Ratio Mathematica Volume 48, 2023

Feebly r-clean ring and feebly $*$-r-clean ring

Saravanan Viswanathan*

Abstract

In this article, we introduce the concept of feebly r-clean ring and feebly $*$-r-clean ring. A ring R is defined to be feebly r-clean, if every element a can be written as $a = r + e - f$, where u is a regular and e, f are orthogonal idempotents and A $*$ ring R is defined to be feebly $*$-r clean if every element a can be written as $a = r + p - q$, where r is regular element and p, q are orthogonal projections. Further we generalise this concept of feebly r-clean ring to feebly $g(x) - r$-clean ring, where $g(x) \in C(R)[x]$ and $C(R)$ is the centre of ring R.
Keywords: r-clean rings, feebly r-clean ring, $*$-clean ring, feebly $*$-clean ring, $*$-r- feebly clean ring.
2020 AMS subject classifications: 13B30, 16N40, 16S60, 16U99. [1]

1 Introduction and preliminaries

Throughout this paper, all rings are assumed to be associative with identity. As defined by Nicholson [1977], an element a in a ring R is clean, if a can be written as sum of an idempotent and a unit. A ring R is clean ring, if every element is clean. Arora and Kundu [2017] defined, an element a of a ring R is called feebly clean if $a = u + e - f$ where u is a unit of R and e, f are orthogonal idempotents of R. A ring R is feebly clean ring if every element of R is feebly clean. Recall that, an element r of a ring R is a regular (Von Neumann), if there exists $y \in R$ such that $r = ryr$. E. Osba and Alkam [2004] defined, a ring R is Von Neumann local ring if for every $a \in R$, at least one of a or $1 - a$ has a Von Neumann inverse. Ashrafi and Nasibi [2013a] defined, an element a of a ring R is r-clean if $a = r + e$ where r is a regular element of R and e is an idempotent of R. A ring R is r-clean

*Department of Mathematics, Periyar University, Salem-636011, Tamilnadu, India; analysis-saravanan@gmail.com.
[1]Received on February 21, 2023. Accepted on June 26, 2023. Published on December 31, 2023. DOI: 10.23755/rm.v48i0.1144. ISSN: 1592-7415. eISSN: 2282-8214.

ring if every element of a ring R is r-clean. Motivated by these ideas we define an element a of a ring R as feebly r-clean if a can be written as $a = r + e - f$ where r is regular element of R and e, f are orthogonal idempotents of R. A ring R is feebly r-clean ring if every element of a ring R is feebly r-clean. For $\emptyset \neq S = \{0, 1\} \subseteq Id(R)$, R is S-feebly r-clean ring if each $a \in R$ can be written as $a = r + e - f$, where r is regular and e, f are orthogonal idemptents from S. In this Paper, we introduce the family of feebly r-clean rings and feebly $*$-r-clean rings. we prove that direct product of feebly r-clean ring is r-clean ring and discussed the properties of feebly r-clean rings with examples. Further we prove that, let $M =_B M_A$ be a bi-module and $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ is a formal triangular matrix is feebly r-clean, then A and B are feebly r-clean ring. We show that incase of an abelian ring this concept of feebly r-clean rings coincides with that of feebly clean rings and hence with that of exchange rings. Also we generalise these results to $g(x)$-r-clean rings. Further we discuss some interesting properties of feebly $g(x)$-r-clean ring, where $g(x)$ is in the center of a ring R.

Recall that, a ring $R[[x, \alpha]]$ is the ring of skew formal power series over R, which means, all formal power series in x with coeffients from R and α is an ring endomorphism. Multiplication is defined by $xr = \alpha(r)r$, for all $r \in R$. In particular $R[[x]] = R[[x; I_R]]$ is the ring of formal power series over R. we prove that, let R be an abelian ring with identity, if R is feebly r-clean, then $R[[x]]$ is feebly r-clean. Consider a ring $R[I, J]$ constructed in Tamer Kosan and Zhou [2016], for any subring J of a ring I, we prove the result, which gives the relation between $R[I, J]$ with I and J about feebly r-cleanness and prove that, if R is a feebly r-clean ring with no non-trivial idempotents, then the center of R is also a feebly r-clean ring.

Next, we introduce the concept of feebly $*$-clean ring and feebly $*$-r-clean ring. Recall that, A ring R is $*$-ring, if there exists an operation $*$:$R \rightarrow R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$, for all $x, y \in R$. An element p of a $*$-ring is projection if $p^2 = p = p^*$. Obviously 0 and 1 are projections of $*$-ring. We define an element a in a $*$-ring R is feebly $*$-clean element, if $a = u + p - q$ where $u \in U(R)$ and p, q are orthogonal projections of R. A ring $*$-ring R is feebly $*$-clean if every element of $*$-ring is feebly $*$-clean. Finally, we define an element a in a $*$-ring R is feebly $*$- r-clean element, if $a = r + p - q$ where $r \in Reg(R)$ and p, q are orthogonal projections of R. A ring $*$-ring R is feebly $*$-r-clean if every element of $*$-ring is feebly $*$-r-clean. Further properties of feebly $*$-r-clean rings are studied, some of them analogous to those for feebly $*$-clean rings.

