d] \in K \) let us define \([a,b] \oplus [(c,d)] = [(ad + bc, bd)] = [(s,t)] \) \( / s \in ad + bc, t \in bd \) and \([a,b] \otimes [(c,d)] = [(ac, bd)] = [(s,t)] / s \in ac, t \in bd \); first
In order to prove the existence of a zero element in \((K, \oplus)\) we first recall that, as a consequence of proposition IV, \([(0,y)]=(0,d)/d \in A^*\), moreover \([(a,b)]\oplus[(0,y)]=\{(s,t) / s \leq ay+bo, t \leq by\}=[(a,b)]; in fact, since \((by)a \leq ay+bo\), we have \(bya=(ay+bo)b\) and, from ii), \(s \leq ay+b0 \ \forall \ t \leq by\) it results \(s=b=0a\). This proves that \([(0,y)]\) is a zero for \(\oplus\); let us now prove that it is unique. To do this let \([(x,y)]\in K\) such that \([(a,b)]\oplus [(x,y)] = [(a,b)]\), that is \(s \in H\) such that \(s=ay+bx, t=by\) then \(s=b=0a\). From this it follows that \(ay+b+bx=bya\), then there exists \(z(ay+b+bx)=bya, z=z^++z\), \(z \in ay+b\) and \(z'' \in bx, z \\in bya\); that is \(z \in bya=bya\) or \((bya-bya)\in bx \neq \emptyset\). Thus \(\{(a-a)y)\}=\{(bya-bya)\}\in bx \neq \emptyset\) and, from proposition II, \(0y=bx\) with \(b=0\); because of i) \(x=0\) and the uniqueness is proved.

We want now to verify the existence and uniqueness, for any \([(a,b)] \in K\), of an element \([(z,w)]\in K\) such that \([(a,b)]\oplus [(z,w)]\in [(0,y)]\). First of all we observe that \([(a,b)]\oplus [(a,b)]=\{(s,t) / s \leq ab, t \leq bb\} and this set trivially contains \([(0,t)]\). As for the uniqueness let \([(z,w)]\in K\) such that \([(0,y)]\in [(a,b)]\oplus [(z,w)]\in [(s,t) / s \leq aw+bz, t \leq bw]\); then \(0 \leq aw+bz\) or \(0 \leq u+v, u \leq aw, v \leq bz\) that is \(v=-ua=-aw\) which implies \(-aw=by\) and this means \((z,w)=[(a,b)]\).

Finally we must prove condition v) in the definition of canonical hypergroup which is equivalent to the following condition: \(-(a,b)\oplus (c,d)=[(a,b)]\oplus [(c,d)]\).

To prove such equality, we first must prove that \(-(a,b)\oplus [(c,d)]\leq [(a,b)]\oplus [(c,d)]\); to do this let \([-s,t)]\) such that \(s \leq ad+bc, t \leq bd\). Thus \(-s=-ad+bc\) that is \(-s=\{-z\}+\{-w\}, z \in ad, w \in bc\); from this, since \(-ad=-a)\) and \(-bc=-b\), \([-s,t)]\in [(a,b)]\oplus [(c,d)]\) follows. Similarly the inverse inclusion can be proved.

From all that has been proved we obtain the requested result. Moreover we can prove the following proposition:

**X.- In** \((K, \oplus, \otimes)\) the following hold:

\[a) \ \forall \ x,y,z \in K : x \otimes (y \otimes z) = (x \otimes y) \otimes z; \]
\[b) \ \exists 1 \in K / \forall \ x \in K : x \times 1 = x; \]
\[c) \ \forall \ x,y,z \in K : x \otimes (y \otimes z) \leq (x \otimes y) \otimes (x \otimes z); \]
\[d) \ \forall \ x,y \in K : -(x \otimes y) = -(x) \otimes y = x \otimes (-y); \]
\[e) \ \forall \ x \in K, x \ different \ from \ zero, \ \exists \ y \in K, y \ different \ from \ zero, \ such \ that \ 1 \in x \otimes y. \]

**Proof.** \(\forall \ [(a,b)], [(c,d)], [(c,f)] \in K \) we have \([(a,b)]\oplus [(c,d)]\oplus [(c,f)]\in [(s,t) / s \leq ac, t \leq bd] \oplus [(c,f)]\in [(x,y)] \in x \leq sc, y \leq df\in [(x,y)] \in x \in (ac)c, y \leq (bd)f; \)

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of all we observe that, because of i), it is always different from zero since b and d are different from zero.