For a ring R, the set of units, the set of idempotents, the set of regular elements, the set of projections and the centre of a ring are denoted by $U(R)$, $Id(R)$, $Reg(R)$, $P(R)$ and $C(R)$, respectively.

2 Feebly r-clean ring

definition 2.1. *An element x in a ring R is called feebly r-clean if there exist a regular element $r \in Reg(R)$ and orthogonal idempotents e, $f \in Id(R)$ such that $x = r + e - f$. A ring R is called feebly r-clean ring if every element of R is feebly r-clean. An element x of a ring R is called $S = \{0, 1\}$ feebly r- clean if $x = r + e - f$, where r is a regular and e, f are orthogonal idempotents from $S = \{0, 1\}$.*

Proposition 2.1. *Every homomorphic image of feebly r-clean ring is feebly r-clean.*

Theorem 2.1. *Let $\{R_i\}$ be a family of rings. Then the direct product R = $\prod R_i$ is feebly r-clean ring if and only if each R_i is feebly r-clean ring ring*

Proof. ($\Rightarrow$) This is immediate from the homomorphic image of a idempotent (resp., regular) is a idempotent (resp., regular).

($\Leftarrow$) Suppose each R_i is feebly r-clean. Let $a = (a_i) \in \prod R_i$. For each i, there exist $r_i \in Reg(R_i)$ and orthogonal idempotents $e_i, f_i \in Id(R_i)$ such that $a_i = r_i + e_i - f_i$. Then $a = r + e - f$, where $r = (r_i) \in Reg(\prod R_i)$ and $e = (e_i), f = (f_i)$ are orthogonal idmepotents of $\prod R_i$. Hence $\prod R_i$ is feebly r-clean ring. □

Lemma 2.1. *Let R be a ring with no zero divisor. Then R is feebly clean if and only if R is feebly r-clean.*

Proof. ($\Rightarrow$) Suppose R is a feebly clean ring. For $a \in R$, then there exist $u \in U(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $a = u + e - f$. Since $u \in Reg(R)$, hence R is feebly r-clean ring.

($\Leftarrow$) Let R be a feebly r-clean ring. For $a \in R$, there exist $r \in Reg(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $a = r + e - f$. Let $r(\neq 0) \in Reg(R)$, then there exists $y \in R$ such that $r = ryr$, which implies $r(1 - yr) = 0$, thus r is a unit. Therefore, R is a feebly clean ring. □

Lemma 2.2. *Let R be a ring and $e^2 = e \in R$. $a = u - f \in eRe$, where $u \in U(eRe)$, $f \in Id(eRe)$. Then there exist element $v \in U(R)$ and $\overline{f} \in Id(R)$ such that $a = v + e - \overline{f}$.*

Proof. Let $u \in U(eRe)$, then there exists $w \in U(eRe)$ such that $uw = e = wu$. Assume $v = u+(1-e)$. Since $v(1-e) = 0 = (1-e)w$, then $v(w+(1-e)) = e = w(1+(1-e))v$. Thus $a-v-e = u+e-f-u-(1-e)-e = -f-(1-e)$. Also $(f+(1-e))^2 = (f+(1-e)$. Hence $a = v+e-\overline{f}$, where $\overline{f} = (f+(1-e))$. □

Proposition 2.2. *If a is feebly clean of a ring R, then $-a$ is also feebly clean in R.*

Proposition 2.3. *Let R be an abelian ring and $a \in R$ is feebly clean element then ae is feebly clean for any idempotent $e \in R$.*

Proof. Let $a \in R$ be a feebly clean element. Then $a = u + e_1 - e_2$, where $u \in U(R)$ and e_1, e_2 are orthogonal of $Id(R)$. Since R is abelian, then $ue \in U(Re)$ and $e_1e, e_2e \in Id(Re)$ with $e_1ee_2e = e_2ee_1e = 0$. Hence $ae = ue + e_1e - e_2e$. Therefore, ae is feebly clean. □

Lemma 2.3. *Let R be an abelian ring such that a and $-a$ both are clean. Then the following hold*

1. *a and $-a$ are feebly clean element of R.*
2. *$a - e$ is feebly clean for any idempotent e of R.*

Proof. (1) Since a is clean element of R, then there exist unit $u \in U(R)$ and idempotent $e \in Id(R)$ such that $a = u + e$. Take $f = 0 \in Id(R)$, then $a = u + e - f$. The similar argument for $-a$, thus $-a = u^{'} + e^{'} - f^{'}$, as we desired.

(2) Let $a = u_1 + e_1$ and $-a = u_2 + e_2$, where $u_1, u_2 \in U(R)$ and $e_1, e_2 \in Id(R)$. Then

$$\begin{aligned} a - e &= (ae + a(1-e)) - e \\ &= ae + a - ae - e \\ &= (a-1)e + a(1-e) \\ &= -(1-a)e + a(1-e) \\ &= -(1 - e_1 - u_1)e + (-u_2 - e_2)(1-e) \\ &= -(1-e_1)e + u_1e - u_2(1-e) - e_2(1-e) \\ &= -\{(1-e_1)e + e_2(1-e)\} + \{u_1e - u_2(1-e)\}. \end{aligned}$$

Therefore, $a - e$ is feebly clean. □

Theorem 2.2. *Let R be an abelian ring. Then R is S-feebly clean if and only if R is S-feebly r-clean.*

Proof. ($\Rightarrow$) This is immediate from feebly clean rings are feebly r-clean.

($\Leftarrow$) Let R be a S-feebly r-clean ring and $x \in R$. So $x = r + e$ or $x = r - f$, where $r = ryr$ for some $y \in R$ and orthogonal idempotents $e, f \in Id(R)$. If $x = r + e$, Assume $yr = e'$, then $(e')^2 = (yr)^2 = yryr = yr = e'$, thus $e' \in Id(R)$. Also

$$\begin{aligned} (ye' + (1-e'))(re' + (1-e')) &= ye're' + ye'(1-e') + (1-e')re' + (1-e')^2 \\ &= ye're' + ye' - ye' + re' - re' + 1 - e' \\ &= yr \\ &= e' + 1 - e' \\ &= 1. \end{aligned}$$

Take $u = (re' + (1 - e'))$, then $u \in U(R)$ and $ue' = r$. Also, $ue' + f' \in U(R)$, where $(1 - e') = f'$. Hence $-r = -(ue' + f') + f'$. Since a and $-a$ are clean, then $a + e$ and $a - e$ are clean. Thus $r + e$ is clean. If $x = r - f$, By Lemma 2.3, $r - f$ is S-feebly clean. Therefore R is S-feebly clean. □

Example 2.1. *For p prime, $R = \mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\}$ is feebly r-clean ring.*

Example 2.2. *For distinct primes p, q, $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)} = \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b, q \nmid b\}$ is feebly r-clean.*

Example 2.3. *Some feebly r-clean rings are not r-clean. Consider $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \{\frac{m}{n} \in \mathbb{Q} \mid p \nmid n, q \nmid n\}$. Here $Id(\mathbb{Z}_{(5)}) = \{0, 1\}$. Take $a \in R$, then $a = \frac{x}{y}$ for some $x, y \in \mathbb{Z}$, where $y \neq 0$, 3 doesnot divides y, 5 doesnot divides y. If $\frac{x}{y}$ is a regular, then $a = \frac{x}{y} = \frac{x}{y} + 0 - 0$, thus a is feebly r clean. If $a = \frac{x}{y}$ is not regular, then $a = \frac{x}{y} - 1$ or $a = \frac{x}{y} + 1$ is regular. Hence $a = \frac{x}{y} - 1 + 1 - 0$ or $a = \frac{x}{y} - 1 + 0 - 1$. Therefore, a is feebly r-clean. But which is not r-clean, because, take $\frac{3}{8} \in R$, then $\frac{3}{8} = \frac{3}{8} + 0$ or $\frac{3}{8} = \frac{-5}{8} + 1$ in both cases $\frac{3}{8}$, $\frac{-5}{8}$ are not units in R.*

Example 2.4. *Let $R = (\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)})[i]$ is feebly r-clean only if $p, q \equiv 3(mod4)$, where p, q are odd primes.*

Theorem 2.3. *Let R be a ring with no non-trivial idempotents, then R is feebly r-clean if and only if R is Von-Neumann local.*

Proof. ($\Rightarrow$) Suppose R is feebly r-clean ring and 0 and 1 are the only idempotents in R. Then for any $x \in R$, either x or $x - 1$ is regular. Hence R is Von-Neumann local.

($\Leftarrow$) Since R is Von-Neumann local, for $a \in R$, then either a or $1 - a$ is regular. If a is regular, then $a = a + 0 - 0$, a is feebly r-clean. If $1 - a$ is regular, then $a - 1$ is also regular, write $a = a - 1 + 1 - 0$. Therefore, a is feebly r-clean. □

Corolary 2.1. *Let R be a feebly r-clean ring with no non-trivial idempotents. Then R is exchange.*

Proof. Since every regular element x of a ring is an exchange element, which means there exists idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Also, von-neumann local rings are exchange rings by Theorem 2.3, as we desired. □

Theorem 2.4. *If R is abelian ring then R is S-feebly r-clean ring if and only if R is an exchange ring*

Proof. ($\Rightarrow$) It is immediate from Corollary 2.1.

($\Leftarrow$) Suppose R is exchange ring. Let $a \in R$, then there exists idempotent $e \in R$ such that $1 - e \in (1 - a)R$. If $e = ax$ we assume $ex = x$ so that $xax = x$.