Let us now prove that Θ and ⊙ are well defined by proving the following two propositions.

VII.- If \((a',b')\in ((a,b)]\) and \((c',d')\in [(c,d)]\) then \([(ad+bc, bd)] = [(a'd'+bc', b'd')]\).

Proof. - In order to prove the requested result we must prove that for any \((s',t')\in [(a'd'+bc', b'd')]\) there exists \((s,t)\in [(ad+bc, bd)]\) such that \((s',t')\in F(s,t)\) and vice-versa. If \((a',b')\in ((a,b)]\) and \((c',d')\in [(c,d)]\) then \(a'b = b'a\) and \(c'd = d'c\) from which we can have \((a'd' + b'c')bd' = b'd'(ad'+bc);\) in fact, as for proposition V, we have \(a'bd' = b'add'\) and \(c'dbb' = d'cbb'\) from which the set \(a'bd' + c'dbb'\) intersects the set \(b'add' + d'cbb'\) by proposition III we have, as requested, \((a'd' + b'c')bd' = b'd'(ad'+bc).\) Now let \((s',t')\in H\) such that \(s'ea'd' + b'c'\) and \(t'eb'd';\) from \(a'd' + b'c')(bd' = b'd'(ad'+bc)\) and ii) there must exist \((s,t)\in H\) with \(s = ad + bc\) and \(t = bd\) with the condition \(s = t'.\) Thus \((s',t')\in F(s,t).\) Similarly it can be proved that for any \((s,t)\in [(ad+bc, bd)]\) there exists \((s',t')\in [(a'd'+bc', b'd')]\) such that \((s',t')\in F(s,t);\) thus Θ is well defined.

VIII.- If \((a',b')\in ((a,b)]\) and \((c',d')\in [(c,d)]\), then \([(ac, bd)] = [(a'c', b'd')]\).

Proof. - As in proposition VII we prove that for any \((s',t')\in [(a'c', b'd')]\) there exists \((s,t)\in [(ac, bd)]\) such that \((s',t')\in F(s,t)\) and vice-versa. If \((a',b')\in ((a,b)]\) and \((c',d')\in [(c,d)]\) then \(a'b = b'a\) and \(c'd = d'c\) that is \(a'bc = b'ad'c\) or \(a'c'bd = b'd'ac;\) from ii) the requested result follows.

At this point we can study the properties of hyperstructure \((K, Θ, ⊙).\)

IX.- \((K, Θ)\) is a canonical hypergroup.

Proof. For associativity we have \(((a,b)⊙ ((c,d)) ⊙ ((e,f))) = ((s,t) / s = ad + bc, t = bd) ⊙ ((e,f)) = ((z,w) / (z,w) = (s,t) ⊙ ((e,f)) = ((z,w) / z ∈ sef ⊕, w ∈ ef, se = ad + bc, t = bd) = ((z,w) / z ∈ (ad + bc) ⊕ (bd)c, w ∈ (bd)f),\) while \(((a,b)⊙((c,d))⊙((e,f))) = (a,b)⊙((u,v)) / u = ecf + de, v ∈ df = ((p,q) / (p,q) = (a,b)⊙((u,v)) = ((p,q) / p = av + bu, q ∈ bv, u = ecf + de, v ∈ df = ((p,q) / p = av(df) + b(cf + de), q = b(df)) and, from associativity and distributivity in A, the two sets are equal. Commutativity of ⊙ follows from commutativity of + and * in A as can be easily proved.
moreover \([a,b] \otimes ((c,d) \otimes (e,f)) = [(a,b) \otimes (c,d)] \otimes (e,f)\] for all \(a, b, c, d, e, f \in K\) with \(a \neq c, b \neq d, e \neq f\).

Thus associativity for \(\otimes\) follows from the analogous property in \((A,\cdot)\). Similarly commutativity in \((K,\otimes)\) follows from commutativity in \((A,\cdot)\).

As for the existence of a unity let us consider the element \([(x,x)] \in K, x \neq 0;\) obviously \([(x,x)] = [(y,y)] \forall y \in A^*\). For this element and for any other element \([(a,b)] \in K\) it results: \([(a,b)] \otimes [(x,x)] = [(s,t)] / s \in ax, t \in bx\}. At this point we observe that, since \(a)bx = \{b)ax,\) from ii) we have that, \(\forall s \in bx, t \in ax\) such that \(ast = bt\) and this implies that \([(a,b)] = [(a,b)] \otimes [(x,x)]\).