Since R is abelian ring, then all idempotents are central, then $xa = x(ax)a = xa(ax) = (xa)ax = a(xa)x = ax$. Also $1 - e = y(1 - a)$, where $(1 - e)y = y$ and $y(1 - a) = (1 - a)y$. Then

$$\begin{aligned} [a - (1 - e)](x - y) &= ax - (1 - e)x - ay + (1 - e)y \\ &= ax - 0 - ay + y \\ &= ax + (1 - a)y \\ &= ax + 1 - e \\ &= ax + 1 - ax \\ &= 1. \end{aligned}$$

Also, $(x - y)[a - (1 - e)] = 1$. Therefore, R is S-feebly r-clean ring. □

Proposition 2.4. *Let R be an abelian ring with identity. If R is feebly r-clean, then $R[[x]]$ is feebly r-clean.*

Proof. ($\Leftarrow$) Since $R = \frac{R[[x]]}{<x>}$, it is immediate from homomorphic image of a idempotent (resp., regular) is a idempotent (resp., regular).

($\Rightarrow$) Suppose R is feebly r-clean. Let $h = a_0 + a_1x + a_2x^2 + ... \in R[[x]]$. By assumption, let $a_0 = r + e - f \in R$, then $r' = r + a_1x + a_2x^2 + ...$ is regular in $R[[x]]$. Hence $h = u' + e - f$ is feebly r-clean in $R[[x]]$

□

Corolary 2.2. *Let R be an abelain ring and α be an endomorphism of R. Then the following are equivalent*

1. *R is feebly r-clean ring.*
2. *The formal power series ring $R[[x]]$ over R is feebly r-clean.*
3. *The skew power series ring $R[[x; \alpha]]$ over R is feebly r-clean.*

Proposition 2.5. *Let J be a subring of a ring I then $R[I, J]$ is feebly r-clean if and only if I is r-clean and J is feebly r-clean ring.*

Proof. ($\Rightarrow$) Since $R[I, J]$ is feebly r-clean, then $R[I, J] = I \otimes I$ by Theorem 2.1, I is r-clean, Clearly J is feebly r-clean ring by Proposition 2.1.

($\Leftarrow$) Assume I is r-clean and J is feebly r-clean. Let $x = (i_1, i_2, ..., i_n, j_1, J_2, ...) \in R[I, J]$. Since I is feebly r-clean ring, then $j = r + e - f$, where $r \in Reg(R)$ and orthogonal idempotents $e, f \in Id(R)$. If $j = r + e - f$, write $j_i = r_i + e_i - f_i$. Then $x = \overline{r} + \overline{e} - \overline{f}$ where $\overline{r} = (r_1, r_2, ...r_n, e, e, ...)$ is regular element of $R[I, J]$ and $\overline{e} = (e_1, e_2, ...e_n, e, e, ...)$, $\overline{f} = (f_1, f_2, ...f_n, e, e, ...)$ are orthogonal idempotents of $R[I, J]$. □

Note that every abelian semi regular ring is feebly clean and hence feebly r-clean. However the converse is not true as shown by the following example.

Example 2.5. *Let $\mathbb{Q}$ be a field of rational number and $\mathbb{Q}'$ the ring of all rational number with odd denominators. Then by [Ashrafi and Nasibi [2013b], Example 2.7] and Theorem 2.2, $R[\mathbb{Q}, \mathbb{Q}']$ commutative exchange ring and hence $R[\mathbb{Q}, \mathbb{Q}']$ is feebly r-clean ring but not semi regular.*

If R is feebly r-clean ring then R/I is feebly r-clean being homomorphic images of feebly r-clean ring. The theorem is partial converse of this statement.

Theorem 2.5. *If I is a regular ideal of ring R with idempotent can be lifted modulo I. Then R is S-feebly r-clean if and only if R/I is S-feebly r-clean.*

Proof. The proof of theorem is follows by [Ashrafi and Nasibi [2013b], Theorem 2.8]. It is enough to prove that for any $a \in R$ if $a+I$ has $a = r - e$ feebly r-clean decomposition in R/I then a is feebly r-clean in R. Let $a+I \in R/I$ has $a = r-e$ feebly r-clean decomposition in R/I, then there exists idempotent $e + I \in R/I$ such that $(a + I) + (e + I) \in Reg(R/I)$, which shows $(a + e) + I \in Reg(R/I)$. Therefore, $((a+e)+I)(x+I)((a+e)+I) = (a+e)+I$ for some $x \in R$, thus $(a + e)x(a + e) - (a + e) \in I$ but I is regular ideal. Hence $(a + e)$ is regular by [Brown and McCoy [1949], Lemma 1]. Since idempotent can be lifted modulo I, Take e is an idempotent of R, as required □

Proposition 2.6. *Let $M =_B M_A$ be a bimodule. If $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ a formal triangular matrix ring is feebly r-clean then A and B are feebly r-clean ring.*

Proof. Let $a \in A$, $b \in B$ and $m \in M$. Take $t = \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \in T$, Then $t = \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ e_2 & e_3 \end{pmatrix} - \begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix}$, where $\begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix} \in Reg(T)$ and $\begin{pmatrix} e_1 & 0 \\ e_2 & e_3 \end{pmatrix}$, $\begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix}$ are orthogonal idempotents of T. Clearly, e_1, f_1 are orthogonal idempotents in A and e_3, f_3 are orthogonal idempotents in B respectively. Also r_1, r_2 regular element in A and B respectively. Then $a = r_1 + e_1 - f_1$ and $b = r_3 + e_3 - f_3$. Hence A and B are both feebly r-clean rings. □

Theorem 2.6. *Let R be a feebly r-clean ring with no non-trivial idempotents. Then the center of R is also a feebly r-clean ring.*

Proof. Let $a \in C(R)$, then there exist regular element $r \in R$ such that either $x = r$ or $x = r + 1$ or $x = r - 1$. If $x = r$, then $r \in Reg(R)$, we write $x = r + 0 - 0$ is feebly r-clean in $C(R)$. If $x = r + 1$, then $x - 1 = r$ is regular and $x-1 \in C(R)$, hence $x = x-1+1-0$ is feebly r-clean in $C(R)$. If $x = r-1$, then $x+1 = r$, Also $x+1 \in C(R)$, we write $x = x+1+0-1$ is feebly r-clean in $C(R)$. Therefore, R is feebly r-clean. □

3 Feebly $g(x)$-r-clean ring

definition 3.1. *Let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $x \in R$ is called feebly $g(x)$-r-clean if $x = r+e-f$, where $r \in Reg(R)$ and $g(e) = g(f) = 0$. We say that R is feebly $g(x)$-r-clean if every element is feebly $g(x)$-r-clean.*

If $g(x) = x^2-x$, then feebly $g(x)$-r-clean ring is similar to feebly r-clean ring. An element s of R is called root of the polynomial $g(x) \in C(R)[x]$ if $g(s) = 0$. For $g(x) = x^2 - x$, the ring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is feebly $g(x)$-r-clean ring.

Let R and S be a rings and $\theta : C(R) \to C(S)$ be a ring homomorphism with $\theta(1) = 1$, then θ induces a map θ' from $C(R)[x]$ to $C(S)[x]$ such that for $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, $\theta'(g(x)) := \sum_{i=0}^{n} \theta(a_i)x^i \in C(S)[x]$.

Proposition 3.1. *Let $\theta : R \to S$ be a ring epimorphism. If R is feebly $g(x)$-r-clean ring then S is feebly $\theta'(g(x))$-r-clean ring.*

Proof. Let $g(x) = a_o + a_1 + ... + a_n x^n \in C(R)[x]$, then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + ... + \theta(a_n)x^n \in C(S)[x]$. Since θ is a ring epimorphism so for any $s \in S$, there exists $x \in R$ such that $\theta(x) = s$. Let $x = r + s_0 - t_0$, where $r \in Reg(R)$ and $g(s_0) = g(t_0) = 0$, as R is feebly $g(x)$-r-clean ring. Now $s = \theta(x) = \theta(r + s_0 - t_0) = \theta(r) + \theta(s_0) - \theta(t_0)$. Clearly $\theta(r) \in Reg(S)$ and

$$\begin{aligned} \theta'(g(\theta(s_0))) &= \theta(a_0) + \theta(a_1)\theta(s_0) + ... + \theta(a_n) + (\theta(s_0))^n \\ &= \theta(a_0 + a_1 s_0 + ... + a_n s_0^n) \\ &= \theta(0) \\ &= 0 \end{aligned}$$

Hence S is feebly $\theta'(g(x))$-r-clean ring. □

Theorem 3.1. *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in 1} R_i$ is feebly $g(x)$-r-clean ring if and only if R_i's are feebly $g(x)$-r-clean ring*

Proof. ($\Rightarrow$) It is immediate from Theorem 2.1.

($\Leftarrow$) Suppose R_i is feebly $g(x)$-r-clean. Let $a = (a_i) \in \prod R_i$. For each i, $a_i = r_i + e_i - f_i$, where $r_i \in Reg(R_i)$ and e_i, f_i are orthogonal idempotent of $Id(R_i)$. Hence $a = r+e-f$, where $r = r_i \in (\prod R_i)$ and $e = e_i, f = f_i \in (\prod R_i)$ with $ef = fe = 0$. Therefore, $\prod R_i$ is feebly $g(x)$-r-clean ring. □

Theorem 3.2. *Let R be a ring and $s, t \in R$, Then R is feebly $(sx^{2n} - tx)$-r-clean ring if and only if R is feebly $(sx^{2n} + tx)$-r-clean ring for $n \in \mathbb{N}$*