Let us now prove property g); \(\forall (a,b), (c,d), (e,f) \in K\) it results \([(a,b)] \otimes [(c,d)] \otimes [(e,f)] = [(a,b)] \otimes [(s,t)] / s \in ac, t \in df\} = \{(x,y) / x \in a(s,t), y \in b(df)\} = \{(x,y) / x \in ac \wedge d(a, b) = 0, t \in b(df)\} = \{(z,w) / z \in ac \wedge d(a, b) = 0, t \in b(df)\} = \{(z,w) / z = b(df) \wedge d(a, b) = 0, t \in b(df)\}. Since \([(x,y)] \otimes [(y,z)] = [(x,z)]\), the previous set contains \(\{(x,y) / x \in ac \wedge d(a, b) = 0, t \in b(df)\}\), that is:

\[
[(a,b)] \otimes [(c,d)] \otimes [(e,f)] = [(a,b)] \otimes [(c,d)] \otimes [(e,f)]
\]

Moreover, in order to prove h), we have \(-[(a,b)] \otimes [(c,d)] = -[(s,t)] / s \in ac, t \in bd\} = \{(s,t) / s \in a(-c), t \in bd\} = \{(s,t) / s \in a(-c), t \in bd\} = \{-[(a,b)] \otimes [(c,d)] = -[(a,b)] \otimes [(c,d)]\}.

Finally, \(\forall (a,b) \in K, a \neq 0,\) it results \([(a,b)] \otimes [(b,a)] = [(s,t)] / s \in ab, t \in ba\} \forall (s,t) \in X, x \neq 0,\) and this ends the proof.

We want now to prove, under particular hypothesis, that there exists a substructure of \(K\) which is weakly isomorphic to \(A\). To do this let us remember that an element \(1 \in A^*\) is called weak unity if and only if \(\forall x \in A\) it results \(1 \cdot x = x\) ([4]). Then, if \((A,+,\cdot)\) is a multiplicative, strongly distributive, commutative hyperring such that: i) if \(a \cdot b = 0, a \neq 0 \Rightarrow b = 0;\) ii) \(\forall X, Y, Z, W \in P^*(A) / XY \approx ZW \Rightarrow (\forall x \in X \wedge (\forall y \in Y \wedge z \in Z) / x \cdot y = z \cdot w);\) iii) \(\exists 1 \in A / \forall x \in A\) it results \(1 \cdot x = x\), it is possible to prove the following two results:

\[\text{XI.-The map } \varphi : A \longrightarrow K \text{ defined as } \varphi(a) = [(a,1)], \forall a \in A, \text{ is a weak monomorphism.}\]

\[\text{Proof. - Because of proposition II } \varphi \text{ is injective; in fact, if } [(a,1)] = [(b,1)], \text{ then } a \cdot 1 = b \cdot 1 \text{ and, from proposition II, this implies } a = b. \text{ Moreover } \varphi(a+b) = [(a+b,1)] \text{ while } \varphi(a) \otimes (b,1) = [(a,1)] \otimes [(1,1)] = [(a,1)] \otimes [(1,1)] = [(a,1)] \otimes [(1,1)] \text{ and, since } a+b = a \cdot 1 + b \cdot 1 = 1 \cdot 1 \text{, then } \varphi(a+b) = \varphi(a) \otimes (b,1).\]
Finally it results \( \varphi(a\ast b) = \{ \varphi(x) / x \in a\ast b \} = \{(x,1) / x \in a\ast b \} \subseteq \varphi(a) \otimes \varphi(b) = \{(a,1) \otimes [(b,1)] = \{(s,t) / s \in a\ast b, t \in 1\ast b \} \).

XII.- Each element of \( K \) belongs to a product \( x \otimes y \) where \( x \in \text{Im} \varphi \) and \( y \) is such that there exists \( y' \in \text{Im} \varphi : y \otimes y' \) contains \([z,z])\.

Proof. - For \([(a,b)] \in K\) it results \([(a,b)] \otimes [(1,b)] = \{(s,t) / s \in a\ast 1, t \in 1\ast b \} \).

As a consequence of what has been proved we will call \((K,\otimes,\otimes)\) the weak hyperfield of quotients for \((A,+,\ast)\).

REFERENCES


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