Proof. Suppose R is feebly $(sx^{2n} - tx)$-r-clean ring. For each $a \in R$, then $a = r + e - f$ where $r \in Reg(R)$ and e, f are orthogonal idempotent of R with $(se^{2n} - te) = (sx^{2n} - tx) = 0$. Since $a \in R$, $-a = r + e - f$, then $a = (-r) - e + f$ with $s(-e)^{2n} - t(-e) = (se^{2n} + ex) = 0$. Hence R is feebly $(sx^{2n} + tx)$-r-clean. The similar argument work for the converse part. □

Proposition 3.2. *Let R be an abelian ring and $g(x) \in C(R)[x]$. Then if a is $g(x)$-clean element in R, then ae^{n-1} is also $g(x)$-clean element in R for any root e in $g(x)$.*

Proof. Suppose a is $g(x)$-r-clean in R. For $a \in R$, then there exist $r \in Reg(R)$ and orthogonal idempotents $e_1, e_2 \in Id(R)$ such that $a = r + e_1 - e_2$ with $g(e_1) = g(e_2) = 0$. Take any e be any root of $g(x)$ in R, then $ae^{n-1} = re^{n-1} + e_1e^{n-1} - e_2e^{n-1}$, hence $re^{n-1} \in Reg(e^{n-1}Re^{n-1})$. Also $(e_1e^{n-1})^n = e_1^ne^{n-1} = e_1e^{n-1}$ and $(e_2e^{n-1})^n = e_2^ne^{n-1} = e_2e^{n-1}$ with $e_1e^{n-1} = e^{n-1}e_2 = 0$. Therefore, ae^{n-1} is $g(x)$-r-clean in R.

□

4 Feebly $*$-clean rings and feebly $*$-r-clean ring

definition 4.1. *An element x of a $*$-ring R is called feebly $*$-clean if $x = u + p - q$ where $u \in U(R)$ and p, q are orthogonal projection of R. A $*$-ring R is called feebly $*$-clean ring if every element of R is feebly $*$-clean*

Proposition 4.1. *Homomorphic image of feebly $*$-clean ring is feebly $*$-clean.*

Lemma 4.1. *Let R ba a boolean $*$-ring. Then R is S-feebly $*$-clean if and only if $* = 1_R$ is the identity map of R.*

Proof. ($\Rightarrow$) Note that boolean rings are feebly clean. Assume that R is S-feebly-$*$-clean, for $a \in R$, then there exist $u \in U(R)$ and orthogonal projection $p, q \in P(R)$ such that $a = u + p - q$. By Assumption, $a = u + p$ or $a = u - q$, hence $a = 1 + p$ or $a = 1 - q$. Also $1 + p, 1 - q \in P(R)$. Thus $a^* = a$, which shows $* = 1_R$.

($\Leftarrow$) Suppose $* = 1_R$ is the identity map of R. Then every idempotent of R is a projection, as desired. □

Example 4.1. *Some feebly clean rings are not feebly $*$-clean. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with involution $*$ defined by $(x, y)^* = (y, x)$.*

Since $Id(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \{(0,0),(0,1),(1,0),(1,1)\}$ and $U(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \{(1,1)\}$, hence every element of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ can be written as sum of units and difference of orthogonal idempotents. Therefore, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is feebly clean but not feebly $*$-clean, because $(0,1)$ is cannot be written as sum units and difference of orthogonal projections.

Corolary 4.1. *Let R be a $*$-ring. If $2 \in U(R)$, the following are equivalent*

1. R is S-feebly clean and every unit of R is self-adjoint.
2. R is S-feebly $*$-clean and $* = 1_R$.

Proof. (1) $\Rightarrow$ (2) Suppose R is S-feebly clean and every unit of R is self-adjoint. Let $a \in R$, then there exist unit $u \in U(R)$ and orthogonal projection $p, q \in P(R)$ sucht that $a = u + p$ or $a = u - q$. Since $1 - 2p, 1 - 2q \in U(R)$, then $(1-2p)^* = (1-2p)$ which implies $p^* = p$ and $(1-2q)^* = (1-2q)$, also $q^* = q$. Note that R be a $*$-ring, If $2 \in U(R)$, then for any $u^2 = 1$, $u^* = u \in R$ if and only if all idempotents are projection. Hence $a^* = (u+p)^* = u + p = a$ and $a^* = (u-q)^* = u - q = a$, as required.

(2) $\Rightarrow$ (1) It is immediate from every S-feebly $*$-clean ring is S-feebly clean. □

For a $*$-ring R, an element $x \in R$ is called self-adjoint square root of 1 if $x^2 = 1$ and $x^* = x$. Also $*$ induces an involution in the power series ring $R[[x]]$ defined by $(\sum_i^\infty a_i x^i)^* = \sum_i^\infty a_i^* x^i$.

Theorem 4.1. *Let R be a $*$-ring, the following are equivalent*

1. *R is S-feebly $*$-clean and $2 \in U(R)$.*
2. *Every element of R is a sum of unit and a self adjoint square root of 1 or an element of the form $2p+1$, where $p^2 = p = p^*$.*

Proof. (1) $\Rightarrow$ (2) Suppose R is S-feebly clean and $2 \in U(R)$. Let $a \in R$, then there exist $u \in U(R)$ and orthogonal projection $p, q \in P(R)$ such that $\frac{a-1}{2} = u+p$ or $\frac{a-1}{2} = u - q$. If $\frac{a-1}{2} = u + p$, then $a = 2u + (1+2p)$, where $p^2 = p = p^*$. If $\frac{a-1}{2} = u - q$, then $a = 2u + (1-2q)$, where $2u \in U(R)$ and $1 - 2q$ is a self-adjoint square root of 1.

(2) $\Rightarrow$ (1) Let $a \in R$, then $a = u + x$ or $a = u + (2p+1)$ where $u \in U(R)$, $p \in P(R)$ and x is self-adjoint square root of 1. Thus $1 = u + x$ or $1 = u + (2p+1)$. If $1 = u + x$ then $(1-u)^2 = x^2 = 1$, which implies $u^2 = 2u$, hence $2 \in U(R)$ by Wang and Cui [2016]. If $1 = u + (2p+1)$, then $u = -2p$, also $2 \in U(R)$. Next we prove that R is feebly $*$-clean ring. For $a \in R$, $2a + 1 = u + x$ or

$2a+1 = u+(2p+1)$ where $u \in U(R)$, x is self-adjoint square root of 1 and $p \in P(R)$. If $2a+1 = u+x$, then $a = \frac{u}{2} - \frac{1-x}{2}$. Also $(\frac{1-x}{2})^2 = \frac{1-x}{2} = (\frac{1-x}{2})^*$ and $2 \in U(R)$, a is feebly $*$ feebly clean. If $2a+1 = u+(2p+1)$, then $a = \frac{u}{2}+p$, a is feebly $*$-clean. □

Lemma 4.2. *Let R be a feebly $*$-clean ring. If I is a $*$-invariant ideal of R Then R/I is feebly $*$-clean. In particular, $R/J(R)$ is feebly $*$-clean ring.*

Proof. Since the homomorphic image of a projection (resp., unit) is a projection (resp., unit), hence the result. Next we prove that $J(R)$ is invariant. Take $x^* \in (J(R))^*$, then $x^* \in J(R)$. Also $x \in J(R)$. For any $a \in R$, then $1 - a^*x \in U(R)$. Hence $1 - a^*x = (1 - a^*x)^*$ is unit of R, as we desired. □

Proposition 4.2. *Let R be a $*$-ring. Then $R[[x]]$ is S-feebly $*$-clean if and only if R is S-feebly $*$-clean.*

Proof. ($\Rightarrow$) Suppose $R[[x]]$ is feebly clean, then as a homomorphic copy of $R[[x]]$, R is feebly $*$-clean.

($\Leftarrow$) Let R be a feebly$*$-clean. Take $f(x) = \sum a_i x^i \in R[[x]]$. For $a_0 \in R$, there exist $r_0 \in Reg(R)$ and orthogonal projection $p_0, q_0 \in R$ such that $a_0 = r_0 + p_0 - q_0$. Then $f(x) = \sum a_i x^i = p_0 - q_0 + r_0 + a_0 + a_1 x + a_2 x^2 + ..$ where $r_0 + a_1 x + a_2 x^2 + ... \in Reg(R[[x]])$ and $p_0, q_0 \in P(R) \subseteq P(R[[x]])$. Therefore, $R[[x]]$ is feebly $*$-clean.

□

Theorem 4.2. *Let $\{R_\alpha\}$ be a colection of $*$-rings. Then the direct product $R = \prod_\alpha R_\alpha$ is feebly $*$-clean if and only if each R_α is feebly $*$-clean ring.*

Proof. Similar to the proof of Theorem 2.1. □

definition 4.2. *An element x in a ring $*$-ring R is called feebly $*$-r-clean if $x = r + p - q$, where $r \in Reg(R)$ and orthogonal projection p, $q \in P(R)$. A $*$-ring R is called feebly $*$-r-clean ring if every element of R is feebly $*$-r-clean.*

Proposition 4.3. *Let R be an abelian $*$-ring. If a is feebly $*$-r-clean element in R. Then ae is feebly $*$-r-clean, for any $e \in P(R)$*

Proof. Let $a = u + p - q$, where $r \in Reg(R)$ and orthogonal projection p, $q \in P(R)$. For $e \in P(R)$, $ae = ue + pe - qe$. Clearly $ue \in U(eRe)$ and pe, qe are orthogonal projection of $P(eRe)$. Hence ae is feebly $*$-clean. □

Proposition 4.4. *Let R ba an abelian $*$-ring and a be a S-feebly $*$-clean element in R and $e \in P(R)$. If $-a$ is S-feebly $*$-clean Then $a + e$ is also S-feebly $*$-clean.*

Proof. Since a and $a+1$ are S-feebly $*$-clean, then $1-a$ is also S-feebly $*$-clean. For a is S-feebly $*$-clean, then $a = u + p$ or $a = u - q$ where $u \in U(R)$ and orthogonal projection $p, q \in P(R)$. If $a = u + p$, then $1 + a = v + q$ where $v \in U(R)$. Now $a+e = (1+a)e+a(1-e) = (ve+u(1-e))+(ge+f(1-e))$. Thus $ve + u(1 - e) \in U(R)$ and $ge + f(1 - e) \in P(R)$, as we required. The similar way will work for $a = u - q$. Hence the result. □

Lemma 4.3. *Let R be an abelian $*$-ring where every idempotent of the form rx or xr is projection for any regular element r, then r is S-feebly $*$-clean.*

Proof. Let $r \in Reg(R)$ then $r = rxr$, for some $x \in R$. So $p = xr \in P(R)$. Take $e = xr$, then

$$\begin{aligned} e &= xr \\ &= xr + r - r \\ &= xr + r - rxr \\ &= xr + rxr - rxrxr \\ &= xr + rp - rpp \\ &= xr + (1 - p)rxr \\ &= (x + (1 - p)rx)r \\ &= ar, \end{aligned}$$

where $a = y + (1 - p)rx$. Here $e^2 = e \in P(R)$ and $(1 - e) = (1 - p)(1 - r) = b(1 - r) \in R(1 - r)$, here $y = (1 - p)$. Assume $ea = a$, then $ara = a$. By R is an abelian $*$-ring, $ra = r(ar)a = ra(ra) = (ra)ra = a(ra)r = ar$ we write $1 - e = y(1 - r)$, where $(1 - e)y = y$ and $y(1 - r) = (1 - r)y$. Then $(1 - r)$ is the inverse of $x - (1 - e)$. as we required. □

Theorem 4.3. *Let R be an abelian $*$-clean, where any idempotent of the form $e = rx$ or xr is projection, for any $r \in Reg(R)$. Then R is S-feebly $*$-r-clean if and only if R is feebly $*$-clean.*

Proof. ($\Leftarrow$) This is immediate from S-feebly $*$-clean rings are S-feebly-$*$-r-clean

($\Rightarrow$) Suppose R is S-feebly $*$-r-clean. Let $a \in R$, then there exist $r \in Reg(R)$ and orthogonal projection $p, q \in P(R)$ such that $a = r + p$ or $a = r - q$. For $r \in Reg(R)$, $r = rbr$ for some $b \in R$. Take $e = rb$, then $(re + (1 - e))(be + (1 - e)) = 1 = (be + (1 - e)(re + (1 - e)) = 1$ and $e \in P(R)$. Thus $u = re + (1 - e)$ is a unit and $r = eu$. Set $p' = 1 - e$, then $-(eu + p')$ is a unit and $p' \in P(R)$. Hence $-r = -(ue + p') + p'$ is S-feebly-$*$-clean by Lemma 4.3, r is S-feebly-$*$-clean. By Proposition 4.4, $x = r + p'$ is S-feebly $*$-clean. The similar argument will work for $x = r - p$. Hence the result. □

5 Conclusions

In this article, feebly $*$-r-clean rings are further discussed with an emphasis on their relations with feebly-r-clean rings and we investigate the properties of them with examples. The future scope of this study is to investigate the ideals of a feebly-r-clean rings and feebly $*$-r-clean rings.

Acknowledgements

The author was partially supported by Periyar University Research Fellowship (Letter No: PU/AD-3/URF/025430/2018), Salem-636011, Tamilnadu, India; analysissaravanan@gmail.com.

References

N. Arora and S. Kundu. Commutative feebly clean rings. *Journal of Algebra and Its Applictions*, 16(5), 2017.

N. Ashrafi and E. Nasibi. On r-clean rings. *Mathematical Report*, 15(2):125–132, 2013a.

N. Ashrafi and E. Nasibi. Rings in which elements are the sum of idempotent and a regular element. *Buttet. Iranian. Math. Soc*, 39(3):579–588, 2013b.

B. Brown and N. McCoy. The maximal regular ideal of a ring. *Proc. Amer. Math. Soc*, 1:165–171, 1949.

M. H. E. Osba and O. Alkam. Combining local and von neumann regular rings. *Comm. in Algebra*, 32(12):2639–2653, 2004.

W. K. Nicholson. Lifting idempotents and exchange rings. *Trans. Amer. Math. Soc*, 229:269–278, 1977.

S. S. Tamer Kosan and Y. Zhou. On weakly clean rings. *Comm. in Algebra*, 2016.

Z. Wang and J. Cui. A note on $*$-clean rings. *Dept. of Math, Southern. Univ.,*, 2016.